# Strength of Materials
# (20A01201T)

## LECTURE NOTES

## I-B.TECH & II-SEM

### Prepared by:

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## VEMU INSTITUTE OF TECHNOLOGY

**(Approved By AICTE, New Delhi and Affiliated to JNTUA, Ananthapuramu)**

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**JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR**

**B.Tech (CE)– II Sem**

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## (20A01201T) STRENGTH OF MATERIALS

**Course Objectives:**

- To make the student understand how to resolve forces and moments in a given system
- To demonstrate the student to determine the centroid and second moment of area
- To impart procedure for drawing shear force and bending moment diagrams for beams.
- To make the student able to analyze flexural stresses in beams due to different loads.
- To enable the student to apply the concepts of strength of materials in engineering applications and design problems.

**Course Outcomes (CO):**

- Understand the basic concepts of forces, Draw Free body Diagrams for forces and determine the centroid and moment of inertia for different cross section areas
- Understand concepts of stresses, strains, elastic moduli and strain energy and Evaluate relations between different moduli
- Draw the shear force and bending moment diagrams for cantilevers, simply supported beams and Overhanging beams with different loads and understand the relationship between shear force and bending moments
- Compute the flexural stresses for different cross sections and Design beam sections for flexure
- Determine shear stresses for different shapes and analyze trusses

**UNIT-I: Introduction to Mechanics:**

Basic Concepts, system of Forces Coplanar Concurrent Forces -Components in Space Resultant - Moment of Forces and its Application - Couples and Resultant of Force Systems. Equilibrium of system of Forces: Free body diagrams, Equations of Equilibrium of Coplanar Systems and Spatial systems-

Center of Gravity and moment of inertia: Introduction – Centroids of rectangular, circular, I, L and T sections - Centroids of built up sections.

Area moment of Inertia: Introduction – Definition of Moment of Inertia of rectangular, circular, I, L and T sections - Radius of gyration. Moments of Inertia of Composite sections.

**UNIT - II :Simple Stresses and Strains:**

Types of stresses and strains – Hooke‘s law – Stress – strain diagram for mild steel – working stress – Factor of safety – lateral strain, Poisson‘s ratio and volumetric strain – Elastic moduli and the relationship between them – Bars of Varying section – Composite bars – Temperature stresses. Strain energy – Resilience – Gradual, Sudden, impact and shock loadings – simple applications.

**UNIT – III Shear Force and Bending Moment:**

Definition of beam – types of beams – Concept of Shear force and bending moment – S.F and B.M diagrams for cantilever, simply supported and over changing beams subjected to point loads, uniformly distributed load, uniformly varying loads and combination of these loads – point of contra flexure – Relation between S.F, B.M and rate of loading at section of a beam.

**UNIT - IV Flexural Stresses:**

Theory of simple bending – Assumptions – Derivation of bending equation: M/I = f/Y = E/R – Neutral axis – Determination of bending stresses – Section modulus of rectangular and circular sections (Solid and Hallow), I, T, Angle and Channel Sections – Design of simple beam sections.

**UNIT - V: Shear Stresses:**

Derivation of formula-Shear stress distribution across various beam sections like rectangular, circular, triangular, I, T and angle sections. Combined bending and shear.

Analysis of trusses by Method of Joints & Sections.

**Textbooks:**

1. S. Timoshenko, D.H. Young and J.V. Rao, "Engineering Mechanics", Tata McGraw-HillCompany.
2. Sadhu Singh, "Strength of Materials", 11th edition 2015, Khanna Publishers.

**Reference Books:**

1. S.S.Bhavikatti, "Strength of materials", Vikas publishing house Pvt. Ltd.
2. R. Subramanian, "Strength of Materials", Oxford University Press.
3. R. K. Bansal, "Strength of Materials", Lakshmi Publications House Pvt. Ltd.
4. Advanced Mechanics of Materials – Seely F.B and Smith J.O. John wiley & Sons inc.,New York.

## Mechanics

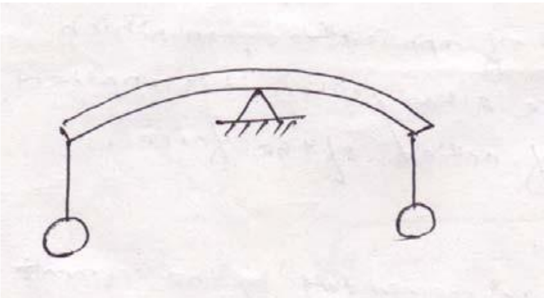
It is defined as that branch of science, which describes and predicts the conditions of rest or motion of bodies under the action of forces. Engineering mechanics applies the principle of mechanics to design, taking into account the effects of forces.

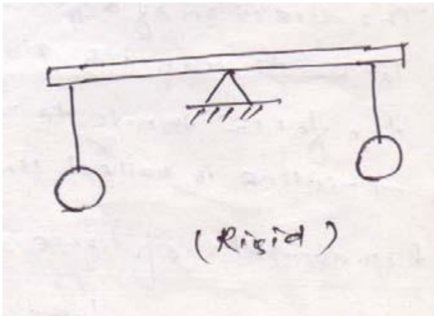
## Statics

Statics deal with the condition of equilibrium of bodies acted upon by forces.

## Rigid body

A rigid body is defined as a definite quantity of matter, the parts of which are fixed in position relative to each other. Physical bodies are never absolutely but deform slightly under the action of loads. If the deformation is negligible as compared to its size, the body is termed as rigid.



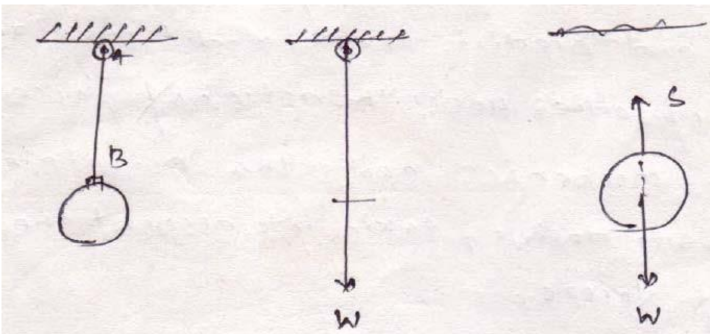



## Force

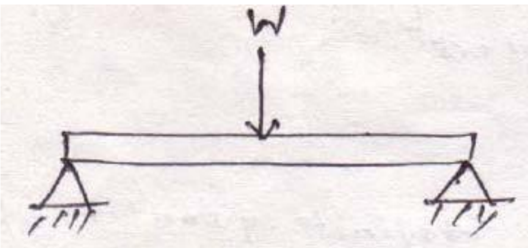
Force may be defined as any action that tends to change the state of rest or motion of a body to which it is applied.

The three quantities required to completely define force are called its specification or characteristics. So the characteristics of a force are:

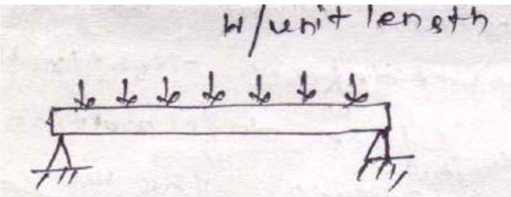
1. Magnitude
2. Point of application
3. Direction of application



**Concentrated force/point load**




**Distributed force**




**Line of action of force**

The direction of a force is the direction, along a straight line through its point of application in which the force tends to move a body when it is applied. This line is called line of action of force.

**Representation of force**

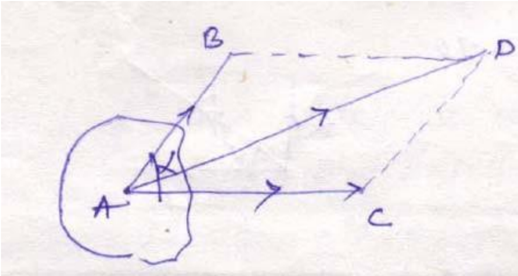
Graphically a force may be represented by the segment of a straight line.

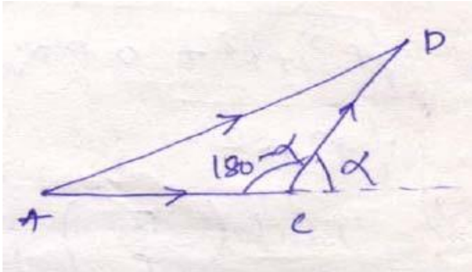

**Composition of two forces**

The reduction of a given system of forces to the simplest system that will be its equivalent is called the problem of composition of forces.

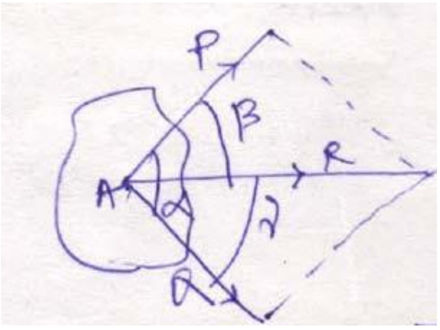
**Parallelogram law**

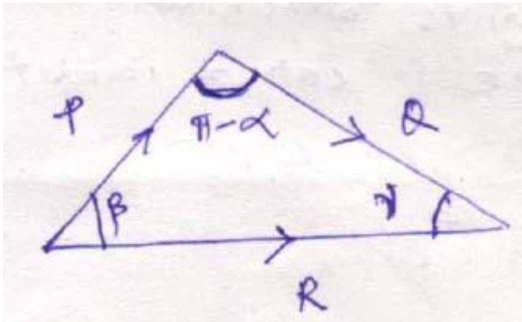
If two forces represented by vectors AB and AC acting under an angle α are applied to a body at point A. Their action is equivalent to the action of one force, represented by vector AD, obtained as the diagonal of the parallelogram constructed on the vectors AB and AC directed as shown in the figure.




Force AD is called the resultant of AB and AC and the forces are called its components.




$$R = \sqrt{\left(P^2 + Q^2 + 2PQ \times Cos\alpha\right)}$$

Now applying triangle law

$$\frac{P}{Sin\gamma} = \frac{Q}{Sin\beta} = \frac{R}{Sin(\pi - \alpha)}$$

**Special cases**

Case-I: If $\alpha = 0^\circ$

$$R = \sqrt{\left(P^2 + Q^2 + 2PQ \times Cos0^\circ\right)} = \sqrt{(P+Q)^2} = P + Q$$

P Q R

R = P+Q

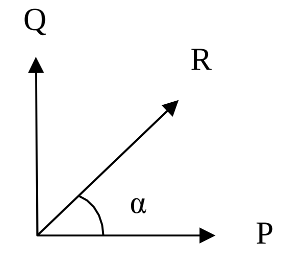
Case- II: If $\alpha = 180^\circ$

$$R = \sqrt{\left(P^2 + Q^2 + 2PQ \times Cos180^\circ\right)} = \sqrt{(P^2 + Q^2 - 2PQ)} = \sqrt{(P-Q)^2} = P - Q$$

Q P R

Case-III: If α = 90˚

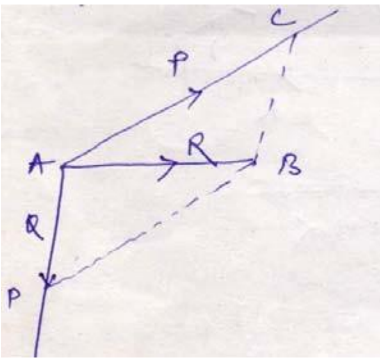
$$R = \sqrt{\left(P^2 + Q^2 + 2PQ \times Cos90^\circ\right)} = \sqrt{P^2 + Q^2}$$




α = tan$^{-1}$ (Q/P)

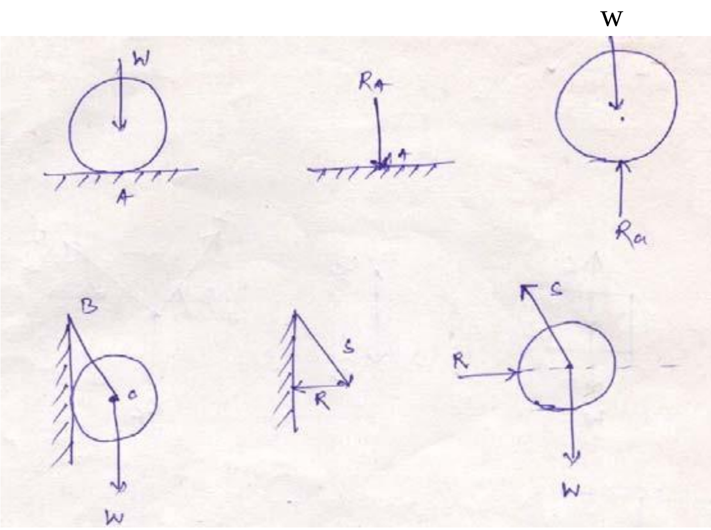
**Resolution of a force**

The replacement of a single force by a several components which will be equivalent in action to the given force is called resolution of a force.




**Action and reaction**

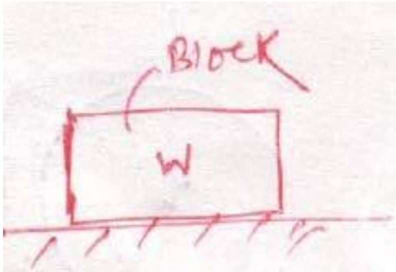
Often bodies in equilibrium are constrained to investigate the conditions.

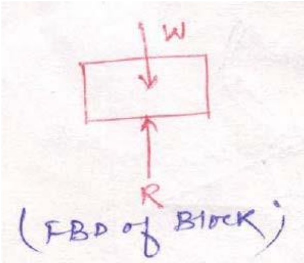


**Free body diagram**

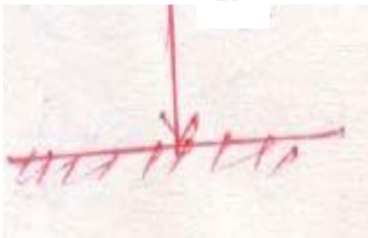
Free body diagram is necessary to investigate the condition of equilibrium of a body or system. While drawing the free body diagram all the supports of the body are removed and replaced with the reaction forces acting on it.

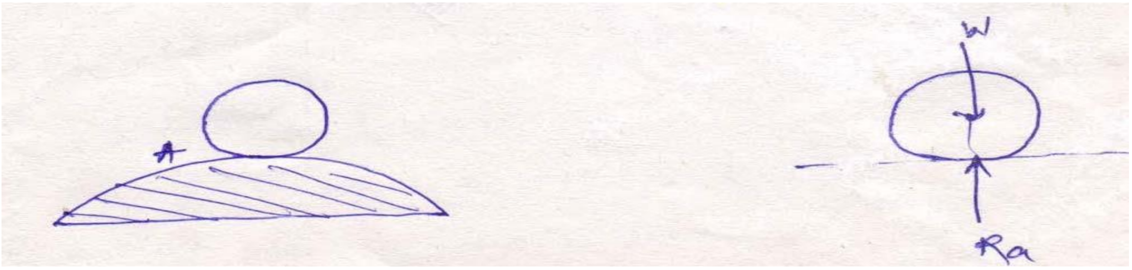
1. Draw the free body diagrams of the following figures.



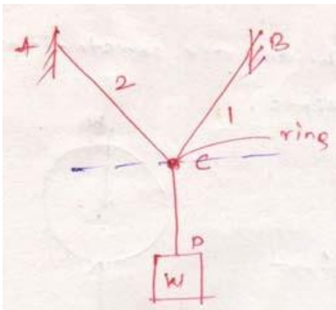



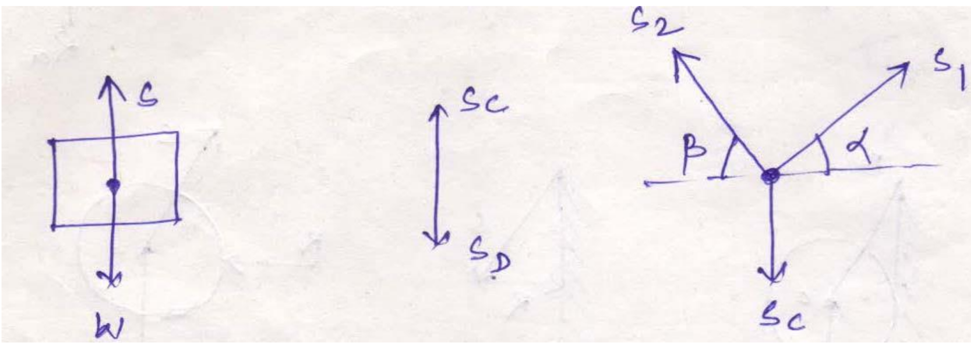

R



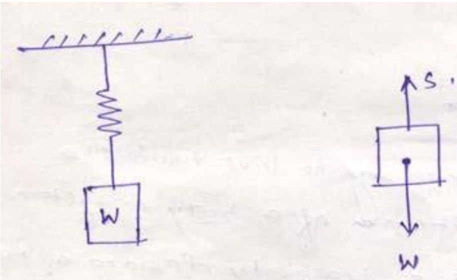


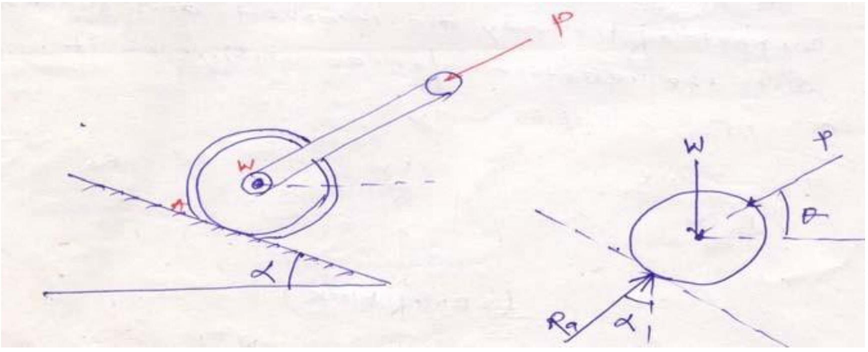

2. Draw the free body diagram of the body, the string CD and the ring.



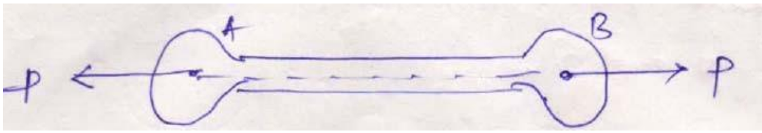
3. Draw the free body diagram of the following figures.



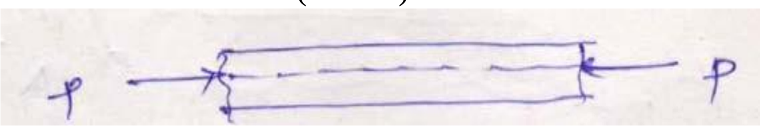




## Equilibrium of colinear forces:

**Equllibrium law:** Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction and collinear in action.



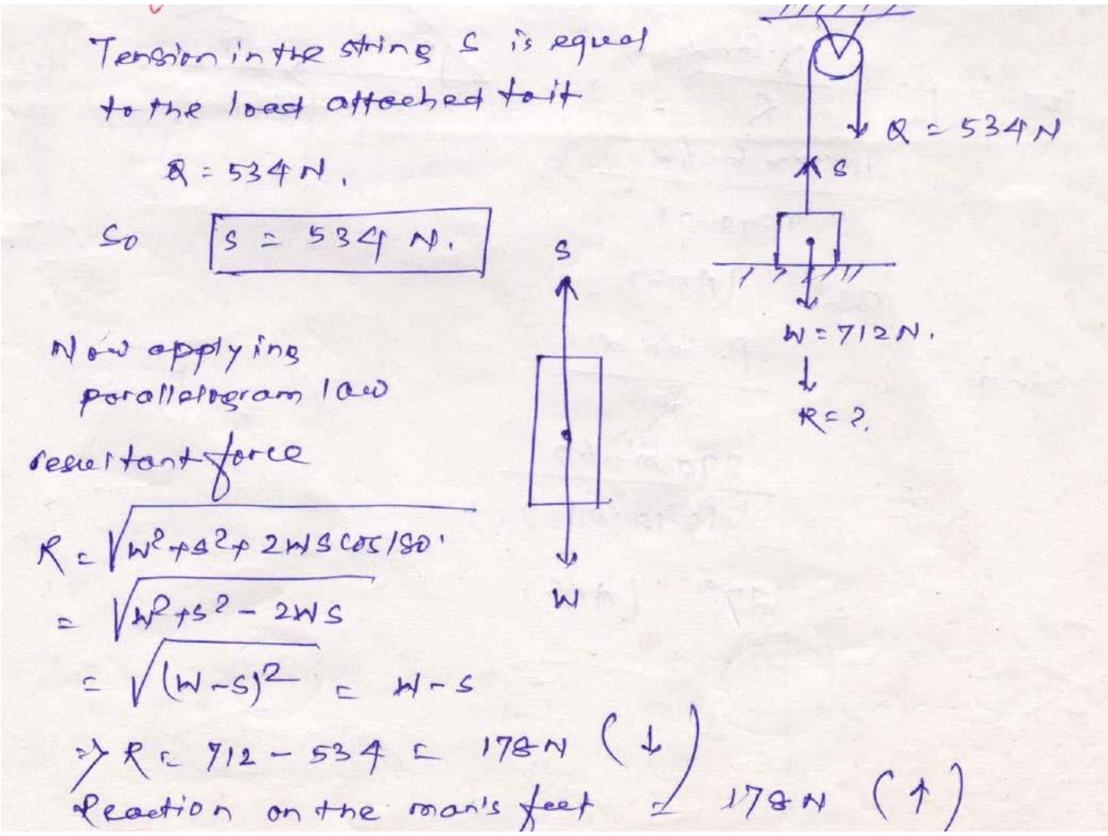

(tension)



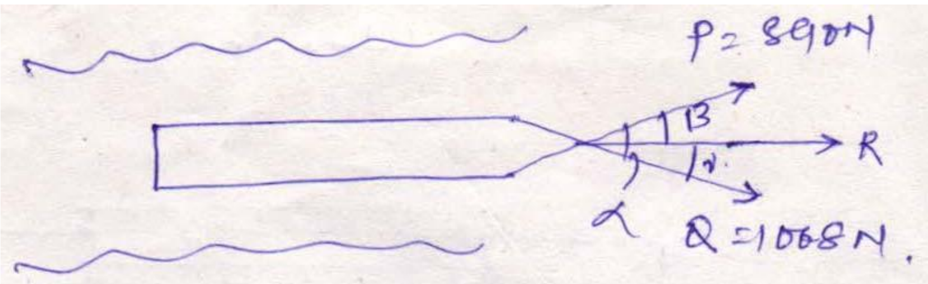

(compression)

## Superposition and transmissibility

**Problem 1:** A man of weight W = 712 N holds one end of a rope that passes over a pulley vertically above his head and to the other end of which is attached a weight Q = 534 N. Find the force with which the man's feet press against the floor.




**Problem 2:** A boat is moved uniformly along a canal by two horses pulling with forces P = 890 N and Q = 1068 N acting under an angle α = 60˚. Determine the magnitude of the resultant pull on the boat and the angles β and ν.



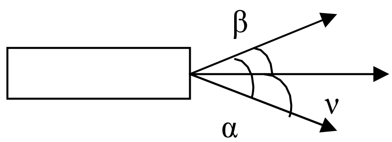

P = 890 N, α = 60˚

Q = 1068 N

$$R = \sqrt{(P^2 + Q^2 + 2PQ\cos\alpha)}$$

$$= \sqrt{(890^2 + 1068^2 + 2 \times 890 \times 1068 \times 0.5)}$$

$$= 1698.01N$$

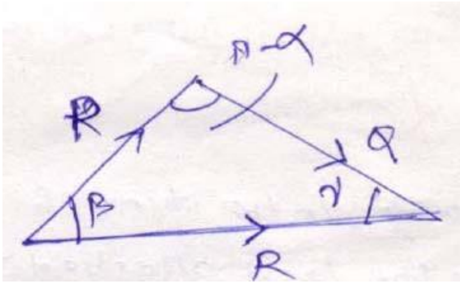


$$\frac{Q}{\sin\beta} = \frac{P}{\sin\nu} = \frac{R}{\sin(\pi - \alpha)}$$

$$\sin\beta = \frac{Q\sin\alpha}{R}$$

$$= \frac{1068 \times \sin 60^{\circ}}{1698.01}$$

$$= 33^{\circ}$$




$$\sin\nu = \frac{P\sin\alpha}{R}$$

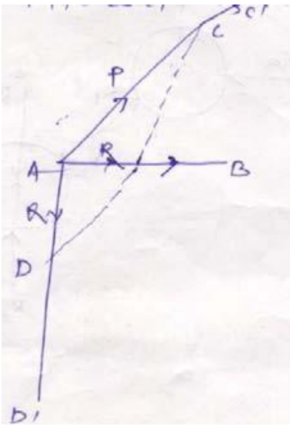
$$= \frac{890 \times \sin 60^{\circ}}{1698.01}$$

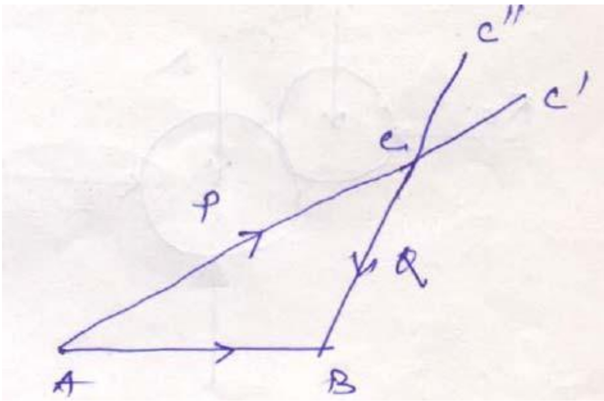
$$= 27^{\circ}$$

**Resolution of a force**

Replacement of a single force by several components which will be equivalent in action to the given force is called the problem of resolution of a force.

By using parallelogram law, a single force R can be resolved into two components P and Q intersecting at a point on its line of action.



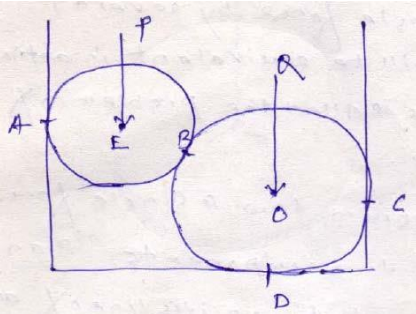

**Equilibrium of collinear forces:**

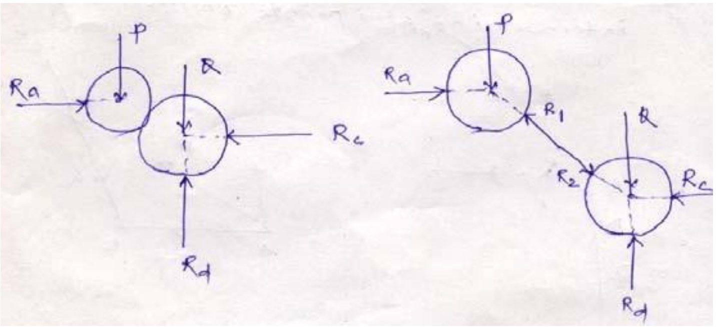
Equilibrium law: Two forces can be in equilibrium only if they are equal in magnitude, opposite in direction and collinear in action.

## Law of superposition

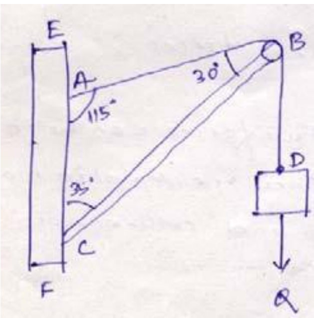
The action of a given system of forces on a rigid body will no way be changed if we add to or subtract from them another system of forces in equllibrium.

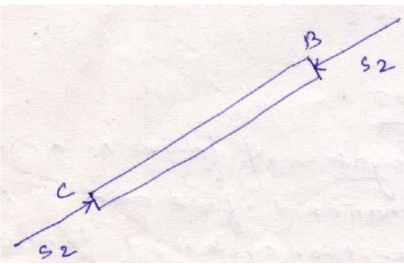
**Problem 3:** Two spheres of weight P and Q rest inside a hollow cylinder which is resting on a horizontal force. Draw the free body diagram of both the spheres, together and separately.

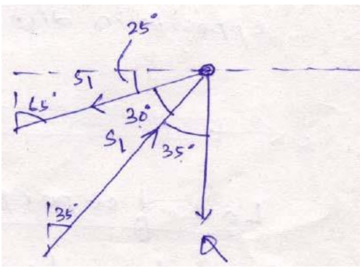


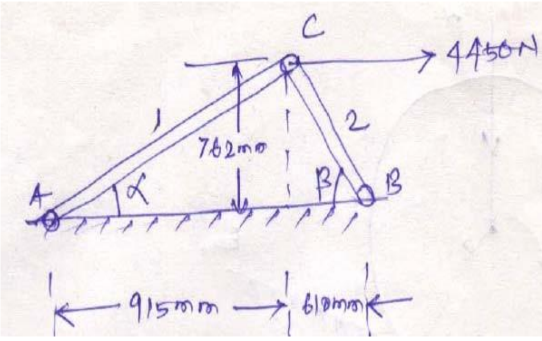

**Problem 4:** Draw the free body diagram of the figure shown below.

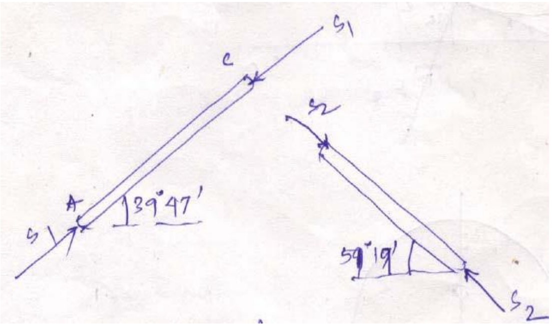


**Problem 5:** Determine the angles α and β shown in the figure.



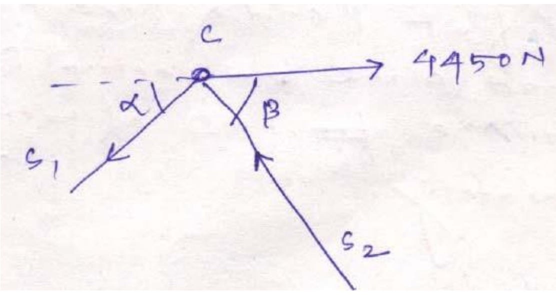




$$\alpha = \tan^{-1}\left(\frac{762}{915}\right)$$

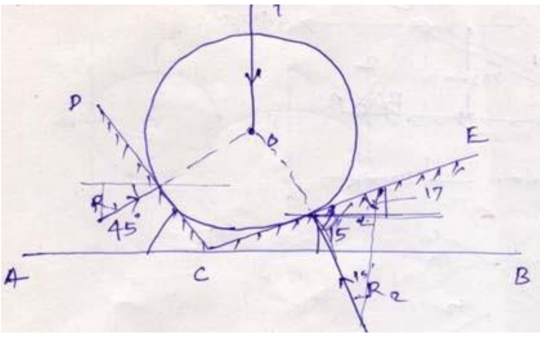
$$= 39^{\circ}47'$$

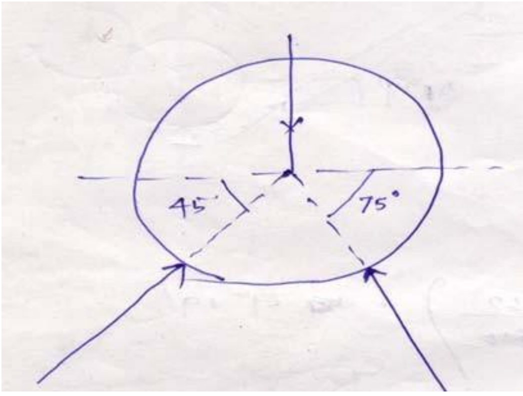
$$\beta = \tan^{-1}\left(\frac{762}{610}\right)$$

$$= 51^{\circ}19'$$

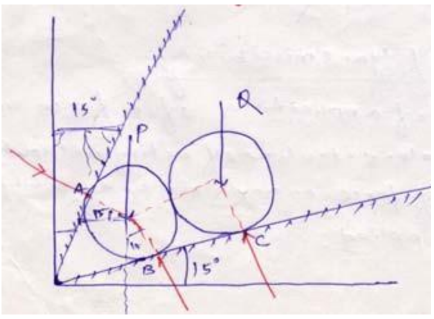


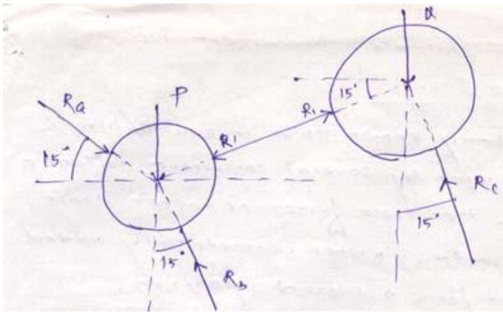
**Problem 6:** Find the reactions $R_1$ and $R_2$.



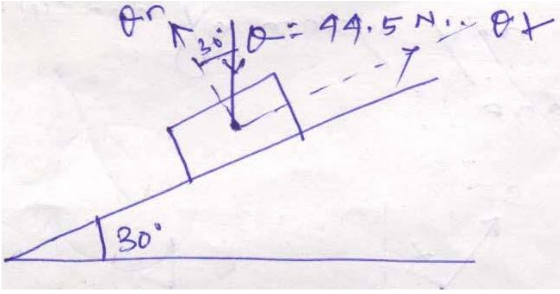




**Problem 7:** Two rollers of weight P and Q are supported by an inclined plane and vertical walls as shown in the figure. Draw the free body diagram of both the rollers separately.

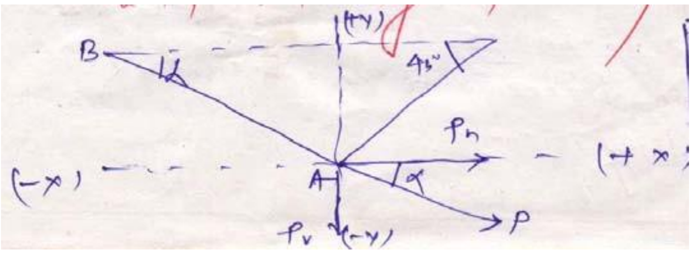


**Problem 8:** Find $\theta_n$ and $\theta_t$ in the following figure.




**Problem 9:** For the particular position shown in the figure, the connecting rod BA of an engine exert a force of P = 2225 N on the crank pin at A. Resolve this force into two rectangular components $P_h$ and $P_v$ horizontally and vertically respectively at A.

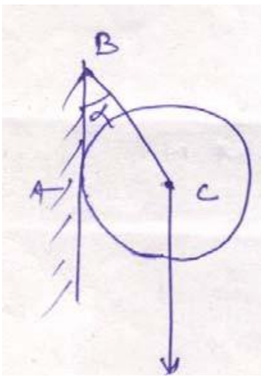



$P_h$ = 2081.4 N
$P_v$ = 786.5 N

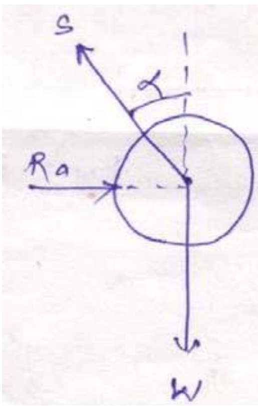
## <u>Equilibrium of concurrent forces in a plane</u>

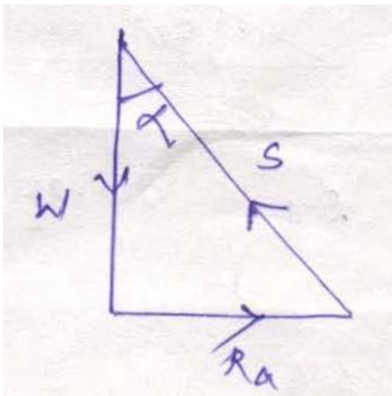
- If a body known to be in equilibrium is acted upon by several concurrent, coplanar forces, then these forces or rather their free vectors, when geometrically added must form a closed polygon.

- This system represents the condition of equilibrium for any system of concurrent forces in a plane.



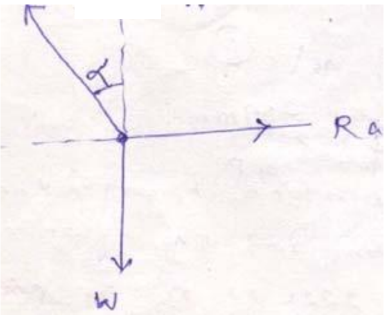

W



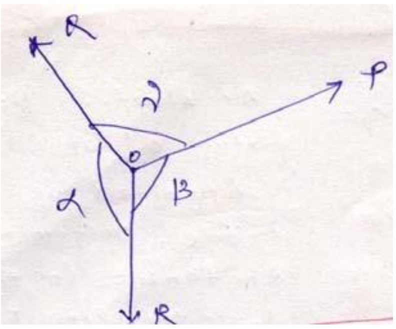
$R_a = w\tan\alpha$
$S = w\sec\alpha$

S

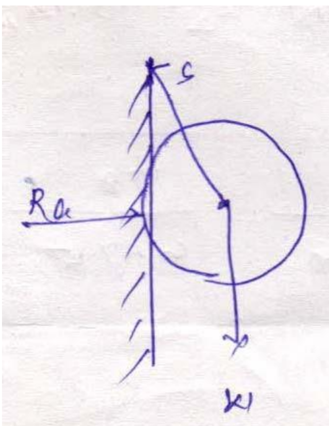


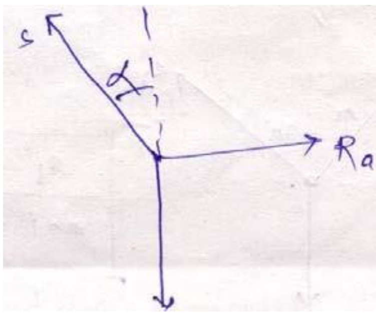

**Lami’s theorem**

If three concurrent forces are acting on a body kept in an equllibrium, then each force is proportional to the sine of angle between the other two forces and the constant of proportionality is same.




$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\upsilon}$$

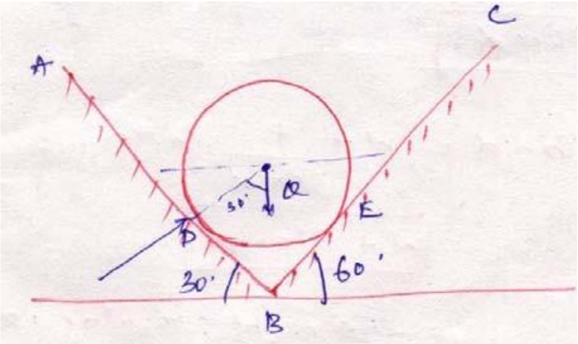


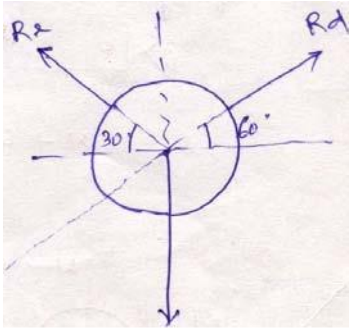




W

$$\frac{S}{\sin 90} = \frac{R_a}{\sin\left(180-\alpha\right)} = \frac{W}{\sin\left(90+\alpha\right)}$$

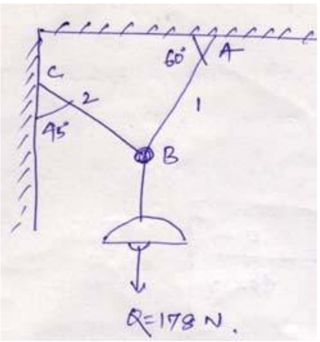
**Problem:** A ball of weight Q = 53.4N rest in a right angled trough as shown in figure. Determine the forces exerted on the sides of the trough at D and E if all the surfaces are perfectly smooth.

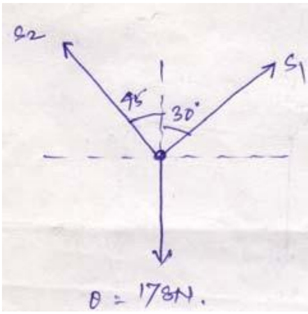


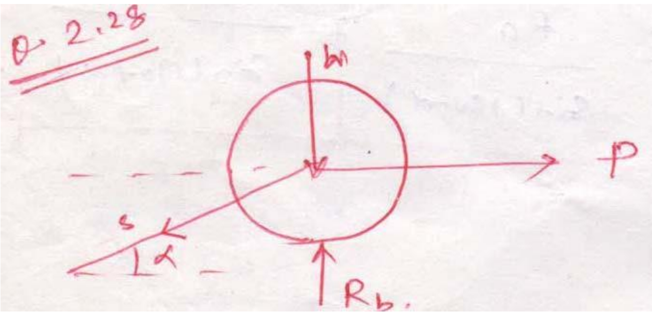

W

**Problem:** An electric light fixture of weight Q = 178 N is supported as shown in figure. Determine the tensile forces $S_1$ and $S_2$ in the wires BA and BC, if their angles of inclination are given.




$$\frac{S_1}{\sin 135} = \frac{S_2}{\sin 150} = \frac{178}{\sin 75}$$

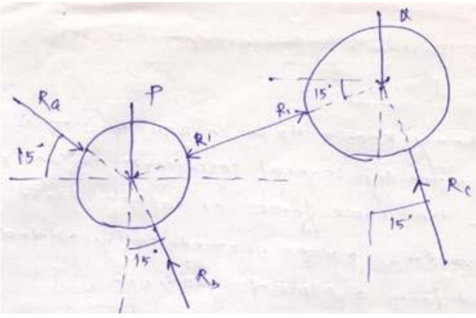



$S_1 \cos\alpha = P$

$\boxed{S = P\sec\alpha}$

$$R_b = W + S\sin\alpha$$
$$= W + \frac{P}{\cos\alpha} \times \sin\alpha$$
$$= W + P\tan\alpha$$

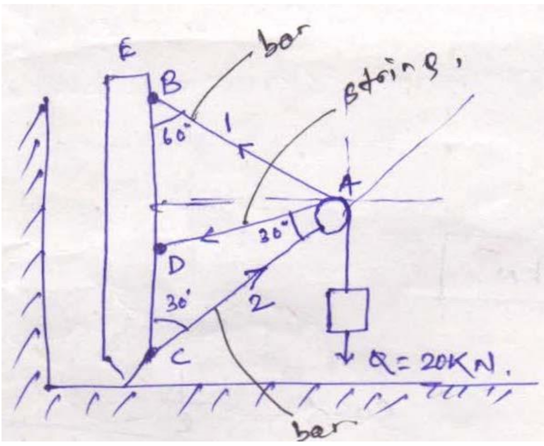
**Problem:** A right circular roller of weight W rests on a smooth horizontal plane and is held in position by an inclined bar AC. Find the tensions in the bar AC and vertical reaction $R_b$ if there is also a horizontal force P is active.

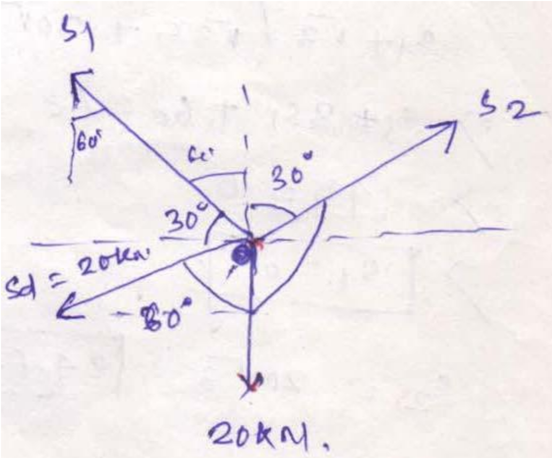



## Theory of transmissiibility of a force:

The point of application of a force may be transmitted along its line of action without changing the effect of force on any rigid body to which it may be applied.

**Problem:**





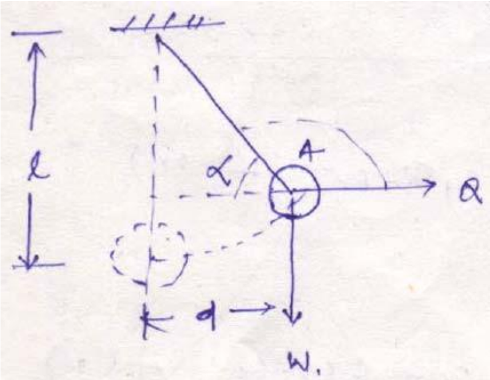

$$\sum X = 0$$
$$S_1 \cos 30 + 20 \sin 60 = S_2 \sin 30$$
$$\frac{\sqrt{3}}{2} S_1 + 20 \frac{\sqrt{3}}{2} = \frac{S_2}{2}$$
$$\frac{S_2}{2} = \frac{\sqrt{3}}{2} S_1 + 10\sqrt{3}$$
$$S_2 = \sqrt{3} S_1 + 20\sqrt{3} \qquad (1)$$

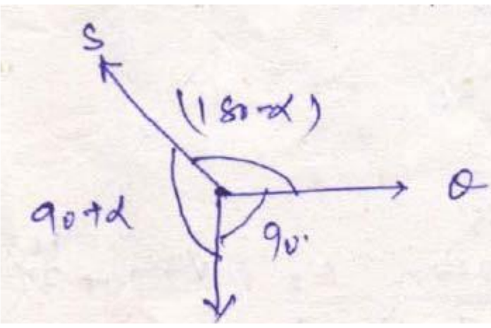
$$\sum Y = 0$$
$$S_1 \sin 30 + S_2 \cos 30 = S_d \cos 60 + 20$$
$$\frac{S_1}{2} + S_2 \frac{\sqrt{3}}{2} = \frac{20}{2} + 20$$
$$\frac{S_1}{2} + \frac{\sqrt{3}}{2} S_2 = 30$$
$$S_1 + \sqrt{3} S_2 = 60 \qquad (2)$$

Substituting the value of $S_2$ in Eq.2, we get

$$S_1 + \sqrt{3}\left(\sqrt{3} S_1 + 20\sqrt{3}\right) = 60$$
$$S_1 + 3S_1 + 60 = 60$$
$$4S_1 = 0$$
$$S_1 = 0KN$$
$$S_2 = 20\sqrt{3} = 34.64KN$$

**Problem:** A ball of weight W is suspended from a string of length l and is pulled by a horizontal force Q. The weight is displaced by a distance d from the vertical position as shown in Figure. Determine the angle α, forces Q and tension in the string S in the displaced position.




W

$$\cos\alpha = \frac{d}{l}$$

$$\alpha = \cos^{-1}\left(\frac{d}{l}\right)$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\Rightarrow \sin\alpha = \sqrt{(1-\cos^2\alpha)}$$

$$= \sqrt{1-\frac{d^2}{l^2}}$$

$$= \frac{1}{l}\sqrt{l^2 - d^2}$$

Applying Lami's theorem,

$$\frac{S}{\sin 90} = \frac{Q}{\sin(90+\alpha)} = \frac{W}{\sin(180-\alpha)}$$

$$\frac{Q}{\sin(90+\alpha)} = \frac{W}{\sin(180-\alpha)}$$

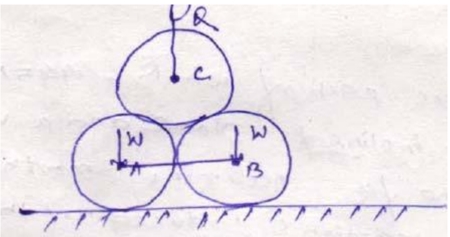
$$\Rightarrow Q = \frac{W\cos\alpha}{\sin\alpha} = \frac{W\left(\frac{d}{l}\right)}{\frac{1}{l}\sqrt{l^2-d^2}}$$

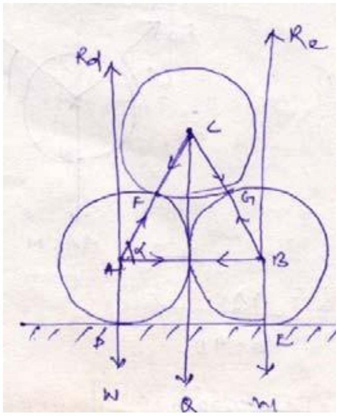
$$\Rightarrow Q = \frac{Wd}{\sqrt{l^2-d^2}}$$

$$S = \frac{W}{\sin\alpha} = \frac{W}{\frac{1}{l}\sqrt{l^2-d^2}}$$

$$= \frac{Wl}{\sqrt{l^2-d^2}}$$

**Problem:** Two smooth circular cylinders each of weight W = 445 N and radius r = 152 mm are connected at their centres by a string AB of length l = 406 mm and rest upon a horizontal plane, supporting above them a third cylinder of weight Q = 890 N and radius r = 152 mm. Find the forces in the string and the pressures produced on the floor at the point of contact.

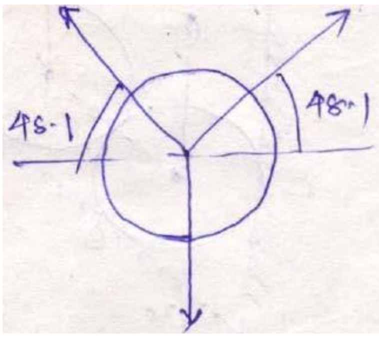



$$\cos\alpha = \frac{203}{304}$$

$$\Rightarrow \alpha = 48.1^\circ$$

Rg Rf




Q

$$\frac{R_g}{\sin 138.1} = \frac{R_e}{\sin 138.1} = \frac{Q}{83.8}$$

$$\Rightarrow R_g = R_e = 597.86N$$

Resolving horizontally

$$\sum X = 0$$

$$S = R_f \cos 48.1$$

$$= 597.86 \cos 48.1$$

$$= 399.27N$$

Resolving vertically

$$\sum Y = 0$$

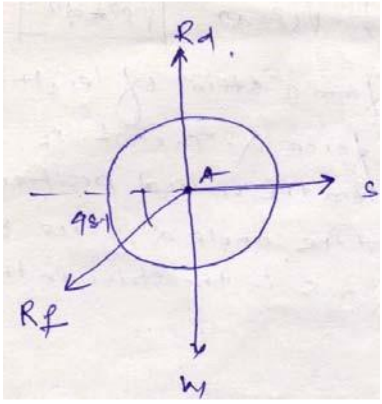
$$R_d = W + R_f \sin 48.1$$

$$= 445 + 597.86 \sin 48.1$$

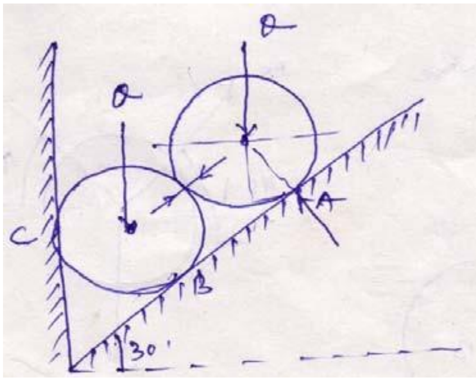
$$= 890N$$

$$R_e = 890N$$

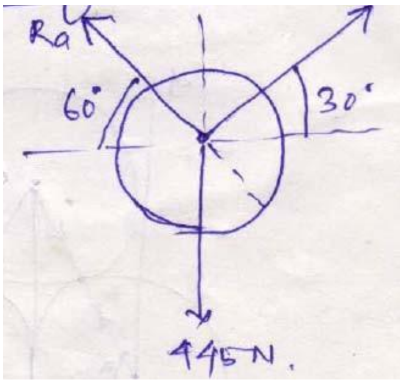
$$S = 399.27N$$




**Problem:** Two identical rollers each of weight Q = 445 N are supported by an inclined plane and a vertical wall as shown in the figure. Assuming smooth surfaces, find the reactions induced at the points of support A, B and C.




S




$$\frac{R_a}{\sin 120} = \frac{S}{\sin 150} = \frac{445}{\sin 90}$$

$$\Rightarrow R_a = 385.38N$$

$$\Rightarrow S = 222.5N$$

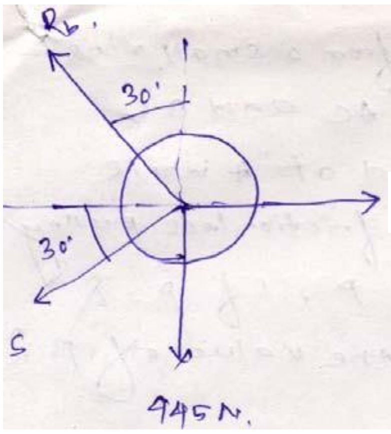
Resolving vertically

$$\sum Y = 0$$

$$R_b \cos 60 = 445 + S \sin 30$$

$$\Rightarrow R_b \frac{\sqrt{3}}{2} = 445 + \frac{222.5}{2}$$

$$\Rightarrow R_b = 642.302N$$




Resolving horizontally

$$\sum X = 0$$

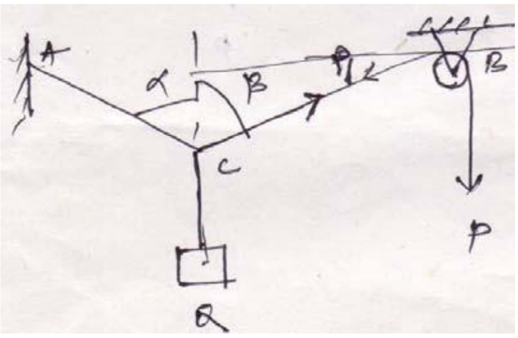
$$R_c = R_b \sin 30 + S \cos 30$$

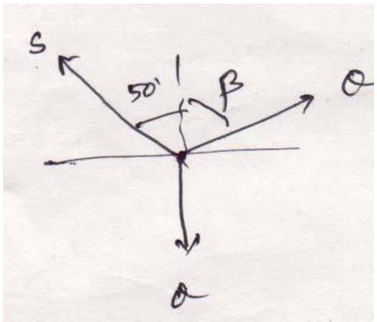
$$\Rightarrow 642.302 \sin 30 + 222.5 \cos 30$$

$$\Rightarrow R_c = 513.84N$$

**Problem:**

A weight Q is suspended from a small ring C supported by two cords AC and BC. The cord AC is fastened at A while cord BC passes over a frictionless pulley at B and carries a weight P. If P = Q and α = 50°, find the value of β.




Resolving horizontally

$$\sum X = 0$$

$$S \sin 50 = Q \sin \beta \qquad (1)$$

Resolving vertically

$$\sum Y = 0$$

$$S \cos 50 + Q \sin \beta = Q$$

$$\Rightarrow S \cos 50 = Q(1 - \cos \beta)$$

Putting the value of S from Eq. 1, we get

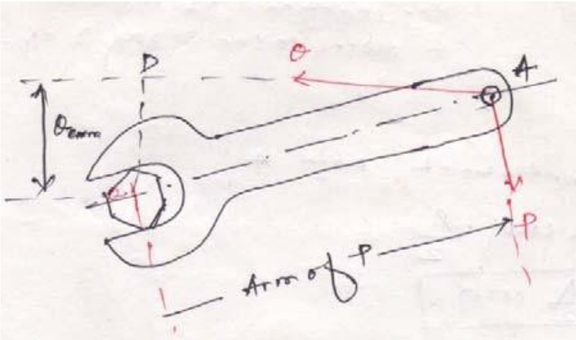
$$S\cos 50 + Q\sin\beta = Q$$
$$\Rightarrow S\cos 50 = Q(1-\cos\beta)$$
$$\Rightarrow Q\frac{\sin\beta}{\sin 50}\cos 50 = Q(1-\cos\beta)$$
$$\Rightarrow \cot 50 = \frac{1-\cos\beta}{\sin\beta}$$
$$\Rightarrow 0.839\sin\beta = 1-\cos\beta$$

Squaring both sides,

$$0.703\sin^2\beta = 1+\cos^2\beta - 2\cos\beta$$
$$0.703(1-\cos^2\beta) = 1+\cos^2\beta - 2\cos\beta$$
$$0.703 - 0.703\cos^2\beta = 1+\cos^2\beta - 2\cos\beta$$
$$\Rightarrow 1.703\cos^2\beta - 2\cos\beta + 0.297 = 0$$
$$\Rightarrow \cos^2\beta - 1.174\cos\beta + 0.297 = 0$$
$$\Rightarrow \beta = 63.13^{\circ}$$

**<u>Method of moments</u>**

**Moment of a force with respect to a point:**




- Considering wrench subjected to two forces P and Q of equal magnitude. It is evident that force P will be more effective compared to Q, though they are of equal magnitude.
- The effectiveness of the force as regards it is the tendency to produce rotation of a body about a fixed point is called the moment of the force with respect to that point.
- Moment = Magnitude of the force × Perpendicular distance of the line of action of force.
- Point O is called moment centre and the perpendicular distance (i.e. OD) is called moment arm.
- Unit is N.m

**Theorem of Varignon:**

The moment of the resultant of two concurrent forces with respect to a centre in their plane is equal to the alzebric sum of the moments of the components with respect to some centre.

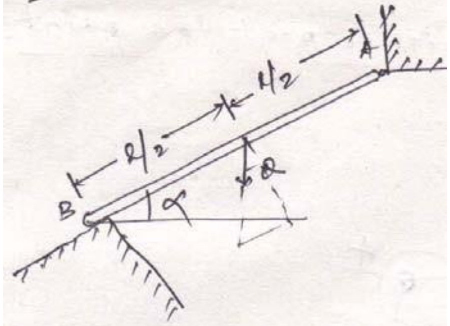
**<u>Problem 1:</u>**

A prismatic clear of AB of length l is hinged at A and supported at B. Neglecting friction, determine the reaction $R_b$ produced at B owing to the weight Q of the bar.

Taking moment about point A,

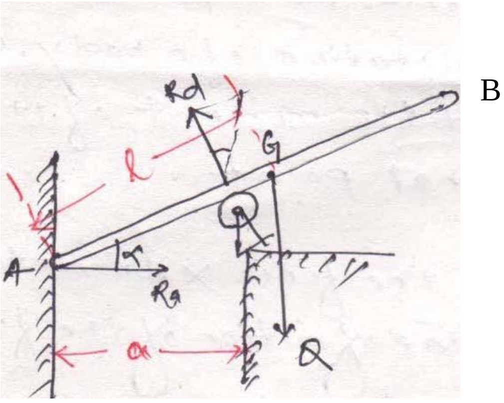
$$R_b \times l = Q\cos\alpha.\frac{l}{2}$$

$$\Rightarrow R_b = \frac{Q}{2}\cos\alpha$$



**Problem 2:**

A bar AB of weight Q and length 2l rests on a very small friction less roller at D and against a smooth vertical wall at A. Find the angle α that the bar must make with the horizontal in equilibrium.




Resolving vertically,

$$R_d \cos\alpha = Q$$

Now taking moment about A,

$$\frac{R_d . a}{\cos\alpha} - Q.l\cos\alpha = 0$$

$$\Rightarrow \frac{Q.a}{\cos^2\alpha} - Q.l\cos\alpha = 0$$

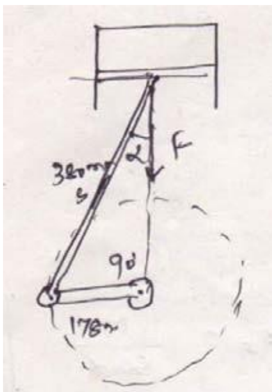
$$\Rightarrow Q.a - Q.l\cos^3\alpha = 0$$

$$\Rightarrow \cos^3\alpha = \frac{Q.a}{Q.l}$$

$$\Rightarrow \alpha = \cos^{-1}\sqrt[3]{\frac{a}{l}}$$

**Problem 3:**

If the piston of the engine has a diameter of 101.6 mm and the gas pressure in the cylinder is 0.69 MPa. Calculate the turning moment M exerted on the crankshaft for the particular configuration.



Area of cylinder

$$A = \frac{\pi}{4}(0.1016)^2 = 8.107 \times 10^{-3} m^2$$

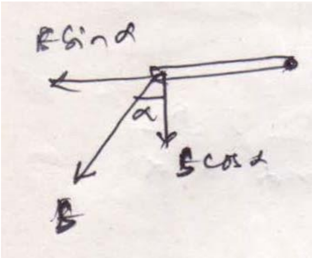
Force exerted on connecting rod,

F = Pressure × Area
 $= 0.69 \times 10^6 \times 8.107 \times 10^{-3}$
 = 5593.83 N

Now $\alpha = \sin^{-1}\left(\frac{178}{380}\right) = 27.93^\circ$

$S\cos\alpha = F$

$$\Rightarrow S = \frac{F}{\cos\alpha} = 6331.29N$$

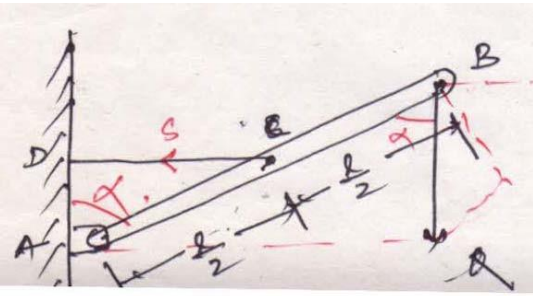



Now moment entered on crankshaft,

$S\cos\alpha \times 0.178 = 995.7N = 1KN$

**Problem 4:**

A rigid bar AB is supported in a vertical plane and carrying a load Q at its free end. Neglecting the weight of bar, find the magnitude of tensile force S in the horizontal string CD.




Taking moment about A,

$\sum M_A = 0$

$$S.\frac{l}{2}\cos\alpha = Q.l\sin\alpha$$

$$\Rightarrow S = \frac{Q.l\sin\alpha}{\frac{l}{2}\cos\alpha}$$

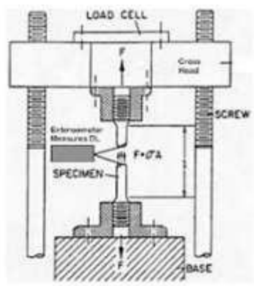
$$\Rightarrow S = 2Q.\tan\alpha$$

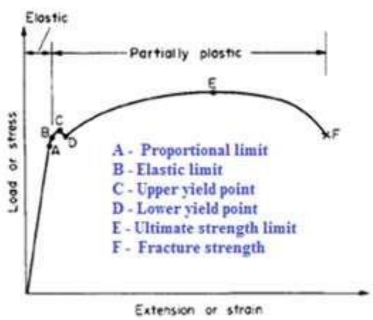
# Strength of Materials (15CV 32)

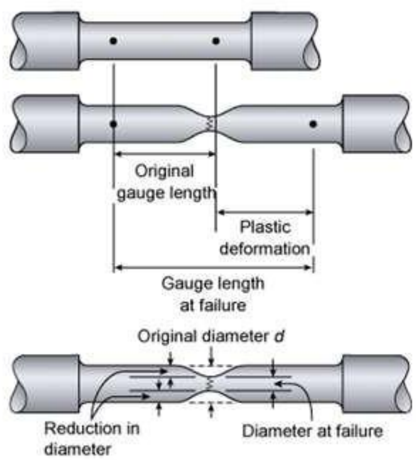
## Module 1 : Simple Stresses and Strains

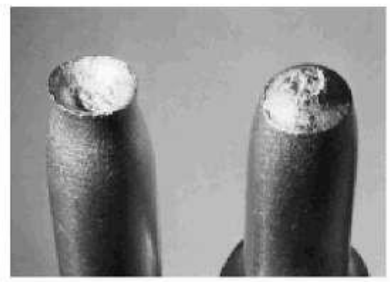
**Dr. H. Ananthan, Professor, VVIET,MYSURU**

**8/21/2017**



**Cup and Cone fracture**

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Introduction, Definition and concept and of stress and strain. Hooke's law, Stress-Strain diagrams for ferrous and non-ferrous materials, factor of safety, Elongation of tapering bars of circular and rectangular cross sections, Elongation due to self-weight. Saint Venant's principle, Compound bars, Temperature stresses, Compound section subýected to temperature stresses, state of simple shear, Elastic constants and their relationship.

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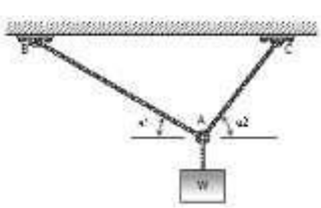
## 1.1 Introduction

In civil engineering structures, we frequently encounter structural elements such as *tie members, cables, beams, columns and struts* subjected to external actions called *forces or loads.* These elements have to be designed such that they have adequate *strength, stiffness and stability*.

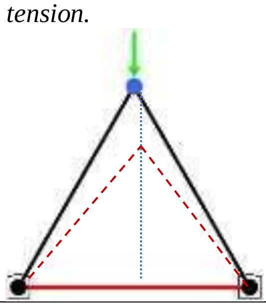
The *strength* of a structural component is its ability to withstand applied forces without failure and this depends upon the *sectional dimensions and material characteristics*. For instance a steel rod can resist an applied tensile force more than an aluminium rod with similar diameter. Larger the sectional dimensions or stronger is the material greater will be the force carrying capacity.

*Stiffness* influences the deformation as a consequence of *stretching, shortening, bending, sliding, buckling, twisting and warping* due to applied forces as shown in the following figure. In a *deformable* body, the distance between two points changes due to the action of some kind of forces acting on it.

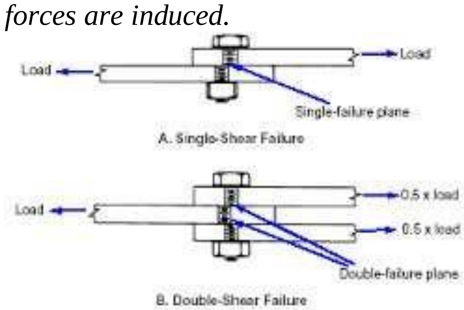
A weight suspended by two cables causes stretching of the cables. Cables are in *axial tension.*



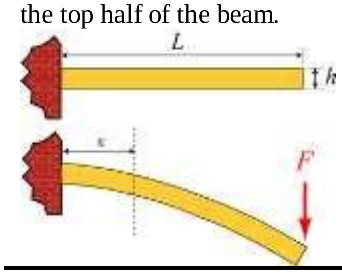

Inclined members undergo shortening, and stretching will be induced in the horizontal member. Inclined members are in *axial compression* and horizontal member is in *axial tension.*



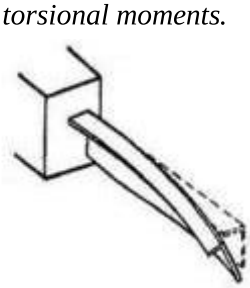
Bolt connecting the plates is subjected to sliding along the failure plane. *Shearing forces are induced.*



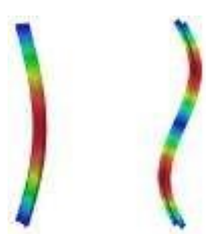

Cantilever beam subjected to *bending* due to transverse loads results in shortening in the bottom half and stretching in the top half of the beam.




Cantilever beam subjected to twisting and warping due to *torsional moments.*



Buckling of long compression members due to *axial load.*



Such deformations also depend upon *sectional dimensions, length and material characteristics.* For instance a steel rod undergoes less of stretching than an aluminium rod with similar diameter and subjected to same tensile force.

*Stability* refers to the ability to maintain its original configuration. This again depends upon *sectional dimensions, length and material characteristics.* A steel rod with a larger length will buckle under a compressive action, while the one with smaller length can remain stable even though the sectional dimensions and material characteristics of both the rods are same.

The subject generally called *Strength of Materials* includes the study of the distribution of internal forces, the stability and deformation of various elements. It is founded both on the results of experiments and the application of the principles of mechanics and mathematics. The results obtained in the subject of strength of materials form an important part of the basis of scientific and engineering designs of different structural elements. Hence this is treated as subject of fundamental importance in design engineering. The study of this subject in the context of civil engineering refers to various methods of analyzing deformation behaviour of structural elements such as plates, rods, beams, columns, shafts etc.,.

## 1.2 Concepts and definitions

A load applied to a structural member will induce internal forces within the member called *stress resultants* and if computed based on unit cross sectional area then they are termed as *intensity of stress or simply stress* in the member.

The stresses induced in the structural member will cause different types of deformation in the member. If such deformations are computed based on unit dimensions then they are termed as *strain* in the member.

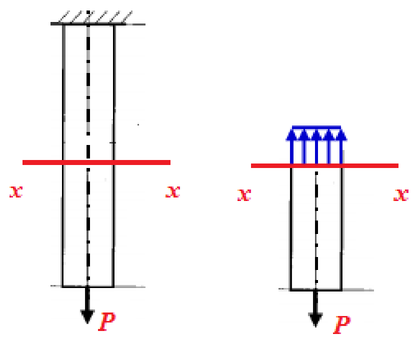
The stresses and strains that develop within a structural member must be calculated in order to assess its strength, deformations and stability. This requires a complete description of the geometry, constraints, applied loads and the material properties of the member.

The calculated stresses may then be compared to some measure of the strength of the material established through experiments. The calculated deformations in the member may be compared with respect limiting criteria established based on experience. The calculated buckling load of the member may be compared with the applied load and the safety of the member can be assessed.

It is generally accepted that analytical methods coupled with experimental observations can provide solutions to problems in engineering with a fair degree of accuracy. Design solutions are worked out by a proper analysis of deformation of bodies subjected to surface and body forces along with material properties established through experimental investigations.

## 1.3 Simple Stress

Consider the suspended bar of original length $\mathbf{L_0}$ and uniform cross sectional area $\mathbf{A_0}$ with a force or load **P** applied to its end as shown in the following figure (a). Let us imagine that the bar is cut in to two parts by a section ***x-x*** and study the equilibrium of the lower portion of the bar as shown in figure (b). At the lower end, we have the applied force P



(a) (b)

It can be noted that, the external force applied to a body in equilibrium is reacted by internal forces set up within the material. If a bar is subjected to an axial tension or compression, **P,** then the internal forces set up are distributed uniformly and the bar is said to be subjected to a *uniform direct or normal or simple stress*. The stress being defined as

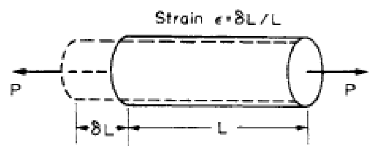
$$stress\ (\sigma) = \frac{Load\ (P)}{Sectional\ Area\ (A)}$$

Note

i. This is expressed as $N/mm^2$ or MPa.
ii. Stress may thus be compressive or tensile depending on the nature of the load.
iii. In some cases the stress may vary across any given section, and in such cases the stress at any point is given by the limiting value of $\delta P/\delta A$ as $\delta A$ tends to zero.

## 1.4 Simple Strain

If a bar is subjected to a direct load, and hence a stress, the bar will change in length. If the bar has an original length L and changes in length by an amount δL as shown below,




then the strain produced is defined as follows:

$$strain\ \mathrm{s} = \frac{change\ \text{in}\ length\ (\eth L)}{original\ length\ (L)}$$

This strain is also termed as *longitudinal strain* as it is measured in the direction of application of load.

*Note:*

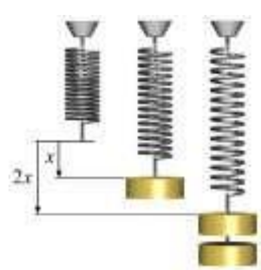
i. Strain is thus a measure of the deformation of the member. It is simply a ratio of two quantities with the same units. It is non-dimensional, i.e. it has no units.

ii. The deformations under load are very small. Hence the strains are also expressed as *strain x 10* $^{-6}$. In such cases they are termed as *microstrain (*με).

iii. Strain is also expressed as a percentage strain : ε (%) = (δL/L)100.

## *1.5* Elastic limit – *Hooke's law*

A structural member is said to be within elastic limit, if it returns to its original dimensions when load is removed. Within this load range, the deformations are proportional to the loads producing them. Hooke's law states that, "*the force needed to extend or compress a spring by some distance is proportional to that distance*". This is indicated in the following figure.




Since loads are proportional to the stresses they produce and deformations are proportional to the strains, the Hooke"s law also implies that, "*stress is proportional to strain within elastic limit*".

$$stress\ (\sigma)\ \ \mathrm{a}\ \ strain(\mathrm{s})\quad \textbf{or}\quad \sigma/\varepsilon = \textbf{\textit{constant}}$$

This law is valid within certain limits for most ferrous metals and alloys. It can even be assumed to apply to other engineering materials such as concrete, timber and non-ferrous alloys with reasonable accuracy.

*The law is named after 17th-century British physicist Robert Hooke. He first stated the law in 1676 as a Latin anagram. He published the solution of his anagram in 1678 as: "uttensio, sic vis" ("as the extension, so the force" or "the extension is proportional to the force").*

## 1.6 Modulus of elasticity or Young's modulus

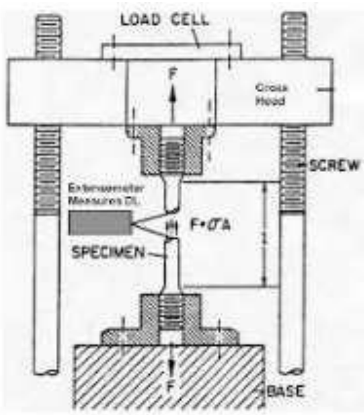
Within the elastic limits of materials, i.e. within the limits in which Hooke's law applies, it has been found that stress/strain = constant. This is termed the *modulus of elasticity or Young's modulus*. This is usually denoted by letter E and has the same units of stress. With $\sigma = P/A$ and $\varepsilon = \delta L/L$, the following expression for E can be derived.

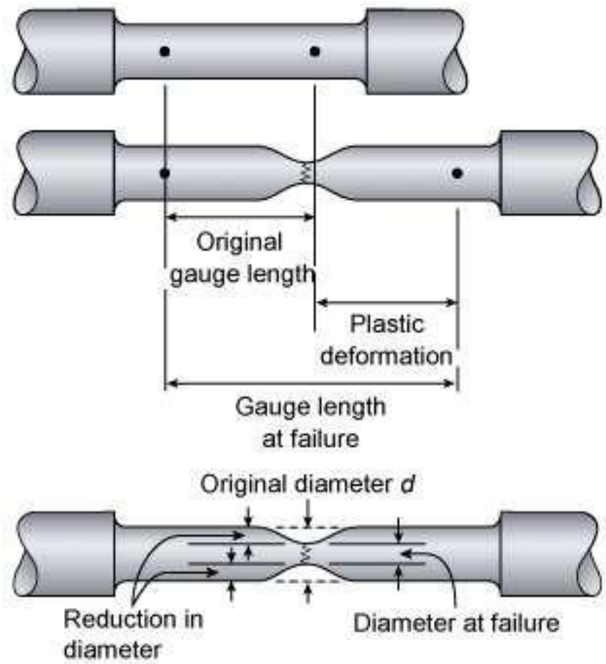
$$E = \frac{\sigma}{s} = \frac{P}{A}\frac{L}{\delta L}$$

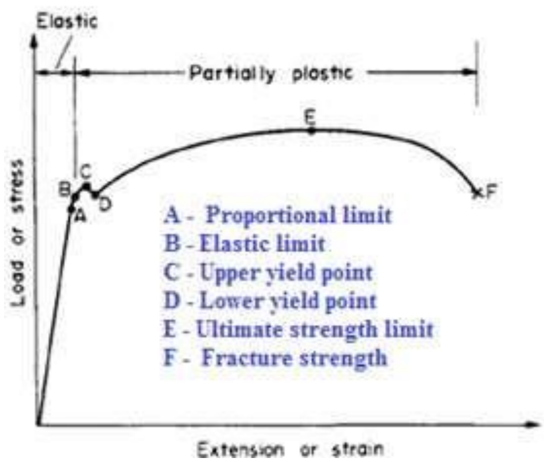
Young's modulus E is generally assumed to be the same in tension or compression and for most engineering materials has a high numerical value. Typically, E = 200000 MPa for steel. This is determined by conducting tension or compression test on specimens in the laboratory.

## 1.7 Tension test

In order to compare the strengths of various materials it is necessary to carry out some standard form of test to establish their relative properties. One such test is the standard tensile test. In this test a circular bar of uniform cross-section is subjected to a gradually increasing tensile load until failure occurs. Measurements of the change in length of a selected gauge length of the bar are recorded throughout the loading operation by means of extensometers. A graph of load against extension or stress against strain is produced.

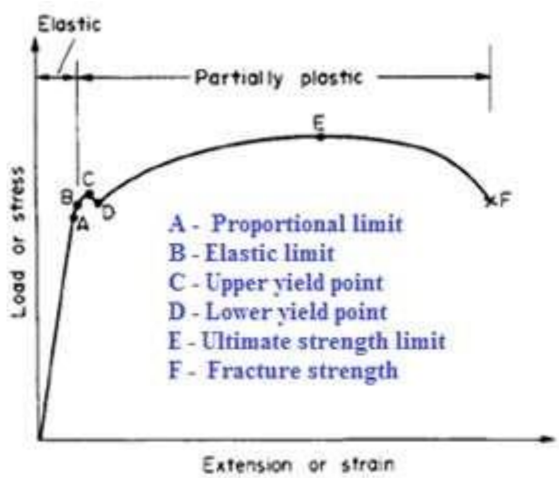



## 1.8 Stress – Strain diagrams for ferrous metals

The typical graph for a test on a mild (low carbon) steel bar is shown in the figure below. Other materials will exhibit different graphs but of a similar general form. Following salient points are to be noted:



i. In the initial stages of loading it can be observed that Hooke's law is obeyed, i.e. the material behaves elastically and stress is proportional to strain. This is indicated by the straight-line portion in the graph up to point A. Beyond this, some nonlinear nature of the graph can be seen. Hence this point (A) is termed the *limit of proportionality*. This region is also called *linear elastic range* of the material.

ii. For a small increment in loading beyond A, the material may still be elastic. Deformations are completely recovered when load is removed but Hooke's law does not apply. The limiting point B for this condition is termed the *elastic limit*. This region refers to *nonlinear elastic range*. It is often assumed that points A and B are coincident.

*iii.* Beyond the elastic limit (A or B), *plastic deformation* occurs and strains are not totally recoverable. Some *permanent deformation* or *permanent set* will be there when the specimen is unloaded. Points C, is termed as the *upper yield point*, and D, as the *lower yield point.* It is often assumed that points C and D are coincident. Strength corresponding to *this* point is termed as the *yield strength* of the material. Typically this strength corresponds to the load carrying capacity.

iv. Beyond point (C or D), strain increases rapidly without proportionate increases in load or stress. The graph covers a much greater portion along the strain axis than in the elastic range of the material. The capacity of a material to allow these large plastic deformations is a measure of *ductility* of the material.

v. Some increase in load is required to take the strain to point E on the graph. Between D and E the material is said to be in the *elastic-plastic state. S*ome of the section remaining elastic and hence contributing to recovery of the original dimensions if load is removed, the remainder being plastic.

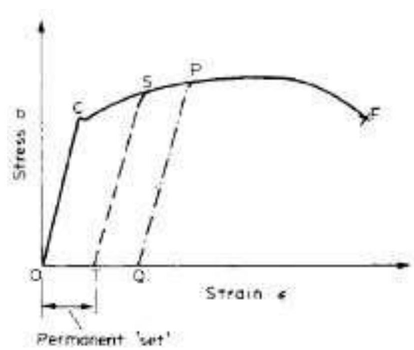
vi. Beyond E, the cross-sectional area of the bar begins to reduce rapidly over a relatively small length. This result in the formation of *necking* accompanied with reduction in load and *fracture (cup and cone)* of the bar eventually occurs at point F.

vii. The nominal stress at failure, termed the *maximum or ultimate tensile stress*, is given by the load at E divided by the original cross-sectional area of the bar. This is also known as the *ultimate tensile strength* of the material.

viii. Owing to the large reduction in area produced by the necking process the actual stress at fracture is often greater than the ultimate tensile strength. Since, however, designers are interested in maximum loads which can be carried by the complete cross-section, the stress at fracture is not of any practical importance.

## 1.9 Influence of Repeated loading and unloading on yield strength

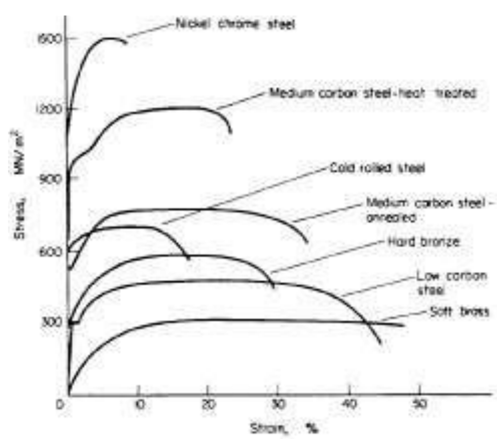
If load is removed from the test specimen after the yield point C has been passed, e.g. to some position S, as shown in the adjoining figure the unloading line ST can, for most practical purposes, be taken to be linear. A second load cycle, commencing with the permanent elongation associated with the strain OT, would then follow the line TS and continue along the previous




curve to failure at F. It can be observed, that the repeated load cycle has the effect of increasing the elastic range of the material, i.e. raising the effective yield point from C to S. However, it is important to note that the tensile strength is unaltered. The procedure could be repeated along the line PQ, etc., and the material is said to have been work hardened. Repeated loading and unloading will produce a yield point approaching the ultimate stress value but the elongation or strain to failure will be very much reduced.

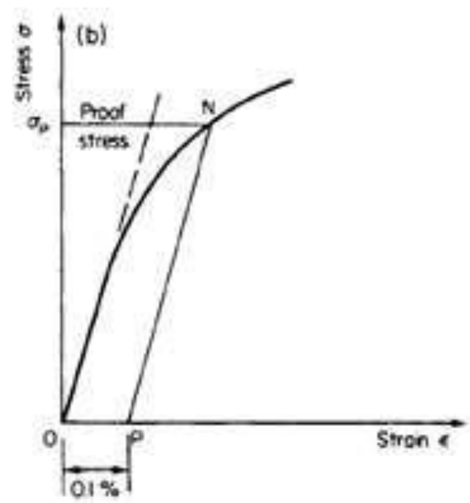
## 1.10 Non Ferrous metals

Typical stress-strain curves resulting from tensile tests on other engineering materials are shown in the following figure.



For certain materials, for example, high carbon steels and non-ferrous metals, it is not possible to detect any difference between the upper and lower yield points and in some cases yield point may not exist at all. In such cases a proof stress is used to indicate the onset of plastic strain. The 0.1% proof stress, for example, is that stress which, when removed, produces a permanent strain of 0.1% of the original gauge length as shown in the following figure.

The 0.1% proof stress can be determined from the tensile test curve as listed below.




i. Mark the point P on the strain axis which is equivalent to 0.1% strain.
ii. From P draw a line parallel with the initial straight line portion of the tensile test curve to cut the curve in N.
iii. The stress corresponding to N is then the 0.1% proof stress.
iv. A material is considered to satisfy its specification if the permanent set is no more than 0.1% after the proof stress has been applied for 15 seconds and removed.

## 1.11 Allowable working stress-factor of safety

The most suitable strength criterion for any structural element under service conditions is that some maximum stress must not be exceeded such that plastic deformations do not occur. This value is generally known as the *maximum allowable working stress*. Because of uncertainties of loading conditions, design procedures, production methods etc., it is a common practice to introduce a *factor of safety* into structural designs. This is defined as follows:

$$Factor\ of\ safety = \frac{Yield\ stress\ (or\ proof\ stress)}{Allowable\ woking\ stress}$$

## 1.12 Ductile materials

The capacity of a material to allow large extensions, i.e. the ability to be drawn out plastically, is termed its *ductility*. A quantitative value of the ductility is obtained by measurements of the percentage elongation or percentage reduction in area as defined below.

$$\%\, elongation = \frac{increase\ in\ gauge\ length\ to\ fracture}{original\ gauge\ length} \times 100$$

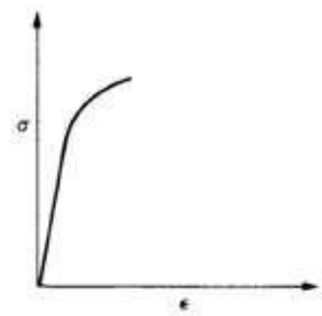
$$\%\, reduction\ in\ area = \frac{cross\ sectional\ area\ of\ necked\ portion}{original\ area} \times 100$$

*Note:*

*A property closely related to ductility is malleability, which defines a material's ability to be hammered out into thin sheets. Malleability thus represents the ability of a material to allow permanent extensions in all lateral directions under compressive loadings.*

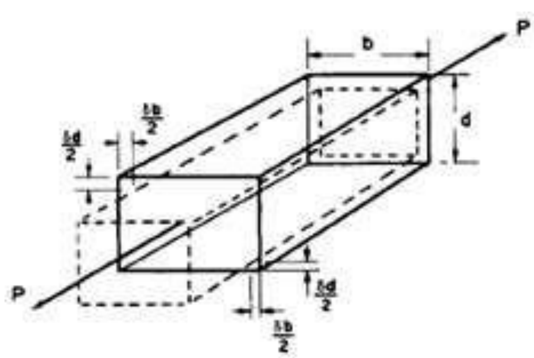
## 1.13 Brittle materials

A brittle material is one which exhibits relatively small extensions to fracture so that the partially plastic region of the tensile test graph is much reduced. There is little or no necking at fracture for brittle materials. Typical tensile test curve for a brittle material could well look like the one shown in the adjoining figure.




## 1.14 Lateral strain and Poisson's ratio

Till now we have focused on the longitudinal strain induced in the direction of application of the load. It has been observed that deformations also take place in the lateral direction. Consider the rectangular bar shown in the figure below and subjected to a tensile load.




Under the action of this load the bar will increase in length by an amount δL giving a longitudinal strain in the bar: $\varepsilon_L = \delta L/L$**.** The bar will also exhibit, however, a reduction in dimensions laterally, i.e. its breadth and depth will both reduce. The associated lateral strains will both be equal, and are of opposite sense to the longitudinal strain. These are computed as : $\varepsilon_{lat} = \delta b/b = \delta d/d$**.**

It has been observed that within the elastic range the ratio of the lateral and longitudinal strains will always be constant. This ratio is termed *Poisson's ratio* (ν).

$$\nu = \frac{\varepsilon_{lat}}{\varepsilon_L}$$

The above equation can also be written as :

$$\varepsilon_{lat} = \nu\varepsilon_L = \nu\frac{\sigma}{E}$$

For most of the engineering materials the value of ν is found to be between 0.25 and 0.33.

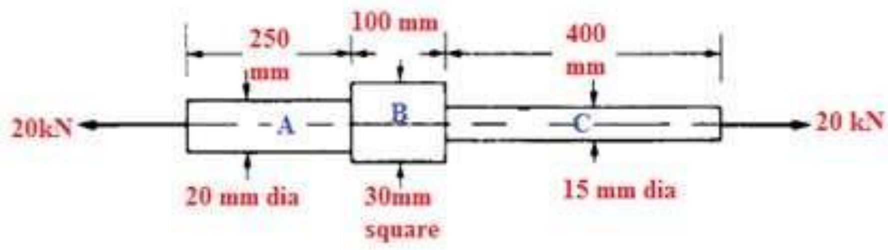
## Example 1

A bar of a rectangular section of 20 mm × 30 mm and a length of 500 mm is subjected to an axial compressive load of 60 kN. If $E$ = 102 kN/mm$^2$ and $\nu$ = 0.34, determine the changes in the length and the sides of the bar.

- Since the bar is subjected to compression, there will be decrease in length, increase in breadth and depth. These are computed as shown below
- L = 500 mm, b = 20 mm, d = 30 mm, P = 60 x1000 = 60000 N, E = 102000 N/mm$^2$
- Cross-sectional area A = 20 × 30 = 600 mm$^2$
- Compressive stress σ = P/A = 60000/600 = 100 N/mm$^2$
- Longitudinal strain $\varepsilon_L$ = σ/E = 100/102000 = 0.00098
- Lateral strain $\varepsilon_{lat} = \nu\ \varepsilon_L$ = 0.34 x 0.00098 = 0.00033
- Decrease in length δL = $\varepsilon_L$ L = 0.00098 x 500 = 0.49 mm
- Increase in breadth δb = $\varepsilon_{lat}$ b = 0.00033 x 20 = 0.0066 mm
- Increase in depth δd = $\varepsilon_{lat}$ d = 0.00033 x 30 = 0.0099 mm

## Example 2

Determine the stress in each section of the bar shown in the following figure when subjected to an axial tensile load of 20 kN. The central section is of square cross-section; the other portions are of circular section. What will be the total extension of the bar? For the bar material E = 210000MPa.




*The bar consists of three sections with change in diameter. Loads are applied only at the ends. The stress and deformation in each section of the bar are computed separately. The total extension of the bar is then obtained as the sum of extensions of all the three sections. These are illustrated in the following steps.*

The bar is in equilibrium under the action of applied forces

<u>Stress in each section of bar = P/A and P = 20000N</u>

i. Area of Bar A = $\pi$ x $20^2/4$ = 314.16 $mm^2$
ii. Stress in Bar A : $\sigma_A$ = 20000/ 314.16 = 63.66MPa
iii. Area of Bar B = 30 x30 = 900 $mm^2$
iv. Stress in Bar B : $\sigma_B$ = 20000/ 900 = 22.22MPa
v. Area of Bar C = $\pi$ x $15^2/4$ = 176.715 $mm^2$
vi. Stress in Bar C : $\sigma_C$ = 20000/ 176.715 = 113.18MPa

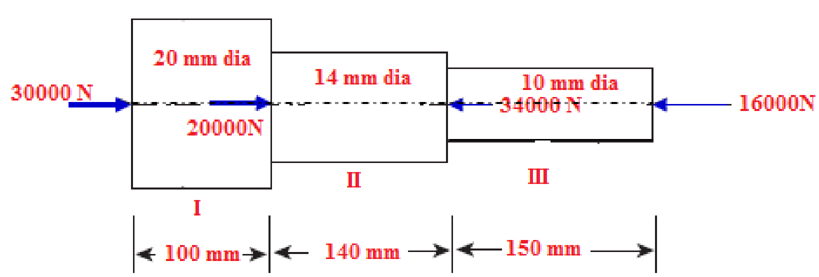
<u>Extension of each section of bar = $\sigma$L/E and E = 210000 MPa</u>

i. Extension of Bar A = 63.66 x 250 / 210000= 0.0758 mm
ii. Extension of Bar B = 22.22 x 100 / 210000= 0.0106 mm
iii. Extension of Bar C = 113.18 x 400 / 210000= 0.2155 mm

Total extension of the bar = <u>**0. 302mm**</u>

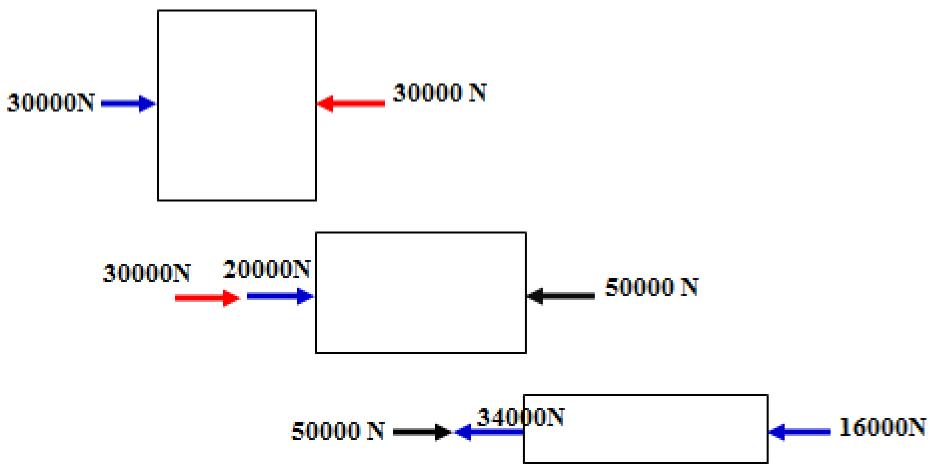
## Example 3

Determine the overall change in length of the bar shown in the figure below with following data: E = 100000 $N/mm^2$



*The bar is with varying cross-sections and subjected to forces at ends as well as at other interior locations. It is necessary to study the equilibrium of each portion separately and compute the change in length in each portion. The total change in length of the bar is then obtained as the sum of extensions of all the three sections as shown below.*

<u>Forces acting on each portion of the bar for equilibrium</u>




<u>Sectional Areas</u>

$A_I = \frac{\pi \times 20^2}{4} = 314.16\ mm^2$ ; $A_{II} = \frac{\pi \times 14^2}{4} = 153.94\ mm^2$; $A_{III} = \frac{\pi \times 10^2}{4} = 78.54\ mm^2$

<u>Change in length in Portion I</u>

Portion I of the bar is subjected to an axial compression of 30000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_I L_I}{A_I E} = \frac{30000 \times 100}{314.16 \times 100000} = 0.096\ mm$$

<u>Change in length in Portion II</u>

Portion II of the bar is subjected to an axial compression of 50000N ( 30000 + 20000). This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{II} L_{II}}{A_{II} E} = \frac{50000 \times 140}{153.94 \times 100000} = 0.455 mm$$

<u>Change in length in Portion III</u>

Portion III of the bar is subjected to an axial compression of (50000 – 34000) = 16000N. This results in *decrease* in length which can be computed as

$$\delta L_I = \frac{P_{III} L_{III}}{A_{III} E} = \frac{16000 \times 150}{78.54 \times 100000} = 0.306 mm$$

*Since each portion of the bar results in decrease in length, they can be added without any algebraic signs.*

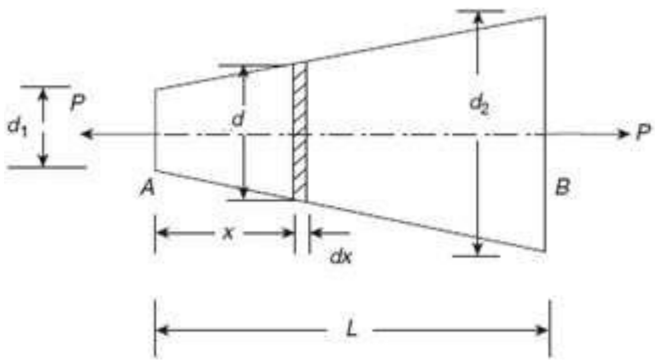
Hence Total decrease in length = 0.096 + 0.455 + 0.306 = **0.857mm**

*Note:*

*For equilibrium, if some portion of the bar may be subjected to tension and some other portion to compression resulting in increase or decrease in length in different portions of the bar. In such cases, the total change in length is computed as the sum of change in length of each portion of the bar with proper algebraic signs. Generally positive sign (+) is used for increase in length and negative sign (-) for decrease in length.*

## 1.15 Elongation of tapering bars of circular cross section

Consider a circular bar uniformly tapered from diameter **$d_1$** at one end and gradually increasing to diameter **$d_2$** at the other end over an axial length **L** as shown in the figure below.




Since the diameter of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of diameter ***d*** and length ***dx*** at a distance of ***x*** from end ***A***.

Using the principle of similar triangles the following equation for d can be obtained

$$d = d_1 + \frac{d_2 - d_1}{L}x = d_1 + kx\ , where\ k = \frac{d_2 - d_1}{L}$$

Cross–sectional area of the bar at *x* : $A_x = \frac{\pi\ (d_1+kx)^2}{4}$

Axial stress at *x*: $\sigma_x = \frac{P}{A_x} = \frac{4P}{\pi\ (d_1+kx)^2}$

Change in length over *dx* : $\delta dx = \frac{\sigma_x\, dx}{E} = \frac{4P\, dx}{\pi E\ (d_1+kx)^2}$

Total change in length: $\delta L = \int_0^L \frac{4P\,dx}{\pi E\,(d_1+kx)^2} = \frac{4P}{\pi E}\left[\frac{(d_1+kx)^{-1}}{-k}\right]_0^L$

*After rearranging the terms*: $\delta L = -\frac{4P}{\pi Ek}\left[\frac{1}{(d_1+kx)}\right]_0^L$

*Upon substituting the limits* : $\delta L = -\frac{4P}{\pi Ek}\left[\frac{1}{(d_1+kL)} - \frac{1}{d_1}\right]$

*After rearranging the terms*: $\delta L = \frac{4P}{\pi Ek}\left[\frac{1}{d_1} - \frac{1}{(d_1+kL)}\right]$

*But* $(d_1 + kL) = d_1 + \frac{d_2 - d_1}{L} L = d_2$

With *the above substitution*: $\delta L = \frac{4P}{\pi Ek}\left[\frac{1}{d_1} - \frac{1}{d_2}\right] = \frac{4P}{\pi Ek}\left[\frac{d_2 - d_1}{d_1 d_2}\right]$

Substituting for $k = \frac{d2-d1}{L}$ in the above expression, following equation for elongation of tapering bar of circular section can be obtained

Total change in length: $\delta L = \frac{4P\,L}{\pi E\,d_1 d_2}$

## Example 4

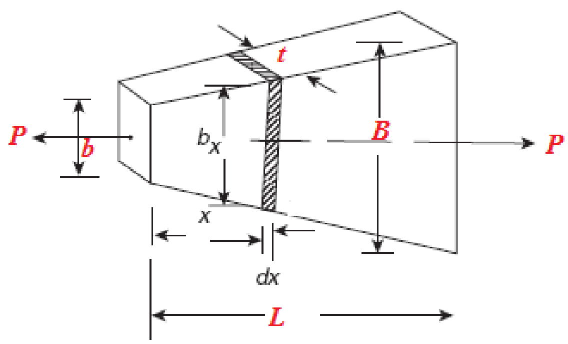
A bar uniformly tapers from diameter 20 mm at one end to diameter 10 mm at the other end over an axial length 300 mm. This is subjected to an axial compressive load of 7.5 kN. If $E$ = 100 kN/mm$^2$, determine the maximum and minimum axial stresses in bar and the total change in length of the bar.

P = 7500 N, E = 100000 N/mm$^2$, $d_1$ = 10mm, $d_2$ = 20mm, L = 300mm

- Minimum compressive stress occurs at $d_2$ = 20mm as the sectional area is maximum.
- Area at $d_2 = \frac{\pi \times 20^2}{4} = 314.16mm^2$
- $\sigma_{min} = \frac{7500}{314.16} = 23.87MPa$
- Maximum compressive stress occurs at $d_1$ = 10mm as the sectional area is minimum.
- Area at $d_1 = \frac{\pi \times 10^2}{4} = 78.54mm^2$
- $\sigma_{min} = \frac{7500}{78.54} = 95.5MPa$
- Total decrease in length: $\delta L = \frac{4P\,L}{E\,d_1 d_2} = \frac{4 \times 7500 \times 300}{\pi \times 100000 \times 10 \times 20} = 0.143$mm

## 1.16 Elongation of tapering bars of rectangular cross section

Consider a bar of same thickness **t** throughout its length, tapering uniformly from a breadth **B** at one end to a breadth **b** at the other end over an axial length **L**. The flat is subjected to an axial force **P** as shown in the figure below.




Since the breadth of the bar is continuously changing, the elongation is first computed over an elementary length and then integrated over the entire length. Consider an elementary strip of breadth $b_x$ and length $dx$ at a distance of $x$ from left end.

Using the principle of similar triangles the following equation for $b_x$ can be obtained

$$b_x = b + \frac{B-b}{L}x = b + kx \text{ , where } k = \frac{B-b}{L}$$

Cross–sectional area of the bar at $x$ : $A_x = b_x t = (b + kx)t$

Axial stress at $x$: $\sigma_x = \frac{P}{A_x} = \frac{P}{(b+kx)t}$

Change in length over $dx$ : $\delta dx = \frac{\sigma_x\, dx}{E} = \frac{P\, dx}{Et(b+kx)}$

Total change in length: $\delta L = \int_0^L \frac{P\, dx}{Et(b+kx)} = \frac{P}{Etk}[ln(b + kx)]_0^L$

*Upon substituting the limits* : $\delta L = \frac{P}{Etk}[\ln(b + kL) - \ln(b)]$

$$But\ (b + kL) = b + \frac{B-b}{L}L = B$$

*With the above substitution*: $\delta L = \frac{P}{Etk}[\ln(B) - \ln(b)] = \frac{P}{Etk}\ln(B/b)$

Substituting for $k = \frac{B-b}{L}$ in the above expression, following equation for elongation of tapering bar of rectangular section can be obtained

$$\delta L = \frac{P\, L}{Et(B-b)}\ln(B/b)$$

## Example 5

An aluminium flat of a thickness of 8 mm and an axial length of 500 mm has a width of 15 mm tapering to 25 mm over the total length. It is subjected to an axial compressive force *P*, so that the total change in the length of flat does not exceed 0.25 mm. What is the magnitude of *P*, if $E$ = 67,000 N/mm$^2$ for aluminium?

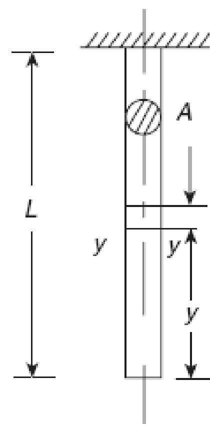
t = 8mm, B = 25mm,b = 15mm, L = 500 mm, δL = 0.25 mm, E = 67000MPa, P =?

$$P = \frac{Et(B-b)\delta L}{ln(B/b)\,L} = \frac{67000\ \times\ 8\ \times(25-15)\times 0.25}{ln(25/15)\times 500} = 5.246kN$$

*Note:*

*Instead of using the formula, this problem can be solved from first principles as indicated in section 1.16.*

## 1.17 Elongation in Bar Due to Self-Weight




Consider a bar of a cross-sectional area of **A** and a length **L** is suspended vertically with its upper end rigidly fixed as shown in the adjoining figure. Let the weight density of the bar is ρ. Consider a section y- y at a distance y from the lower end.

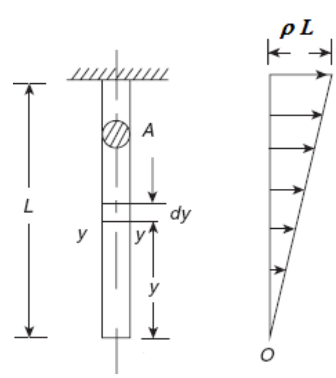
Weight of the portion of the bar below y-y = ρ A y
Stress at y-y : $\sigma_y$ = ρ A y /A = ρ y
Strain at y-y : $\varepsilon_y$ = ρ y / E
Change in length over dy: δdy = ρ y dy / E

Total change in length : $\delta L = \int_0^L \frac{\rho\, y\, dy}{E} = [\frac{\rho y^2}{2E}]_0^L = \frac{\rho L^2}{2E}$

This can also be written as : $\delta L = \frac{(\rho AL)L}{2AE} = \frac{WL}{2AE}$

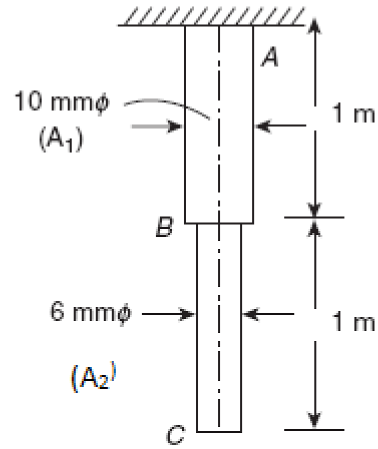
*W = ρ A L represents the total weight of the bar*

*Note:*
*The stress in the bar gradually increases linearly from zero at bottom to ρ L at top as shown below.*



**Example 6**

A stepped steel bar is suspended vertically. The diameter in the upper half portion is 10 mm, while the diameter in the lower half portion is 6 mm. What are the stresses due to self-weight in sections B and A as shown in the figure. E = 200 kN/mm$^2$. Weight density, $\rho$ = 0.7644x10$^{-3}$ N/mm$^3$. What is the change in its length if E = 200000 MPa?




Stress at B will be due to weight of portion of the bar BC

Sectional area of BC: $A_2 = \pi \times 6^2/4 = 28.27$ mm$^2$

Weight of portion BC: $W_2 = \rho A_2 L_2 = 0.7644 \times 10^{-3} \times 28.27 \times 1000 = 21.61$N

Stress at B: $\sigma_B = W_2/A_2 = 21.61/28.27 =$ **0.764 MPa**

Stress at A will be due to weight of portion of the bar BC + AB

Sectional area of AB: $A_1 = \pi \times 10^2/4 = 78.54$ mm$^2$

Weight of portion AB: $W_1 = \rho A_1 L_1 = 0.7644 \times 10^{-3} \times 78.54 \times 1000 = 60.04$N

Stress at A: $\sigma_c = (W_1+W_2)/A_1 = (60.04+ 21.61) / 78.54 =$ **1.04 MPa**

Change in Length in portion BC

This is caused due to weight of BC and is computed as:

$$\delta L_{BC} = \frac{W_2 L_2}{2A_2E} = \frac{21.61\times1000}{2\times28.27\times200000} = 0.00191\text{mm}$$

Change in Length in portion AB

This is caused due to weight of AB and due to weight of BC acting as a concentrated load at B and is computed as:

$$\delta L_{AB} = \frac{W_1 L_1}{2A_1E} + \frac{W_2 L_1}{E\,A_1} = \frac{60.04\times1000}{2\times78.54\times200000} + \frac{21.61\times1000}{200000\times78.54} = 0.0033\text{mm}$$

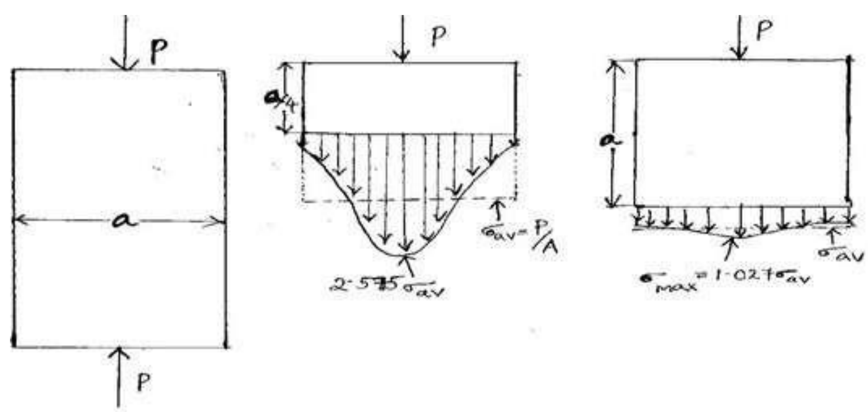
Total change in length = 0.00191+ 0.0033 = **0.00521mm**

## 1.18 Saint Venant's principle

In 1855, the French Elasticity theorist *Adhemar Jean Claude Barre de Saint-Venant* stated that the difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from the load. The stresses and strains in a body at points that are sufficiently remote from points of application of load depend only on the static resultant of the loads and not on the distribution of loads.

Stress concentration is the increase in stress along the cross-section that maybe caused by a point load or by any another discontinuity such as a hole which brings about an abrupt change in the cross sectional area.

In St.Venant"s Principle experiment, we fix two strain gages, one near the central portion of the specimen and one near the grips of the Universal Testing Machine"s (UTM) upper (stationary) holding chuck.. The respective strain values obtained from both the gages are measured and then plotted with respect to time. Since stress is proportional to strain, as per St.Venant"s principle, the stress will be concentrated near the point of application of load. Although the average stress along the uniform cross section remains constant, at the point of application of load, the stress is distributed as shown in figure below with stress being concentrated at the load point. The further the distance from the point of application of load, the more uniform the stress is distributed across the cross section.



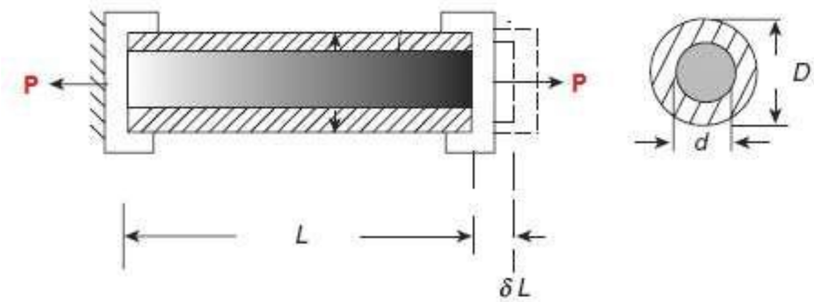

## 1.19 Compound or composite bars

A composite bar can be made of two bars of different materials rigidly fixed together so that both bars strain together under external load. As the strains in the two bars are same, the stresses in the two bars will be different and depend on their respective modulus of elasticity. A stiffer bar will share major part of external load.

In a composite system the two bars of different materials may act as suspenders to a third rigid bar subjected to loading. As the change in length of both bars is the same, different stresses are produced in two bars.

### 1.19.1 Stresses in a Composite Bar

Let us consider a composite bar consisting of a solid bar, of diameter ***d*** completely encased in a hollow tube of outer diameter ***D*** and inner diameter ***d*,** subjected to a tensile force **P** as shown in the following figure.




Let the extension of composite bar of length ***L*** be ***δL***. Let $E_S$ and $E_H$ be the modulus of elasticity of solid bar and hollow tube respectively. Let $\sigma_S$ and $\sigma_H$ be the stresses developed in the solid bar and hollow tube respectively.

Since change in length of solid bar is equal to the change in length of hollow tube, we can establish the relation between the stresses in solid bar and hollow tube as shown below :

$$\frac{\sigma_S L}{E_S} = \frac{\sigma_H L}{E_H} \text{ or } \sigma_S = \sigma_H \frac{E_S}{E_H}$$

Area of cross section of the hollow tube : $A_H = \dfrac{\pi(D^2 - d^2)}{4}$

Area of cross section of the solid bar : $A_S = \dfrac{\pi d^2}{4}$

Load carried by the hollow tube : $P_H = \sigma_H A_H$ and Load carried by the solid bar : $P_S = \sigma_S A_S$

But $P = P_S + P_H = \sigma_S A_S + \sigma_H A_H$

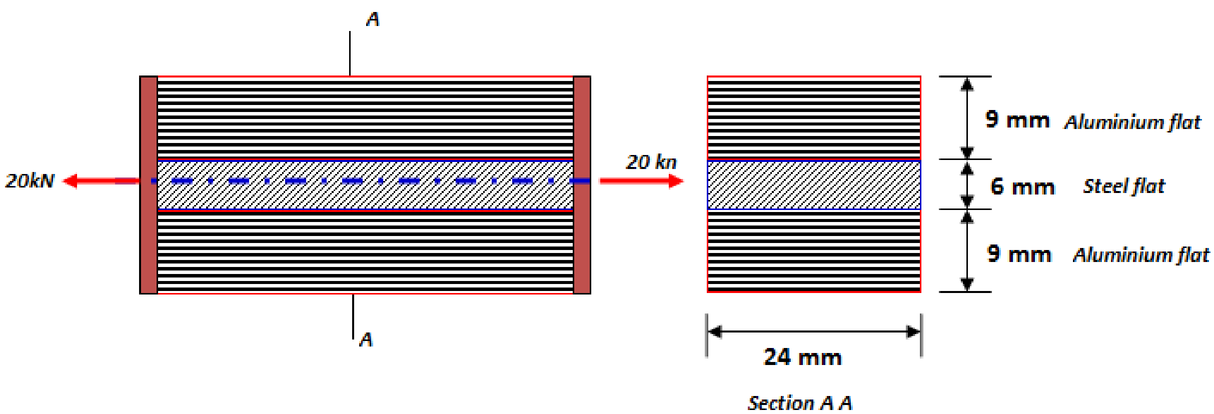
With $\sigma_S = \sigma_H \frac{E_S}{E_H}$, the following equation can be written

$$P = \sigma_H \frac{E_S}{E_H} A_s + \sigma_H A_H = \sigma_H \left(A_H + \frac{E_S}{E_H} A_s\right)$$

$E_S/E_H$ is called *modular ratio*. Using the above equation stress in the hollow tube can be calculated. Next, the stress in the solid bar can be calculated using the equation $P = \sigma_S A_S + \sigma_H A_H$.

## Example 7

A flat bar of steel of 24 mm wide and 6 mm thick is placed between two aluminium alloy flats 24 mm × 9 mm each. The three flats are fastened together at their ends. An axial tensile load of 20 kN is applied to the composite bar. What are the stresses developed in steel and aluminium alloy? Assume $E_S$ = 210000 MPa and $E_A$ = 70000MPa.




Area of Steel flat: $A_S = 24 \times 6 = 144$ mm$^2$

Area of Aluminium alloy flats: $A_A = 2 \times 24 \times 9 = 432$ mm$^2$

Since all the flats elongate by the same extent, we have the condition that $\frac{\sigma_S L}{E_S} = \frac{\sigma_A L}{E_A}$.

The relationship between the stresses in steel and aluminum flats can be established as:

$$\sigma_S = \sigma_A \frac{E_S}{E_A} = 3\,\sigma_A$$

Since $P = P_S + P_A = \sigma_S A_S + \sigma_A A_A$. This can be written as

$$P = 3\sigma_A A_s + \sigma_A A_A = \sigma_A (3A_s + A_A)$$

From which stress in aluminium alloy flat can be computed as:

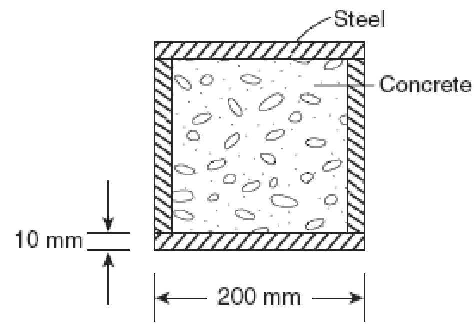
$$\sigma_A = \frac{P}{(3A_s + A_A)} = \frac{20 \times 1000}{(3 \times 144 + 432)} = 23.15MPa$$

Stress in steel flat can be computed as:

$$\sigma_S = 3 \times 23.15 = 69.45MPa$$

## Example 8

A short post is made by welding steel plates into a square section and then filling inside with concrete. The side of square is 200 mm and the thickness $t$ = 10 mm as shown in the figure. The steel has an allowable stress of 140 N/mm$^2$ and the concrete has an allowable stress of 12 N/mm$^2$. Determine the allowable safe compressive load on the post. $E_C$ = 20 GPa, $Es$ = 200 GPa.




Since the composite post is subjected to compressive load, both concrete and steel tube will shorten by the same extent. Using this condition following relation between stresses in concrete and steel can be established.

$$\frac{\sigma_C L}{E_C} = \frac{\sigma_S L}{E_S} \text{ or } \sigma_S = \sigma_C \frac{E_S}{E_C} = 10\,\sigma_C$$

Assume that load is such that $\sigma_s$ = 140 N/mm$^2$. Using the above relationship, the stress in concrete corresponding to this load can be calculated as follows:

$$140 = 10\,\sigma_C \;\; or \;\; \sigma_C = 14\, N/mm^2 > 12 \text{ N/mm}^2$$

*Hence the assumed load is not a safe load.*

Instead assume that load is such that $\sigma_c$ = 12 N/mm$^2$. The stress in steel corresponding to this load can be calculated as follows:

$$\sigma_s = 12 \times 10 \;\; or \;\; \sigma_s = 120\, N/mm^2 < 140 \text{ N/mm}^2$$

*Hence the assumed load is a safe load which is calculated as shown below.*

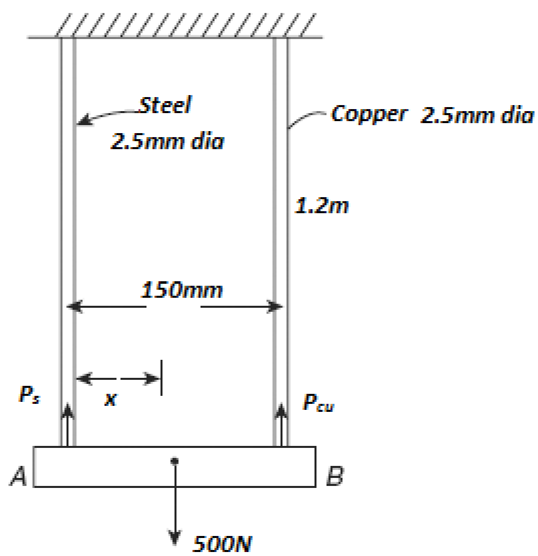
Area of concrete section Ac = 180 x180 = 32400mm$^2$.

Area of steel tube As = 200 x 200 – 32400 = 7600 mm$^2$.

$$P = \sigma_C A_C + \sigma_s A_s = 12 \times 32400 + 120 \times 7600 = 1300.8kN$$

**Example 9**

A rigid bar is suspended from two wires, one of steel and other of copper, length of the wire is 1.2 m and diameter of each is 2.5 mm. A load of 500 N is suspended on the rigid bar such that the rigid bar remains horizontal. If the distance between the wires is 150 mm, determine the location of line of application of load. What are the stresses in each wire and by how much distance the rigid bar comes down? Given $E_s$ = 3$E_{cu}$= 201000 N/mm$^2$.




i. Area of copper wire (Acu) = Area of steel wire(As) = π x 2.5$^2$/4 = 4.91 mm$^2$

ii. For the rigid bar to be horizontal, elongation of both the wires must be same. This condition leads to the following relationship between stresses in steel and copper wires as:

$$\sigma_s = \frac{E_s}{E_{cu}} \sigma_{cu} = 3\sigma_{cu}$$

iii. Using force equilibrium, the stress in copper and steel wire can be calculated as:

$$P = P_s + P_{cu} = \sigma_s A_s + \sigma_{cu} A_{cu} = 3\, \sigma_{cu} A_s + \sigma_{cu} A_{cu} = \sigma_{cu} (3A_s + A_{cu})$$

$$\sigma_{cu} = \frac{P}{(A_{cu} + 3A_S)} = \frac{500}{(4.91 + 3 \times 4.91)} = 25.46\ MPa$$

$$\sigma_s = 3 \times 25.46 = 76.37\ MPa$$

iv. Downward movement of rigid bar = elongation of wires

$$\delta L_s = \frac{\sigma_s}{E_s} L = \frac{76.37}{201000} \times 1200 = 0.456\ mm$$

v. Position of load on the rigid bar is computed by equating moments of forces carried by steel and copper wires about the point of application of load on the rigid bar.

$$P_s\, x = P_c\,(150 - x)$$
$$(76.37 \times 4.91)x = (25.46 \times 4.91)\,(150 - x)$$
$$\frac{x}{150 - x} = 0.333$$

$$x = 37.47mm\ from\ steel\ wire$$

Note:

*If the load is suspended at the centre of rigid bar, then both steel and copper wire carry the same load. Hence the stress in the wires is also same. As the moduli of elasticity of wires are different, strains in the wires will be different. This results in unequal elongation of wires causing the rigid bar to rotate by some magnitude. This can be prevented by offsetting the load or with wires having different length or with different diameter such that elongation of wires will be same.*

## Example 10

A load of 2MN is applied on a column 500mm x 500mm. The column is reinforced with four steel bars of 12mm dia, one in each corner. Find the stresses in concrete and steel bar. Es = 2.1 x$10^5$ N/mm$^2$ and Ec = 1.4 x $10^4$ N/mm$^2$.

i. Area of steel bars: As= 4 x (π x $12^2$/4) = 452.4 mm$^2$

ii. Area of concrete: Ac = 500 x500 – 452.4 = 249547.6 mm$^2$

iii. Relation between stress in steel and concrete : $\sigma_s = \frac{E_S}{E_c}\sigma_c = \frac{2.1\times10^5}{1.4\times10^4}\sigma_c = 15\sigma_c$

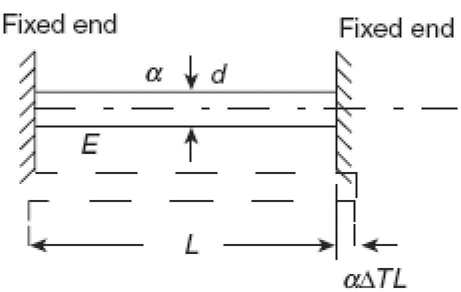
iv. $P = P_s + P_c = \sigma_s A_s + \sigma_c A_c = 15\,\sigma_c A_s + \sigma_c A_c = \sigma_c\,(15A_s + A_c)$

v. $Stress\ in\ concrete\ \sigma_c = \frac{P}{(A_c + 15A_S)} = \frac{2\times10^6}{(\,249547.6 + 15\times452.4)} = 7.8\ MPa$

vi. $Stress\ in\ steel\ \sigma_s = 15\sigma_c = 15 \times 7.8 = 117MPa$

## 1.20 Temperature stresses in a single bar

If a bar is held between two unyielding (rigid) supports and its temperature is raised, then a compressive stress is developed in the bar as its free thermal expansion is prevented by the rigid supports. Similarly, if its temperature is reduced, then a tensile stress is developed in the bar as its free thermal contraction is prevented by the rigid supports. Let us consider a bar of diameter $\boldsymbol{d}$ and length $\boldsymbol{L}$ rigidly held between two supports as shown in the following figure. Let $\boldsymbol{\alpha}$ be the coefficient of linear expansion of the bar and its temperature is raised by $\Delta \boldsymbol{T}$ (°C)




- Free thermal expansion in the bar = $\alpha \ \Delta T \ L$.
- Since the supports are rigid, the final length of the bar does not change. The fixed ends exert compressive force on the bar so as to cause shortening of the bar by $\alpha \ \Delta T \ L$.
- Hence the compressive strain in the bar = $\alpha \ \Delta T \ L \ / \ L = \alpha \ \Delta T$
- Compressive stress = $\alpha \ \Delta T \ E$
- Hence the thermal stresses introduced in the bar = $\alpha \ \Delta T \ E$

*Note:*

*The bar can buckle due to large compressive forces generated in the bar due to temperature increase or may fracture due to large tensile forces generated due to temperature decrease.*

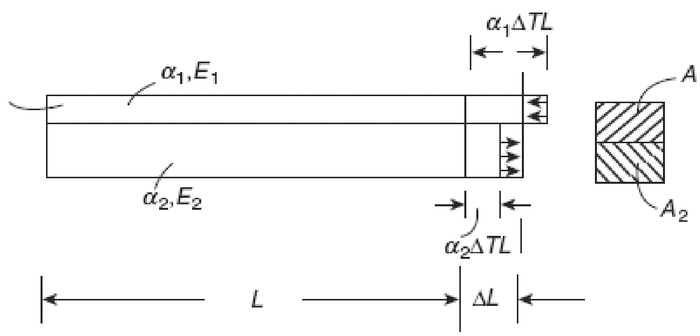
## Example 11

A rail line is laid at an ambient temperature of 30°C. The rails are 30 m long and there is a clearance of 5 mm between the rails. If the temperature of the rail rises to 60°C, what is the stress developed in the rails?. Assume $\alpha = 11.5 \times 10^{-6}$/°C, E = $2{,}10{,}000 \ N/mm^2$

- $L$ = 30,000 mm, $\alpha = 11.5 \times 10^{-6}$/°C, Temperature rise $\Delta T$ = 60-30 = 30°C
- Free expansion of rails = $\alpha\, \Delta T\, L = 11.5 \times 10^{-6} \times 30 \times 30000$ = 10.35mm
- Thermal expansion prevented by rails = Free expansion – clearance = 10.35 – 5 = 5.35mm
- Strain in the rails $\varepsilon$ = 5.35/30000 = 0.000178
- Compressive stress in the rails = $\varepsilon$ x E = 0.000178 x 210000 =**<u>37.45N/mm²</u>**.

## 1.21 Temperature Stresses in a Composite Bar

A composite bar is made up of two bars of different materials perfectly joined together so that during temperature change both the bars expand or contract by the same amount. Since the coefficient of expansion of the two bars is different thermal stresses are developed in both the bars. Consider a composite bar of different materials with coefficients of expansion and modulus of elasticity, as $\boldsymbol{\alpha_1}$, $\boldsymbol{E_1}$ and $\boldsymbol{\alpha_2}$, $\boldsymbol{E_2}$, respectively, as shown in the following figure. Let the temperature of the bar is raised by $\Delta \boldsymbol{T}$ *and* $\boldsymbol{\alpha_1} > \boldsymbol{\alpha_2}$




Free expansion in bar 1 = $\alpha_1\, \Delta T\, L$ and Free expansion in bar 2 = $\alpha_2\, \Delta T\, L$. Since both the bars expand by $\Delta L$ together we have the following conditions:

- Bar 1: $\Delta L < \alpha_1\, \Delta T\, L$. The bar gets compressed resulting in compressive stress
- Bar 2: $\Delta L > \alpha_2 \Delta T\, L$. The bar gets stretched resulting in tensile stress.

Compressive strain in Bar 1 : $s_1 = \frac{\alpha 1 \Delta TL - \Delta L}{L}$

Tensile strain in Bar 2 : $s_2 = \frac{\Delta L - \alpha 2 \Delta TL}{L}$

$$s_1 + s_2 = \frac{\alpha_1 \Delta TL - \Delta L}{L} + \frac{\Delta L - \alpha_2 \Delta TL}{L} = (\alpha_1 - \alpha_2)\Delta T$$

Let $\sigma_1$ and $\sigma_2$ be the temperature stresses in bars. The above equation can be written as:

$$\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2)\Delta T$$

In the absence of external forces, for equilibrium, compressive force in Bar 1 = Tensile force in Bar 2. This condition leads to the following relation

$$\sigma_1 A_1 = \sigma_2 A_2$$

Using the above two equations, temperature stresses in both the bars can be computed. This is illustrated in the following example.

*Note:*

*If the temperature of the composite bar is reduced, then a tensile stress will be developed in bar 1 and a compressive stress will be developed in bar 2 , since* $\boldsymbol{\alpha_1} > \boldsymbol{\alpha_2}$.

## Example 12

A steel flat of 20 mm × 10 mm is fixed with aluminium flat of 20 mm × 10 mm so as to make a square section of 20 mm × 20 mm. The two bars are fastened together at their ends at a temperature of 26°C. Now the temperature of whole assembly is raised to 55°C. Find the stress in each bar. $E_s$ = 200 GPa, $E_a$ = 70 GPa, $\alpha_s = 11.6 \times 10^{-6}$/°C, $\alpha_a = 23.2 \times 10^{-6}$/°C.

- Net temperature rise, $\Delta T = 55 - 26 = 29$°C.
- Area of Steel flat (As) = Area of Aluminium flat (Aa) = 20 x10 =200 mm2
- For equilibrium, $\sigma_s A_s = \sigma_a A_a$ ; $\sigma_s = \sigma_a$ will be one of the conditions to be satisfied by the composite assembly.
- $But\ \frac{\sigma a}{E_a} + \frac{\sigma s}{E_s} = (\alpha_a - \alpha_s)\Delta T = (23.2 - 11.6) \times 29 \times 10^{-6} = 0.000336$
- $\frac{\sigma s}{200000} + \frac{\sigma a}{70000} = 0.000336$
- $270000\, \sigma_s = 4709600$ ;
- $\sigma_s(tensile) = \sigma_a(compressive) = 17.44MPa$ as $\alpha_a > \alpha_s$

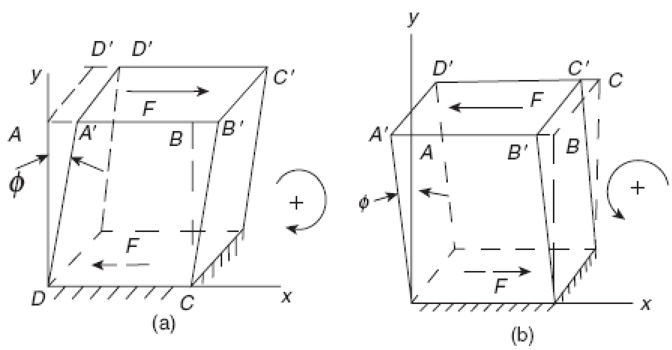
## Example 13

A flat steel bar of 20 mm × 8 mm is placed between two copper bars of 20 mm × 6 mm each so as to form a composite bar of section of 20 mm × 20 mm. The three bars are fastened together at their ends when the temperature of each is 30°C. Now the temperature of the whole assembly is raised by 30°C. Determine the temperature stress in the steel and copper bars. $E_s$ = 2$E_{cu}$= 210 kN/mm$^2$, $\alpha_s$ = 11 × 10$^{-6}$/°C, $\alpha_{cu}$ = 18 × 10$^{-6}$/°C.

- Net temperature rise, $\Delta T$ = 30°C.
- Area of Steel flat ($A_s$) = 20 x 8 = 160 mm$^2$
- Area of Copper flats ($A_{cu}$) = 2 x 20 x 6 =240 mm$^2$
- For equilibrium, $\sigma_s A_s = \sigma_{cu} A_{cu}$ ; $\sigma_s = 1.5\,\sigma_{cu}$ will be one of the conditions to be satisfied by the composite assembly.
- $But\,\frac{\sigma_{cu}}{E_{cu}} + \frac{\sigma_s}{E_s} = (\alpha_{cu} - \alpha_s)\Delta T = (18 - 11) \times 30\ \times 10^{-6} = 0.00021$
- $\frac{\sigma cu}{105000} + \frac{1.5\sigma cu}{210000} = 0.00021$
- $\sigma_{cu}$ = 12.6MPa (compressive) and $\sigma_s$ = 18.9MPa (tensile) as $\alpha_{cu} > \alpha_s$

## 1.22 Simple Shear stress and Shear Strain

Consider a rectangular block which is fixed at the bottom and a force $F$ is applied on the top surface as shown in the figure (a) below.




Equal and opposite reaction $F$ develops on the bottom plane and constitutes a couple, *tending to rotate the body in a clockwise direction*. This type of shear force is a *positive shear force* and the shear force per unit surface area on which it acts is called *positive shear stress* ($\tau$). If force is applied in the opposite direction as shown in Figure (b), then they are termed as negative shear force and shear stress.

The *Shear Strain* ($\phi$) = AA''/AD = tan$\phi$. Since $\phi$ is a very small quantity, tan$\phi \approx \phi$. Within the elastic limit, $\tau \propto \phi$ *or* $\tau$ = $\boldsymbol{G}\ \phi$

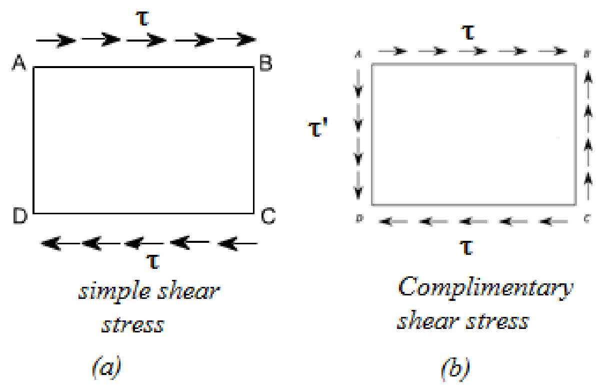
The constant of proportionality $\boldsymbol{G}$ is called *rigidity modulus or shear modulus.*

*Note:*

*Normal stress is computed based on area perpendicular to the surface on which the force is acting, while, the shear stress is computed based on the surface area on which the force is acting. Hence shear stress is also called tangential stress.*

## 1.23 Complementary Shear Stresses

Consider an element ABCD subjected to shear stress ($\tau$) as shown in figure (a). We cannot have equilibrium with merely equal and opposite tangential forces on the faces AB and CD as these forces constitute a couple and induce a turning moment. The statical equilibrium demands that there must be tangential components ($\tau''$) along AD and CB such that that can balance the turning moment. These tangential stresse ($\tau''$) is termed as *complimentary shear stress*.




Let t be the thickness of the block. Turning moment due to $\tau$ will be $(\tau \text{ x t x } L_{AB})\, L_{BC}$ and Turning moment due to $\tau'$ will be $(\tau'' \text{ x t x } L_{BC})\, L_{AB.}$ Since these moments have to be equal for equilibrium we have:

$$(\tau \text{ x t x } L_{AB})\, L_{BC} = (\tau'' \text{ x t x } L_{BC})\, L_{AB.}$$

From which it follows that $\tau = \tau''$ , that is, intensities of shearing stresses across two mutually perpendicular planes are equal.

## 1.24 Volumetric strain

This refers to the slight change in the volume of the body resulting from three mutually perpendicular and equal direct stresses as in the case of a body immersed in a liquid under pressure. This is defined as the *ratio of change in volume to the original volume* of the body.

Consider a cube of side „**a**" strained so that each side becomes „**a** ± δ**a'.**

- Hence the linear strain = δa/a.
- Change in volume = $(a \pm \delta a)^3 - a^3 = \pm 3a^2\delta a$. (ignoring small higher order terms)
- Volumetric strain $\varepsilon_v = \pm 3a^2\, \delta a/a^3 = \pm 3\, \delta a/a$
- *The volumetric strain is three times the linear strain*

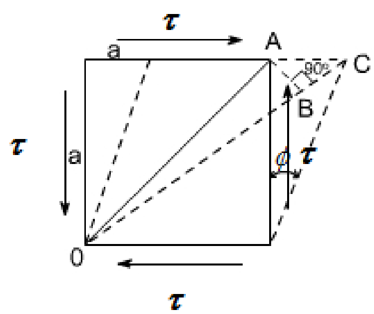
## 1.25 Bulk Modulus

This is defined as the ratio of the normal stresses (p) to the volumetric strain ($\varepsilon_v$) and denoted by **'K'.** Hence **K = p/$\varepsilon_v$** . This is also an elastic constant of the material in addition to E, G and ν.

## 1.26 Relation between elastic constants

### 1.26.1 Relation between E,G and ν

Consider a cube of material of side „a' subjected to the action of the shear and complementary shear stresses and producing the deformed shape as shown in the figure below.

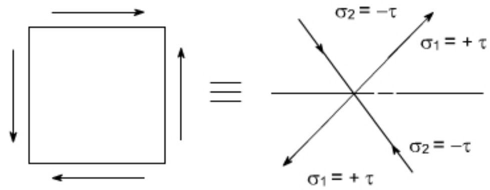



- Since, within elastic limits, the strains are small and the angle ACB may be taken as $45^0$.
- Since angle between OA and OB is very small hence OA ≈ OB. BC, is the change in the length of the diagonal OA
- Strain on the diagonal OA = Change in length / original length = BC/OA

$$= AC \cos 45/ (a/\sin 45) = AC/ 2a = a\,\phi / 2\,a \;= \phi / 2$$

- It is found that *strain along the diagonal is numerically half the amount of shear stain*.
- But from definition of rigidity modulus we have, $G = \tau / \phi$
- Hence, Strain on the diagonal OA = $\tau / 2G$

The shear stress system is equivalent or can be replaced by a system of direct stresses at $45^0$ as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear stress.



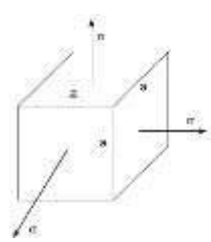

Strain in diagonal OA due to direct stresses = $\frac{\sigma_1}{E} - \frac{\sigma_2}{E} = \frac{c}{E} + \frac{c}{E} = \frac{c}{E}(1 + \ )$

Equating the strain in diagonal OA we have $\frac{c}{2G} = \frac{c}{E}(1 + \ )$

**Relation between E,G and ν can be expressed as :** $E = 2G(1 + v)$

### 1.26.2 Relation between E,K and ν

Consider a cube subjected to three equal stresses a shown in the figure below.



Strain in any one direction = $\frac{\sigma}{E} - \frac{\sigma}{E} - \frac{\sigma}{E} = \frac{\sigma}{E}(1 - 2\ )$

Since the volumetric strain is three times the linear strain: $s_v = 3\frac{\sigma}{E}(1 - 2\ )$

From definition of bulk modulus : $s_v = \frac{\sigma}{K}$

$$3\frac{\sigma}{E}(1 - 2\ ) = \frac{\sigma}{K}$$

**Relation between E,K and ν can be expressed as :** $E = 3K(1 - 2v)$

*Note: Theoretically* ν *< 0.5 as E cannot be zero*

### 1.26.3 Relation between E, G and K

We have E = 2G(1+ν) from which ν = (E - 2G) / 2G
We have E = 3K(1-2ν) from which ν = (3K -E) / 6K

(E - 2G) / 2G = (3K -E) / 6K or (6EK - 12GK) = (6GK - 2EG) or 6EK+2EG = (6GK +12GK)

**Relation between E,G and K can be expressed as:** $E = \frac{9GK}{(3K+G)}$

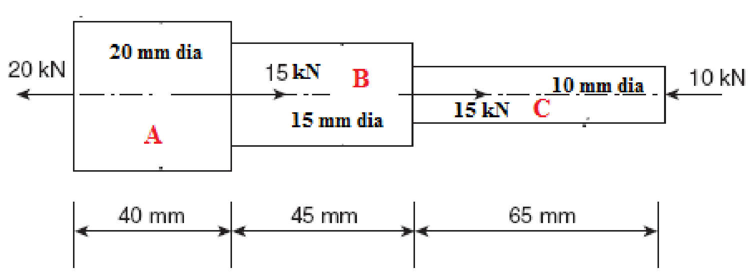
## 1.27 Exercise problems

1. A steel bar of a diameter of 20 mm and a length of 400 mm is subjected to a tensile force of 40 kN. Determine (a) the tensile stress and (b) the axial strain developed in the bar if the Young‟s modulus of steel $E$ = 200 kN/mm$^2$
   *Answer: (a) Tensile stress = 127.23MPa, (b) Axial strain = 0.00064*

2. A 100 mm long bar is subjected to a compressive force such that the stress developed in the bar is 50 MPa. (a) If the diameter of the bar is 15 mm, what is the axial compressive force? (b) If $E$ for bar is 105 kN/mm$^2$, what is the axial strain in the bar?
   *Answer: (a) Compressive force = 8.835 kN, (b) Axial strain = 0.00048*

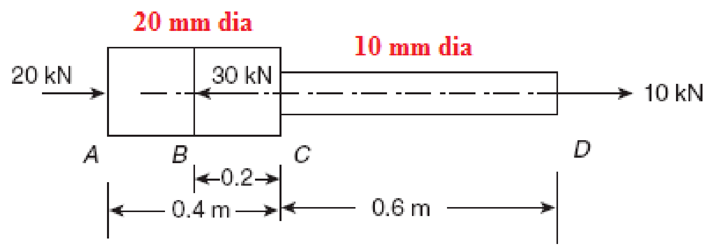
3. A steel bar of square section 30 × 30 mm and a length of 600 mm is subjected to an axial tensile force of 135 kN. Determine the changes in dimensions of the bar. $E$ = 200 kN/mm$^2$, $\nu$ = 0.3.
   *Answer: Increase in length* $\delta l$ = 0.45 mm, Decrease in breadth $\delta b = 6.75 \times 10^{-3}$ mm,

4. A stepped circular steel bar of a length of 150 mm with diameters 20, 15 and 10 mm along lengths 40, 50 and 65 mm, respectively, subjected to various forces is shown in figure below. If E = 200 kN/mm$^2$, determine the total change in its length.




   *Answer : Total decrease in length = 0.022mm*

5. A stepped bar is subjected to axial loads as shown in the figure below. If $E$ = 200 GPa, calculate the stresses in each portion $AB$, $BC$ and $CD$. What is the total change in length of the bar?

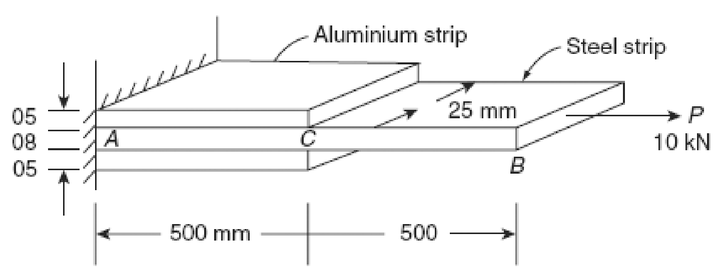



*Answer: Total increase in length = 0.35mm*

6. A 400-mm-long aluminium bar uniformly tapers from a diameter of 25 mm to a diameter of 15 mm. It is subjected to an axial tensile load such that stress at middle section is 60 MPa. What is the load applied and what is the total change in the length of the bar if E = 67,000 MPa? (*Hint: At the middle diameter = (25+15)/2 = 20 mm*).
*Answer: Load = 18.85kN, Increase in length = 0.382 mm*

7. A short concrete column of 250 mm × 250 mm in section strengthened by four steel bars near the corners of the cross-section. The diameter of each steel bar is 30 mm. The column is subjected to an axial compressive load of 250 kN. Find the stresses in the steel and the concrete. Es = 15 Ec = 210 GPa. If the stress in the concrete is not to exceed 2.1 $N/mm^2$, what area of the steel bar is required in order that the column may support a load of 350 kN?
*Answer: Stress in concrete = 2.45*$N/mm^2$, *Stress in steel = 36.75*$N/mm^2$, *Area of steel = 7440* $mm^2$

8. Two aluminium strips are rigidly fixed to a steel strip of section 25 mm × 8 mm and 1 m long. The aluminium strips are 0.5 m long each with section 25 mm × 5 mm. The composite bar is subjected to a tensile force of 10 kN as shown in the figure below. Determine the deformation of point B. Es = 3EA = 210 $kN/mm^2$. *Answer: 0.203mm*
(*Hint: Portion CB is a single bar, Portion AC is a composite bar. Compute elongation separately for both the portions and add*)

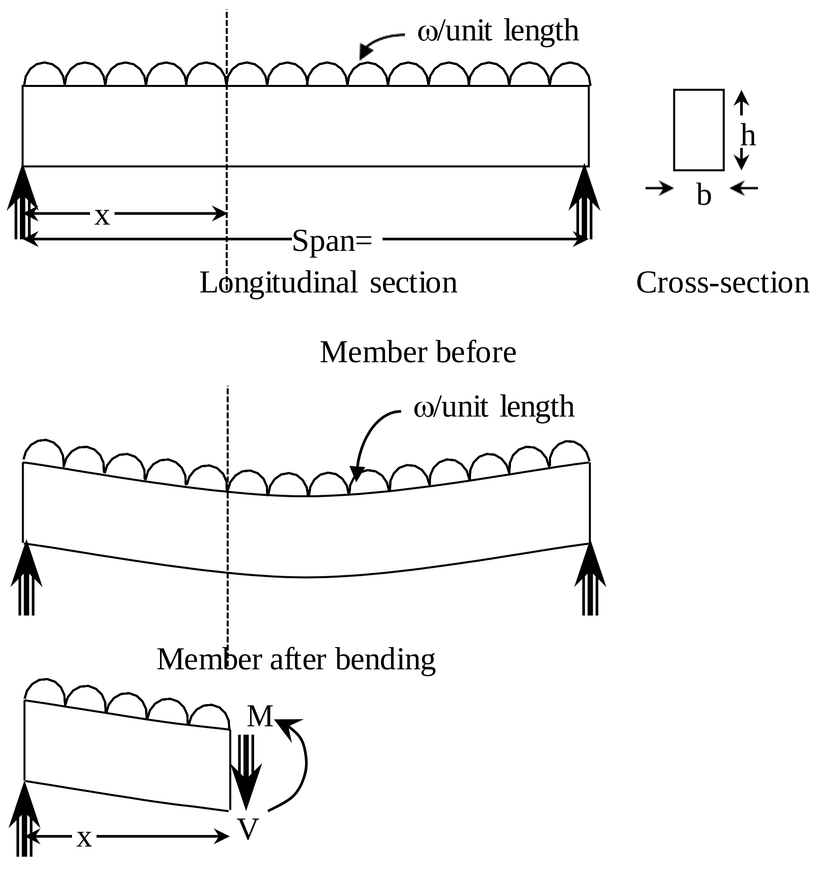


## BENDING MOMENT AND SHEAR FORCES

**INTRODUCTIO N**

Beam is a structural member which has negligible cross- section compared to its length. It carries load perpendicular to the axis in the plane of the beam. Due to the loading on the beam, the beam deforms and is called as deflection in the direction of loading. This deflection is due to bending moment and shear force generated as resistance to the bending. Bending Moment is defined as the internal resistance moment to counteract the external moment due to the loads and mathematically it is equal to algebraic sum of moments of the loads acting on one side of the section. It can also be defined as the unbalanced moment on the beam at that section.

Shear force is the internal resistance developed to counteract the shearing action due to external load and mathematically it is equal to algebraic sum of vertical loads on one side of the section and this act tangential to cross section. These two are shown in Fig 3.01 (a).




Unbalanced Moment = Bending Moment (M) & Unbalanced Force = Shear Force

Fig. 3.01 (a)

For shear force Left side Upward force to the section is Positive (LUP) and Right side Upward force to the section is Negative (RUN) as shown in Fig. 3.01 (b).

For Bending Moment, Moment producing sagging action to the beam or clockwise moment to the left of the section and anti- clockwise moment to the right of the section is treated as positive and Moment producing hogging action to the beam or anti- clockwise moment to the left of the section and clockwise moment to the right of the section is treated as Negative as shown in Fig. 3.01(b).

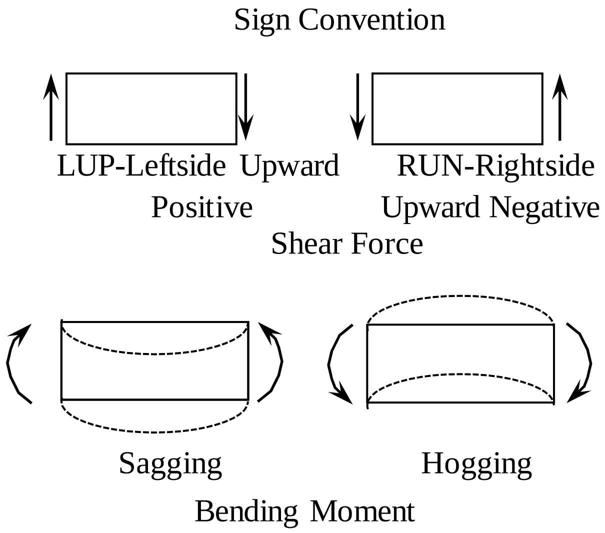



Fig. 3.01 (b)

**Elastic Curve**

Generally the beam is represented by a line and the beam bends after the loading. The depiction of the bent portion of the beam is known as elastic curve.

The shape of the elastic curve is the best way to find the sign of the Bending Moment as shown in the Fig. 3.02

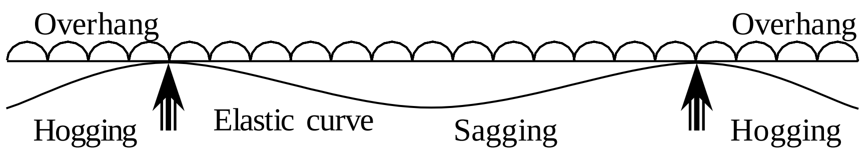



Fig. 3.02 Elastic Curve

**Support Reactions:**

The various structural members are connected to the surroundings by various types of supports .The structural members exert forces on supports known as action. Similarly supports exert forces on structural members known as reaction.

A beam is a horizontal member, which is generally placed on supports.

The beam is subjected to the vertical forces known as action. Supports exe rt forces on beam known as reaction.

**Types of supports:**

1) Simple supports
2) Roller supports
3) Hinged or pinned supports
4) Fixed supports

**1) Simple supports:**

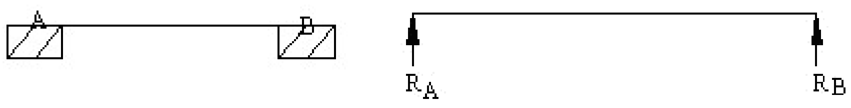



Fig. 3.03

Simple supports are those supports, which exert reactions perpendicular to the plane of support. It restricts the translation of body in one direction only, but not rotation.

**2) Roller supports:**

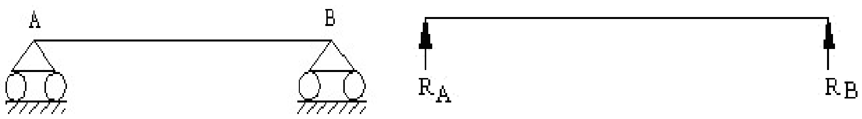



Fig. 3.04

Roller supports are the supports consisting of rollers which exert reactions perpendicular to the plane of the support. They restrict translation along one direction and no rotation.

**3) Hinged or Pinned supports:**

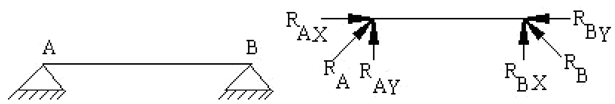



Fig. 3.05

Hinged supports are the supports which exert reactions in any direction but for our convenient point of view it is resolved in to two components. Therefore hinged supports restrict translation in both directions. But rotation is possible.

**4) Fixed supports:**

Fixed supports are those supports which restricts both translation and rotation of the body. Fixed supports develop an internal moment known as restraint moment to prevent the rotation of the body.

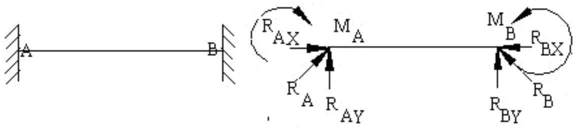



Fig. 3.06

**Types of Beams:-**

**1) Simply supported Beam:**


Fig. 3.07

It is a beam which consists of simple supports. Such a beam can resist forces normal to the axis of the beam.

2) **Continuous Beam**:

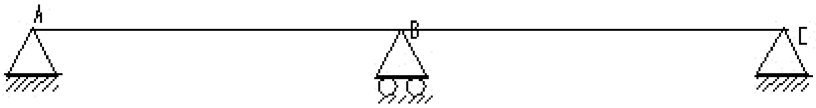



Fig. 3.08

It is a beam which consists of three or more supports.

3) **Cantilever beam**:

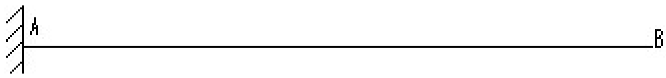



Fig. 3.09

It is a beam whose one end is fixed and the other end is free.

**3) Propped cantilever Beam**:

It is a beam whose one end is fixed and other end is simply supported.

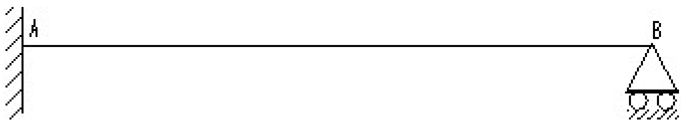



Fig. 3.10

**4) Overhanging Beam**:

It is a beam whose one end is exceeded beyond the support.

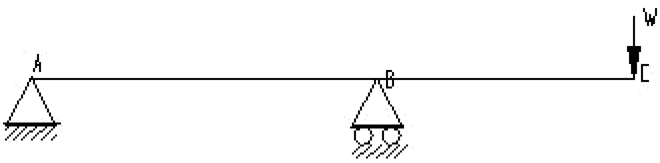



Fig.3.11

**Types of loads:**

**1) Concentrated load**: A load which is concentrated at a point in a beam is known as concentrated load.

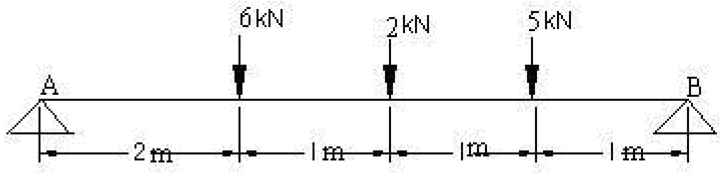



Fig. 3.12

**2) Uniformly Distributed load**: A load which is distributed uniformly along the entire length of the beam is known as Uniformly Distributed Load.

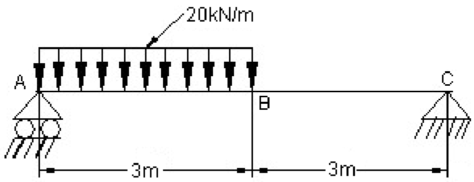



Fig. 3.13

Convert the U.D.L. into point load which is acting at the centre of particular span

Magnitude of point load=20KN/mx3m=60kN

**3) Uniformly Varying load**: A load which varies with the length of the beam is known as Uniformly Varying load

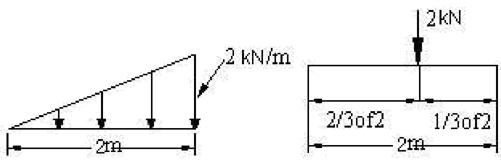



Fig. 3.14

Magnitude of point load=Area of triangle and which is acting at the C.G. of triangle.

**Problems on Equilibrium of coplanar non concurrent force system.**

**Tips to find the support reactions:**

1) In coplanar concurrent force system, three conditions of equilibrium can be applied namely

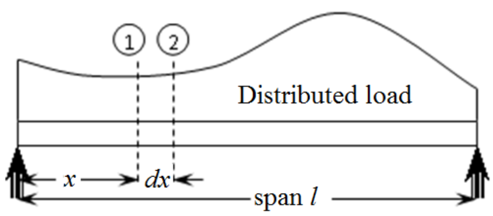
$$\sum Fx = 0, \quad \sum Fy = 0 \text{ and } \Sigma M = 0$$

2) Draw the free body diagram of the given beam by showing all the forces and reactions acting on the beam

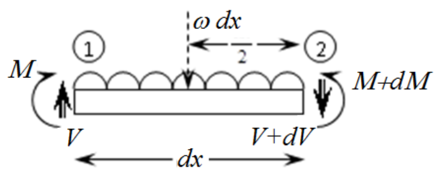
3) Apply the three conditions of equilibrium to calculate the unknown reactions at the supports. **Determinate structures** are those which can be solved with the fundamental equations of equilibrium. i.e. the 3 unknown reactions can be solved with the three equations of equilibrium.

***Relationship between Uniformly distributed load (udl), Shear force and Bending Moment.***

Consider a simply supported beam subjected to distributed load ω which is a function of *x* as shown in Fig. 3.15(a). Consider section 11 at a distance *x* from left support and another section 22 at a small distance *dx* from section 11. The free body diagram of the element is as shown in Fig. 3.15(b). To the left of the section 11 the internal force *V* and the moment *M* acts in the +ve direction. To the right of the section 22 the internal force and the moment are assumed to increase by a small amount and are respectively *V*+*dV* and *M*+*dM* acting in the +ve direction.




Longitudinal section of the loaded beam




Free body diagram of the element of the beam

For the equilibrium of the system, the algebraic sum of all the vertical forces must be zero.

$\rightarrow +\text{ve}\sum V = 0;$

$V - \omega dx - (V + dV) = 0$

$-\omega dx - dV = 0$

$$-\omega = \frac{dV}{dx} \qquad \text{...(01)}$$

Eq. 01 the udl at any section is given by the negative slope of shear force with respect to distance $x$ or negative udl is given by the rate of change of shear force with respect to distance $x$.

Within a limit of distributed force $\omega_1$ and $\omega_2$ over a distance of $a$, shear force is written as

$V = \int_{\omega}^{\omega_2} -\omega dx$

For the equilibrium of the system, the algebraic sum Moments of all the forces must be zero. Taking moment about section 22

$\sum M = 0;$

$M + Vdx - (\omega dx)\left(\frac{dx}{2}\right) - (M + dM) = 0$

Ignoring the higher order derivatives, we get

$Vdx - dM = 0$

$$\text{or } V = \frac{dM}{dx} \qquad 02$$

Eq. 02 shows the shear force at any section is given by rate of change in bending moment with respect to distance $x$.

Within a limit of distributed force $\omega_1$ and $\omega_2$ and shear force $V_1$ and $V_2$ over a distance of $a$, we can write bending moment as

$M = \int_{V_1}^{V_2} Vdx$

**Point of contra flexure or point of inflection.**

These are the points where the sign of the bending moment changes, either from positive to negative or from negative to positive. The bending moment at these points will be zero.

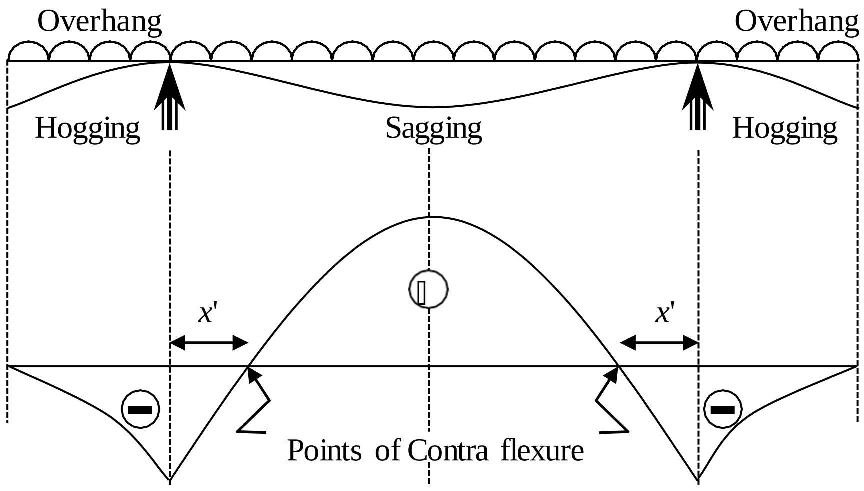



Fig. 3.16 Bending Moment Diagram

**Procedure to draw Shear Force and Bending Moment Diagram**

- Determine the reactions including reactive moments if any using the conditions of equilibrium viz. $\sum H = 0$; $\sum V = 0$; $\sum M = 0$

**Shear Force Diagram (SFD)**

- Draw a horizontal line to represent the beam equal to the length of the beam to some scale as zero shear line.
- The shear line is vertical under vertical load, inclined under the portion of uniformly distributed load and parabolic under the portion of uniformly varying load. The shear line will be horizontal under no load portion. Remember that the shear force diagram is only concerned with vertical loads only and not with horizontal force or moments.
- Start from the left extreme edge of the horizontal line (For a cantilever from the fixed end), draw the shear line as per the above described method. Continue until all the loads are completed and the check is that the shear line should terminate at the horizontal line.
- The portion above the horizontal line is positive shear force and below the line is negative shear force.
- To join the shear line under the portion of uniformly varying load, which is a parabola, it is to be remembered that the parabola should be tangential to the horizontal if the

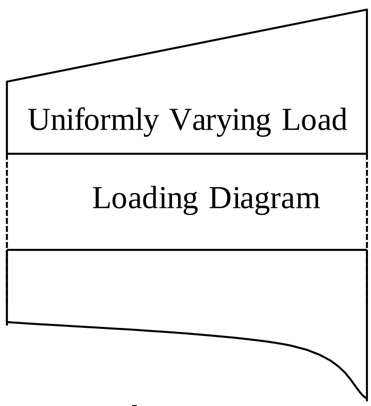



Fig. 3.17 Shear Force Diagram

corresponding load at the loading diagram is lesser and will be tangential to vertical if the corresponding load at the loading diagram is greater.

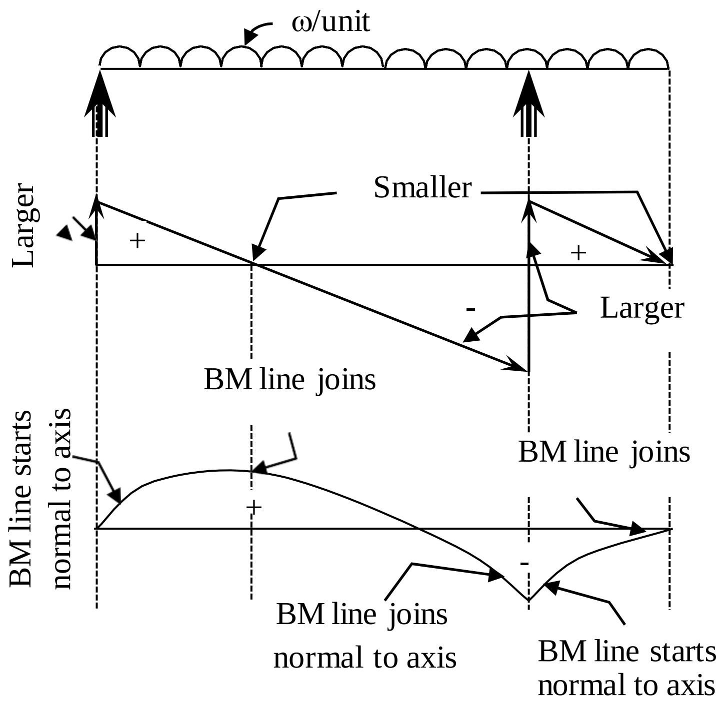



Fig. 3.18 SFD, BMD and Loading Diagrams

**Bending Moment Diagram (BMD)**

- Draw a horizontal line to represent the beam equal to the length of zero shear line under the SFD.
- The Bending Moment line is vertical under the applied moment, inclined or horizontal under the no load portion, parabolic under the portion of uniformly distributed load and cubic parabola under the portion of uniformly varying load.
- Compute the Bending Moment values as per the procedure at the salient points.
- Bending Moment should be computed just to the left and just to the right under section where applied moment is acting. i.e. $M_{AL}$ and $M_{AR}$. Once the applied moment is to be ignored and next the moment is to be considered as per the sign convention.
- Draw these values as vertical ordinates above or below the horizontal line corresponding to positive or negative values.

- Start the Bending Moment line from the left extreme edge of the horizontal line, draw as per the above described method under prescribed loading conditions. Continue until the end of the beam and the check is that the line should terminate at the horizontal line.
- The portion above the horizontal line is positive Bending Moment and below the line is negative Bending Moment.
- Locate the point of Maximum Bending Moment. It occurs at the section where Shear Force is zero.
- Locate the Point of Contra flexure where the Bending Moment line crosses the horizontal line. i.e. the sign of Bending Moment line changes its sign.

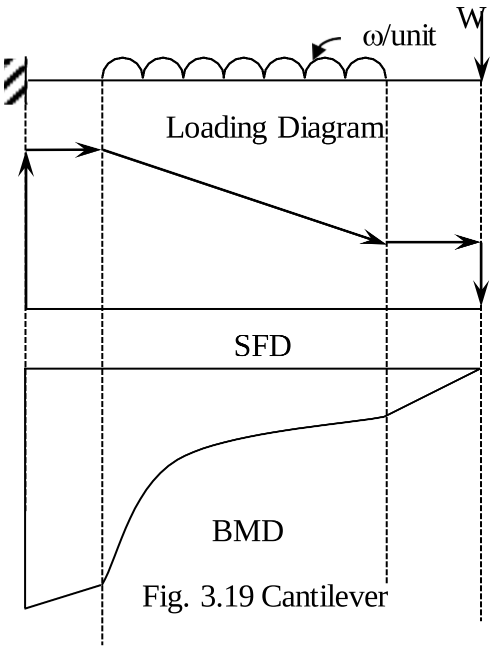


Fig. 3.19 Cantilever

To join the Bending Moment line under the portion of uniformly distributed load which is a parabola, it is to be remembered that the parabola should be tangential to the horizontal if the corresponding shear force value at the loading diagram is lesser and will be tangential to vertical if the corresponding shear force line at the shear force diagram is greater as shown in Fig. 3.17.

In case of the beam being a ***Cantilever***, start the Shear force from the fixed end. i.e. arrange the cantilever such that the fixed end is towards left end.

**Problems**

**S TANDARD PROBLEMS**

***Eccentric Concentrated Load***

Consider a simply supported beam of span *l* with an eccentric point load W acting at a distance *a* from support as shown in Fig. 3.20

The reactions can be obtained from the equations of equilibrium

(Write the Upward acting forces on one side and downward acting forces on the other side of the equation to avoid confusion among sign convention).

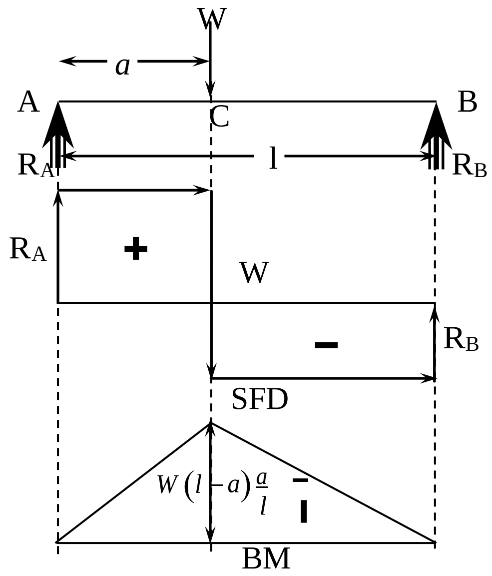



Fig. 3.20 SS with Point load

$\sum V_A = 0$; $R_A + R_B = W$ (01)

Taking moments about A,

$\sum M_A = 0$;

(Write the clockwise moments on one side and anti-clockwise moments on the other side of the equation to avoid confusion among sign convention).

$(R_B)(l) = (W)(a)$

$$R_B = \frac{Wa}{l}$$

Similarly Taking moments about B,

$\sum M_B = 0$;

$(R_A)(l) = (W)(l—a)$

$$R_A = \frac{W(l-a)}{l}$$

**Check**

To check the computations, substitute in Eq. 01, we have

$$R_A + R_B = \frac{Wa}{l} + \frac{W(l-a)}{l} = W\left[\frac{a+l-a}{l}\right] = W$$ and hence OK.

**Shear Force Values**

$$V_A = 0 + R_A = \frac{W(l-a)}{l}$$

$$V_C = \frac{W(l-a)}{l}$$

$V_C = \frac{W(l-a)}{l} - W = -\frac{Wa}{l}$

$V_B = -\frac{Wa}{l}$

$V_B = -\frac{Wa}{l} + \frac{Wa}{l} = 0$

**Bending Moment Values**

***Note:*** The Bending Moment will always will be zero at the end of the beam unless there is an applied moment at the end.

$M_A = 0$

$M_B = 0$

$M_C = (R_A)a = \frac{W(l-a)}{l} \times a = W(l-a)\frac{a}{l}$ also

$M_C = (RB)(l-a) = \left(\frac{Wa}{l}\right)\times(l-a) = W(l-a)\frac{a}{l}$

***Uniformly Distributed Load***

Consider a simply supported beam of span $l$ with an uniformly distributed load ω/m acting over the entire span as shown in Fig. 3.35

The reactions can be obtained from the conditions of equilibrium.

As the loading is symmetrical

$R_A = R_B$ and hence

$\sum V_A = 0 ; R_A + R_B = 2R_A = 2R_B = \omega \, x l$

$R_A = R_B = \frac{\omega l}{2}$

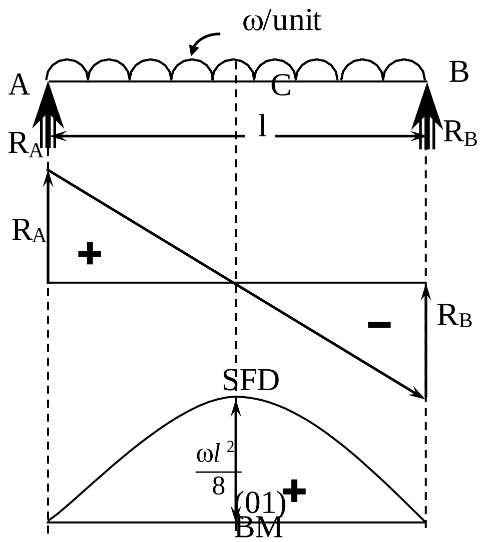



Fig. 3.21 SS with UDL

**Shear Force Values**

$V_A = R_A = \frac{\omega l}{2}$

$V_B = \frac{\omega l}{2} - \omega l = -\frac{\omega}{2}$

Shear Force at Midsection will be

$V_C = \frac{\omega l}{2} - \frac{\omega l}{2} = 0$

**Bending Moment Values**

$M_A = 0$

$M_B = 0$

$$M_C = (R_A)\frac{l}{2} = \left[\frac{\omega l}{2}\right] \times \left[\frac{l}{2}\right] = \frac{\omega l^2}{4}$$

***Uniformly Varying Load***

Consider a simply supported beam of span *l* with an uniformly varying load ω/m acting over the entire span as shown in Fig. 3.24

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;$

$$R_A + R_B = \left(\frac{\omega l}{2}\right) \qquad (01)$$

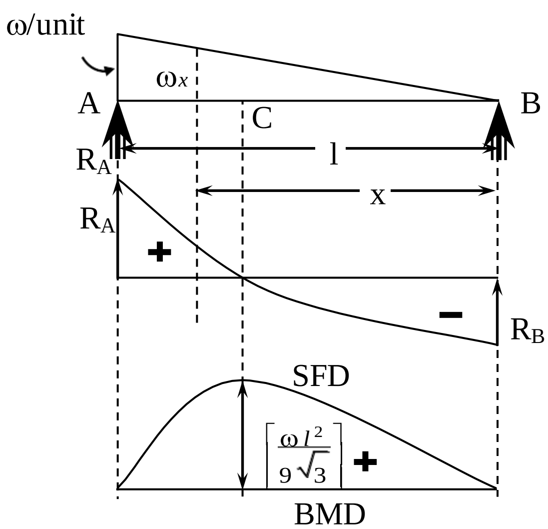


Fig. 3.22 SS with UVL

Taking moments about A,

$\sum M_A = 0;$

$$R_B \times l = \left(\frac{\omega l}{2}\right)\left(\frac{l}{3}\right) = \frac{\omega l^2}{6}$$

$$R_B = \frac{\omega l}{6}$$

Taking moments about B,

$\sum M_B = 0;$

$$R_A \times l = \left(\frac{\omega l}{2}\right)\left(\frac{2l}{3}\right) = \frac{\omega l^2}{3}$$

$$R_A = \frac{\omega l}{3}$$

**Check**

To check the computations, substitute in Eq. 01, we have

$$R_A + R_B = \left(\frac{\omega l}{6}\right) + \left(\frac{\omega l}{3}\right) = \frac{\omega l}{2}$$

Hence O.K.

**Shear Force Values**

$$V_A = R_A = \frac{\omega l}{3}$$

$$V_B = \frac{\omega l}{3} - \frac{\omega l}{2} = -\frac{\omega l}{6} \text{ and } V_B = -\frac{\omega l}{6} + \frac{\omega l}{6} = 0$$

**Location of Zero Shear Force**

Consider a section at a distance *x* from left support and load intensity at that section $\omega_x$ is given by $\omega_x = \left(\frac{x}{l}\right)\omega$

and Shear Force at that section is given by

$$V_x = \frac{1}{2}\omega_x \times x - R_B = \left(\frac{\omega x^2}{2l}\right) - \left(\frac{\omega l}{6}\right) \Rightarrow 0 \text{ or } x = \frac{l}{\sqrt{3}}$$

**Bending Moment Values**

$M_A = 0$

$M_B = 0$

Bending Moment will be maximum at Zero Shear Force and

$$M_c = (R_B)\,x - \left[\frac{1}{2} \times \omega_x \times x\right]\left(\frac{x}{3}\right) = \left[\frac{\omega l}{6}\right] \times x - \left[\frac{\omega x^3}{6l}\right]$$

$$= \left[\frac{\omega l}{6}\right] \times \left[\frac{l}{\sqrt{3}}\right] - \left[\frac{\omega}{6l}\right]\left[\frac{l}{\sqrt{3}}\right]^3$$

$$= \left[\frac{\omega l^2}{6\sqrt{3}}\right]\left(1 - \frac{1}{3}\right) = \left[\frac{\omega l^2}{9\sqrt{3}}\right]$$

**Cantilever with Point Load**

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A = W$

Taking moments about A,

$M_A = -W(l - a)$

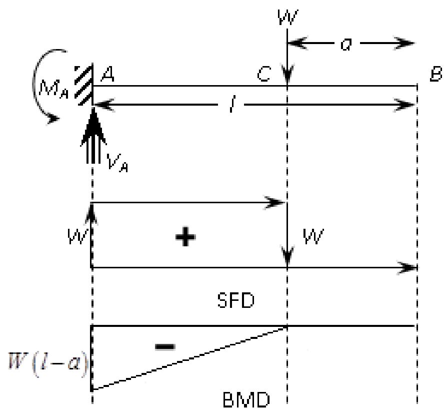


Fig. 3.33 Cantilever with Point Load

Shear Force Values

$V_B = 0$

$V_C = 0$

$V_C = 0 - W = -W$

$V_A = -W$

$V_A = -W + W = 0$

Bending Moment Values

$M_B = 0$

$M_C = 0$

$M_A = -W(l - a)$

**Cantilever with Uniformly Distributed Load (UDL)**

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A = \omega l$

Taking moments about A,

$$M_A = -\omega l \times \left(\frac{l}{2}\right) = -\frac{\omega l^2}{2}$$

Shear Force Values

$V_B = 0$

$V_A = -\omega l$

$V_A = -\omega l + \omega l = 0$

Bending Moment Values

$M_B = 0$

$$M_A = -\omega l \times \left(\frac{l}{2}\right) = -\frac{\omega l^2}{2}$$

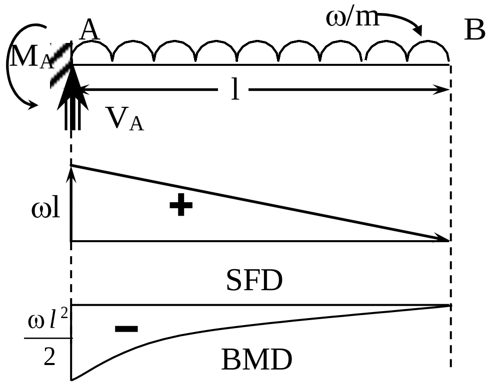



Fig. 3.34 Cantilever with UDL

**Cantilever with Uniformly Varying Load (UVL)**

**Case (*i*)**

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A = \frac{\omega l}{2}$

Taking moments about A,

$$M_A = -\left(\frac{\omega l}{2}\right) \times \left(\frac{l}{3}\right) = -\frac{\omega l^2}{6}$$

Shear Force Values

$V_B = 0$

$$V_A = \frac{\omega l}{2}$$

$$V_A = \frac{\omega l}{2} - \frac{\omega l}{2} = 0$$

Bending Moment Values

$M_B = 0$

$$M_A = -\left(\frac{\omega l}{2}\right) \times \left(\frac{l}{3}\right) = -\frac{\omega l^2}{6}$$

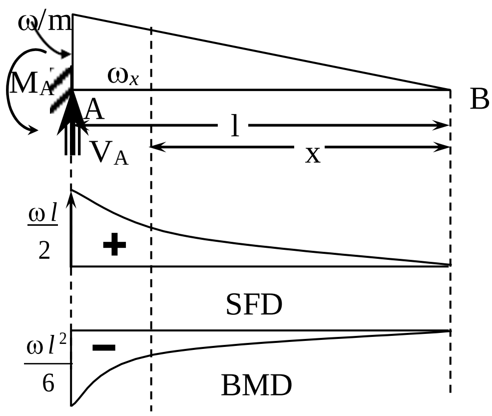



Fig. 3.35 Cantilever with UVL

Consider a section at a distance $x$ from free end and load intensity at that section $\omega_x$ is given by

$$\omega_x = \left(\frac{x}{l}\right)\omega$$

Shear Force at that section is given by

$$V_x = \frac{1}{2}\omega_x \times x = \left(\frac{\omega x^2}{2l}\right)$$

Bending Moment at that section is given by

$$M_x = -\left[\frac{1}{2}\omega_x \times x\right]\left(\frac{x}{3}\right) = -\left(\frac{\omega x^3}{6l}\right)$$

**Case (*ii*)**

The reactions can be obtained from the conditions of equilibrium.

$$\sum V_A = 0;\ R_A = \frac{\omega l}{2}$$

Taking moments about A,

$$M_A = \left(\frac{\omega}{2}l\right)\times\left(\frac{2l}{3}\right) = \frac{\omega l^2}{3}$$

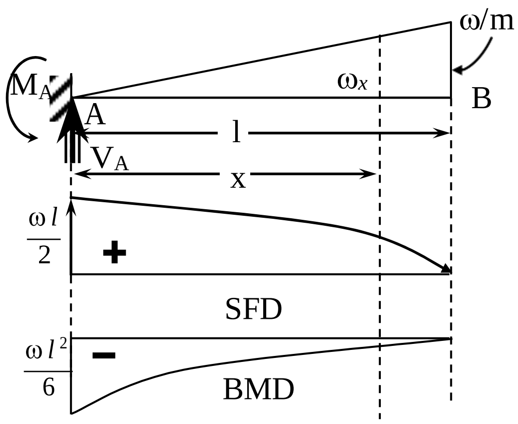



Fig. 3.36 Cantilever with UVL

Shear Force Values

$V_B = 0$

$$V_A = \frac{\omega l}{2}$$

$$V_A = \frac{\omega l}{2} - \frac{\omega l}{2} = 0$$

Bending Moment Values

$M_B = 0$

$$M_A = \left(\frac{\omega l}{2}\right)\times\left(\frac{2l}{3}\right) = \frac{\omega l^2}{3}$$

Consider a section at a distance $x$ from free end and load intensity at that section $\omega_x$ is given by

$$\omega_x = \left(\frac{x}{l}\right)\omega$$

Shear Force at that section is given by

$$V_x = R_A - \frac{1}{2}\omega_x \times x = \left(\frac{\omega l}{2}\right) - \left(\frac{\omega x^2}{2l}\right)$$

Bending Moment at that section is given by

$$M_x = R_A \times x - \left[\frac{1}{2}\omega_x \times x\right]\left(\frac{x}{3}\right) - M_A = \left(\frac{\omega l}{2}\right)x - \left(\frac{\omega x^3}{6l}\right) - \left(\frac{\omega l^2}{3}\right)$$

**Cantilever with Partial Uniformly Distributed Load (UDL)**

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0$; $R_A = \omega b$ Taking moments about A,

$$M_A = -\omega b \times \left( a + \frac{b}{2} \right)$$

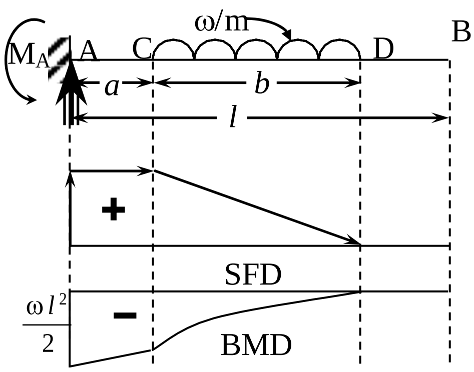


Fig. 3.37 Cantilever with Partial

Shear Force Values

$V_B = 0$

$V_D = 0$

$V_C = — \omega b$

$V_A = — \omega b$

$V_A = — \omega b + \omega b = 0$

Bending Moment Values

$M_B = 0$

$M_D = 0$

$$M_C = -\omega b \times \left( \frac{b}{2} \right) = -\frac{\omega b^2}{2}$$

$$M_A = -\omega\, b \times \left( a + \frac{b}{2} \right)$$

3.01. Draw the Shear Force and Bending Moment Diagram for a Cantilever beam subjected to concentrated loads as shown in Fig. 3.38.

From the conditions of equilibrium

$\sum V = 0$; $R_A = 10 + 20 + 30 = 60$ kN (↑)

$\sum M = 10 \times 6 + 20 \times 3 + 30 \times 2 = 180$ kN-m.

**Shear Force Values at Salient Points**

$V_D = 0 – 10 = -10$ kN

$V_C = -10 – 20 = -30$ kN

$V_B = -30 – 30 = -60$ kN

$V_A = -60 + 60 = 0$kN

**Bending Moment Values at Salient Points**

$M_D = 0$ kN-m

$M_C = -10 \times 3 = -30$ kN-m

$M_B = -10 \times 4 – 20 \times 1 = - 60$ kN-m

$M_A = -10 \times 6 – 20 \times 3 – 30 \times 2 = - 180$ kN-m

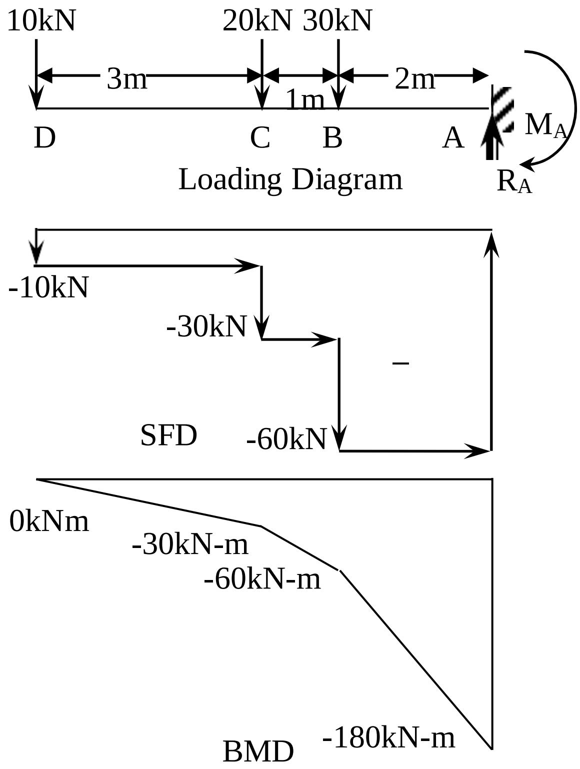


Fig.3.38 Cantilever

3.02. A cantilever beam is subjected to loads as shown in Fig. 3.39. Draw SFD and BMD.

From the conditions of equilibrium

$\sum V_A = 0$; $R_A = 10 + 30 + 20 \text{ x } 5 = 140$ kN (↑)

$\sum M_A = 30 \text{ x } 2 + 10 \text{ x } 3 + (20 \text{ x } 5)\left(\frac{5}{2}\right) + 40 = 380$ kN-m.

**Shear Force Values at Salient Points**

$V_D = 0$ kN

$V_C = 0 - 20 \text{ x } 2 = -40$ kN

$V_C = -40 - 10 = -50$ kN

$V_B = -50 - 20 \text{ x } 1 = -70$ kN

$V_B = -70 - 30 = -100$ kN

$V_A = -100 - 20 \text{ x } 2 = -140$ kN

$V_A = -140 + 140 = 0$kN

**Bending Moment Values at Salient Points**

As there is applied moment at section D, there will be two moments at that section and hence

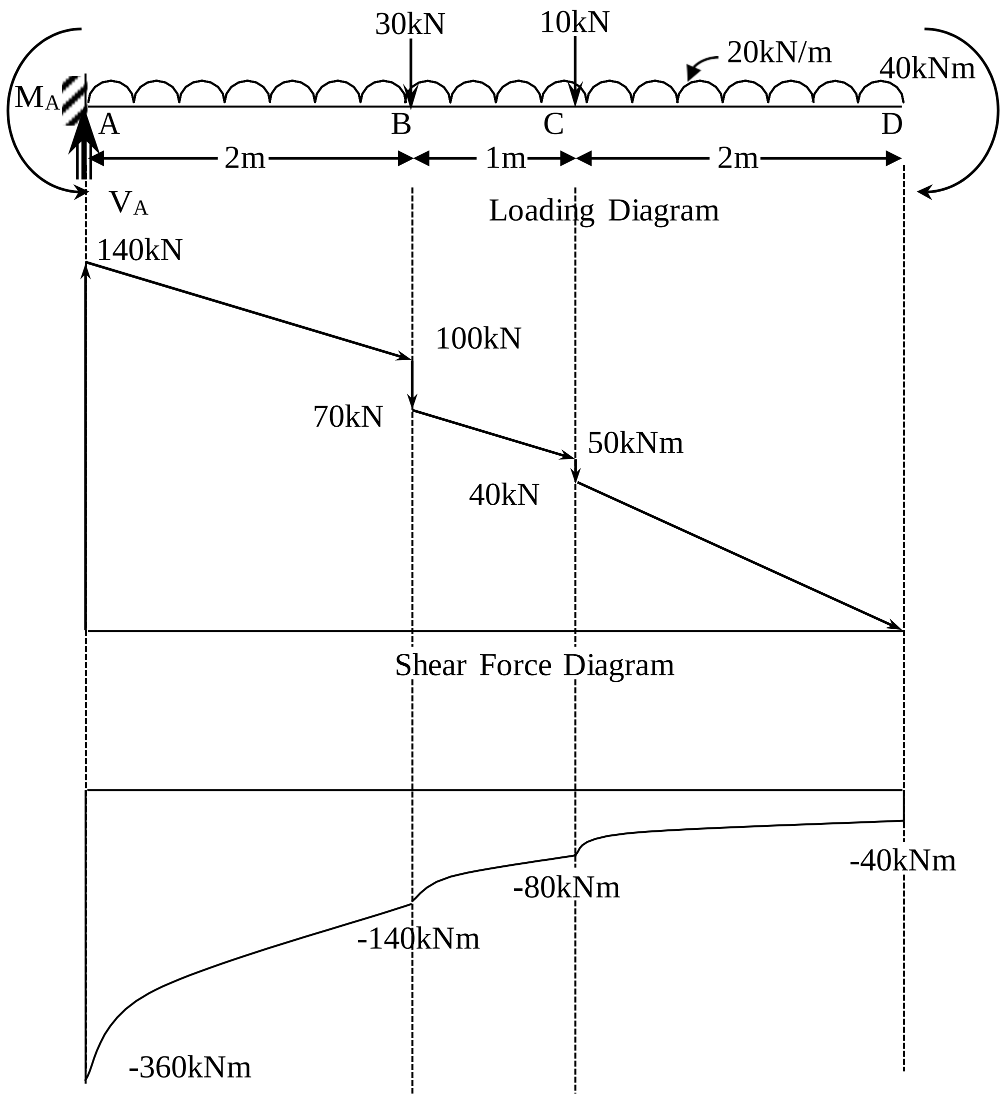
$M_{DR} = 0$

$M_{DL} = 0 - 40 = -40$kN-m

$M_C$ = –20 x 2 x 1 – 40 = –80 kN-m

$M_B$ = –20 x 3 x 1.5 – 10 x 1 – 40 = – 140 kN-m

$M_A$ = –20 x 5 x 2.5 – 10 x 3 – 20 x 2– 40 = – 360 kN-m




Bending Moment Diagram

Fig. 3.39 BMD & SFD - Cantilever

3.03. Draw BMD and SFD for the cantilever beam shown in Fig. 3.40. Locate the point of contra flexure if any,

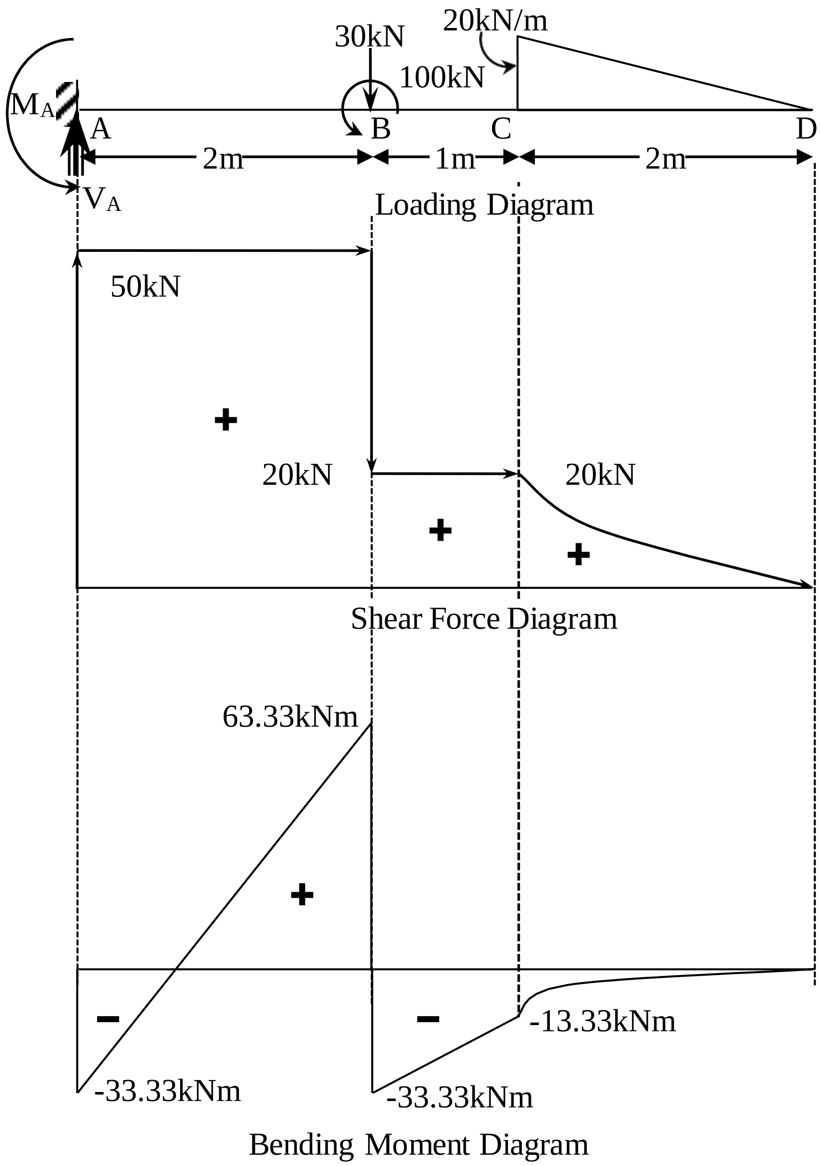



Fig. 3.40 BMD & SFD - Cantilever

From the conditions of equilibrium

$\Sigma V_A = 0$; $R_A = 30 + \left(\frac{1}{2}\right)$ x 20 x 2 = 50 kN (↑)

$\Sigma M_A = 30 \times 2 + \left(\frac{1}{2}\right)(20 \times 2)\left(3 + \frac{2}{3}\right) - 100 = 33.33$ kN-m

**Shear Force Values at Salient Points**

$V_D = 0$ kN

$V_C = 0 - \left(\frac{1}{2}\right)(20 \times 2) = -20$ kN

$V_B = -20$ kN

$V_B = –20 – 30 = –50$ kN

$V_A = –50$ kN

$V_A = –50 + 50 = 0$kN

**Bending Moment Values at Salient Points**

As there is applied moment at section B, there will be two moments at that section and hence

$M_D = 0$ kN

$M_C = -\left(\frac{1}{2}\right)(20 \text{ x } 2)\left(\frac{2}{3}\right) = –13.33$ kN-m

$M_{BR} = -\left(\frac{1}{2}\right)(20 \text{ x } 2)\left(1+\frac{2}{3}\right) = –33.33$ kN-m

$M_{BL} = –33.33 + 100 = + 66.67$kN-m

$M_A = -\left(\frac{1}{2}\right)(20 \text{ x } 2)\left(3+\frac{2}{3}\right) – 30 \text{ x } 2 + 100 = –33.33$ kN-m

**Points of contraflexure:**

$\frac{x}{33.33} = \frac{(2-x)}{66.67}$ or $x = 0.67$m

It lies at 0.67m and 2m right of the left support.

**Bracket Connections**

There can be following types of bracket connections which can be converted to load and moment.

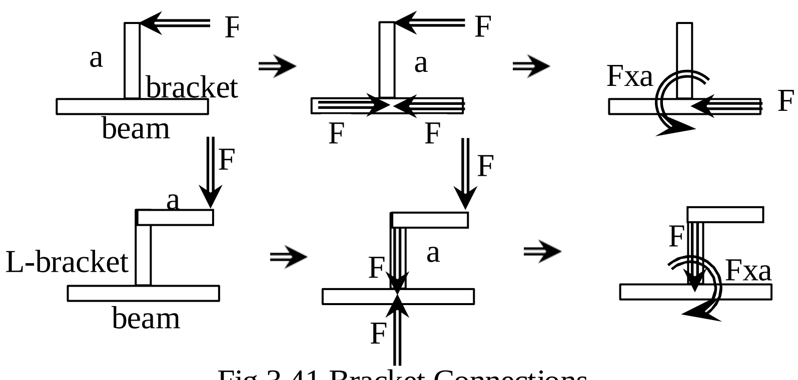



Fig.3.41 Bracket Connections

The types of brackets are vertical and L bracket as shown in Fig. 3.41. Apply two equal, opposite and collinear forces at the joint where the load gets transferred to the beam. The two forces ($F$) acting equal and opposite separated by a distance will form a couple equal to the product of Force and the distance between the forces along with the remaining Force.

3.04. An overhanging beam ABC is loaded as shown in Fig. 3.42. Draw the shear force and bending moment diagrams. Also locate point of contraflexure. Determine maximum +ve and —ve bending moments. (Jan-06)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 2\times6+2 = 14\text{kN}$

Taking moments about A,

$$\Sigma M_A = 0; \qquad 4R_B = (2\times6)\left(\frac{6}{2}\right)+2\times6 \text{ or } R_B = \frac{48}{4} = 12\text{kN}$$

Similarly taking moments about B,

$$\Sigma M_B = 0;\ 4R_B + 2\times2+(2\times2)\left(\frac{2}{2}\right) = (2\times4)\left(\frac{4}{-}\right) \text{ or } R_A = \frac{8}{4} = 2\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_A + R_B = 2 + 12 = 14$ kN (O.K.)

**Zero Shear Force**

Consider a section at a distance $x$ where Shear Force is zero as shown in Fig.3,42, From similar triangles, we have

$$\frac{2}{x} = \frac{6}{(4-x)}$$

$$x = 1\text{m}$$

**Bending Moment Values**

$M_A = 0$

$$M_B = -2\times2-2\times2\times\left(\frac{2}{2}\right) = -8\text{kN}$$ (Negative because Sagging)

$M_C = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = 2x - \frac{2\times x^2}{2} = 2x - x^2 = 1\text{kNm}$$

Maximum positive BM is 1kNm at 1 m to right of left support and negative BM is 8kNm at right support.

**Point of Contraflexure:** Bending Moment equation at section $y$ is

$$M_y = 2y - \frac{2\times y^2}{2} = 2y - y^2 \Rightarrow 0 \text{ or } y = 2\text{m}$$

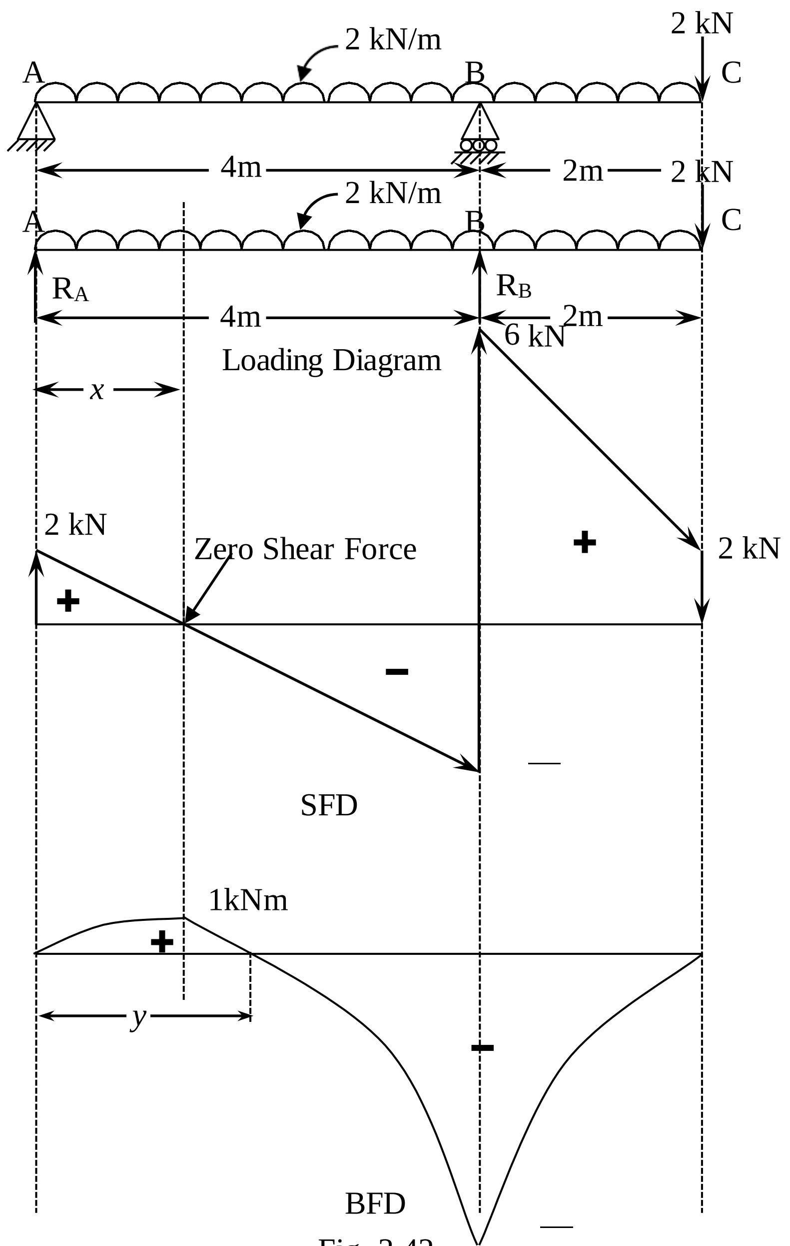


Fig. 3.42

3.05. Draw the Shear Force and Bending Moment Diagram for the loaded beam shown in Fig. 3.43. Find the Maximum bending moment.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 40 \times 4 = 160\text{kN}$ (01)

Taking moment about A,

$\Sigma M_A = 0; 8R_B = (40\times 4)\left(1+\frac{4}{2}\right) \text{or } R_B = \frac{480}{8} = 60\text{kN}$

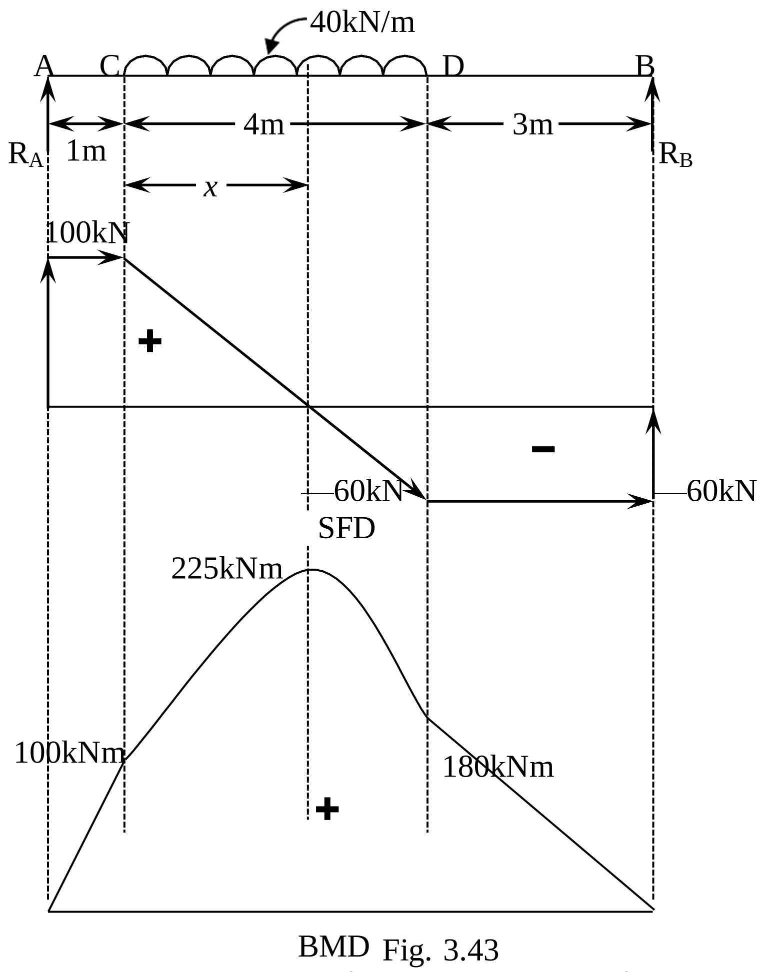


Fig. 3.43

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 40\times4 = 160\text{kN}$ (01)

Taking moment about A,

$$\Sigma M_A = 0; 8R_B = (40\times4)\left(1+\frac{4}{2}\right) \text{or } R_B = \frac{480}{8} = 60\text{kN}$$

Similarly taking moment about B,

$$\Sigma M_B = 0;\ 8R_A = (40\times4)\left(3+\frac{4}{2}\right) \text{or } R_A = \frac{800}{8} = 100\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_A + R_B = 100 + 60 = 160$ kN (O.K.)

**Zero Shear Force**

Consider a section at a distance $x$ where Shear Force is zero as shown in Fig. 3.43

From similar triangles, we have

$$\frac{100}{x} = \frac{60}{(4-x)} \text{ or } x = 2.5\text{m}$$

$\boldsymbol{V_o = 1 + 2.5 = 3.5}$**m** from right support.

**Bending Moment Values**

$M_B = 0$

$M_D = 60 \times 3 = 180\text{kN}$

$M_C = 60 \times 7 - (40 \times 4)\left(\frac{4}{2}\right) = 100\text{kN}$

$M_A = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$M_x = 100 \times (1 + x) - \frac{40 \times x^2}{2} = 100 \times (1 + x) - 20 \times x^2 = 225\text{kNm}$

3.06. Draw the Shear Force and Bending Moment Diagram for the loaded beam shown in Fig. 3.44. Also locate the Point of Contraflexure. Find and locate the Maximum +ve and —ve Bending Moments.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_C + R_D = 40 + 20 = 60\text{kN}$ (01)

Taking moment about C,

$\Sigma M_C = 0; 4R_D + 2 \times 40 = 20 \times 6 \text{ or } R_D = \frac{40}{4} = 10\text{kN}$

Similarly taking moments about D,

$\Sigma M_D = 0;\ 4R_C + 20 \times 2 = 40 \times 6 \text{ or } R_C = \frac{200}{4} = 50\text{kN}$

**Check**

Substituting in Eq. 01, we have $R_C + R_D = 50 + 10 = 60$ kN (O.K.)

**Zero Shear Force is at right support**

**Bending Moment Values**

$M_B = 0$

$M_D = -20 \times 2 = -40\text{kN-m}$

$M_C = -40 \times 2 = -80\text{kNm}$

$M_A = 0$

**Maximum Moments:** Maximum negative BM is 80 kNm at the left support.

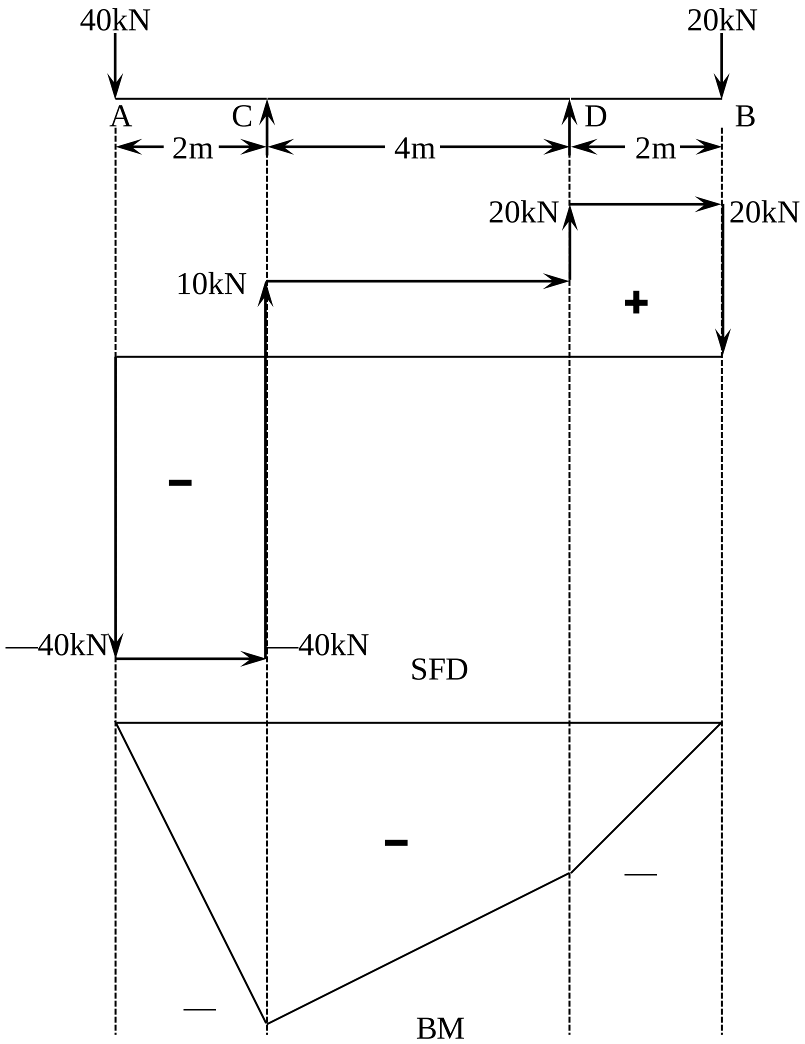



Fig. 3.44

3.07. Draw BMD and SFD for the loaded beam shown in Fig. 3.45. Also locate the Point of contraflexure and Maximum +ve and —ve Bending Moment

The reactions can be obtained from the conditions of equilibrium.

Taking moment about A,

$\sum V_A = 0;\ R_A + R_B = 3+5+2\times 6 = 20\text{kN}$ (01)

$$\Sigma M_A = 0; 6R_B + 3\times 2 = (2\times 6)\left(\frac{6}{2}\right) + 5\times 8 \text{ or } R_B = \frac{70}{6} = 11.67\text{kN}$$

Similarly taking moment about B,

$$\Sigma M_B = 0;\ 6R_A + 5\times 2 = (2\times 6)\left(\frac{6}{2}\right) + 3\times 8 \text{ or } R_A = \frac{50}{6} = 8.33\text{kN}$$

**Check:** Substituting in Eq. 01, we have $R_A + R_B = 11.67 + 8.33 = 20$ kN (O.K.)

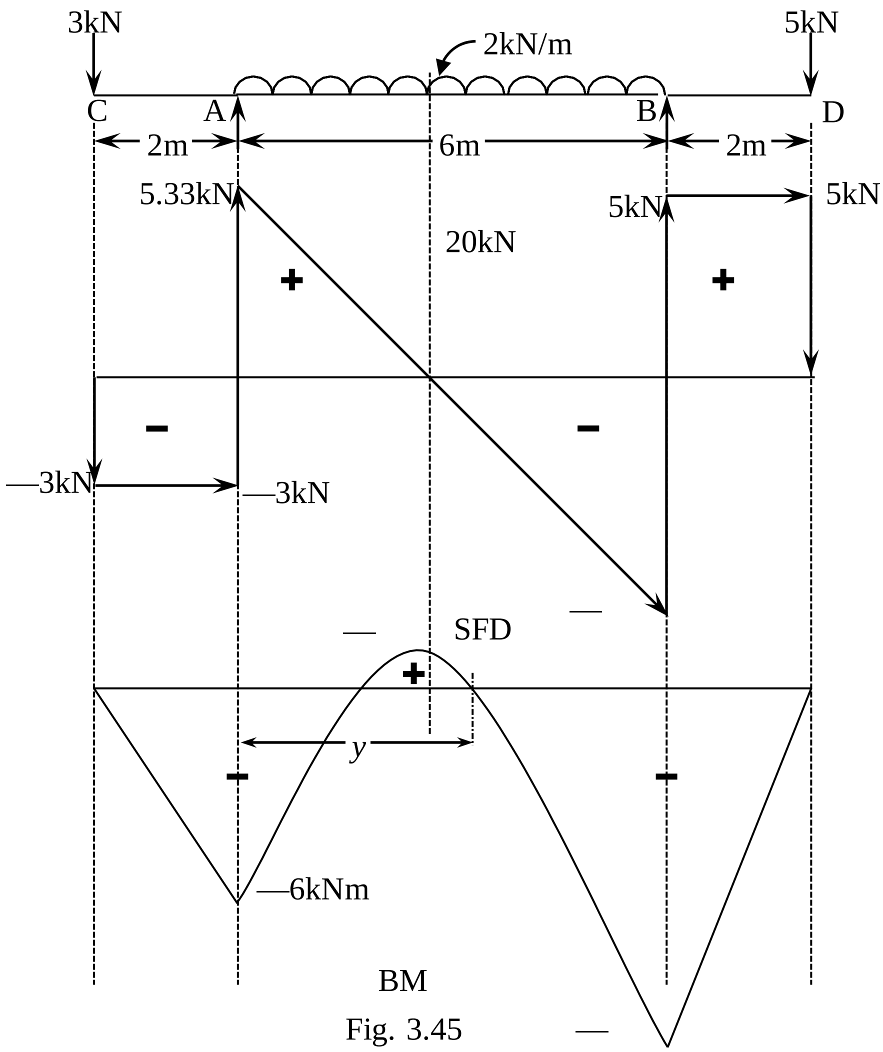



Fig. 3.45

**Check:** Substituting in Eq. 01, we have $R_A + R_B = 11.67 + 8.33 = 20$ kN (O.K.)

**Zero Shear Force**

Consider a section at a distance $x$ where Shear Force is zero as shown in Fig. 3.45.

From similar triangles, we have

$$\frac{5.33}{x} = \frac{6.67}{(6-x)} \quad \text{or } x = 2.67\text{m}$$

**Bending Moment Values**

$M_D = 0$

$M_B = -5\times 2 = -10\text{kN}$

$M_A = -3\times 2 = -6\text{kN}$

$M_C = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = 8.33\times x - 3(2+x) - \frac{2\times x^2}{2} = 8.33\times x - 3(2+x) - \frac{2\times x^2}{2} = 1.11\text{kNm}$$

**Points of Contraflexure:**

Bending moment at section $y$ from the left support is given by

$$M_y = 8.33y - 3\times(2+y) - \frac{2y^2}{2} \text{ or } y^2 - 5.33y + 6 = 0 \text{ and } y = 1.61\text{m and } 3.72\text{m}$$

Hence the points at 1.61m and 3.72m to right of left support.

3.08. Draw the BMD and SFD for the loaded beam shown in Fig. 3.46.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 20\text{kN}$

Taking moment about A,

$$\Sigma M_A = 0; 3R_B = 20\times 4 + 10$$

$$R_B = \frac{90}{3} = 30\text{kN}$$

Similarly taking moments about B,

$$M_B = 0;\ 3R_A + 10 + (20\times 1) = 0$$

$$R_A = -\frac{30}{3} = -10\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_A + R_B = -10 + 30 = 20$ kN (O.K.)

**Bending Moment Values**

$M_D = 0$

$M_B = -20\times 1 = -20\text{kNm}$ (Negative because Sagging)

$M_{C_R} = -20\times 2 + 30\times 1 = -10\text{kNm}$

$M_{C_L} = -10 - 10 = -20\text{kNm}$ or (By considering right side forces)

$M_{C_L} = -10\times 2 = 20\text{kNm}$ (By considering left side forces)

$M_A = 0$

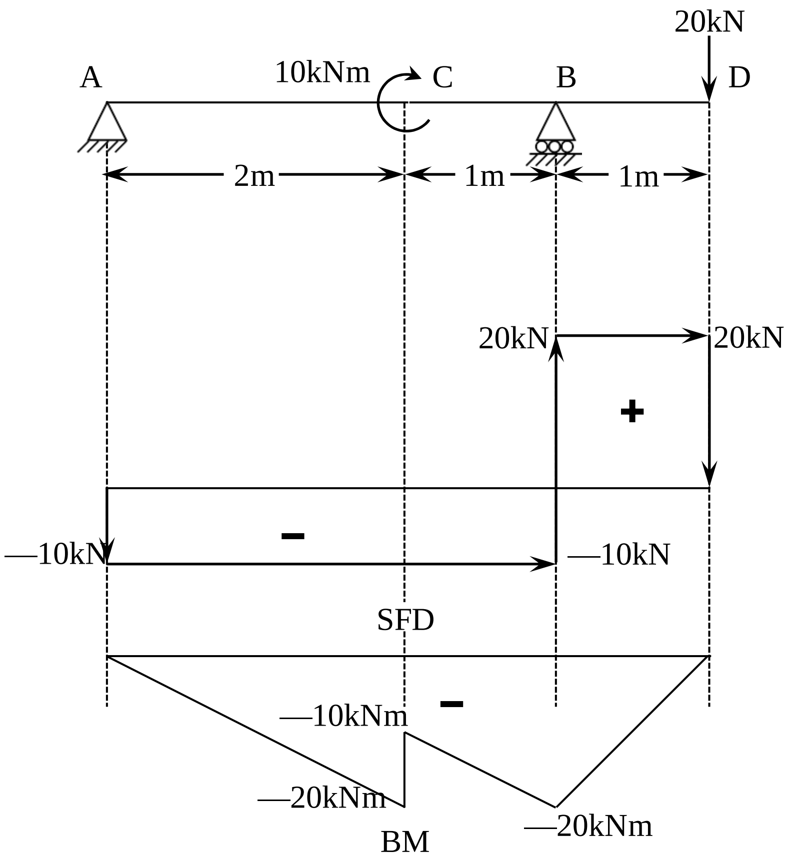



Fig. 3.46

An overhang beam ABC is loaded as shown in Fig. 3.47. Draw BMD and SFD.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 4\times3+12 = 24\text{kN}$

Taking moment about A,

$\Sigma M_A = 0; 6R_B = 12\times9 + (4\times3)\left(3+\frac{3}{2}\right) \text{or } R_B = \frac{162}{6} = 27\text{kN}$

Similarly taking moments about B,

$M_B = 0;\ 6R_A + 12\times3 = (4\times3)\left(\frac{3}{2}\right) \text{ or } R_A = -\frac{18}{6} = -3\text{kN}$

**Check**

Substituting in Eq. 01, we have $R_A + R_B = —3 + 27 = 24$ kN (O.K.)

**Bending Moment Values**

$M_D = 0$

$M_B = -12\times3 = -36\text{kNm}$ (Negative because Sagging)

$M_C = -3\times3 = -6\text{kNm}$

$M_A = 0$

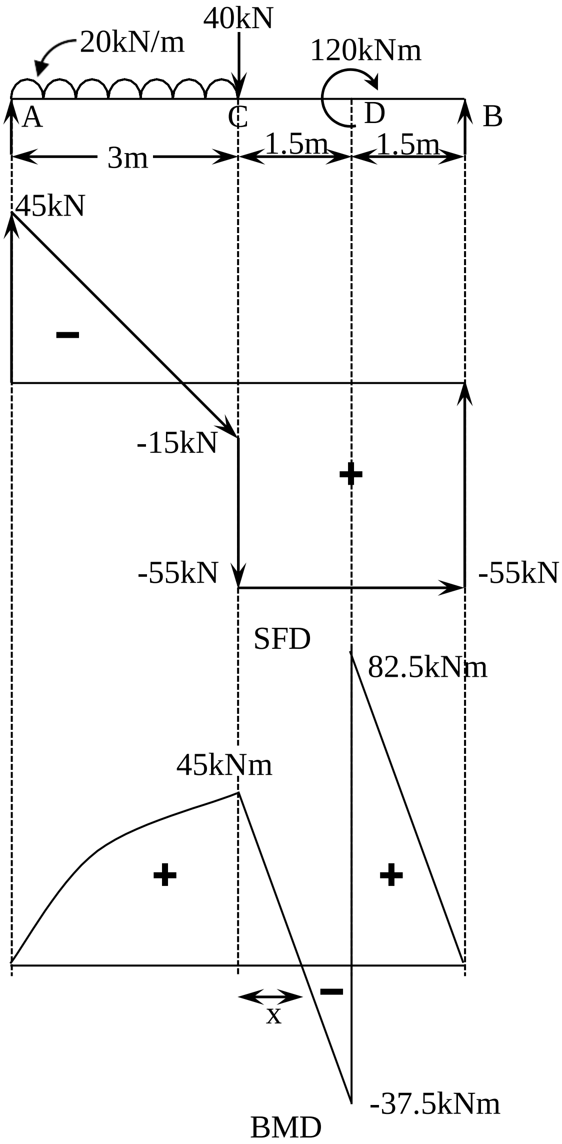



Fig. 3.48

3.09. Draw SFD and BMD for the beam shown in Fig. 3.48. Determine the maximum BM and its location. Locate the points of contraflexure. (July 02)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 20\times3 + 40 = 100\text{kN}$

Taking moment about A,

$$\Sigma M_A = 0; 6R_B = (20\times3)\left(\frac{3}{2}\right) + 40\times3 + 120 \text{ or } R_B = \frac{330}{6} = 55\text{kN}$$

Similarly taking moments about B,

$$M_B = 0;\ 6R_A = 40\times3 + (20\times3)\left(3 + \frac{3}{2}\right) - 120 \text{ or } R_A = \frac{270}{6} = 45\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_A + R_B = 45 + 55 = 100$ kN (O.K.)

**Bending Moment Values**

$M_B = 0$

$M_{D_R} = 55\times1.5 = 82.5\text{kNm}$

$M_{D_L} = 82.5 - 120 = -37.5\text{kNm}$ (By considering right side forces)

$M_{D_L} = 45\times 4.5 - (20\times3)\left(1.5+\frac{3}{2}\right) - 40\times1.5 = -37.5\text{kNm}$ (By left side forces)

$M_C = 55\times3 - 120 = 45\text{kNm}$ (By considering right side forces)

$M_C = 45\times3 - (20\times3)\left(\frac{3}{2}\right) = 45\text{kNm}$ (By left side forces)

$M_A = 0$

**Points of Contraflexure**

Consider a section at a distance $x$ where BM is changing its sign as shown in Fig. 3.49. From similar triangles, we have

$$\frac{45}{x} = \frac{37.5}{(1.5-x)}$$

$x = 0.818\text{m}$

The Points of contraflexure are located at 3.818m and 4.5m from the left support.

3.10. A beam ABCDE is 12m long simply supported at points B and D. Spans AB=DE=2m is overhanging. BC=CD=4m. The beam supports a udl of 10kN/m over AB and 20kN/m over CD. In addition it also supports concentrated load of 10kN at E and a clockwise moment of 16kNm at point C. Sketch BMD and SFD. (Aug 05)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_D = 10\times2 + 20\times4 + 10 = 110\text{kN}$ (01)

Taking moment about B,

$\Sigma M_B = 0; 8R_D + (10\times2)\left(\frac{2}{2}\right) = 10\times10 + (20\times4)\left(4+\frac{4}{2}\right) + 16 \text{ or } R_D = \frac{576}{8} = 72\text{kN}$

Similarly taking moment about D,

$\Sigma M_D = 0;\ 8R_B + 10\times2 + 16 = (10\times2)\left(8+\frac{2}{2}\right) + (20\times4)\left(\frac{4}{2}\right) \text{ or } R_B = \frac{304}{8} = 38\text{kN}$

**Check**

Substituting in Eq. 01, we have $R_B + R_D = 38 + 72 = 110$ kN (O.K.)

**Zero Shear Force**

Consider a section at a distance $x$ where Shear Force is zero as shown in Fig. 3.50.

From similar triangles, we have

$$\frac{12}{x} = \frac{68}{(4-x)} \quad \text{or } x = 0.6\text{m}$$

**Bending Moment Values**

$M_E = 0$

$$M_D = -10 \times 2 = -20\text{kN}$$

$$M_{C_R} = 72 \times 4 - 10 \times 6 - (20 \times 4)\left(\frac{4}{2}\right) = 68\text{kNm}$$

$$M_{C_L} = 68 - 16 = 52\text{kNm}$$ (From right side forces)

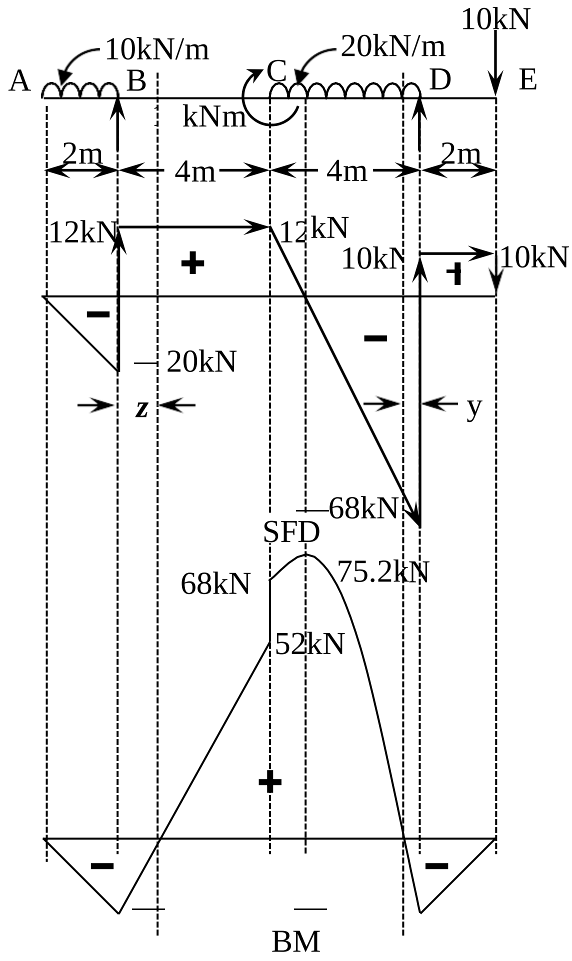



Fig. 3.49

$$M_{C_L} = 38 \times 4 - (10 \times 2)\left(4 + \frac{2}{2}\right) = 52\text{kNm}$$ (From left side forces)

$$M_B = -(10 \times 2)\left(\frac{2}{2}\right) = -20\text{kNm}$$

$M_A = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = 72\times(4-x)-10(2+4-x)-\frac{20\times(4-x)^2}{2}$$
$$= 72\times(4-0.6)-10(2+4-0.6)-10(4-0.6)^2 = 75.2\text{kNm}$$

**Point of Contraflexures**

Consider a section at a distance $z$ where Bending Moment is zero as shown in Fig. 3.49. From similar triangles, we have

$$\frac{20}{z}=\frac{52}{(4-z)} \text{ and } z=1.1\text{m}$$

Bending Moment at Section $y$ from point D is zero and can be written as

$$M_y = 72\times y-10(2+y)-\frac{20\times y^2}{2}=0$$
$$= 72\times y-10(2+y)-10\times y^2 = 62y-10y^2-20=0 \text{ and } y=0.341\text{m}$$

3.11. Draw the Shear Force and Bending Moment Diagrams for the beam shown in Fig. 3.50. Locate the point of contraflexure if any. (Feb 04)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_D = (10\times5)+80+80+(16\times2.5) = 250\text{kN}$

Taking moment about A,

$$\Sigma M_A = 0; 12.5R_D = (10\times5)\left(\frac{5}{2}\right)+80\times5+80\times7.5+(16\times2.5)\left(12.5+\frac{2.5}{2}\right)$$
$$R_D = \frac{1675}{12.5}=134\text{kN}$$

Similarly taking moments about B,

$$\Sigma M_D = 0; 12.5R_A+(16\times2.5)\left(\frac{2.5}{2}\right)=(10\times5)\left(7.5+\frac{5}{2}\right)+80\times7.5+80\times5=$$
$$R_A = \frac{1450}{12.5}=116\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_A + R_B = 116 + 134 = 250$ kN (O.K.)

**Bending Moment Values**

$M_E = 0$
$$M_D = -(16\times2.5)\left(\frac{2.5}{2}\right) = -50\text{kNm}$$
$$M_C = 134\times5-(16\times2.5)\left(5+\frac{2.5}{2}\right)= 425\text{kNm}$$
$$M_B = 116\times5-(10\times5)\left(\frac{5}{2}\right)= 455\text{kNm}$$

$M_A = 0$

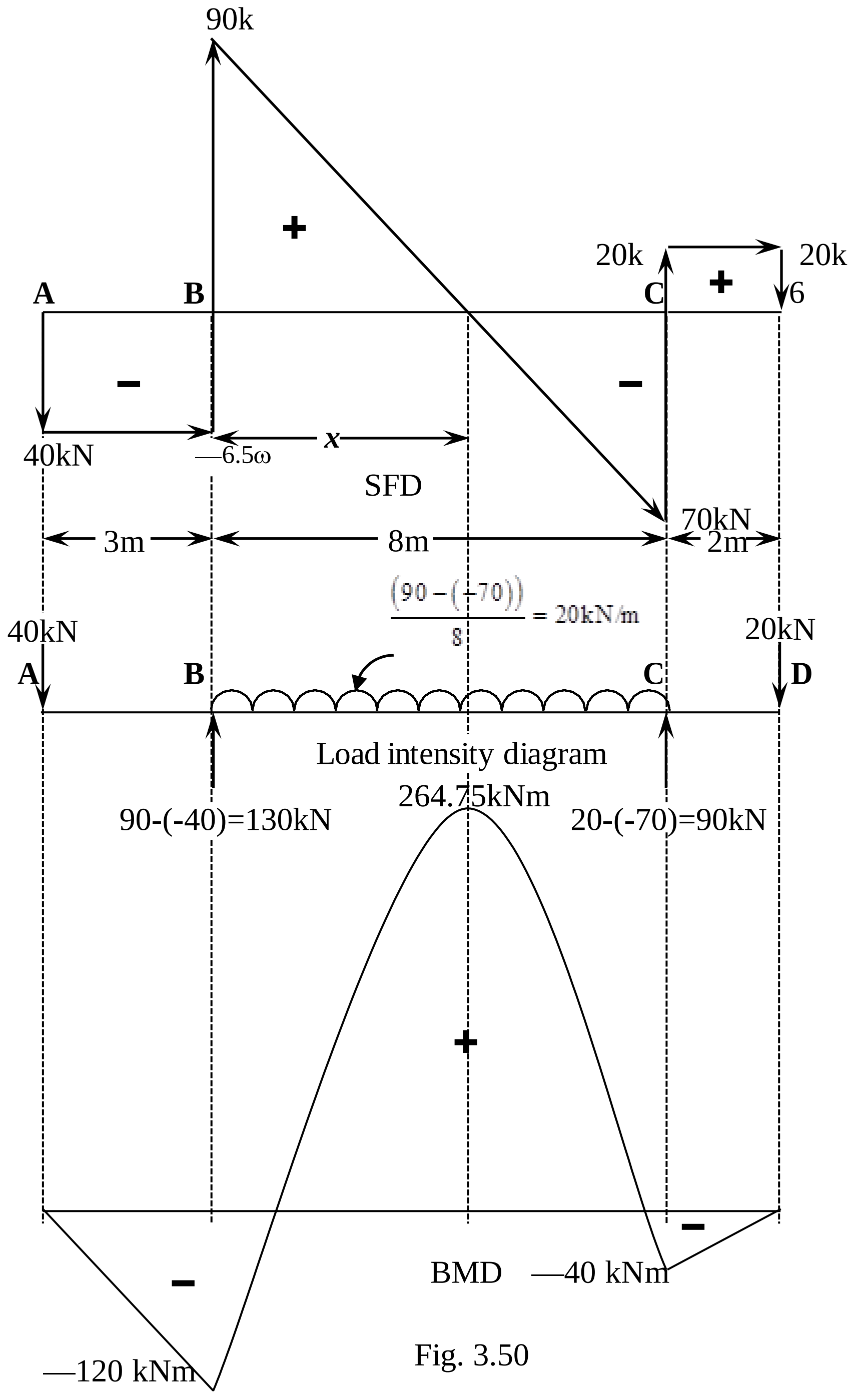



Fig. 3.50

**Point of Contraflexure**

Consider a section at a distance *y* from the right support where Bending Moment is zero as shown in Fig. From similar triangles, we have

$$\frac{50}{y} = \frac{425}{(5-y)} \text{ and } z = 0.526\text{m}$$

3.12. From the given shear force diagram shown in the Fig. 3.50, develop the load intensity diagram and draw the corresponding bending moment diagram indicating the salient features. (Jan 08)

The vertical lines in Shear force diagram represent vertical load, horizontal lines indicate generally no load portion, inclined line represents udl and parabola indicates uniformly varying load.

To generate load intensity diagram, the computations are shown in Fig. 3.50. The vertical line from the horizontal line below the line indicates negative value and vice versa. To check whether the applied moments are there in the loading diagram, we can take algebraic sum of moments of all the loads about any point and if there is a residue from the equation it indicates the applied moment in the opposite rotation to be applied anywhere on the beam.

**Check**

Taking Moments about B, we have

$$\Sigma M_B = 0;\ 40\times 3 + 90\times 8 - 20\times 10 - (20\times 8)\left(\frac{8}{2}\right) = 0$$

***Note:*** Hence there is no applied moment or couple and if there is any residue from the equation like $+M$ kNm then there is an applied moment of $M$ kNm clockwise and vice versa.

**Bending Moment Values**

$M_D = 0$

$M_C = -20 \times 2 = -40$ kNm (Negative due to hogging moment)

$M_B = -40 \times 3 = -120$ kNm (Negative due to hogging moment)

$M_A = 0$

Maximum Bending Moment occurs at zero shear force which is located at a distance $x$ from the left support as shown in Fig. From similar triangles, we have

$$\frac{90}{x} = \frac{70}{(8-x)} \text{ or } x = 4.5\text{m}$$

Maximum Bending Moment at the section $x$ is

$$M_x = 130x - 40\times(3+x) - \frac{20x^2}{2} = 130x - 40\times(3+x) - x^2$$

$$= 130\times 4.5 - 40\times(3+4.5) - 4.5^2 = 264.75\text{kNm}$$

3.13. A beam 6m long rests on two supports with equal overhangs on either side and carries a uniformly distributed load of 30kN/m over the entire length of the beam as shown in Fig. 3.51. Calculate the overhangs if the maximum positive and negative bending moments are to be same. Draw the SFD and BMD and locate the salient points. (Jan 07)

The reactions can be obtained from the conditions of equilibrium.

As the loading is symmetrical $R_A = R_B$ and hence

$\sum V_A = 0$; $R_B + R_C = 2\,R_B = 2R_C = 30 \text{ x } (6+2a)$

$$R_B = R_C = \frac{30\times 6}{2} = 90\text{kN}$$

Bending Moment at any section $x$ from the left end is given by

$$M_x = 90(x-a) - \frac{30x^2}{2} \text{ or } 90(x-a) - 15x^2 \qquad 01$$

From the given problem, maximum positive and negative bending moments are to be same, which occurs at zero shear force sections. From the above loading diagram, it can be seen that the zero shear force occurs at support and at centre (as the loading

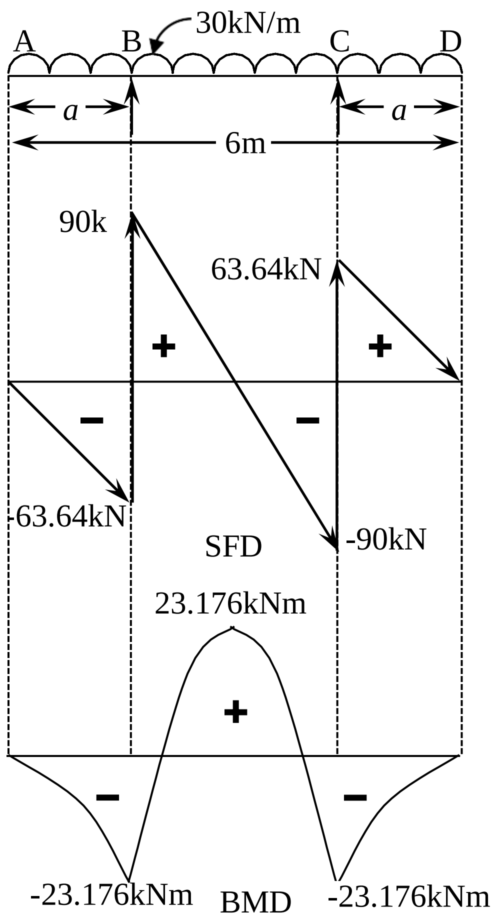


Fig. 3.51

is symmetrical). Hence substituting $x = a$ and 3, we get maximum +ve and —ve Bending Moment.

$$M_B = -15a^2$$

$$M_E = 90(3-a) - 15(3)^2 = 90(3-a) - 135$$

Equating the absolute values of above two equations, we have

$$15a^2 = 90(3-a) - 135 \text{ or } a^2 + 6a - 9 = 0 \text{ and } a = 1.243\text{m}$$

**Bending Moment Values**

$M_D = 0$

$$M_C = -\frac{30\times1.243^2}{2} = -23.176\text{kNm}$$

$$M_B = -\frac{30\times1.243^2}{2} = -23.176\text{kNm}$$

$M_A = 0$

$$M_E = 90(3-1.243) - \frac{30\times1.243^2}{2} = 23.176\text{kNm}$$

**Points of Contraflexure:**

$$M_x = 90(x-1.243) - 15x^2 = 6(x-1.243) - x^2 \Rightarrow 0 \text{ or } x = 1.76\text{m and } 4.24\text{m}$$

The points of contraflexure are at 1.76m and 4.24m from left end.

3.14. Draw the Shear Force and Bending Moment Diagram for a simply supported beam subjected to uniformly varying load shown in Fig. 3.52.

The trapezoidal load can be split into udl and uvl (triangular load) as shown in Fig. 3.43.

$$\sum V_A = 0;\ R_A + R_B = (15\times6) + \left(\frac{1}{2}\right)(10\times6) = 120\text{kN} \qquad 01$$

Taking moment about A,

$$\Sigma M_A = 0; 6R_B = (15\times6)\left(\frac{6}{2}\right) + \left(\frac{1}{2}\right)(10\times6)\left(\frac{2}{3}\times6\right) \text{ or } R_D = \frac{390}{6} = 65\text{kN}$$

Similarly taking moments about B,

$$\Sigma M_B = 0; 6R_A = (15\times6)\left(\frac{6}{2}\right) + \left(\frac{1}{-}\right)(10\times6)\left(\frac{6}{-}\right) + 80\times7.5 + 80\times5 \text{ or } R_A = \frac{330}{6} = 55\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_A + R_B = 55 + 65 = 120$ kN (O.K.)

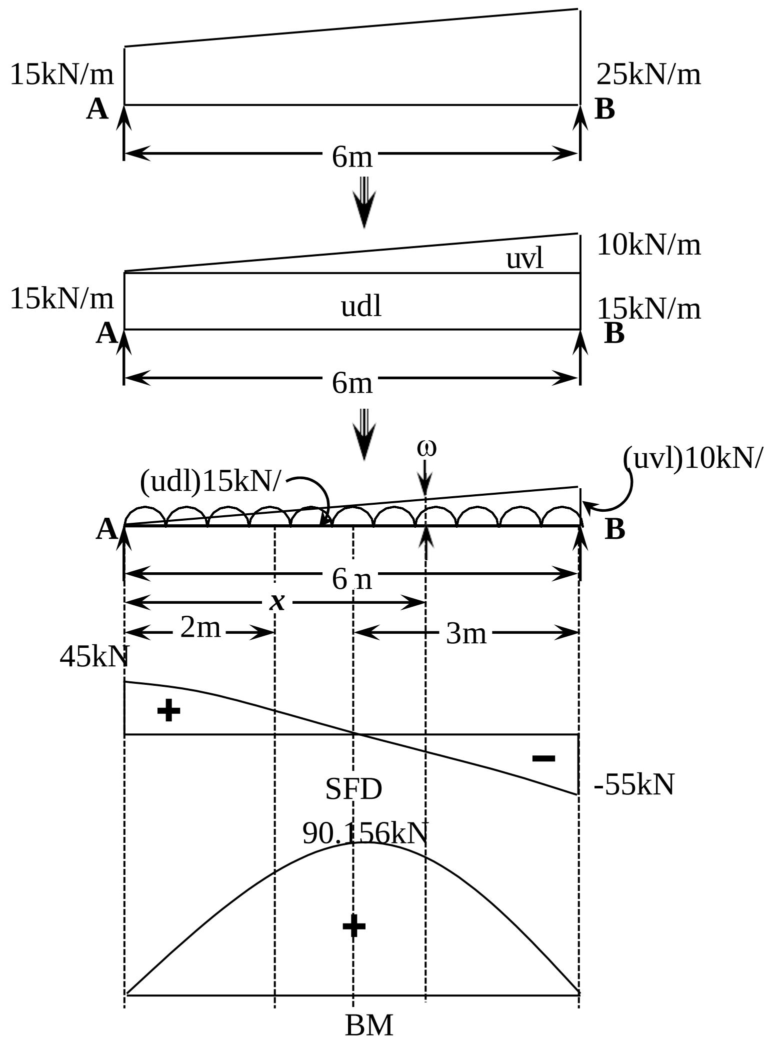



Fig.

**Shear Force Equation at any section *x* from left support**

Consider a section $x$ at a distance $x$ from the left support as shown.

The intensity of uvl at $x$ is given by

$$\omega_x = \left(\frac{10 \times x}{6}\right) = 1.67x \text{ kN/m}$$

$$V_x = 55 - 15x - \frac{1.67x^2}{2} = 55 - 15x - \frac{5}{6}x^2 \text{ kN}$$

At $x$ = 2m, $V_2 = 55 - 15 \times 2 - \frac{5}{6} \times 2^2 = 21.67\text{kN}$

At $x$ = 3m, $V_3 = 55 - 15 \times 3 - \frac{5}{6} \times 3^2 = 2.5\text{kN}$

At $x$ = 5m, $V_5 = 55 - 15 \times 5 - \frac{5}{6} \times 5^2 = -40.83\text{kN}$

Zero Shear Force = $V_o = 55 - 15 \times x - \frac{5}{6} \times x^2 = 0$ solving we get, $x = 3.124\text{m}$

**Bending Moment Values**

**Bending Moment Equation at any section $x$ from left support**

Consider a section $x$ at a distance $x$ from the left support as shown.

$$M_x = 55x - \frac{15x^2}{2} - \left(\frac{1.67x^2}{2}\right)\left(\frac{x}{3}\right) = 55x - 7.5x^2 - \frac{5}{18}x^3 \text{ kNm}$$

$$M_x = 55\times -7.5x^2 - \frac{5}{18}x^3$$

$M_B = 0$

$M_A = 0$

Maximum Bending Moment occurs at SF = 0, i.e. $x = 3.124$m

$$M_x = 55\times 3.124 - 7.5\times 3.124^2 - \left(\frac{5}{18}\right)\times 3.124^3 = 90.156\text{kNm}$$

3.15. A beam ABCD 20m long is loaded as shown in Fig. 3.53. The beam is supported at B and C with a overhang of 2m to the left of B and a overhang of $a$m to the right of support C. Determine the value of $a$ if the midpoint of the beam is point of inflexion and for this alignment plot BM and SF diagrams indicating the important values.

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_C = 5\omega + \omega\times 20 = 25\omega$ kN (01)

Taking moment about B,

$$\Sigma M_B = 0; (18-a)R_C + (5\omega)\times 2 + \left(\frac{\omega\times 2^2}{2}\right) = \left(\frac{\omega\times(20-2)^2}{2}\right)$$

$$(18-a)R_C = 150\omega \text{ or } R_C = \frac{150\omega}{(18-a)}$$

Similarly taking moment about C,

$$\Sigma M_C = 0; (18-a)R_B + \left(\frac{\omega a^2}{2}\right) = (5\omega)(20-a) + \left(\frac{\omega\times(20-a)^2}{2}\right)$$

$$(18-a)R_B = 300\omega - 25a\omega \text{ or } R_B = \frac{\omega(300-25a)}{(18-a)}$$

**Check**

Substituting in Eq. 01, we have

$$R_B + R_C = \frac{150\omega}{(18-a)} + \frac{\omega(300-25a)}{(18-a)} = 25\omega \text{ (O.K.)}$$

**Point of contraflexure**

Consider a section at a distance $x$ from left support as shown in Fig. 3.53. Bending moment at this section is given by

$$M_x = R_B \times (x-2) - 5\omega \times x - \frac{\omega x^2}{2} = \left[\frac{\omega(300-25a)}{(18-a)}\right] \times (x-2) - 5\omega \times x - \frac{\omega x^2}{2}$$

From the given data, this is zero at $x$ = 10m. Hence

$$\left[\frac{\omega(300-25a)}{(18-a)}\right] \times (x-2) - 5\omega \times x - \frac{\omega x^2}{2} = 0$$

$$\left[\frac{(300-25a)}{(18-a)}\right] \times 8 - 5 \times 10 - \frac{10^2}{2} = 0$$

$$\left[\frac{(300-25a)}{(18-a)}\right] = 12.5$$

$$300 - 25a = 225 - 12.5a \text{ or } a = 6\text{m}$$

$$R_B = \frac{\omega(300-25a)}{(18-a)} = \frac{\omega(300-25\times 6)}{(18-6)} = 12.5\omega$$

$$R_C = \frac{150\omega}{(18-a)} = \frac{150\omega}{(18-6)} = 12.5\omega$$

**Zero Shear Force**

Consider a section at a distance $y$ where Shear Force is zero as shown in Fig. 3.53. From similar triangles, we have

$$\frac{5.5}{y} = \frac{6.5}{(12-y)} \text{ or } y = 5.5\text{m}$$

**Bending Moment Values**

$\boldsymbol{M_D = 0}$

$$M_C = -\omega \times \frac{6^2}{2} = -18\omega$$

$$M_B = -5\omega \times 2 - \omega \times \frac{2^2}{2} = -12\omega$$

$\boldsymbol{M_A = 0}$

$$M_E = 12.5\omega \times 5.5 - 5\omega \times (5.5+2) - \frac{\omega(5.5+2)^2}{2} = 3.125\omega$$

Another point of contraflexure is

$$M_x = \left[\frac{\omega(300-25\times 6)}{(18-6)}\right] \times (6-2) - 5\omega \times 6 - \frac{\omega 6^2}{2}$$

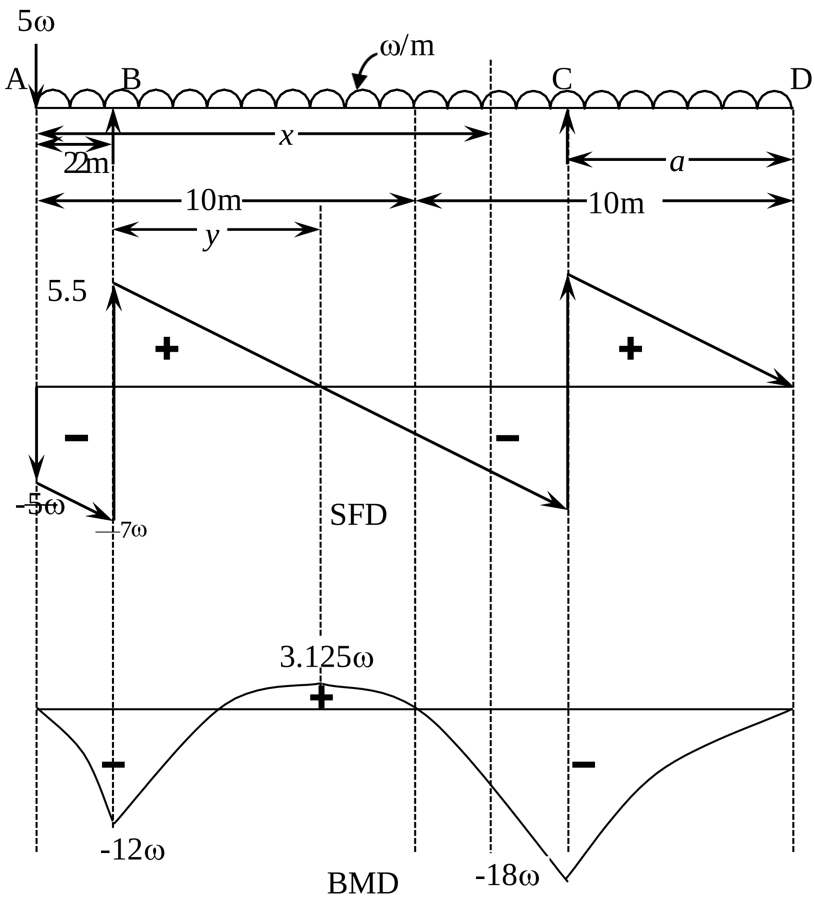



Fig. 3.53

3.16 For the beam AC shown in Fig. 3.54, determine the magnitude of the load $P$ acting at C such that the reaction at supports A and B are equal and hence draw the Shear force and Bending moment diagram. Locate points of contraflexure. (July 08)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_A + R_B = 45\times4+P$ 01

From the given data, $R_A = R_B$ and substituting in Eq. 01, $2R_A = 2R_B = 180 + P$

Taking moment about A,

$$\Sigma M_A = 0; 6R_B = 7P + (45\times4)\left(\frac{4}{2}\right) + 30 \text{ or } 6R_B = 7P + 390$$

Substituting from Eq. 01,

$$3(180+P) = 7P + 390 \text{ or } P = 37.5\text{kN}$$

**Check**

Similarly taking moments about B,

$$\Sigma M_B = 0; 6R_A + P\times1 + 30 = (45\times4)\left(2+\frac{4}{2}\right)$$

$$6R_A = 690 - P$$

Substituting from Eq. 01, $3(180+P) = 690 - P$ or $P = 37.5\text{kN}$

Hence O.K.

$2R_A = 2R_B = 180 + 37.5 = 217.5\text{kN}$

$R_A = R_B = 108.75\text{kN}$

**Zero Shear Force**

Consider a section at a distance $x$ where Shear Force is zero as shown in Fig. 3.54. From similar triangles, we have

$$\frac{108.75}{x} = \frac{71.25}{(4-x)} \text{ or } x = 2.417\text{m}$$

**Bending Moment Values**

$M_C = 0$

$M_B = -37.5 \times 1 = -37.5\text{kNm}$ (Negative because Sagging)

$M_{D_R} = 108.75 \times 2 - 37.5 \times 3 = 105\text{kNm}$

$M_{D_L} = 108.75 \times 4 - (45 \times 4)\left(\frac{4}{2}\right) = 75\text{kNm}$ (From left side forces)

$M_{D_L} = 105 - 30 = 75\text{kNm}$ (From Right side forces)

$M_A = 0$

Maximum Bending moment occurs at zero shear force. i.e. at $x = 2.417$

$$M_x = 108.75 \times x - \frac{45 \times x^2}{2} = 108.75 \times 2.417 - \frac{45 \times 2.417^2}{2} = 131.41\text{kNm}$$

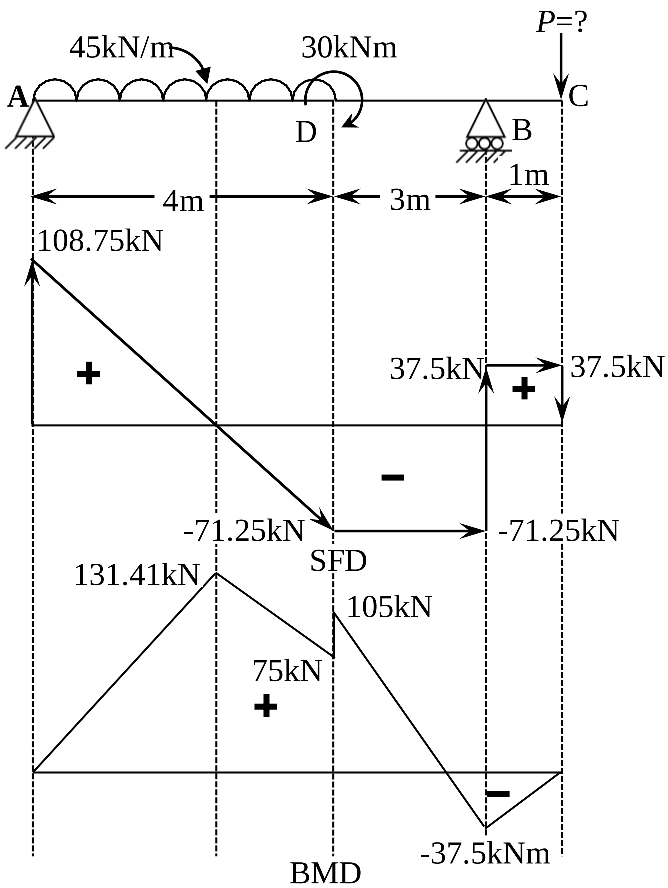



Fig. 3.54

3.16. Draw the bending moment and shear force diagrams for a prismatic simply supported beam of length $L$, subjected to a clockwise moment $M$ at the centre of the beam and a uniformly distributed load of intensity $q$ per unit length acting over the entire span. (Jan 09)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0$; $R_A + R_B = q \times L$ kN (01)

Taking moment about A,

$$\Sigma M_A = 0; R_B \times L + M = \frac{q \times L^2}{2}$$

$$R_B = \frac{q \times L}{2} - \frac{M}{L}$$

Similarly taking moment about B,

$$\Sigma M_B = 0; R_A \times L = \frac{q \times L^2}{2} + M$$

$$R_A = \frac{q \times L}{2} + \frac{M}{L}$$

**Check**

Substituting in Eq. 01, we have

$$R_A + R_B = \frac{q \times L}{2} + \frac{M}{L} + \frac{q \times L}{2} - \frac{M}{L} = qL \text{ (O.K.)}$$

**Zero Shear Force**

Consider a section at a distance $x$ where Shear Force is zero as shown in Fig. 3.55. From similar triangles, we have

$$\frac{\left[\frac{qL}{2} + \frac{M}{L}\right]}{x} = \frac{\left[\frac{qL}{2} - \frac{M}{L}\right]}{(L-x)} \text{ or } x = \left[\frac{L}{2} + \frac{M}{qL}\right]$$

$$= \frac{qL^2}{8} + \frac{M}{2} + \frac{M^2}{2qL^2}$$

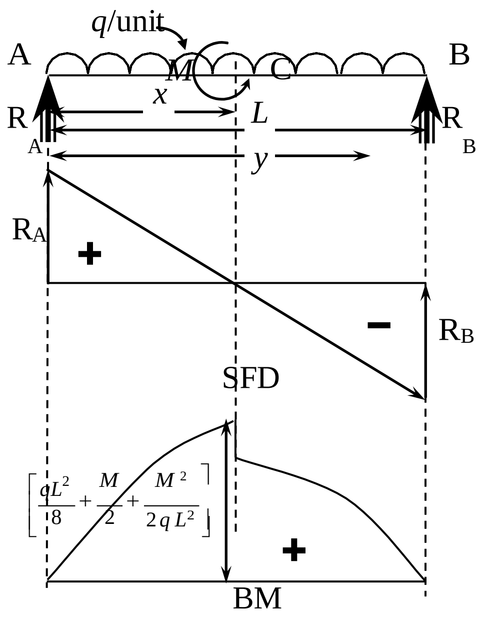



Fig. 3.55 SS with UDL &

**Bending Moment Values**

$M_B = 0$

$M_A = 0$

Bending Moment at zero Shear Force will be either Maximum or Minimum.

$$M_x = \left[\frac{q \times L}{2} + \frac{M}{L}\right] \times x - \frac{q}{2} \times x^2 = \left[\frac{q \times L}{2} + \frac{M}{L}\right] \times \left[\frac{L}{2} + \frac{M}{qL}\right] - \frac{q}{2} \times \left[\frac{L}{2} + \frac{M}{qL}\right]^2$$

$$M_{\max} = \frac{qL^2}{8} + \frac{M}{2} + \frac{M^2}{2qL^2}$$

3.17. For the loaded beam shown in Fig. 3.56, Draw the Shear Force and Bending Moment Diagram. Find and locate the Maximum +ve and —ve Bending Moments. Also locate the Point of Contraflexures. Detail the procedure to draw the SFD and BMD. (July 09)

It can be seen the loading is symmetrical and the Reactions are equal. From the conditions of equilibrium

$\sum V_A = 0;$

$$R_A + R_B = 2R_A = 2R_B = 2\times\left(20+\frac{1}{2}\times(10\times 2)\right)+20\times 2 \text{ or } R_A = R_B = 50\text{kN}$$

**Bending Moment Values**

$M_F = M_C = 0$

$$M_A = M_B = -20\times 2 = -40\text{kNm}$$

$$M_{D_L} = M_{E_R} = 50\times 2 - 20\times 4 - \left(\frac{1}{2}\times 10\times 2\right)\left(\frac{2\times 2}{3}\right) = 6.67\text{kNm}$$

$$M_{D_R} = M_{E_L} = 6.67 - 10 = -3.33\text{kNm}$$

**Maximum Bending Moment and Points of Contraflexure**

**Maxumum Bending Moment**

Bending Moment at any section $x$ in the region DE is given by

$$M_x = 50x - 20(x+2) - \left[\left(\frac{1}{2}\times 10\times 2\right)\left(x-\frac{2}{3}\right)\right] - 20\frac{(x-2)^2}{2} - 10$$

The Maximum bending moment occurs at zero shear force.

i.e. $x = (5\text{-}2) = 3$ m

$$M_x = 50\times 3 - 20(3+2) - \left[\left(\frac{1}{2}\times 10\times 2\right)\left(3-\frac{2}{3}\right)\right] - 20\frac{(3-2)^2}{2} - 10 = 6.67\text{kNm}$$

**Shear Force Diagram**

1. Draw a horizontal line $C_1F_2$ equal to the length of the beam 10m to some scale, under the beam CF as shown.
2. Start the Shear force line from left extreme edge $C_1$. Draw $C_1C_2$ under the vertical load 20kN acting at C downward equal to some scale. To start with, the shear force at $C_1$=0 and at $C_2$, the Shear force = 0 – 20 (-ve as it is acting downward) = -20 kN.
3. There is no load in the region CA and hence under this region, the shear force line $C_2A_1$ will be a horizontal line parallel to beam axis.

4. At A, there is a reaction $R_A$ which is treated as vertical load = 50kN and hence the shear force line $A_1A_2$ = 50kN to some scale and the shear force at $A_2$ = -20 + 50 (+ as it is upward) = +30 kN.
5. There is a uvl in the region AD and the shear force line will be a parabola in this region. The parabola will be tangential to vertical at $A_2$ as there is relatively higher load intensity at A and will be parallel to horizontal at $D_1$ as the load intensity is lesser at D. Hence the curve is sagging. The vertical distance from $A_2$ to $D_1$ is equal to the total load equivalent to uvl, i.e. ½ x 10 x 2 = 10kN and the shear force at $D_1$ = 30 - 10 (- as it is downward) = +20 kN.
6. There is an udl in the region DE and hence the shear force line is inclined from $D_1$ to $E_1$. The vertical distance from $D_1$ to $E_1$ is equal to the total load equivalent to udl, i.e. 20 x 2 = 40kN and the shear force at $E_1$ = 20 - 40 (- as it is downward) = -20 kN.
7. There is a uvl in the region EB and the shear force line will be a parabola in this region. The parabola will be tangential to horizontal at $E_1$ as there is relatively lower load intensity at E and will be parallel to vertical at $B_1$ as the load intensity is higher at B. Hence the curve is hogging. The vertical distance from $E_1$ to $B_1$ is

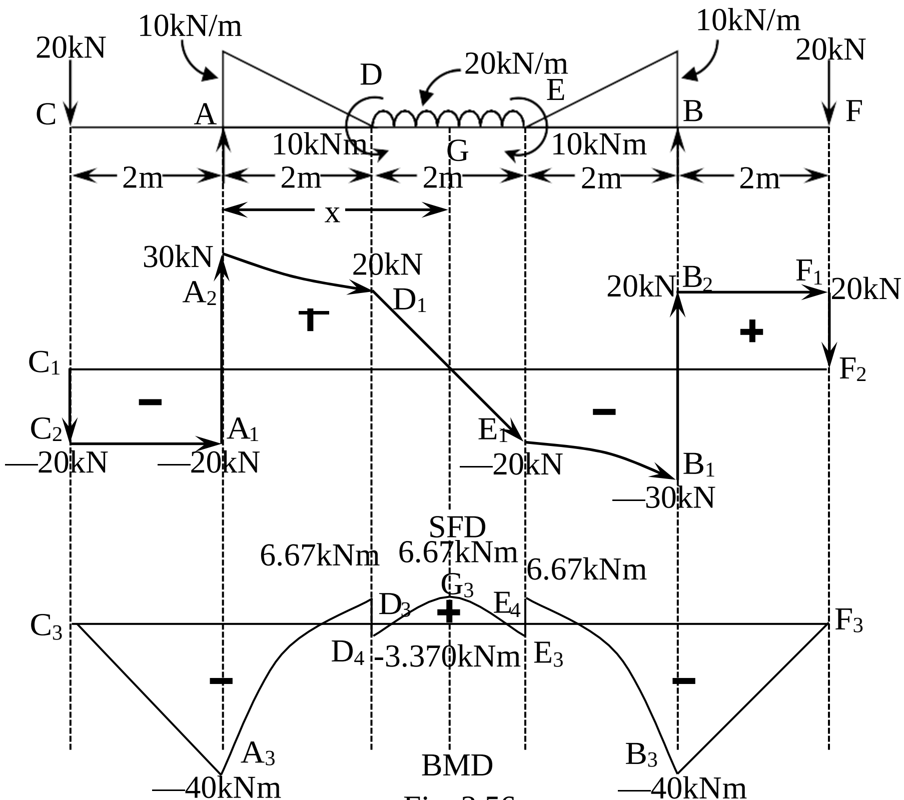


Fig. 3.56

equal to the total load equivalent to uvl, i.e. ½ x 10 x 2 = 10kN and the shear force at $B_1$ = -20 - 10 (- as it is downward) = -30 kN.

8. At B, there is a reaction $R_B$ which is treated as vertical load = 50kN and hence the shear force line $B_1B_2$ = 50kN to same scale and the shear force at $B_2$ = -30 + 50 (+ as it is upward) = +20 kN.
9. There is no load in the region BF and hence under this region, the shear force line $B_2F_1$ will be a horizontal line parallel to beam axis.
10. Draw $F_1F_2$ under the vertical load 20kN acting at F downward equal to same scale. The shear force at $F_2$ = 20 – 20 = 0 (-ve as it is acting downward). Note that for the Shear Force Diagram to be precise, the shear force line must finally join the horizontal axis. If there is any shortage or surplus, the shear force diagram must be redrawn.
11. The portion of the shear force diagram above the horizontal axis is +ve and the one below the horizontal axis is –ve.

**Bending Moment Diagram**

1. The Bending Moment is zero at the extreme edges of the beam unless there is an applied moment or couple acting at the edges, Hence the Moment at C = $M_C$ = 0 i.e. at $C_3$.
2. The Bending moment at A is -40 kNm and hence the bending moment line is inclined under the no load portion CA (it can be either horizontal or inclined depending on the moments at the corresponding ends of the portion in the region).
3. The region AD has a uvl and hence the bending moment line will be a cubic parabola (the index of BM is always one more than SF at any section and hence bending moment line is inclined under horizontal shear force line, parabola under inclined shear force line and cubic parabola under parabolic shear force line). The parabola joins the bending moment values at $A_3$ is -40kNm and at $D_3$ is +6.67kNm (Bending moment to the left of D). The cubic parabola will be parallel to vertical at $A_3$ and parallel to horizontal at $D_3$ as the absolute value of shear force at $A_2$ = 30kN (more) compared to that at $D_1$ = 20kN.
4. The bending moment line is always a vertical line under the applied moment or couple. There is an clockwise applied moment of 10kNm acting at D and hence it is hogging. The vertical line $D_3D_4$ is downward and equal to the applied

moment to the same scale = 10kNm. The Bending moment value at $D_4$ = -3.37 kNm

5. The region DG is acted upon by udl, the shear force line is inclined and the bending moment line will be a parabola from $D_4$ to $G_3$. The parabola is joining Bending moment at $D_4$ = -3.37 to that at $G_3$ = 6.67kNm. The bending moment line will be tangential to vertical at $D_4$ and tangential to horizontal at $G_3$ as the shear force at $D_1$ = 20kN which is relatively higher than at G which is 0.
6. The region GE is acted upon by udl, the shear force line is inclined and the bending moment line will be a parabola from $G_3$ to $E_3$. The parabola is joining Bending moment at $G_3$ = 6.67 to that at $E_3$ = -3.37kNm. The bending moment line will be tangential to horizontal at $G_3$ and tangential to vertical at $E_3$ as the absolute shear force at G = 0kN which is relatively lesser than at $E_3$ =3.37kNm.
7. There is an anti-clockwise applied moment of 10kNm acting at E and hence it is sagging. The vertical line $E_3E_4$ is upward and equal to the applied moment to the same scale = 10kNm. The Bending moment value at $E_4$ = 6.67 kNm
8. The region EB has a uvl and hence the bending moment line will be a cubic parabola. The parabola joins the bending moment values at $E_4$ is 6.67kNm (Bending moment to the right of E) and at $B_3$ is -40kNm. The cubic parabola will be tangential to horizontal at $E_4$ and parallel to vertical at $B_3$ as the absolute value of shear force at $E_1$ = 20kN (less) compared to that at $B_1$ = 30kN.
9. The Bending moment at B is -40 kNm and hence the bending moment line is inclined under the no load portion BF to join the horizontal axis at $F_3$ where the bending moment is zero.

3.19 Draw the shear force and bending moment diagrams for a overhanging beam shown in Fig. 3.57. Find and locate the points of contraflexure. (July 09)

The reactions can be obtained from the conditions of equilibrium.

$$\sum V_A = 0;\ R_B + R_D = 10\times 2 + 40 + \frac{1}{2}\times 20\times 2 + 20 = 100\text{kN} \qquad (01)$$

Taking moment about B,

$$\Sigma M_B = 0; 4R_D + (10\times 2)\left(\frac{2}{2}\right) = 40\times 2 + \left(\frac{1}{2}\times 20\times 2\right)\left(2 + \frac{2\times 2}{3}\right) + 20\times 6$$

$$R_D = \frac{246.67}{4} = 61.67\text{kN}$$

Similarly taking moment about D,

$$\Sigma M_D = 0; 4R_B + (20\times 2) = (10\times 2)\left(4 + \frac{2}{2}\right) + 40\times 2 + \left(\frac{1}{2}\times 20\times 2\right)\left(\frac{2\times 1}{3}\right)$$

$$R_B = \frac{153.33}{4} = 38.33\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_B + R_D = 38.33 + 61.67 = 100$ kN (O.K.)

**Bending Moment Values**

$M_E = 0$

$M_D = -20\times 2 = -40\text{kN}$

$$M_C = 61.67\times 2 - 20\times 4 - \left(\frac{1}{2}\times 20\times 2\right)\left(\frac{2\times 2}{3}\right) = 16.67\text{kNm}$$

$$M_B = -(10\times 2)\left(\frac{2}{2}\right) = -20\text{kNm}$$

$M_A = 0$

**Points of Contraflexures**

*Bending moment* at any section $x$ from the left support

For region CD

$$M_x = 38.33x - (10\times 2)(x+1) - 40(x-2) - \left(\frac{1}{2}\times 20\times\frac{(x-2)^2}{2}\right)\left(\frac{2}{3}\right)(x-2)$$

For Point of contraflexure, $M_x = 0$, solving, we get $x = 2.713$m

For region BC $M_x = 38.33x - (10\times 2)(x+1)$

For Point of contraflexure, $M_x = 0$, solving, we get $x = 1.09$m

From second method, consider the similar triangles between BC,

$$\frac{x}{20}=\frac{2-x}{16.67} \text{ or } x = 1.09\text{m}$$

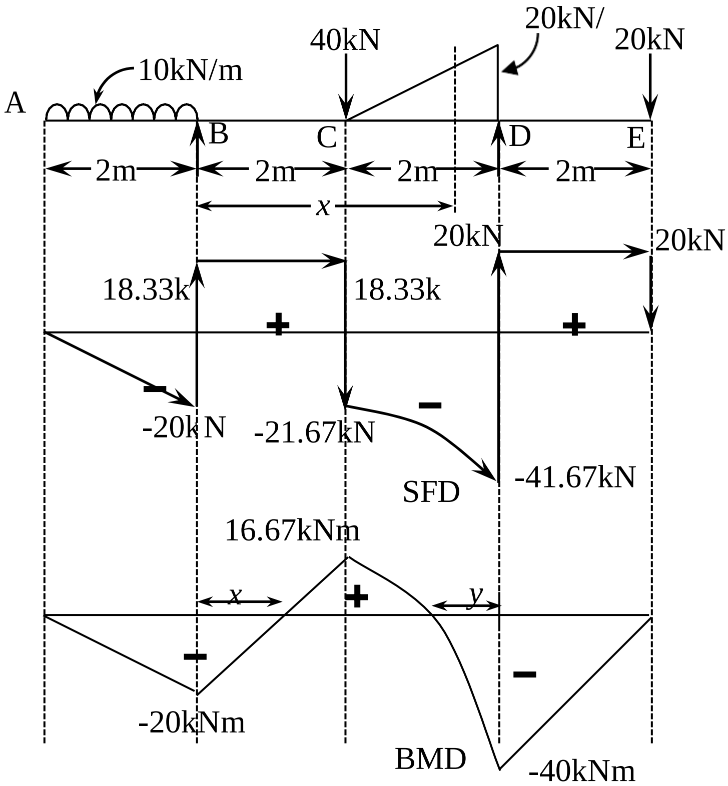



Fig. 3.57

3.20 For the beam shown in Fig.3.58, draw the shear force and bending moment diagram and locate the Point of contraflexure if any. (Jan 09)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_D = 10\times 2 + 30 + 40 + 20\times 4 = 170\text{kN}$ (01)

Taking moment about B,

$$\Sigma M_B = 0; 6R_D = (10\times 2)\left(\frac{2}{2}\right) + 30\times 2 + 40\times 4 + (20\times 4)\left(4+\frac{4}{2}\right) \text{ or } R_D = \frac{720}{6} = 120\text{kN}$$

Similarly taking moment about D,

$$\Sigma M_D = 0; 6R_B = (10\times 2)\left(4+\frac{2}{2}\right) + 30\times 4 + 40\times 2 \text{ or } R_B = \frac{300}{6} = 50\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_B + R_D = 50 + 120 = 170$ kN (O.K.)

**Bending Moment Values**

$M_E = 0$

$$M_D = -(20\times 2)\left(\frac{2}{2}\right) = -40\text{kN}$$

$$M_C = 120 \times 2 - (20 \times 4)\left(\frac{4}{2}\right) = 80\text{kNm}$$

$$M_B = 50 \times 2 - (10 \times 2)\left(\frac{2}{2}\right) = 80\text{kNm}$$

$M_A = 0$

**Points of Contraflexures**

*Bending moment* at any section $x$ from the left support

For region CD

$$M_x = 38.33x - (10 \times 2)(x+1) - 40(x-2) - \left(\frac{1}{2} \times 20 \times \frac{(x-2)^2}{2}\right)\left(\frac{2}{3}\right)(x-2)$$

For Point of contraflexure, $M_x = 0$, solving, we get $x = 2.713$m

For region BC $M_x = 38.33x - (10 \times 2)(x+1)$

For Point of contraflexure, $M_x = 0$, solving, we get $x = 1.09$m

From second method, consider the similar triangles between BC,

$$\frac{x}{20} = \frac{2-x}{16.67} \text{ or } x = 1.09\text{m}$$

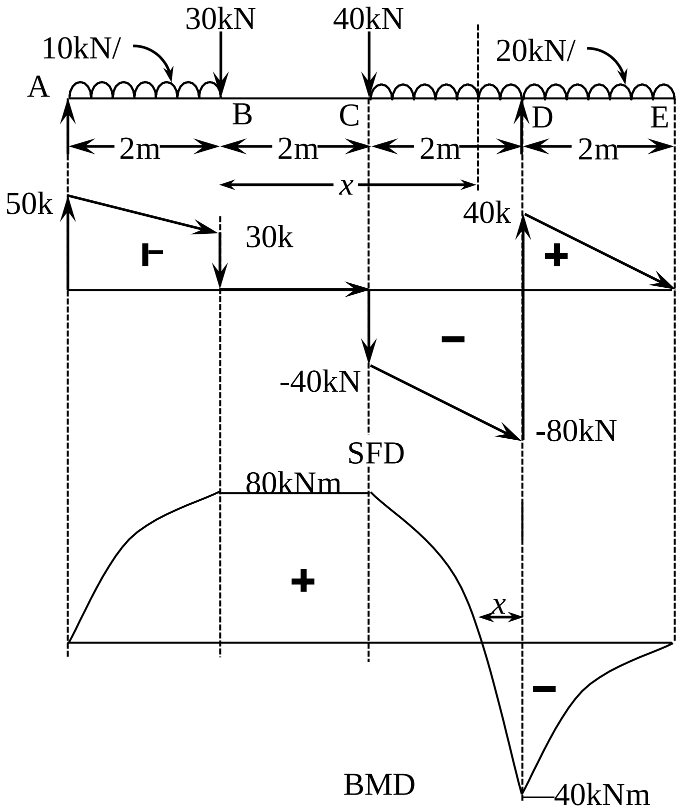



Fig. 3.58

3.21 For the beam shown in Fig. 3.59, obtain SFD and BMD. Locate Points of contraflexure, if any. (July 09)

The reactions can be obtained from the conditions of equilibrium.

$\sum V_A = 0;\ R_B + R_D = 5\times 8 + 50 = 90\text{kN}$ (01)

Taking moment about B,

$$\Sigma M_B = 0; 16R_D + 120 = (5\times 8)\left(\frac{8}{2}\right) + 50\times 12 + 160 \text{ or } R_D = \frac{800}{16} = 50\text{kN}$$

Similarly taking moment about D,

$$\Sigma M_D = 0; 16R_B + 160 = (5\times 8)\left(8 + \frac{8}{2}\right) + 50\times 4 + 120 \text{ or } R_D = \frac{640}{16} = 40\text{kN}$$

**Check**

Substituting in Eq. 01, we have $R_B + R_D = 40 + 50 = 90$ kN (O.K.)

**Bending Moment Values**

$M_{DR} = 0$

$M_{AL} = -160\text{kNm}$

$M_C = 50\times 4 - 160 = 40\text{kNm}$

$M_B = 50\times 8 - 50\times 4 - 160 = 40\text{kNm}$

$M_{AR} = -120\text{kNm}$

$M_{AL} = 0$

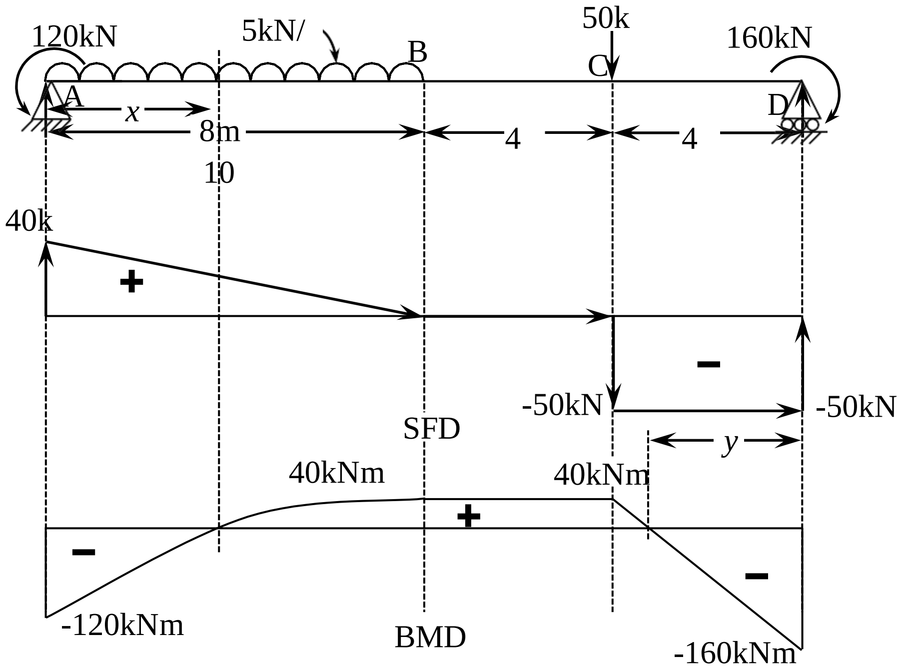



Fig. 3.59

**Points of Contraflexures**

*Bending moment* at any section *x* from the left support

For region AB

$$M_x = 40x - \left(\frac{5x^2}{2}\right) - 120 = 0 \text{ or } x = 4\text{m}$$

Point of contraflexure is $x = 4$m from the left support.

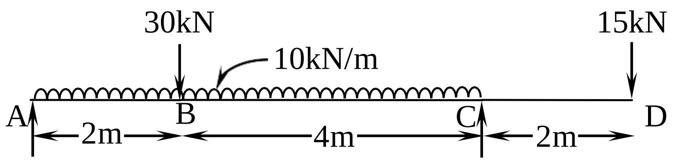
For region CD $M_y = 50y - 160 = 0$ or $y = 3.2$m

For Point of contraflexure is $y = 3.2$m from the right support.

From second method, consider the similar triangles between CD

$\frac{y}{160} = \frac{4-y}{40}$ or $y = 3.2$m

A beam ABCD, 8m long has supports at **A** and at **C** which is 6m from point **A.** The beam carries a **UDL** of 10kN/m between **A** and **C.** At point **B** a 30kN concentrated load acts 2m from the support **A** and a point load of 15kN acts at the free end **D**. Draw the SFD and BMD giving salient values. Also locate the point of contra-flexure if any. (14)(July 2015)




From the conditions of equilibrium, we have algebraic sum of vertical forces to be zero.

$\uparrow +\Sigma V = 0;\qquad R_A + R_C = 30 + 15 + (10)(6) = 105 \text{ kN } (\uparrow)$

Algebraic sum of moments about any point is zero. Taking moments about A, we get

$$\Sigma M_A = 0;\qquad 6R_C = (30)(2) + (15)(8) + \left[(10)(6)\right]\left(\frac{6}{2}\right) = 360 \text{ kN}$$

$$R_C = 60 \text{ kN}(\uparrow)$$

Taking moments about C, we get

$$\Sigma M_C = 0;\qquad 6R_A + (15)(2) = (30)(4) + \left[(10)(6)\right]\left(\frac{6}{2}\right) = 270 \text{ kN}$$

$R_A = 45 \text{ kN}(\uparrow)$

Check: $R_A + R_C = 45 + 60 = 105 \text{ kN } (\uparrow)$

Shear Force Diagram can be directly drawn.

**Bending Moment values**:

Unless there are end moments of the beam, the Moments are zero at ends of the beam.

$M_A = 0$ and $M_D = 0$

$$M_B = (45)(2) - \left[(10)(2)\right]\left(\frac{2}{2}\right) = 70\text{kNm}$$

$$M_C = -(15)(2) = -30\text{kNm}$$

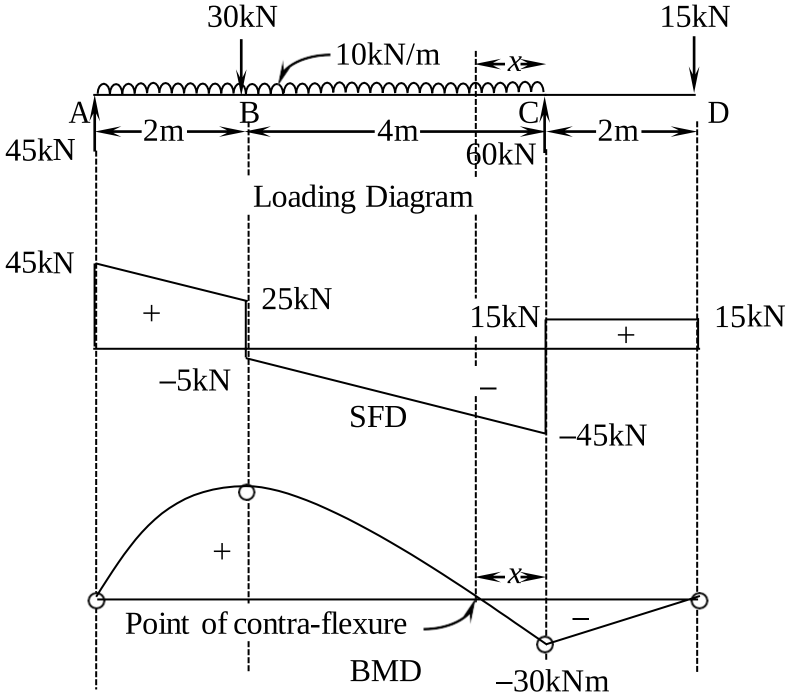
To locate the point of contra-flexure where the bending moment changes its sign, consider the section to be at a distance $x$ towards left of the right support as shown. The bending moment at the section is given by

$$M_x = 60x - (15)(2+x) - (10)(x)\left(\frac{x}{2}\right) \Rightarrow 0$$

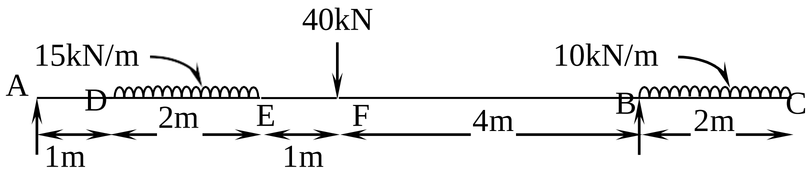
$$45x - 30 - 5x^2 = 0$$

Solving, $x = 0.725\text{m}$ and $8.275\text{m}$

Hence the point of contra-flexure is at 0.725m to left of right support.




Draw the Shear force and bending moment diagrams for the Fig. shown (10) July 2016




From the conditions of equilibrium, we have algebraic sum of vertical forces to be zero.

$\uparrow +\Sigma V = 0;\qquad R_A + R_B = (15)(2) + 40 + (10)(2) = 90\ \text{kN}\ (\uparrow)$

Algebraic sum of moments about any point is zero. Taking moments about A, we get

$$\Sigma M_A = 0;\qquad 8R_B = \left[(15)(2)\right]\left(1+\frac{2}{2}\right) + (40)(1+2+1) + \left[(10)(2)\right]\left(8+\frac{2}{2}\right) = 400\ \text{kN}$$

$$R_B = 50\,\text{kN}(\uparrow)$$

Taking moments about B, we get

$$\Sigma M_B = 0; \qquad 8R_A + \left[(10)(2)\right]\left(\frac{2}{2}\right) = (40)(4) + \left[(15)(2)\right]\left(4 + 1 + \frac{2}{2}\right) = 340 \text{ kN}$$

$R_A = 40 \text{ kN} (\uparrow)$

Check: $R_A + R_B = 40 + 50 = 90 \text{ kN} (\uparrow)$

Shear Force Diagram can be directly drawn.

**Bending Moment values**:

Unless there are end moments of the beam, the Moments are zero at ends of the beam.

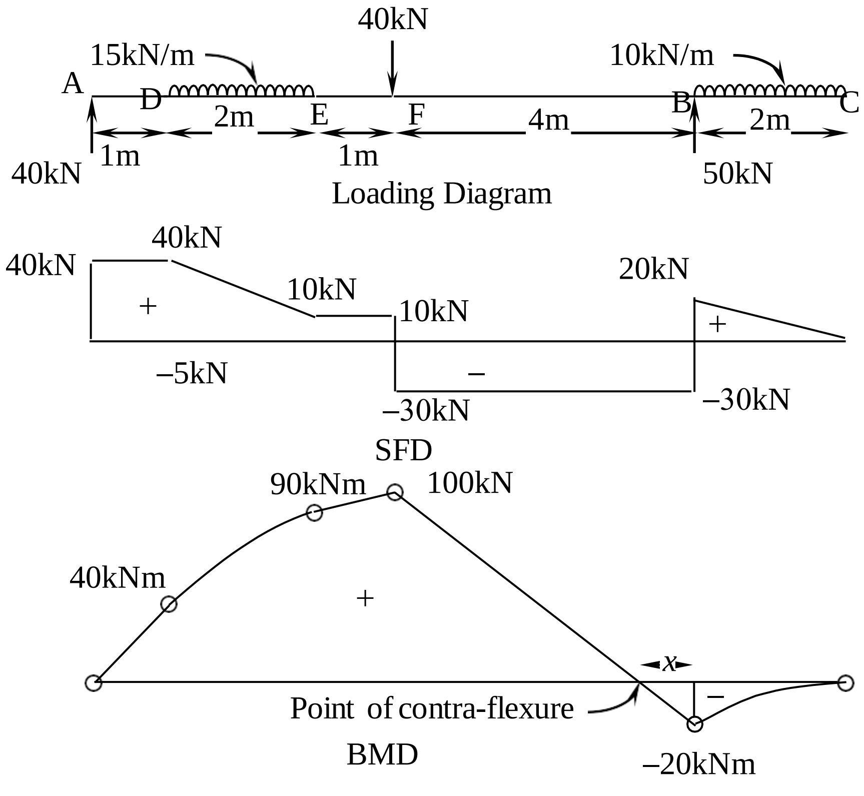
$M_A = 0$ and $M_C = 0$

$M_D = (40)(1) = 40\text{kNm}$

$$M_E = (40)(3) - \left[(15)(2)\right]\left(\frac{2}{2}\right) = 90\text{kNm}$$

$$M_F = (40)(4) - \left[(15)(2)\right]\left(1 + \frac{2}{2}\right) = 100\text{kNm}$$

$$M_B = -\left[(10)(2)\right]\left(\frac{2}{2}\right) = -20\text{kNm}$$

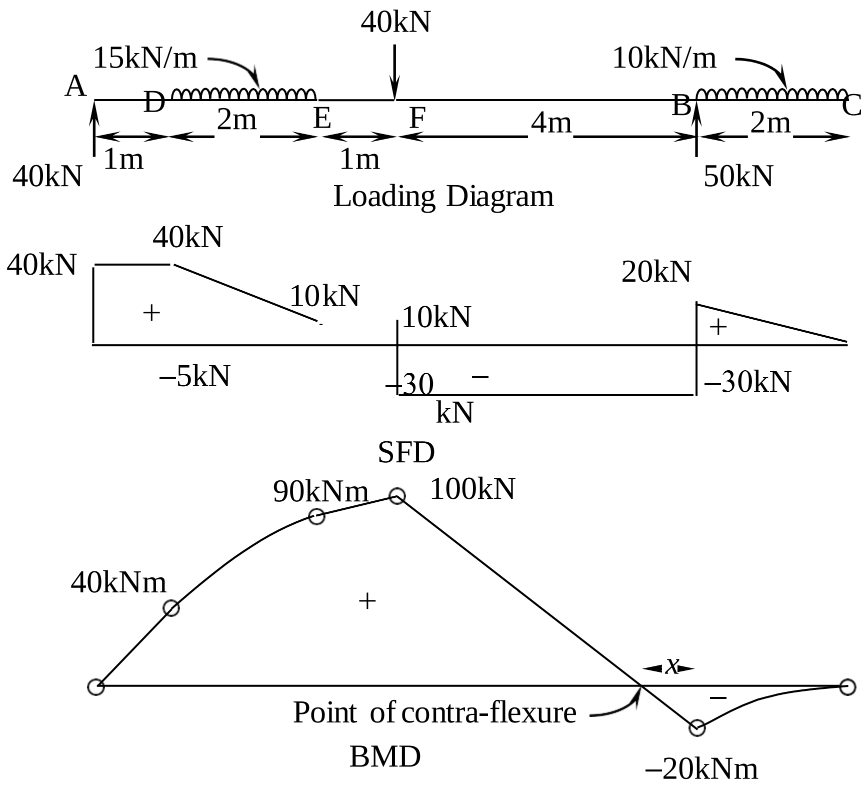



To locate the point of contra-flexure where the bending moment changes its sign, consider the section to be at a distance $x$ towards left of the right support as shown. Bending moment inclined line is crossing zero line as a straight line forming two alternate triangles which are similar. Hence using similar triangle properties

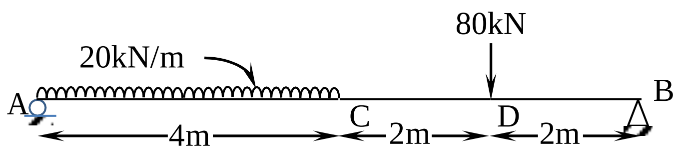
$$\frac{4-x}{x}=\frac{100}{20}$$

Solving, $x = 0.67\text{m}$

Hence the point of contra-flexure is at 0.67m to left of right support.




Draw Shear force and Bending moment Diagram for the beam shown in Fig.




Fromthe conditions of equilibrium, we have algebraic sum of vertical forces to be zero.

$\uparrow +\Sigma V = 0;$ $\qquad R_A + R_B = (20)(4) + 80 = 160\ \text{kN}\ (\uparrow)$

$\Sigma M_A = 0;$ $\qquad 8R_B = \left[(20)(4)\right]\left(\frac{4}{2}\right) + (80)(4+2) = 640\ \text{kN}$

$R_B = 80\,\text{kN}(\uparrow)$

Algebraic sum of moments about any point is zero. Taking moments about A, we get

$$\Sigma M_A = 0; \qquad 8R_B = \left[(20)(4)\right]\left(\frac{4}{2}\right) + (80)(4+2) = 640 \text{ kN}$$

$$R_B = 80 \text{ kN}(\uparrow)$$

Taking moments about B, we get

$$\Sigma M_B = 0; \qquad 8R_A = \left[(20)(4)\right]\left(4+\frac{4}{2}\right) + (80)(2) = 640 \text{ kN}$$

$$R_A = 80 \text{ kN}(\uparrow)$$

Check: $R_A + R_B = 80 + 80 = 160 \text{ kN } (\uparrow)$

Shear Force Diagram can be directly drawn.

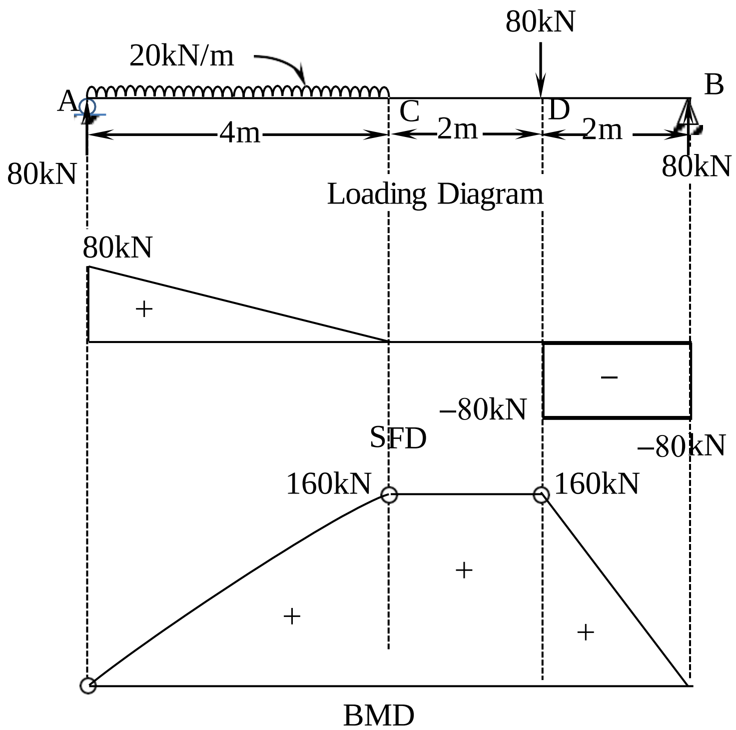
**Bending Moment values**:

Unless there are end moments of the beam, the Moments are zero at ends of the beam.

$M_A = 0$ and $M_B = 0$

$$M_C = (80)(4) - \left[(20)(4)\right]\left(\frac{4}{2}\right) = 160\text{kNm}$$

$$M_D = (80)(2) = 160\text{kNm}$$



## *5.2 Bending Stress*

### *a. Simplifying assumptions*

- The stresses caused by the bending moment are known as ***bending stress***, or ***flexure stresses***. The relationship between these stresses and the bending moment is called the ***flexure formula*****.**

- In deriving the flexure formula, make the following assumptions:

  - The beam has an axial plane of symmetry, which we take to be the *xy*-plane (see Fig. 5.1).

**Figure 5.1 Symmetrical beam with loads lying in the plane of symmetry.**

- The applied loads (such as $F_1$,$F_2$ and $F_3$ in Fig.5.1) lie in the plane of the symmetry and are perpendicular to the axis of the beam (the *x*-axis).The axis of the beam **bends but does not stretch** ( the axis lies some where in the plane of symmetry; its location will be determined later).

- Plane sections of the beam remain plane (do not warp ) and perpendicular to the deformed axis of the beam. Change in the cross-sectional dimensions of the beam are negligible.

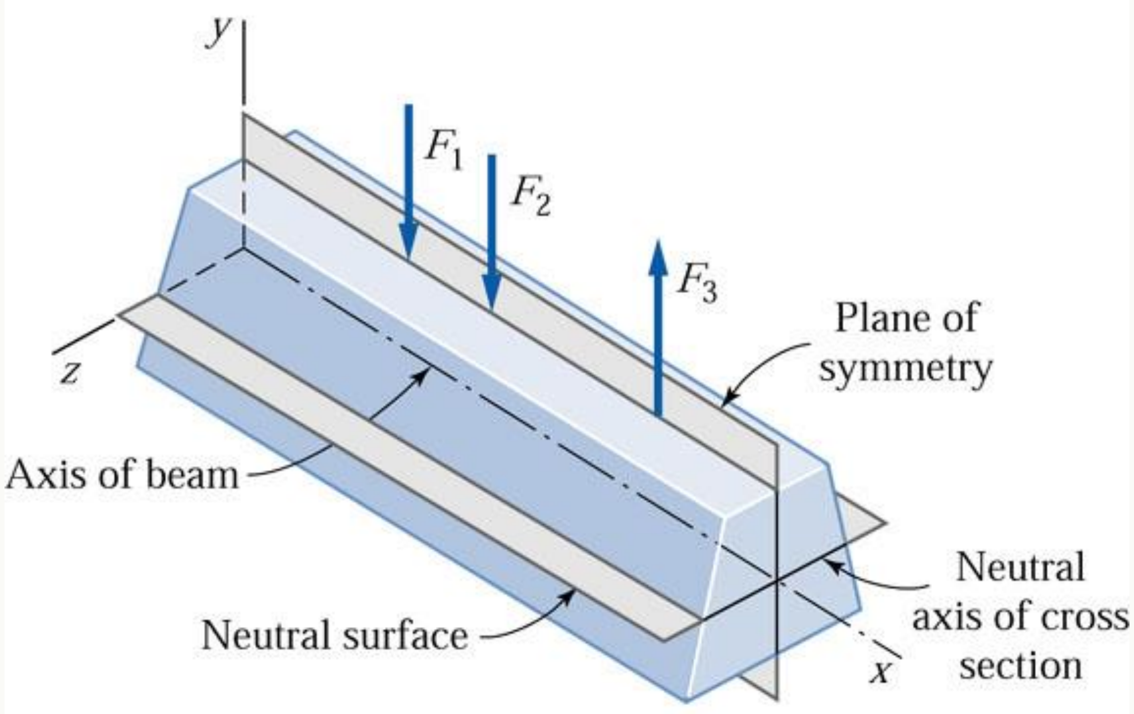
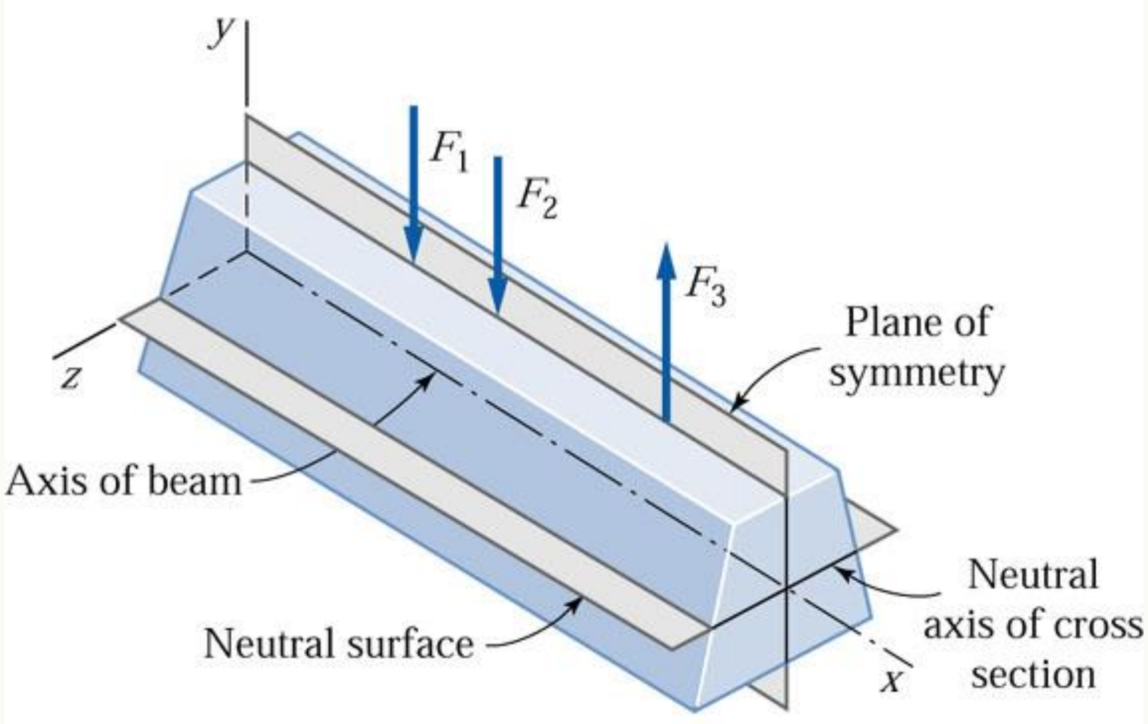



**Figure 5.1 Symmetrical beam**

- Because the shear stresses caused by the vertical shear force will distort (warp) an originally plane section, we are limiting our discussion here to the deformations caused by the bending moment alone.
- the deformations due to the vertical shear force are negligible in the slender beams compared to the deformations caused by bending .

- The above assumptions lead us to the following conclusion: *Each cross section of the beam rotates as a rigid entity about a line called the neutral axis of the cross section*.
- The ***neutral axis*** passes through the axis of the beam and is perpendicular to **the plane of symmetry**, as shown in Fig. 5.1. The *xz*-plane that contains the neutral axes of all the cross sections is known as **the *neutral surface* of the beam**.



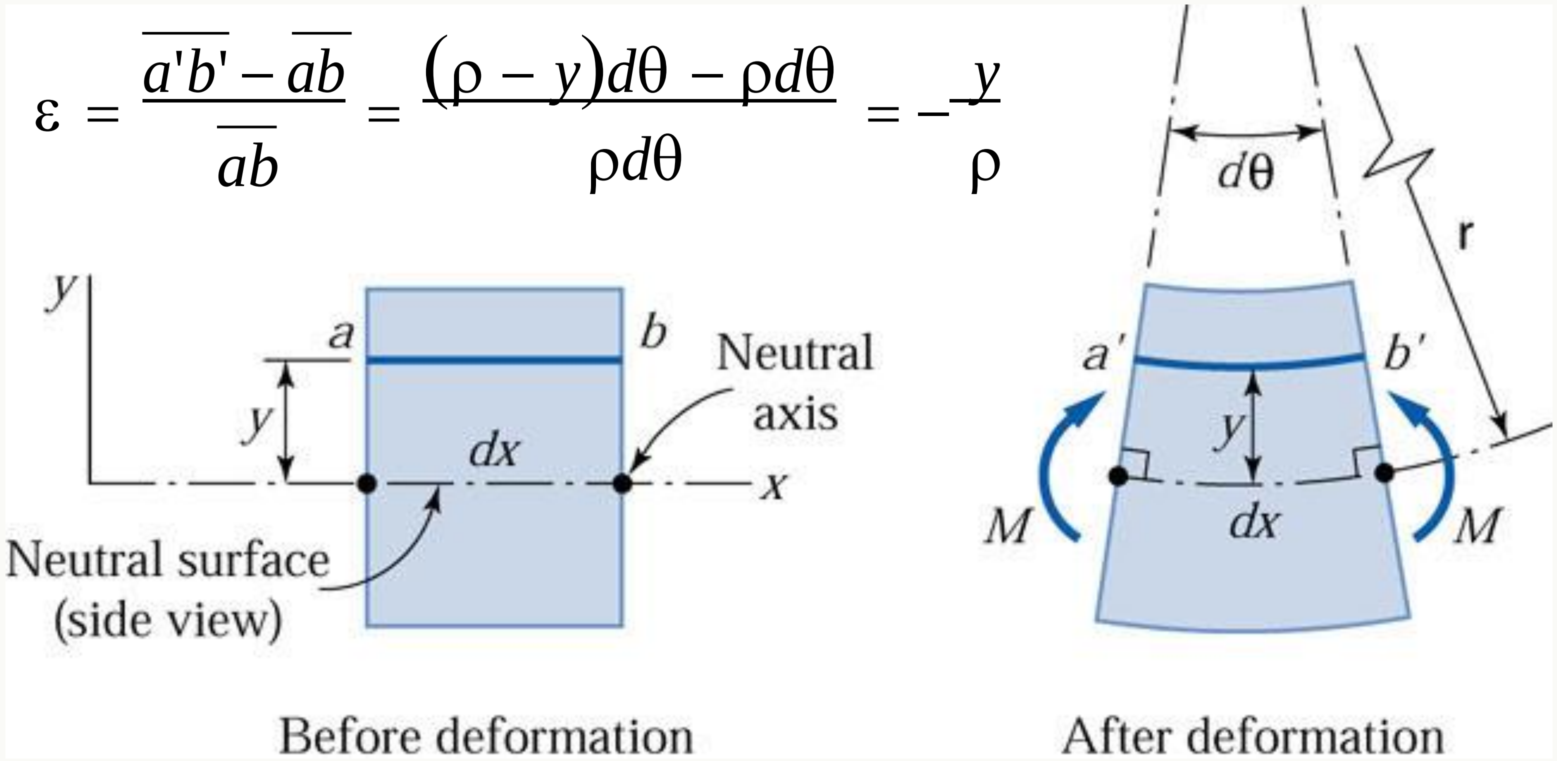
***b. Compatibility***

- The neutral surface becomes curved upon deformation, as indicated in Fig.5.2.
- The longitudinal fibers lying on the neutral surface are undeformed, whereas the fibers above the surface are compressed and the fibers below are stretched.

**Figure 5.2 Deformation of an infinitesimal beam segment.**

- The fiber form are arc *a′ b′* of radius (ρ-y) , subtended by the angle d θ , its deformed length is $\overline{a'b'} = (\rho - y)d\theta$
- The original length of this fiber is $\overline{ab} = dx = \rho d\theta$ . The normal strain ε of the fiber

$$\varepsilon = \frac{\overline{a'b'} - \overline{ab}}{\overline{ab}} = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} = -\frac{y}{\rho}$$



- Assuming that the stress is less than the proportional limit of the material we can obtain the normal stress in the fiber ab from Hook′ s law:

$$\boxed{\sigma = E\varepsilon = -\frac{E}{\rho}y} \tag{5.1}$$

Equation (5.1) shown that the normal stress of a longitudinal fiber is proportional to the distance y of the fiber from the neutral surface.

- The negative sign indicates that positive bending moment causes compressive stress when $y$ is positive (fiber above the neutral surface) and tensile stress when $y$ is negative (fiber below the neutral surface).

### *c. Equilibrium*

- Figure 5.3 shows the normal force acting on the infinitesimal area dA of the cross section is $dP = \sigma\, dA$. Substituting $\sigma = -(E/\rho)y$,

$$dP = -\frac{E}{\rho} y dA \qquad \text{(a)}$$

Where *y* is the distance of dA from the neutral axis (NA).

- The resultant of the normal stress distribution over the cross section must be equal to the bending moment M acting about the neutral axis (z-axis).

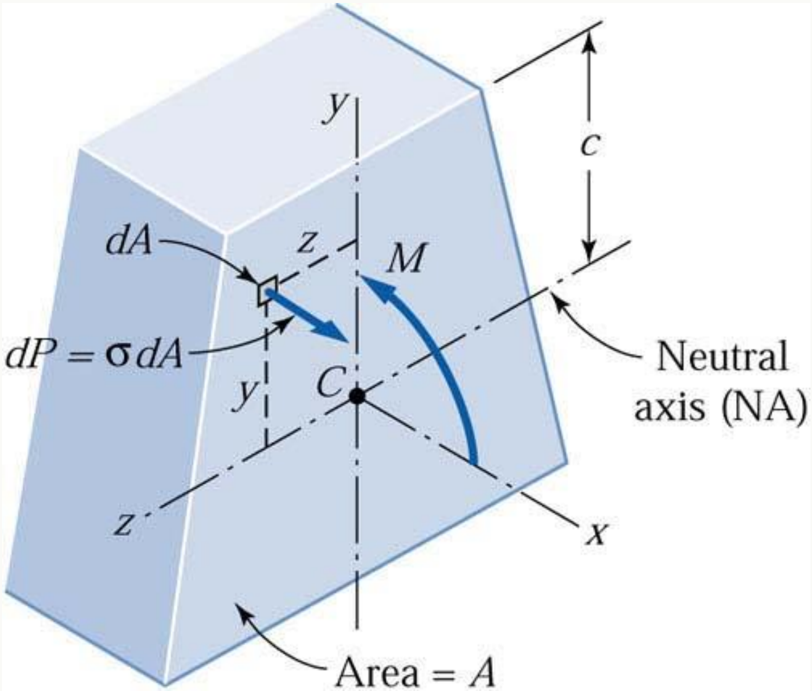



**Figure 5.3 Calculating the resultant of the Normal stress acting on the cross section. Resultant is a couple Equal to the internal bending moment of *M*.**

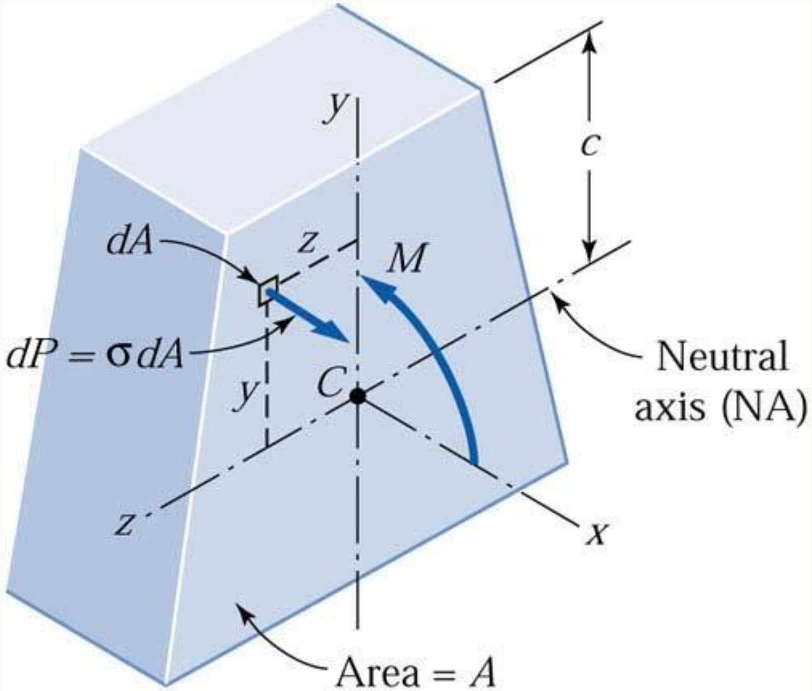
❑In other work, $\boxed{-\int_A y dp = M}$

where the integral is taken over the entire cross-sectional area $A$

❑the resultant axial force and the resultant bending moment about the y-axis must be zero; that is,

$$\boxed{\int_A dP = 0} \quad \text{and} \quad \boxed{\int_A z dP = 0}$$

These three equilibrium equations are developed in detail below.




***Resultant Axial Force Must Vanish*** The condition for zero axial force is

$$\int_A dp = -\frac{E}{\rho}\int_A y dA = 0$$

- Because $E / \rho \neq 0$, this equation can be satisfied only if

$$\boxed{\int_A y\,dA = 0} \qquad \text{(b)}$$

The integral in Eq.(b) is the first moment of the cross-sectional area about the neutral axis. It can be zero only if the neutral axis passes through **centroid C of the cross-sectional area**.

***Resultant Moment About y-Axis Must Vanish***

This condition is

$$\boxed{\int_A z\,dP = -\frac{E}{\rho}\int_A zy\,dA = 0} \qquad \text{(c)}$$

The integral in Eq.(b) is **the product of inertia of the cross-sectional area**.

***Resultant Moment About the Neutral Axis Must Equal M***

Equating the resultant moment about the *z*-axis to *M*

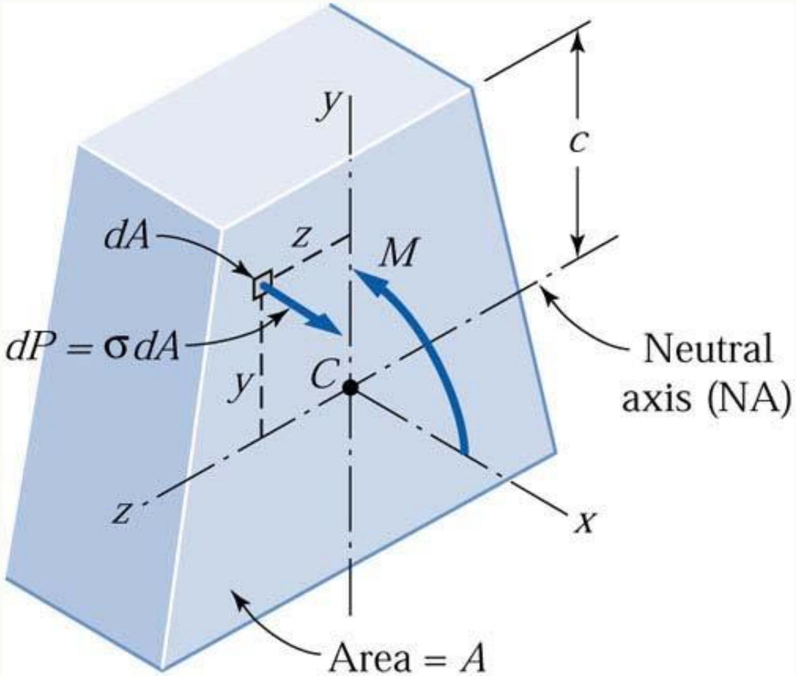
$$\boxed{-\int_A y\,dp = M}\qquad\qquad \boxed{-\int_A y\,dp = \frac{E}{\rho}\int_A y^2 dA = M}$$

Recognizing that $\boxed{\int_A y^2 dA = I}$ is the moment of inertia of the cross-sectional area about the neutral axis ( the *z*-axis), we obtain the ***moment curvature relationship***

$$\boxed{M = \frac{EI}{\rho}} \qquad (5.2a)$$

A convenient form of this equation is

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}} \qquad (5.2b)$$



### *d. Flexure formula; section modulus*

- Substituting the expression for $1/\rho$ from Eq.(5.2) into Eq. (5.1), we get the *flexure formula* :

$$\boxed{\sigma = -\frac{My}{I}} \qquad (5.3)$$

Note that a positive bending moment M causes negative (compressive) stress above the neutral axis and positive ( tensile) stress below the neutral axis

- The **maximum value** of bending stress without regard to its sign is given by

$$\boxed{\sigma_{\max} = \frac{[M_{\max}]c}{I}} \qquad (5.4a)$$

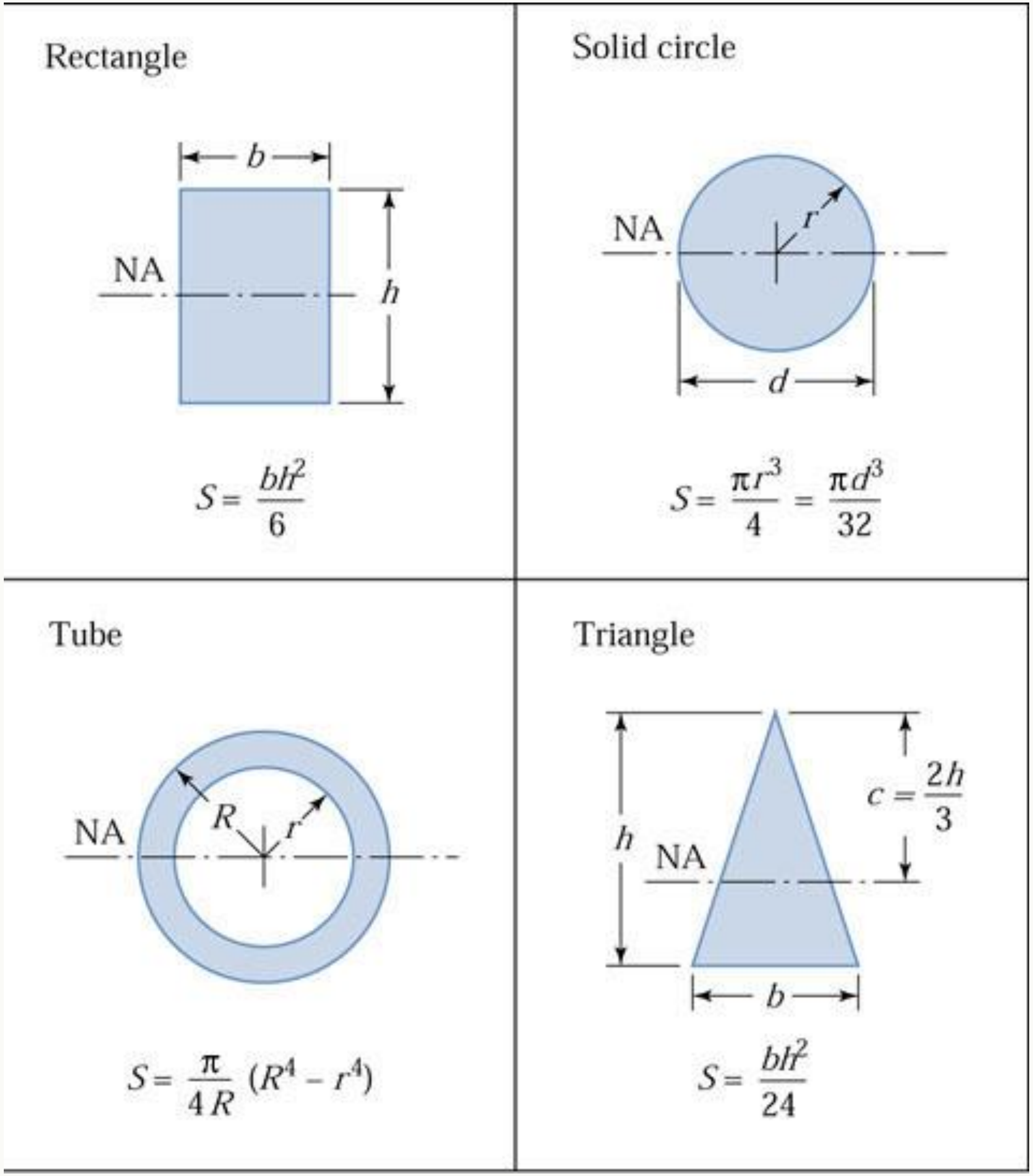
where $c$ is the distance from the neutral axis to the outermost point of the cross section.

Equation (5.4a) is frequently written in the form

$$\boxed{\sigma_{\max} = \frac{[M_{\max}]}{S}} \tag{5.4b}$$

where $S = I / c$ is called **the *section modulus* of the beam**. The dimension of $S$ is $[L^3]$, so that its units are in.$^3$, mm$^3$, and so on. The formulas for the section moduli of common cross sections are given in Fig. 5.4.

**Figure 5.4 Section moduli of simple cross sectional shapes.**




- The section moduli of standard structural shapes are listed in various handbooks; an abbreviated list is given in Appendix B.

**Figure 5.4 Section moduli of simple cross sectional shapes.**

## e. Procedures for determining bending stresses

### Stress at a Given Point

- Use the method of sections to determine the bending moment M at the cross section containing the given point.
- Determine the location of the neutral axis.
- Compute the moment of inertia I of the cross- sectional area about the neutral axis. ( If the beam is standard structural shape, its cross- sectional properties are listed in Appendix B. P501)
- Determine the y-coordinate of the given point. Note that y is positive if the point lies above the neutral axis and negative if it lies below the neutral axis.
- Compute the bending stress from $\sigma = -My / I$. If correct sign are used for *M* and *y*, the stress will also have the correct sign (tension positive compression negative).

## ***Maximum Bending Stress: Symmetric Cross Section***

If the neutral axis is an axis of symmetric of the cross section, the maximum tensile and compression bending stresses are equal in magnitude and occur at the section of the largest bending moment. The following procedure is recommended for determining the maximum bending stress in a prismatic beam:

- Draw the bending moment diagram by one of the methods described in Chapter 4. Identify the bending moment $M_{max}$ that has the largest magnitude (disregard the sign)
- Compute the moment of inertia I of the cross- sectional area about the neutral axis. ( If the beam is a standard structural shape, its cross- sectional properties are listed in Appendix B.)
- Calculate the maximum bending stress from $\sigma_{max} = [M_{max}]c / I$, where c is the distance from the neutral axis to the top or bottom of the cross section .

## ***Maximum Tensile and Compressive Bending Stresses:***

### **Unsymmetrical Cross Section**

If the neutral axis is not an axis of symmetry of the cross section, the maximum tensile and compressive bending stresses may occur at different sections.

- Draw the bending moment diagram. Identify the largest positive and negative bending moments.
- Determine the location of the neutral axis and record the distances $c_{top}$ and $c_{bot}$ from the neutral axis to the top and bottom of the cross section.
- Compute the moment of inertia I of the cross section about the neutral axis.

- Calculate the bending stresses at the top and bottom of the cross section where the largest positive bending moment occurs from $\sigma = -My / I$.

✓ At the top of the cross section, where $y = c_{top}$,we obtain $\sigma_{top} = -Mc_{top}/ I$.

✓ At the bottom of the cross section, we have $y = - c_{bot}$, so that $\sigma_{bot} = Mc_{bop}/ I$.

- Repeat the calculations for the cross section that carries the largest negative bending moment.
- Inspect the four stresses thus computed to determine the largest tensile (positive) and compressive (negative) bending stresses in the beam.

***Note on Units***

the units of terms in the flexure formula $\sigma = -My / I$.

- In the U.S. Customary system, M is often measured in pound-feet and the cross sectional properties in inches, It is recommended that you convert M into lb·in. and compute $\sigma$ in lb/in.$^2$ (psi). Thus, the units in the flexure formula become

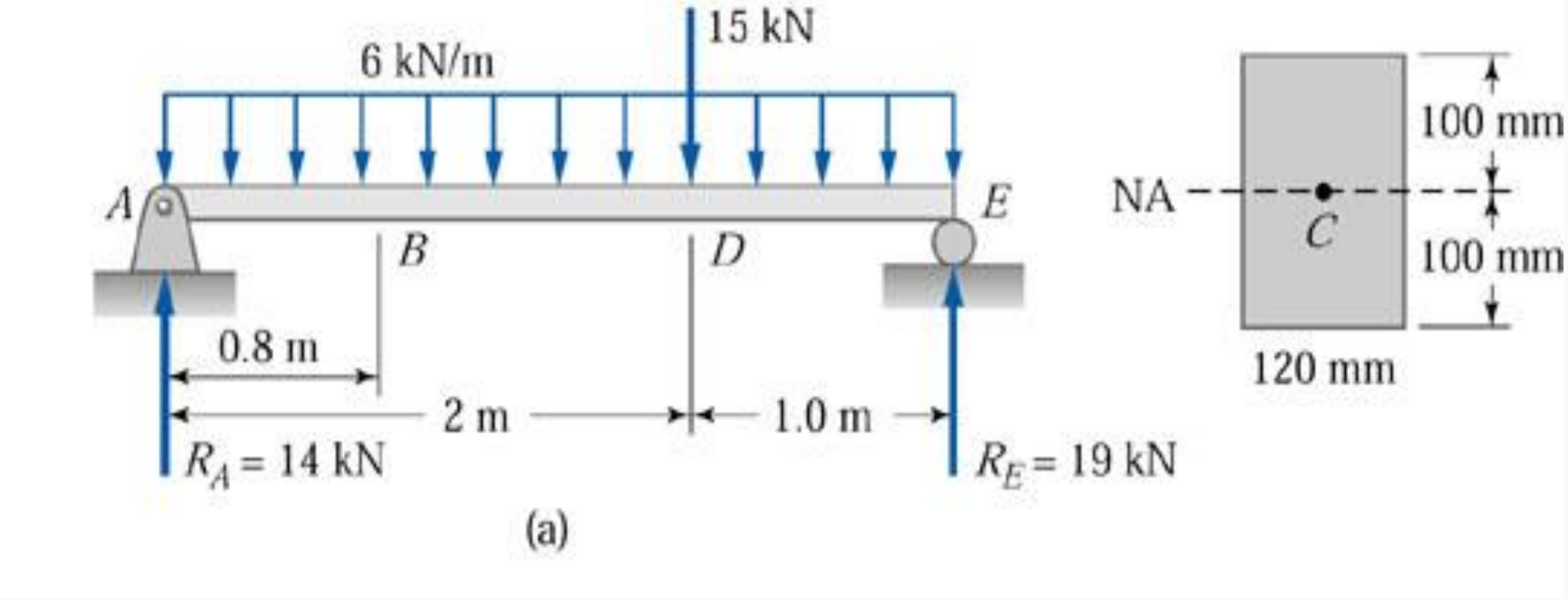
$$\sigma\left[lb / in.^2\right] = \frac{M\left[lb \cdot in.\right] y\left[in.\right]}{I\left[in.^4\right]}$$

- In SI system, M is usually expressed in N · m, whereas the cross-sectional dimensions are in mm. To obtain $\sigma$ in N/m$^2$ (Pa), he cross sectional properties must be converted to meters, so that the units in the flexure equation are

$$\sigma\left[N / m^2\right] = \frac{M\left[N \cdot m\right] y\left[m\right]}{I\left[m^4\right]}$$

## *Sample Problem* 5.1

The simply supported beam in Fig. (a) has a rectangular cross section 120 mm wide and 200 mm high. (1) Compute the maximum bending stress in the beam. (2) Sketch the bending stress distribution over the cross section on which the maximum bending stress occurs. (3) Compute the bending stress at a point on section *B* that is 25 mm below the top of the beam.



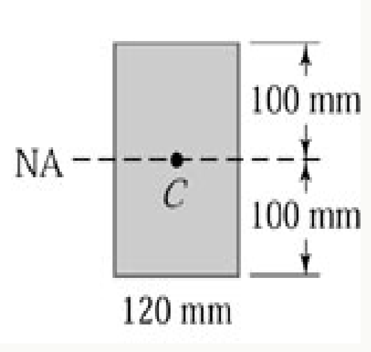
## Solution

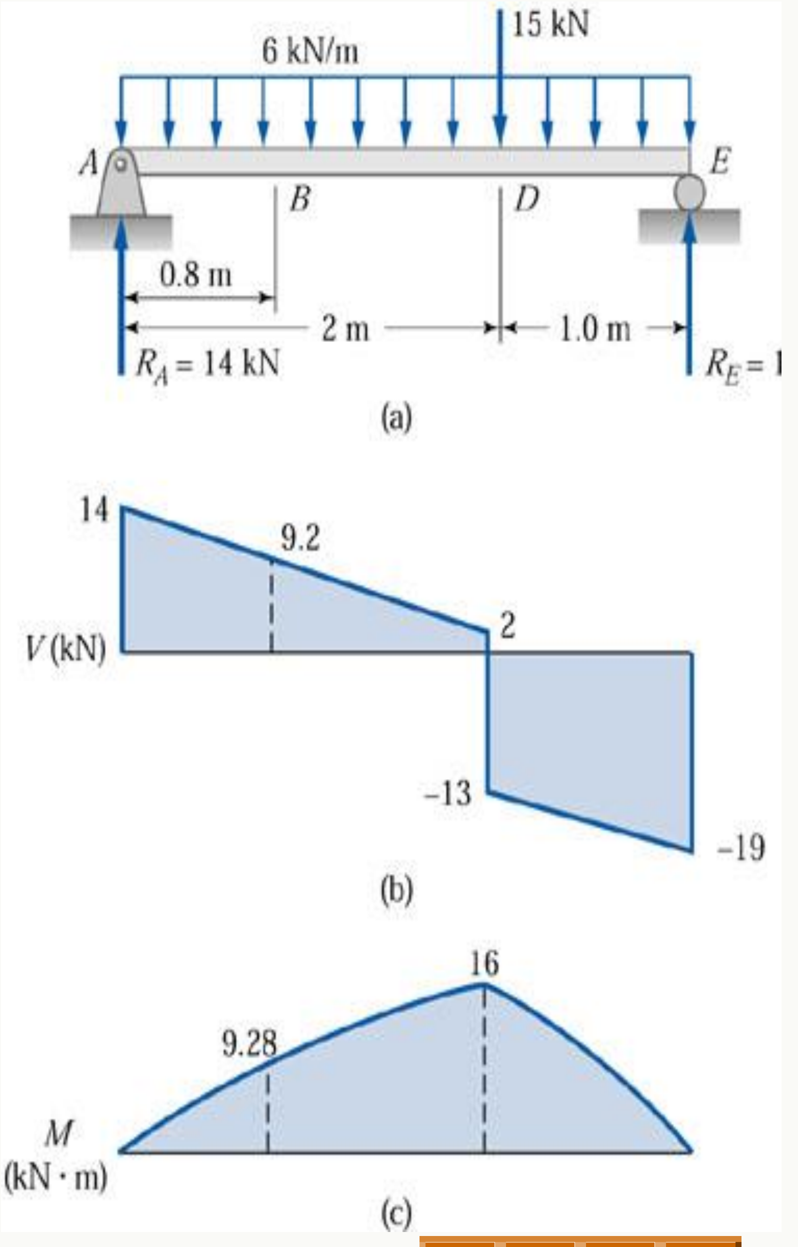
### *Preliminary Calculations*

The shear force and bending moment diagrams. $M_{max} = +16$ kN·m, occurring at *D*. The neutral axis (NA) is an axis of symmetry of the cross section as shown in Fig. (a). The moment of inertia of the cross section about the neutral axis is

$$I = \frac{bh^3}{12} = \frac{0.12(0.2)^3}{12} = 800 \times 10^{-6}\, m^4$$

and the distance c between the neutral axis and the top (or bottom) of the cross section is c = 100 mm = 0.1 m.



**Part 1**

The maximum bending stress in the beam on the cross section that carries the largest bending moment, which is the section at *D*.

$$\sigma_{max} = \frac{[M_{max}]c}{I} = \frac{(16\times10^{3})(0.1)}{80.0\times10^{-6}} = 20.0\times10^{6}\,Pa = 20.0\ MPa \quad \text{Answer}$$

**Part 2**

The stress distribution on the cross section at *D* is shown in Fig. (d)

(i) The bending stress varies linearly with distance from the neutral axis;

(ii) Because $M_{max}$ is positive, the top half of the cross section is in compression and the bottom half is in tension.

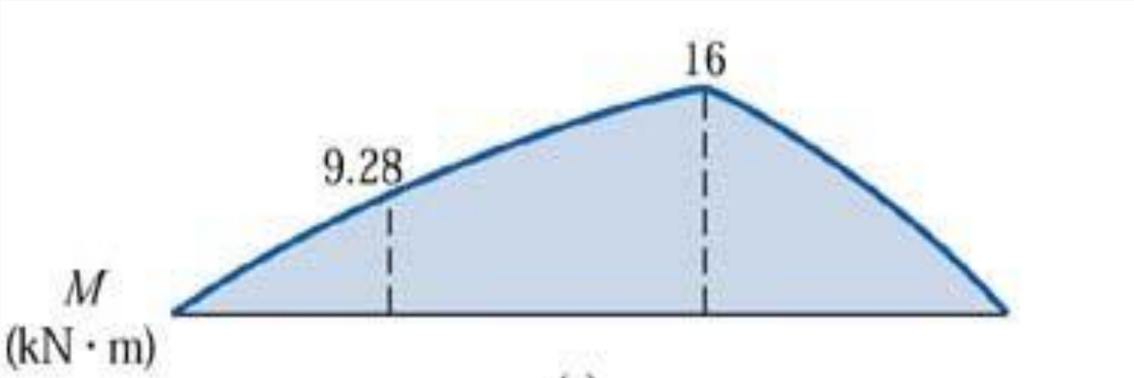
(iii) Due to symmetry of the cross section about the neutral axis, the maximum tensile and compressive stresses are equal in magnitude.

## Part 3

From Fig. (c) we see that the bending moment at section *B* is $M = +9.28$ kN·m. The *y*-coordinate of the point that lies 25 mm below the top of the beam is $y = 100 - 25 = 75$ mm $= 0.075$ m.

$$\sigma = -\frac{My}{I} = -\frac{\left(9.28\times10^{3}\right)(0.075)}{80.0\times10^{-6}} = -8.70\times10^{6}\ Pa = -8.70\,MPa \qquad \text{Answer}$$

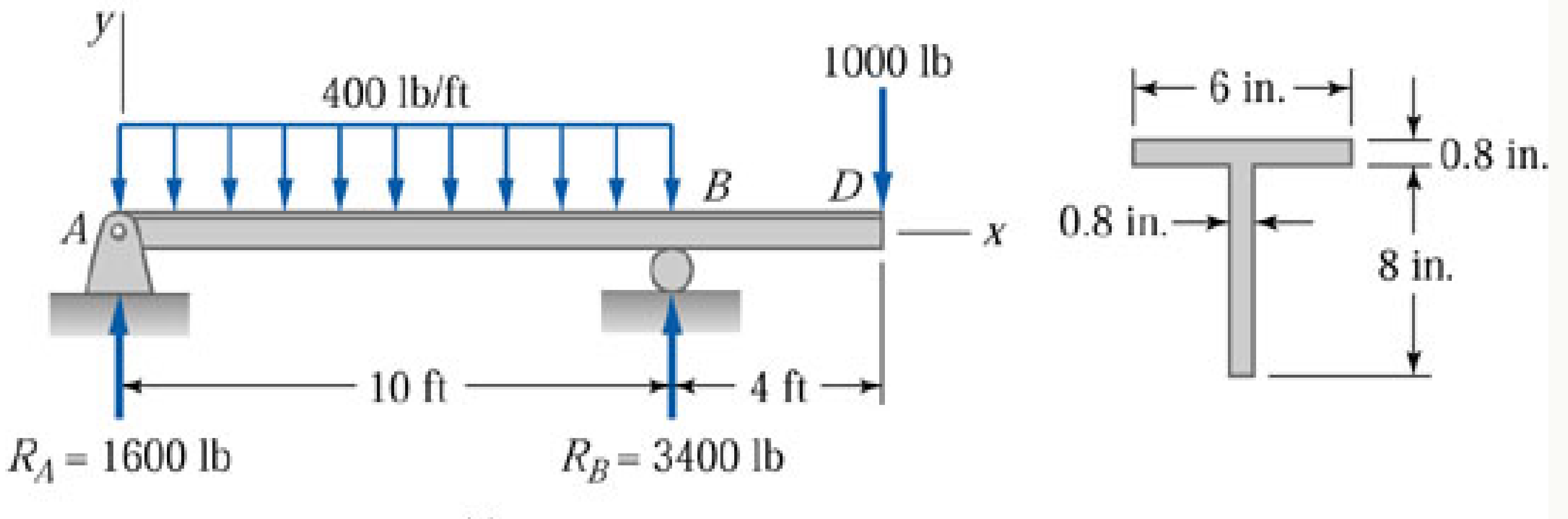
The negative sign indicates that this bending stress is compressive, which is expected because the bending moment is positive and the point of interest lie above the neutral axis.



(c)

## ***Sample Problem*** **5.2**

The simply supported beam in Fig. (a) has the T-shaped cross section shown. Determine the values and locations of the maximum tensile and compressive bending stresses.




(a)

***Solution***

***Preliminary Calculations***

**F**ind the largest positive and negative bending moment. The results are shown in Fig. (a)−(c). From Fig.(c), the largest positive and negative bending moment are 3200 lb·ft and 4000 lb·ft respectively.

As shown in Fig.(d), the cross section to be composed of the two rectangles with areas $A_1 = 0.8(8) = 6.4$ in.$^2$ and $A_2 = 0.8\ (6) = 4.8$ in.$^2$. The centroidal coordinates of the areas are $\overline{y_1} = 4in.$ and $\overline{y_2} = 8.4in.$, , measured from the bottom of the cross section. The coordinate $\overline{y}$ of the centroid $C$ of the cross section is

$$\overline{y} = \frac{A_1\overline{y_1} + A_2\overline{y_2}}{A_1 + A_2} = \frac{6.4(4) + 4.8(8.4)}{6.4 + 4.8} = 5.886in.$$

Compute the moment of inertia $I$ of the cross-sectional area about the neutral axis. Using **the parallel-axis theorem**, $I = \sum\left[\overline{I_i} + A_i\left(\overline{y_i} - \overline{y}\right)^2\right]$, where $\overline{I_i} = b_i h_i^3/12$ is the moment of inertia of a rectangle about its own centroidal axis Thus,

$$I = \left[\frac{0.8(8)^3}{12} + 6.4\left(4 - 5.886\right)^2\right] + \left[\frac{6(0.8)^3}{12} + 4.8\left(8.4 - 5.886\right)^2\right] = 87.49in.^4$$

***Maximum Bending stresses***

The distances from the neutral axis to the top and the bottom of the cross section are $c_{top} = 8.8 - \bar{y} = 8.8 - 5886 = 2.914 in.$ and $c_{bot} = \bar{y} = 5.886 in.$, as shown in Fig.(c). Because these distances are different, we must investigate stresses at two locations: at $x$ = 4 ft (where the largest positive bending moment occurs) and at $x$ = 10 ft (where the largest negative bending moment occurs).

***Stresses at x = 4 ft*** The bending moment at this section is $M$ = +3200 lb． ft causing compression above the neutral axis and tension below the axis. The resulting bending stresses at the top and bottom of the cross section are

$$\sigma_{top} = -\frac{Mc_{top}}{I} = -\frac{(3200 \times 12)(2.914)}{87.49} = -1279\, psi$$

$$\sigma_{bot} = -\frac{Mc_{bot}}{I} = \frac{(3200 \times 12)(-5.886)}{87.49} = 2580\, psi$$

***Stresses at x = 10 ft*** The bending moment at this section is $M = -4000 \text{lb}\cdot\text{ft}$, resulting in tension the neutral axis and compression below the neutral axis. The corresponding bending stresses at the extremities of the cross section are

$$\sigma_{top} = -\frac{Mc_{top}}{I} = \frac{(-4000 \times 12)(2.914)}{87.49} = 1599\, psi$$

$$\sigma_{bot} = -\frac{Mc_{bot}}{I} = \frac{(-4000 \times 12) - (5.886)}{87.49} = -3230\, psi$$

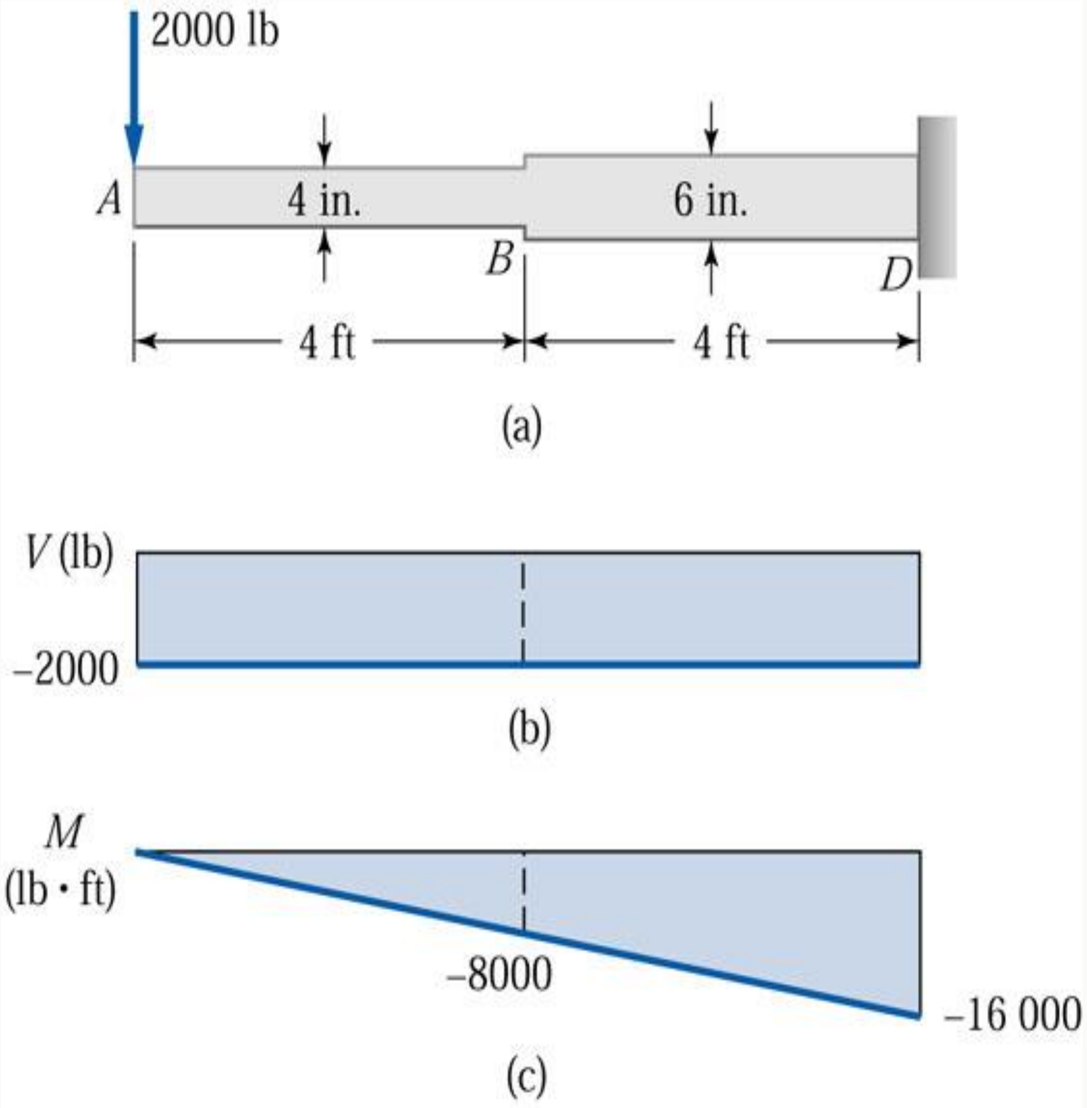
Inspecting the above results, we conclude that the maximum tensile and compressive stresses in the beam are

$(\sigma_T)_{max} =$ 2580 psi ( bottom of the section at $x$ = 4 ft )

$(\sigma_c)_{max} =$ 3230 psi ( bottom of the section at $x$ = 10 ft )

## ***Sample Problem*** **5.3**

The cantilever beam in Fig. (a) is composed of two segments with rectangular cross sections. The width of the each section is 2 in., but the depths are different, as shown in the figure. Determine the maximum bending stress in the beam.



***Solution***

Because the cross section of the beam is not constant, the maximum stress occurs either at the section just to the left of *B* ($M_B$ = - 8000 lb． ft) or at the section at *D* ($M_D$ = - 16000 lb． ft). the section moduli of the two segments are

$$S_{AB} = \frac{bh^2{}_{AB}}{6} = \frac{(2)(4)^2}{6} 5.333 in.^3$$

$$S_{BD} = \frac{bh^2{}_{BD}}{6} = \frac{(2)(6)^2}{6} 12.0 in.^3$$

From Eq. (5.4b) the maximum bending stresses on the two cross sections of the interest are

$$(\sigma_B)_{\max} = \frac{[M_B]}{S_{AB}} = \frac{8000 \times 12}{5.333} = 18000\, psi$$

$$(\sigma_D)_{\max} = \frac{[M_D]}{S_{BD}} = \frac{16000 \times 12}{12.0} = 16000\, psi$$

Comparing the above values, we find that the maximum bending stress in the beam is

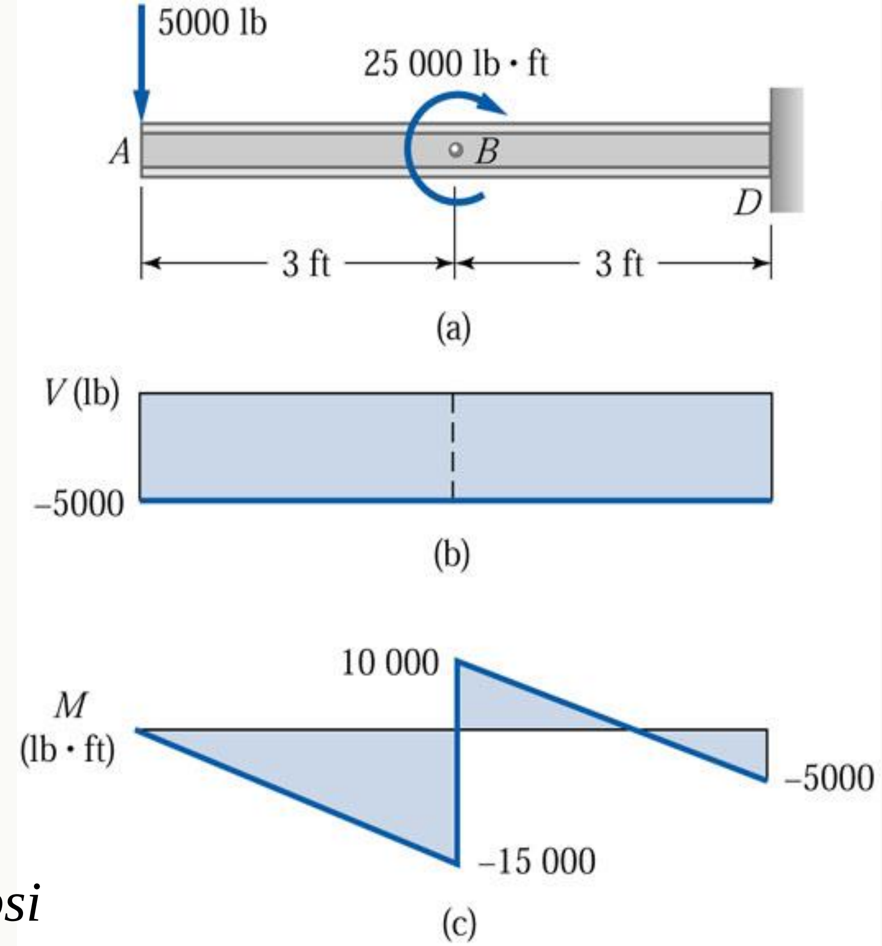
$\sigma_{\max}$ = 18000 psi (on the cross section just to the left of *B*)

*Answer*

❖ This is an example where the maximum bending stress occurs on a cross section at the bending moment is not maximum.

## *Sample Problem* 5.4

The wide- flange section W 14×30 is use as a cantilever beam, as shown in Fig.(a). Find the maximum bending stress in the beam.




### *Solution*

The largest bending moment is $|M_{max}| = 15000 \text{ lb} \cdot \text{ft}$ acting just to the left of section *B*. From the tables in Appendix B, we find that the section modulus of a W14×30 (P520) section is $S = 42.0 \text{ in.}^3$. Therefore, the maximum bending stress in the beam is

$$\sigma_{max} = \frac{|M_{max}|}{S} = \frac{15000 \times 12}{42.0} = 4290\, psi$$

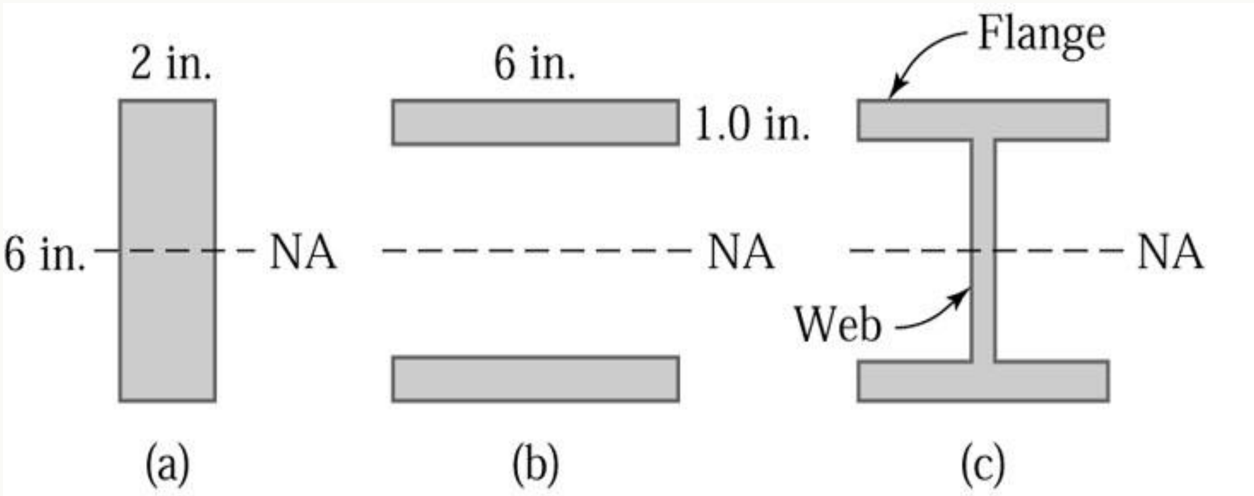
## 5.3 *Economic Sections*

- The portions of a located near the neutral surface are understressed compared with at the top or bottom. Therefore, beams with certain cross- sectional shape ( including a rectangle and circle) utilize the material inefficiently because much of the cross section contributes little to resisting the bending moment.

- Consider, for example, in Fig. 5.5(a) The section modulus has increased to $S = bh^2/6 = 2(6)^2/6 = 12$ in.$^3$. If working stress is $\sigma_w =$ 18 ksi, the maximum safe bending moment for the beam is $M = \sigma_w \cdot S = 18\ (12) = 216$ kip·in.

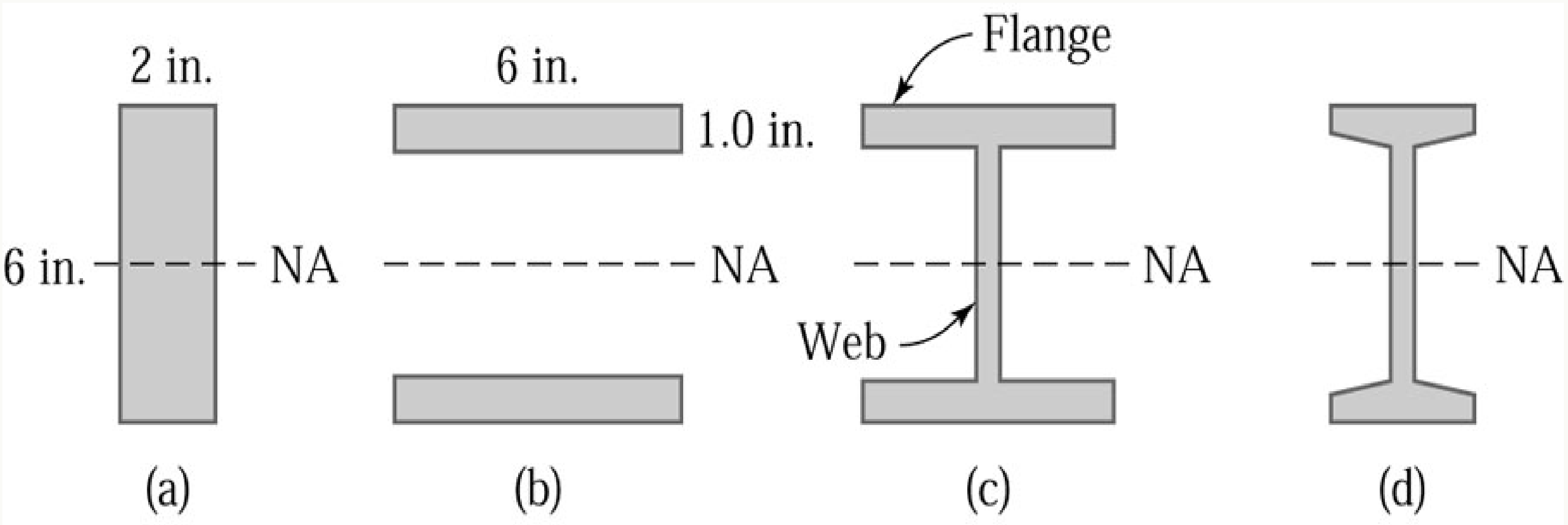
**Figure 5.5 Different ways to distribute the 12-in.$^2$ cross-sectional area in (a) without changing the depth.**

- In Fig. 5.5(b), we have rearranged the area of the cross section but kept the same overall depth. It can be shown that the section that the section modulus has increased to $S$ = 25.3 in.$^3$（the parallel-axis theorem）. Thus, the new maximum allowable moment is M = 18 (25.3) = 455 kip·in., which is more than **twice** the allowable moment for the rectangular section of the same area.
- The section in Fig. 5.5(b) is not practical because its two parts, called the *flanges*. As in Fig. 5.5(c). The vertical connecting piece is known as the *web* of the beam. The web functions as the main shear-carrying component of the beam.



## *a. Standard structural shapes*

- Figure 5.5 (c) is similar to a *wide-flange beam*, referred to as a W-shape. Another “slimmer”version of the shape is the I-beam (referred to as an S-shape) shown in Fig. 5.5(d). The I-beam preceded the wide- flange beam, but because it is not as efficient, it has largely been replaced by the wide- flange beam.



- Properties of W-and S-shapes are given in **Appendix B**.

- in SI units, the designation W610×140 indicates a wide-flange beam with a nominal depth of 610mm and a nominal mass per unit length of 140 kg/m. The tables in **Appendix B** indicates the actual depth of the beam is 617 mm and the actual mass is 140.1 kg/m.
- In U.S. Customary units, a W36×300 is a wide-flange beam with a nominal depth 36 in. that weighs 300 lb/ft. The actual depth of this section is 36.74 in.
- Referring to **Appendix B**, in addition to listing the dimensions, tables of structural shapes give properties of the cross-sectional area, such as moment of inertia ($I$), section modulus ($S$), and radius of gyration ($r$)[4] for each principal axis of the area.

When a structural section is selected to be used as a beam. The section modulus must be equal to or greater than section modulus determined by the flexure equation; that is,

$$S \geq \frac{|M_{\max}|}{\sigma_w} \qquad (5.5)$$

the section modulus of the selected beam must be equal to or greater than the ratio of the bending moment to the working stress.

- If a beam is very slender (large $L/r$), it may fail by *lateral bucking* before the working stress is reached. I-beams are particularly vulnerable to lateral bucking because of their low torsional rigidity and small moment of inertia about the axis parallel to the web.

### *b. Procedure for selecting standard shapes*

A design engineer is often required to select the **lightest** standard structural shape ( such as a W-shape) that can carry a given loading in addition to the weight of the beam. Following is an outline of the selection process;

. Neglecting the weight of the beam, draw the bending moment diagram to find the largest bending moment $M_{max.}$.

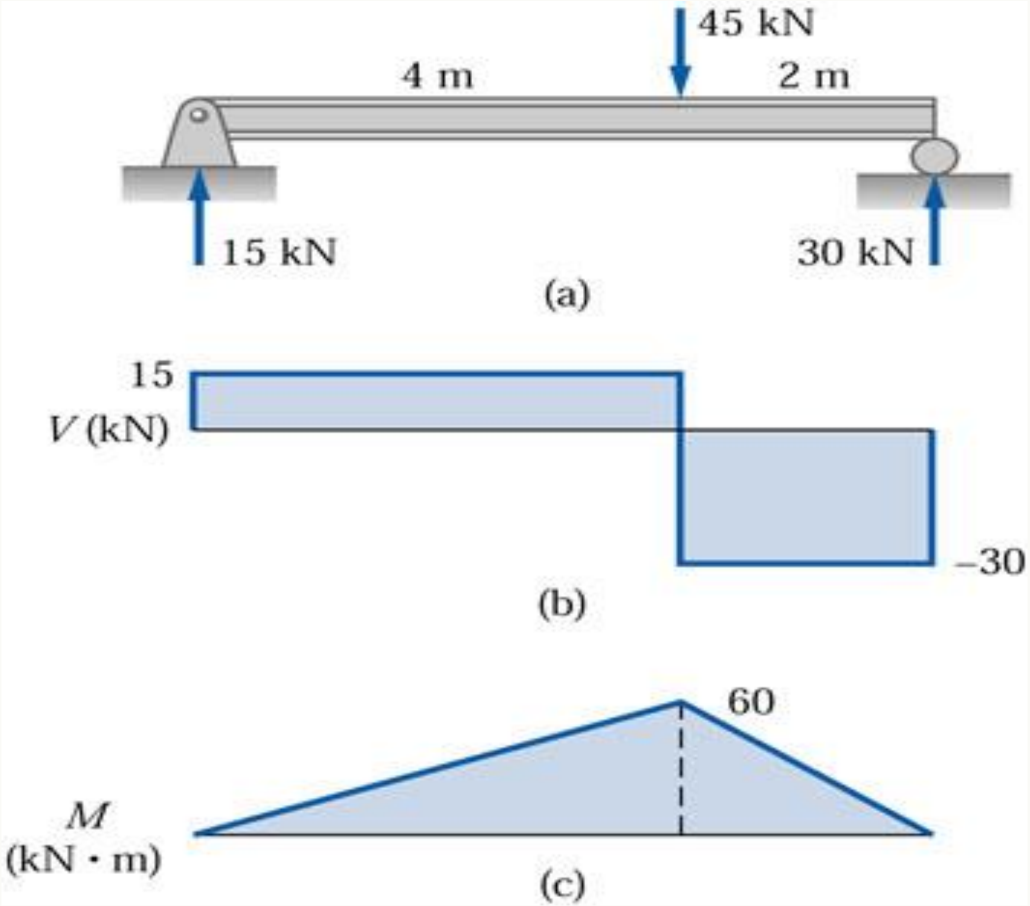
. Determine the minimum allowable section modulus from $S_{\min} = |M_{max.}| / \sigma_w$, is the working stress.

. Choose the lightest shape from the list of structural shapes (such as a Appendix B) for which $S \geq S_{\min}$ and note its weight.

. Calculate the maximum bending stress $\sigma_{\max}$ in the selected beam caused by the prescribed loading plus the *weight of the beam*. If $\sigma_{\max} \leq \sigma_w$, the selection is finished. Otherwise, the second-lightest shape with $S \geq S_{\min}$ must be considered and the maximum bending stress recalculated. The process must be repeated unit a satisfactory shape is found.

## *Sample Problem* 5.5

What is the lightest W-shape beam that will support the 45-kN load shown in Fig. (a) without exceeding a bending stress of 120 MPa？ Determine the actual bending stress in the beam.




***Solution***

Finding the reactions shown in Fig.(a), and sketch the shear force and bending moment diagrams in Figs. (b) and (c).

The minimum bending acceptable section modulus that can carry this moment is

$$S_{\min} = \frac{|M_{\max}|}{\sigma_w} = \frac{60\times10^3}{120\times10^6} = 500\times10^{-6}\,m^3 = 500\times10^3\,mm^3$$

Referring to the table of Properties of W-shape (Appendix B SI Unit) and find that the following are the lightest beams in each size group that satisfy the requirement $S \geq S_{\min}$: (P508)

| **Section** | **S(mm³)** | **Mass(kg/m)** |
|---|---|---|
| W200×52 | $512\times10^3$ | 52.3 |
| W250×45 | $534\times10^3$ | 44.9 |
| W310×39 | $549\times10^3$ | 38.7 |

The reason is that although the lightest beam is the cheapest on the basis of the weight alone, headroom clearances frequently require a beam with less depth than the lightest one.

Our first choice is the W310×39 section with $S = 549\times10^{-6}\,m^3$.

The weight of the beam for the W310×39 section is

$$w_o = (38.7\ \text{kg/m}) \times (9.81\ \text{m/s}^2) = 380\ \text{N/m} = 0.380\ \text{kN/m}$$

From (d) shows the beam supporting **both** the 45-kN load and the weight of the beam. The maximum bending moment is found to be $M_{max}$ = 61.52 kN·m, again occurring under the concentrated load.

Therefore, the maximum bending stress in the selected beam is

$$\sigma_{max} = \frac{|M_{max}|}{S} = \frac{61.52 \times 10^3}{549 \times 10^{-6}} = 112.1 \times 10^6\ pa = 112.1 MPa$$

Because this stress is less than the allowable stress of 120 MPa, the lightest W-shape that can safely support the 45-kN load is

W310×39 (with $\sigma_{max}$ = 112.1MPa) *Answer*

## *5.4 Shear Stress in Beams*

### *a. Analysis of flexure action*

- In Fig. 5.6, The separate layers would slide past one another, and the **total bending strength** of the beam would be the sum of the strength of the individual layers. Such **a built-up beam** would be considerably weaker than a **solid beam** of equivalent dimensions.
- From the above observation, we conclude that the horizontal layers in a solid beam are prevented from sliding by shear stresses that act between the layers.

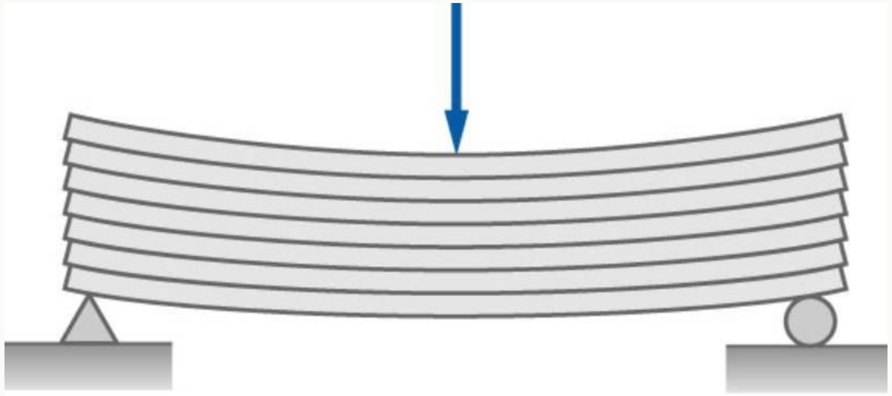


Figure 5.6 Bending of a layered beam with no adhesive between the layers.

- In Fig. 5.7. We isolate the shaded portion of the beam by using two cutting planes: a vertical cut along section 1 and horizontal cut located at the distance *y’* above the neutral axis.

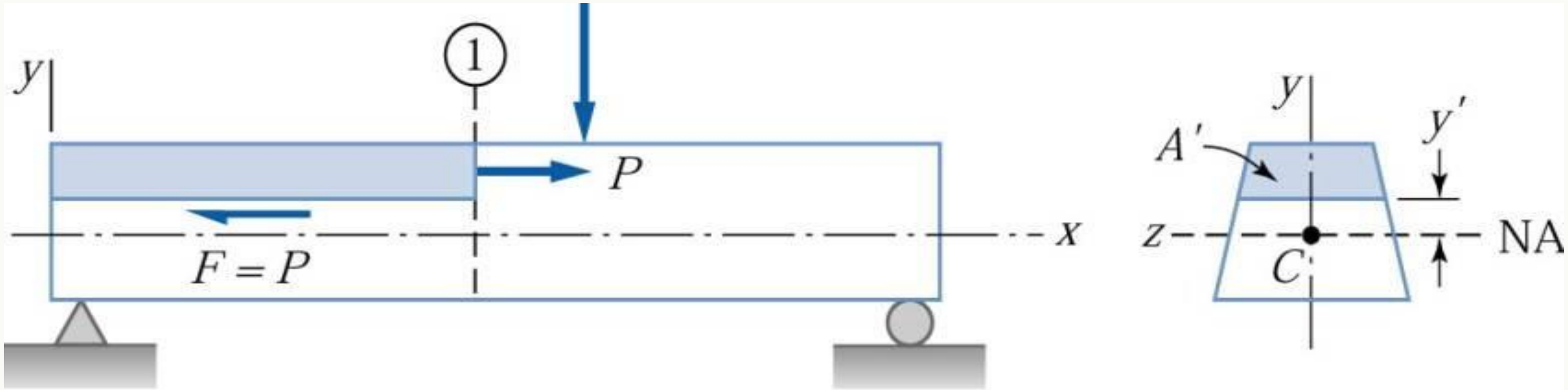



Figure 5.7 Equilibrium of the shaded portion of the beam requires a longitudinal shear force F = P, where P is the resultant of the normal stress acting on area A’ of section (1).

- Calculate $P$ using Fig. 5.8. The axial force acting on the area element $dA$ of the cross section is $dP = \sigma\, dA$.

- If $M$ is the bending moment acting at section 1 of the beam, the bending stress is given by Eq. (5.3): $\sigma = -My/I$, where $y$ is the distance of the element from the neutral axis, and $I$ is the moment of inertia of the *entire cross-sectional area* of the beam about the neutral axis.

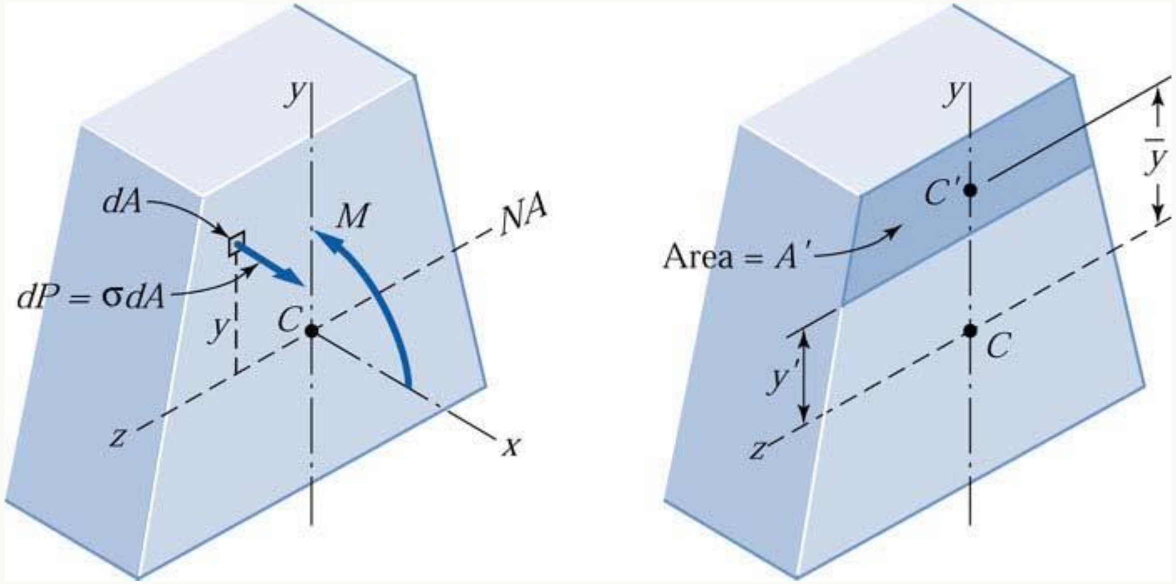



Figure 5.8 Calculating the resultant force of the normal stress over a portion of the cross-sectional area.

$$dP = -\frac{My}{I}\,dA$$

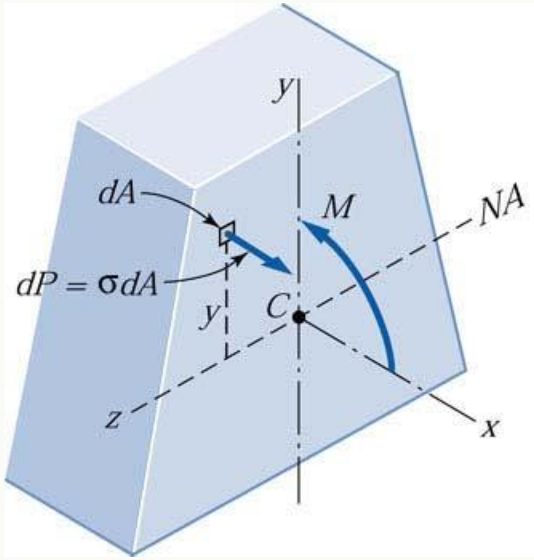
Integrating over the area A’, we get

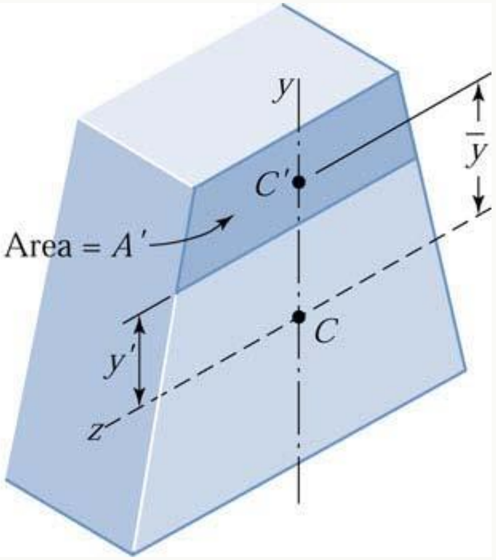
$$P = \int_{A'} dp = -\frac{M}{I}\int_{A'} y\,dA = -\frac{MQ}{I} \qquad (5.6)$$

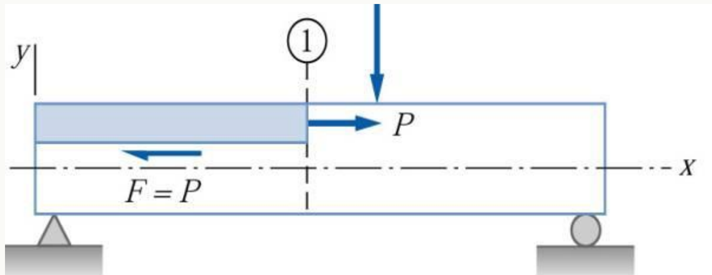
Where

$$\boxed{Q = \int_{A'} y\,dA} \qquad (5.7a)$$

is the first moment of area *A’* about the neutral axis. The negative sign in Eq. (5.6) indicates that positive *M* results in forces *P* and *F* that are directed opposite to those shown in Fig. 5.7.



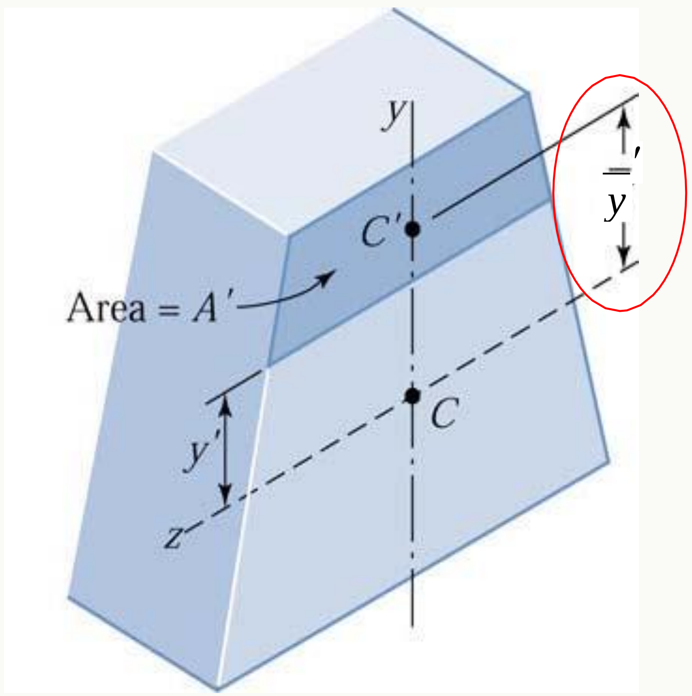



- Denoting the distance between the neutral axis and centroid $C$` of the area $A'$ by $\boxed{\bar{y}'}$, we can write Eq. (5.7) as

$$Q = A`\bar{y}' \qquad (5.7b).$$

- In Eqs. (5.7b), $Q$ represents the first moment of the cross-sectional area that lies *above* $y'$. Because the first moment of the total cross-sectional area about the neutral axis is zero, that first moment of the area *below* $y'$ is $-Q$. Therefore, the magnitude of $Q$ can be computed by using the area either **above** or **below** $y'$, whichever is more convenient.



- The maximum value of $Q$ occurs **at the neutral axis** where $y' = 0$. It follows that horizontal shear force $F$ is largest on the neutral surface. The variation of $Q$ with $y'$ for a rectangular cross section is illustrated in Fig. 5.9.

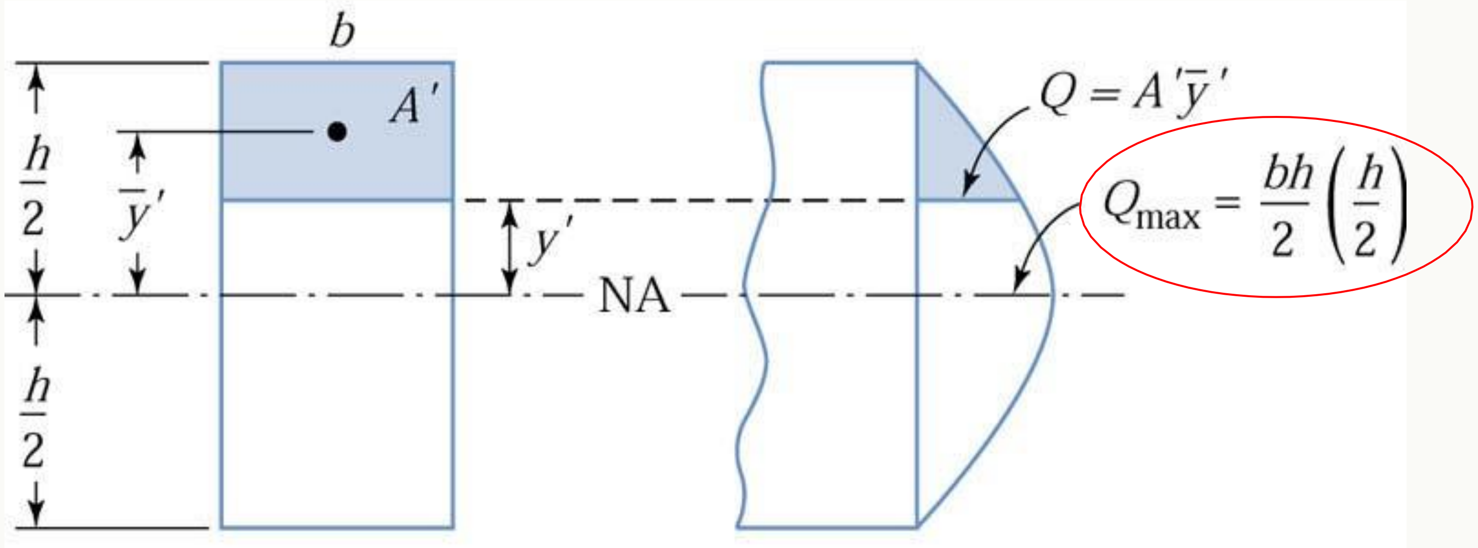



Figure 5.9 Variation of the first moment $Q$ of area $A'$ about the neutral axis for a rectangular cross section.

## *b. Horizontal shear stress*

Consider Fig. 5.10. A horizontal plane located a distance *y'* above the neutral axis of the cross section. If the bending moment at section1 of the beam is *M*, the resultant force acting on face 1 of the body is given by Eq. (5.6):

$$\boxed{P = -M\,\frac{Q}{I}}$$

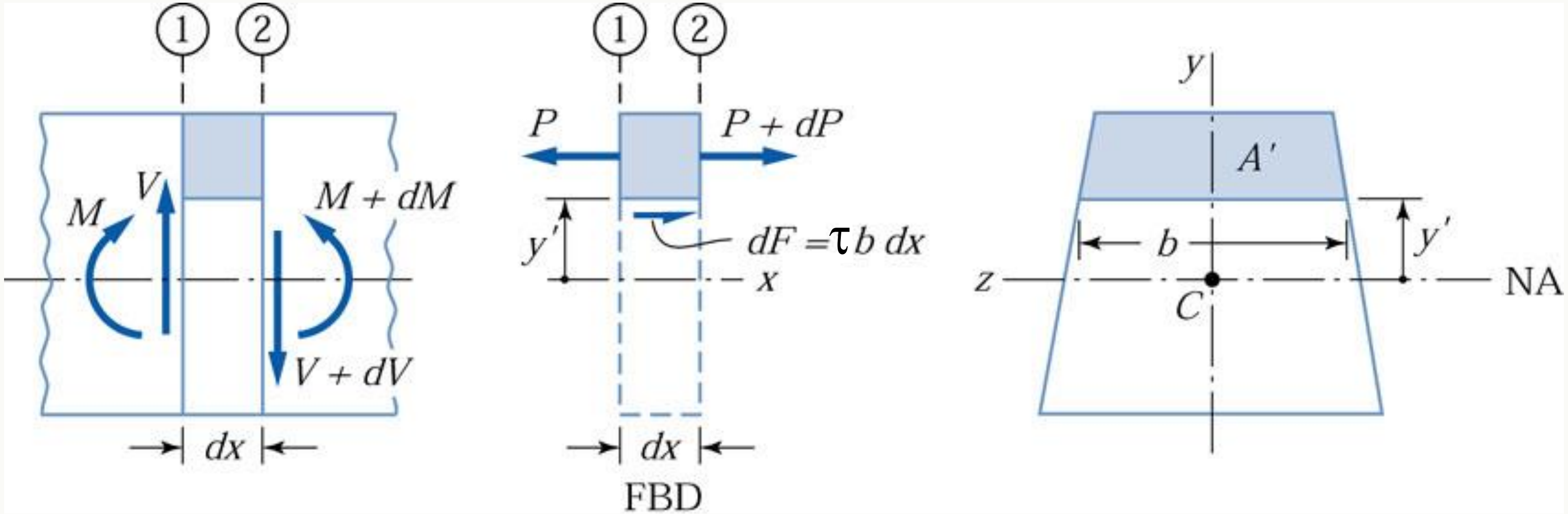


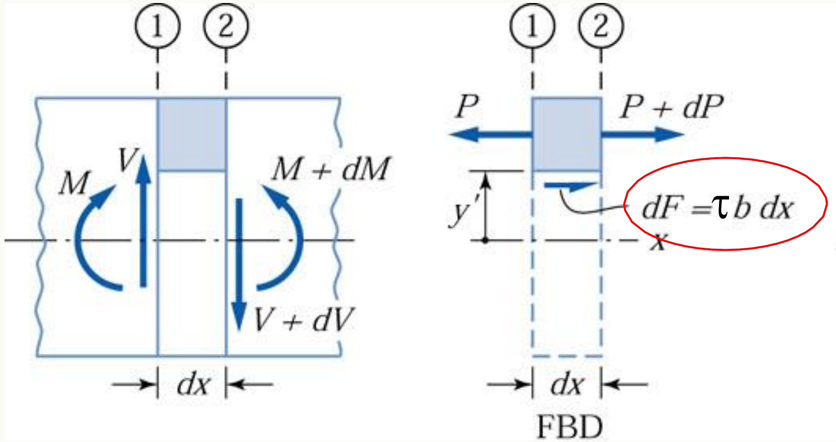

Figure 5.10 Determining the longitudinal shear stress from the free-body diagram of a beam element.

❑ The bending moment acting at section 2 is *M+dM*, where *dM* is the infinitesimal change in *M* over the distance *dx*. Therefore, the resultant normal force acting on face 2 of the body is

$$p + dP = -(M + dM)\frac{Q}{I} \qquad P = -M\frac{Q}{I}$$

$$(P + dP) - P = -(M + dM)\frac{Q}{I} - \left(-M\frac{Q}{I}\right)$$

$$= -dM\frac{Q}{I}$$




❑ Equilibrium can exist only if there is an equal and opposite shear force *dF* acting on the horizontal surface. If we let τ be the *average shear stress* acting on the horizontal surface, its resultant is dF = τ *bdx*. Where *b* is the width of the cross section at *y* = *y*`, as shown in Fig. 5.10. The equilibrium requirement for the horizontal forces is

$$\Sigma F = 0: \ (P + dP) - P + \tau\, b\, dx = 0$$

- Substituting for $(P+dP) - P$ from Eq. (a), we get

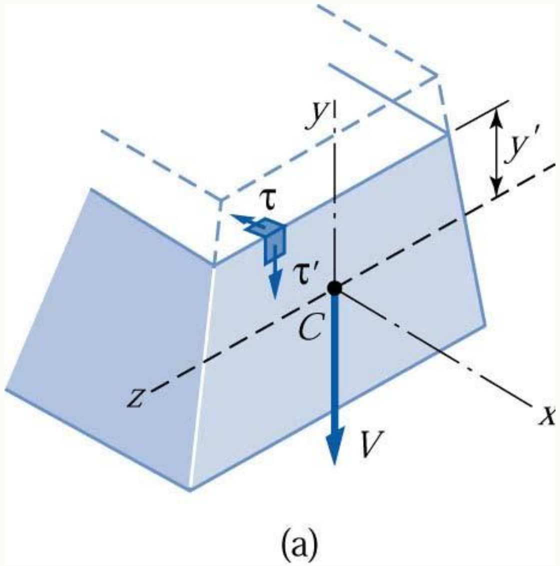
$$-dM\frac{Q}{I} + \tau b dx = 0 \qquad\qquad \tau = \frac{dM}{dx}\frac{Q}{Ib} \qquad \text{(b)}$$

Recalling the relationship $V = dM/dx$ between the shear force and the bending moment we obtain for the average horizontal shear stress $\tau$

$$\boxed{\tau = \frac{VQ}{Ib}} \qquad (5.8)$$

## *c. Vertical shear stress*

- Eq. (5.8) $\tau = \frac{VQ}{Ib}$ ( a plane parallel to the neutral surface ). A shear stress is always accompanied by a complementary shear stress of equal magnitude, the two stresses acting on mutually perpendicular plane.



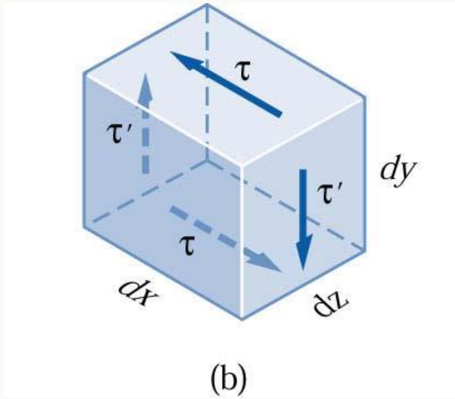
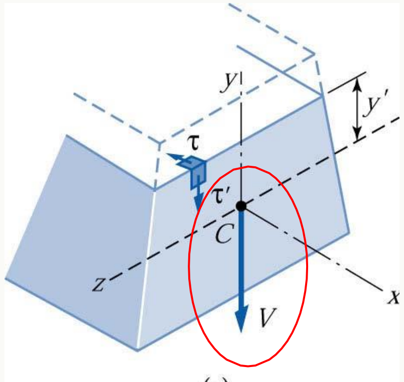




**Figure 5.11 The vertical stress τ' acting at a point on a cross section equals the longitudinal shear stress τ acting at the same point.**

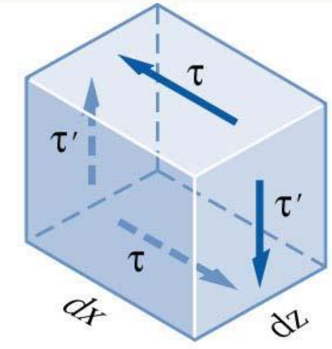
- In a beam, the complementary stress τ ’ is a vertical shear stress that acts on the cross section of the beam, as illustrated in Fig. 5.11 (a). Because τ = τ ’, Eq.(5.8) can be used to compute the vertical as well as the horizontal shear stress at a point in a beam.

- The resultant of the vertical shear stress on the cross-sectional area *A* of the beam is,of course, the shear force *V*;. $V = \int_A \tau dA$



(a)

- To prove that τ = τ `, consider Fig. 5.11(b). The horizontal and vertical forces are τ *dxdz* and τ ’*dydz*, respectively. These forces from two couples of opposite sense. For rotational equilibrium, the magnitudes of the couples must be equal; that is, ( τ *dxdz*) *dy* = ( τ `*dydz*) *dx*, which yields τ = τ ’.



(b)

### *d. Discussion and limitations of the shear stress formula*

- The shear stress formula $\tau = VQ/(Ib)$ predicts that the largest shear stress in a prismatic beam occurs at the cross section that carries the largest vertical shear force $V$.

- The location ( the value of $y'$ ) of the maximum shear stress within that section is determined by the ratio $Q/b$. Because $Q$ is always maximum at $y' = 0$, the neutral axis is usually a candidate for the location of the maximum shear stress.

- However, If the width $b$ at the neutral axis is larger than at other parts of the cross section, it is necessary to compute $\tau$ at two or more values of $y'$ before its maximum value can be determined.

- When deriving the shear stress formula, Eq. (5.8), $\tau = \frac{VQ}{Ib}$ $\tau$ should be considered at the ***average*** shear stress. This restriction is necessary because the variation of the shear stress across the width *b* the cross section is often unknown.

- Equation (5.8) is sufficiently accurate for rectangular cross sections and for cross sections that are composed of rectangles, such as W and S-shapes.

- Let us consider as an example the **circular cross** section in Fig. 5.12.

**Figure 5.12 Shear stress distribution along a horizontal line of a circular cross section.**

$$\boxed{\tau = \frac{VQ}{Ib}}$$

- For other cross- sectional shapes, however, the formula for $\tau$ must be applied with caution. Let us consider as an example the circular cross section in Fig. 5.12.

- It can be shown that the shear stress at the periphery of the section must be tangent to the boundary, as shown in the figure.

- The direction of shear stresses at interior points is unknown, except at the centerline, where the stress is vertical due to symmetry. To obtain an estimate of the maximum shear stress, the stresses are assumed to be directed toward a common center *B*, as shown.

**Figure 5.12 Shear stress distribution along a horizontal line of a circular cross section.**

- The vertical components of these shear stresses are assumed to be **uniform** across the width of the section and are computed from Eq. (5.8). Under this assumption, **the shear stress at the neutral axis is 1.333$V/(\pi r^2)$. (4/3)($V/\pi r^2$)**

- A more elaborate analysis shows that the shear stress actually varies from 1.23 $V/(\pi r^2)$ at the edges to 1.38 $V/(\pi r^2)$ at the center.

- Shear stress, like normal stress, exhibits stress concentrations near **shape corners**, **fillets** and **holes** in the cross section. The junction between the web and the flange of a W-shape is also an area of stress concentration.

## *e.* ***Rectangular and wide-flange sections***

Determine the shear stress as a function of $y$ for a rectangular cross section of base $b$ and height $h$. From Fig. 5.13, the shaded area is $A' = b\,[(h/2)-y]$, its centroidal coordinate being $\bar{y}' = \left[(h/2)+y\right]/2$.

$$Q = A'\bar{y}' = \left[b\left(\frac{h}{2}-y\right)\right]\left[\frac{1}{2}\left(\frac{h}{2}+y\right)\right] = \frac{b}{2}\left(\frac{h^2}{4}-y^2\right)$$

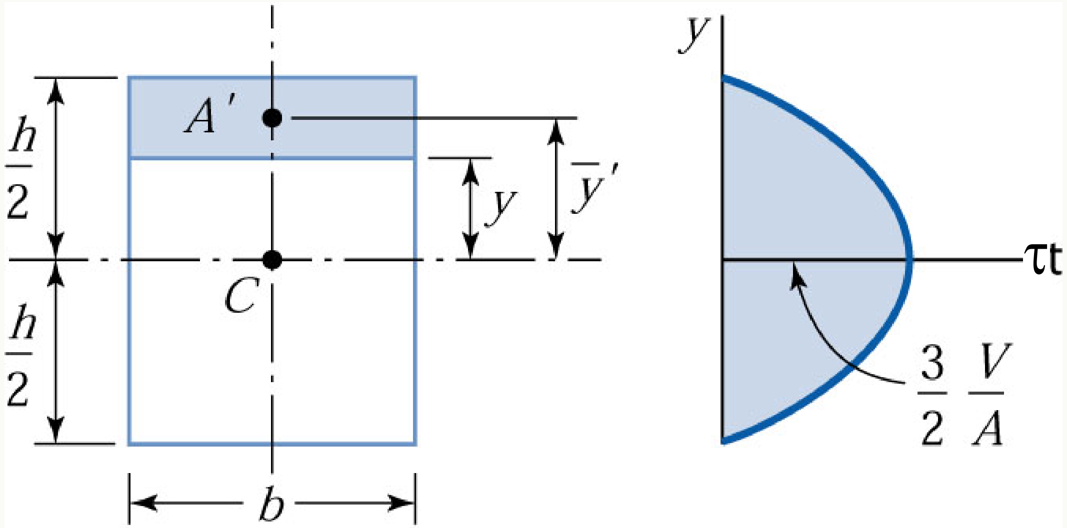



**Figure 5.13 Shear stress distribution on a rectangular cross section.**

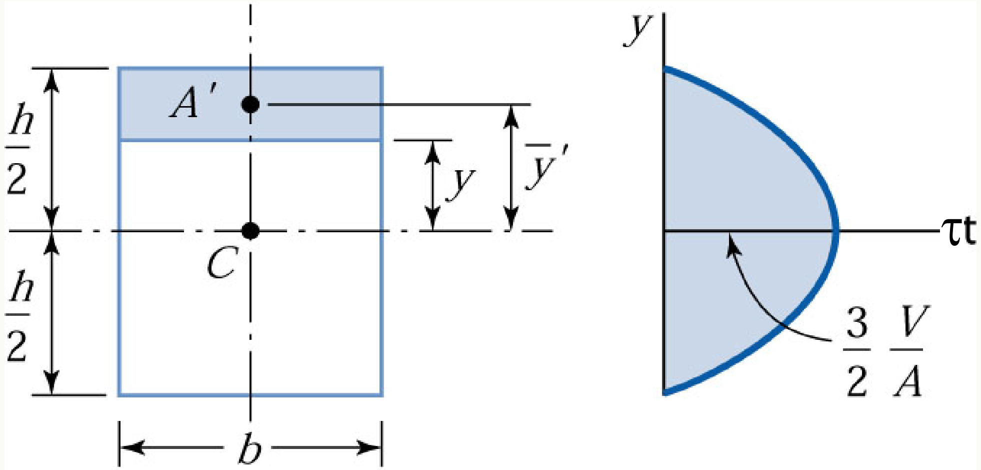
$$\tau = \frac{VQ}{Ib} = \frac{V}{2I}\left(\frac{h^2}{4} - y^2\right) \qquad \text{(c)}$$

The shear stress is distributed **parabolically** across the depth of the section, as shown in Fig.5.13. The maximum shear stress occurs at the neutral axis. If we substitute $y = 0$ and $I = bh^3/12$, Eq. (c) reduces to

$$\tau_{\max} = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A} \qquad (5.9)$$

where $A$ is the cross –sectional area.

The shear stress in rectangular section is 50% greater than the average shear stress on the cross section.



- In wide-flange sections (W-shapes), most of the bending moment is carried by the flanges, whereas the web resists the bulk of the vertical shear force. Figure 5.14. $Q$ is contributed mainly by the flanges of the beam. Consequently, $Q$ does not vary with $y$, so that the shear stress in the web is almost constant.

- In fact $\tau_{max} = V/A_{web}$ can be used as an approximation to the maximum shear stress in most cases, where $A_{web}$ is the cross-sectional area of the web.

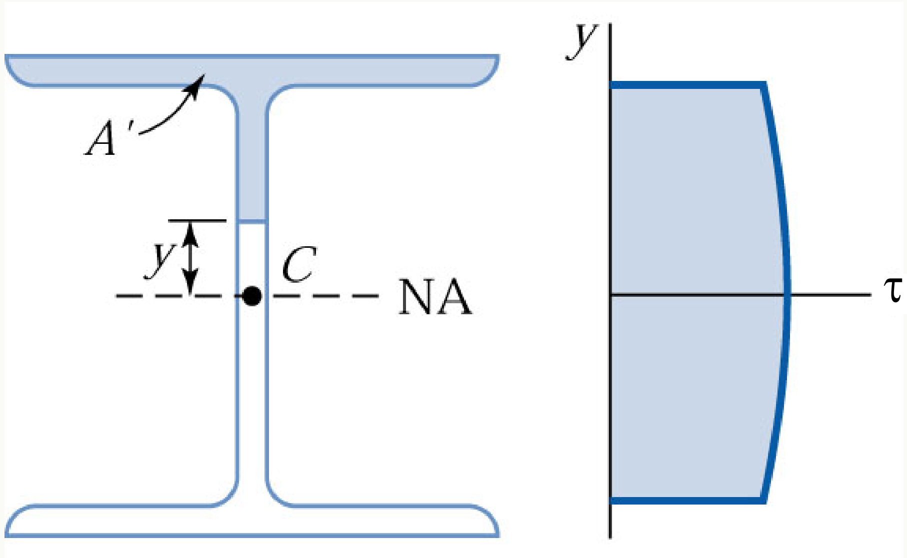



**Figure 5.14 Shear Stress distribution on the web of a wide-flange beam.**

## a. ***Procedure for analysis of shear stress***:

- Use equilibrium analysis to determine the vertical shear force *V* acting on the cross section containing the specified point (the construction of a shear force diagram is usually a good idea).

- Locate the neutral axis and compute the moment of inertia *I* of the cross- sectional area about the neutral axis (If the beam is a standard structural shape, its cross- sectional properties are listed in Appendix B.)

- Compute the first moment Q of the cross- sectional area that lies above (or below)the specified point.

- Calculate the shear stress from $\tau = VQ/(Ib)$, where $b$ is the width of the cross section at the specified point.

- The maximum shear stress $\tau_{max}$ on a given cross section occurs where $Q/b$ is largest.

- If the width $b$ is constant, then $\tau_{max}$ occurs at the neutral axis because that is where Q has its maxmum value.

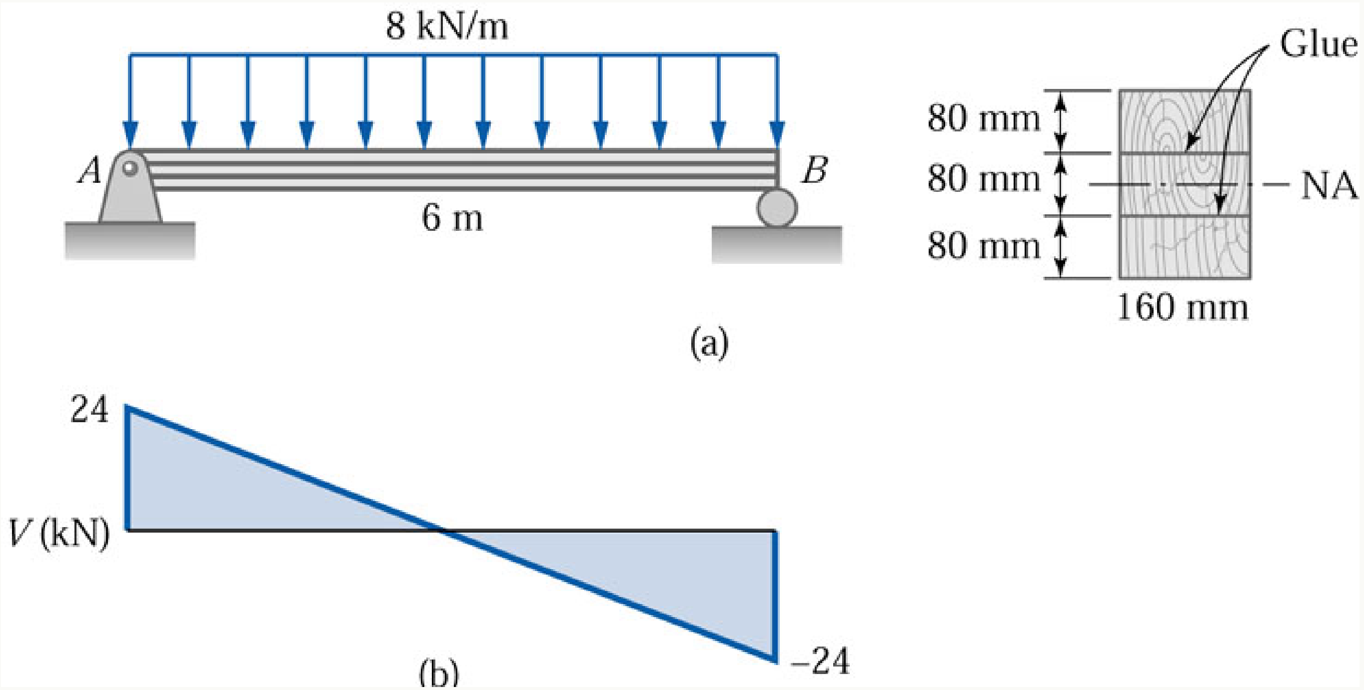
- If $b$ is not constant, it is necessary to compute the shear stress at more than one point in order to determine its maximum value.

In the U.S. Customary system, $\tau\left[lb/in^2.\right] = \dfrac{V[lb]Q\left[in.^3\right]}{I\left[in.^4\right]b[in.]}$

In the SI system, $\tau\left[N/m^2\right] = \dfrac{V[N]Q\left[m^3\right]}{I\left[m^4\right]b[m]}$

***Sample Problem* 5.6**

The simply supported wood beam in Fig.(a) is fabricated by gluing together **three** 160-mm by 80-mm plans as shown. Calculate the maximum shear stress in (1) the glue; and (2) the wood.



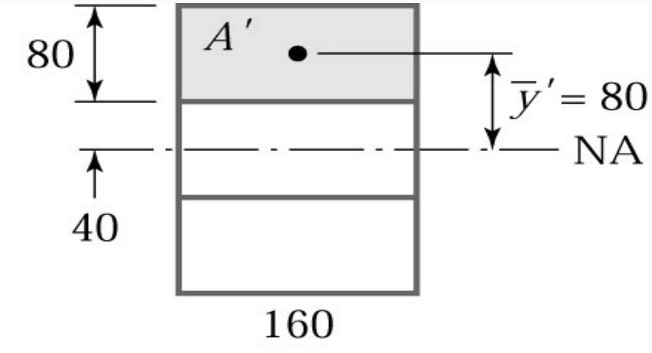
***Solution***

From the shear force diagram in Fig. (b), the maximum shear force in the beam is $V_{max}$ = 24 kN, occurring at the supports. The moment of inertia of the cross-sectional area of the beam about the neutral axis is

$$I = \frac{bh^3}{12} = \frac{160(240)^3}{12} = 184.32\times10^6\,mm^4 = 184.32\times10^{-6}\,m^4$$

**Part 1**

The shear stress is the glue corresponds to the horizontal shear stress. Its maximum value can be computed from Eq. (5.8): $\tau_{max} = V_{max}\,Q/(Ib)$, where $Q$ is the first moment of the area *A*' shown in Fig.(c); that is,



(c)

$$Q = A`\bar{y}` = (160\times80)(80) = 1.024\times10^6 = 1.024\times10^{-3}\,m^3$$

Therefore, the shear stress in the glue, which occurs over either support, is

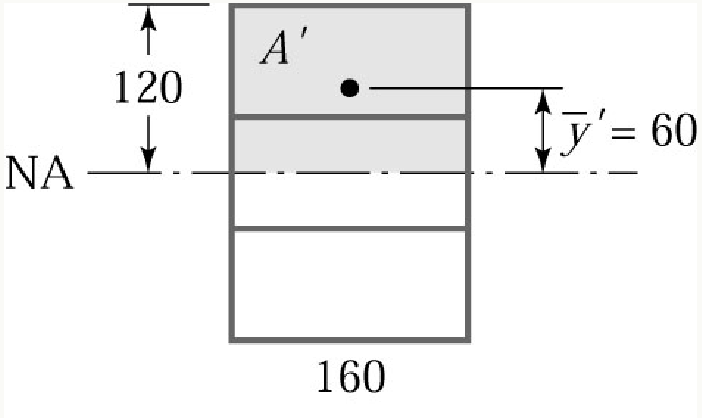
$$\tau_{max} = \frac{V_{max}Q}{Ib} = \frac{\left(24\times10^{3}\right)\left(1.024\times10^{-3}\right)}{\left(184.32\times10^{-6}\right)\left(0.160\right)}$$

$$= 8.33\times10^{3}\,Pa = 8.33\,kPa$$

Answer

**Part 2**

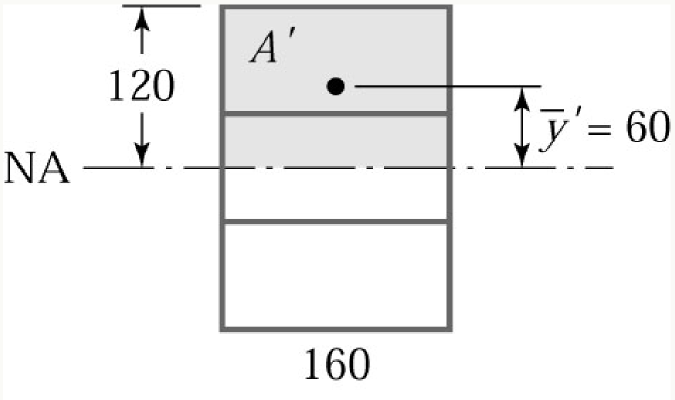
Because the cross section is rectangular, the maximum shear stress in the wood can be calculated from Eq. (5.9):



(d)

$$\tau_{max} = \frac{3}{2}\frac{V_{max}}{A} = \frac{3}{2}\frac{\left(24\times10^{3}\right)}{\left(0.160\right)\left(0.240\right)} = 938\times10^{3}\,Pa = 938\,kPa$$

The same result can be obtained from Eq. (5.8), where now *A'* is the area above the neutral axis, as indicated in Fig. (d). The first moment of this area about the neutral axis is



(d)

$$Q = A'\bar{y}' = (160 \times 120)(60) = 1.152 \times 10^{6}\ mm^{3} = 1.152 \times 10^{-3}\ m^{3}$$

Equation (5.8)this becomes

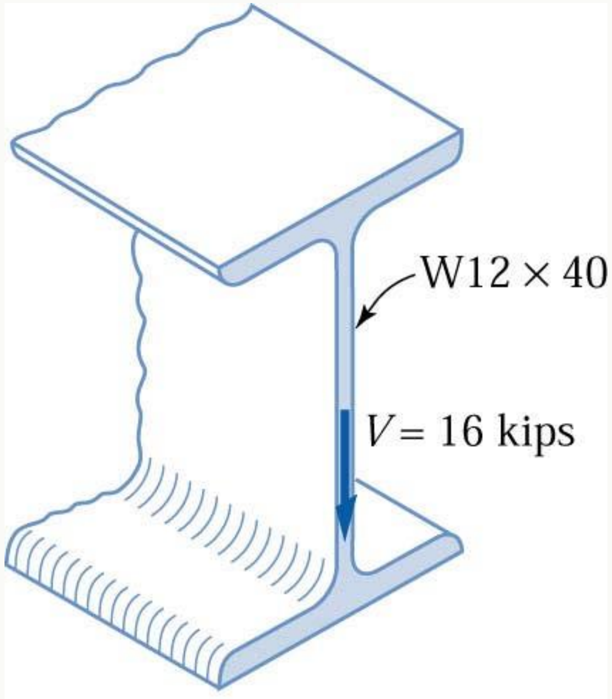
$$\tau_{max} = \frac{V_{max} Q}{Ib} = \frac{(24 \times 10^{3})(1.152 \times 10^{-3})}{(184.32 \times 10^{-6})(0.160)}$$

$$= 938 \times 10^{3}\ Pa = 938 kPa$$

which agrees with the previous result.

***Sample Problem* 5.7**

The W12×40 section in Fig.(a) is used as a beam. If the vertical shear acting at a certain section of the beam is 16 kips, determine the following at that section: (1) the minimum shear stress in the web;(2) the maximum shear stress in the web; and (3)the percentage of the shear force that is carried by the web.




(a)

***Solution***

The W12×40 section is shown in Fig.(b), where the dimensions were obtained from the tables in **Appendix B** (P521). The drawing approximates the web and the flanges by rectangles, thereby ignoring the small fillets and rounded corners present in the actual section. The **tables** also list the moment of inertia of the section about the neutral axis as $I = 310$ in.$^4$.

**Part 1**

**The minimum shear stress** in the web occurs at the junction with **the flange**, where $Q/b$ is **smallest** (note that $b = 0.295$ in. is constant within the web). $Q$ is the first moment of the area $A'1$ shown in Fig.(b) about the neutral axis:

$$Q = A`_1 \bar{y}` = (8.005 \times 0.515)\frac{11.94 - 0.515}{2} = 23.55 in.^3$$

The minimum shear stress in thus becomes

$$\tau_{\min} = \frac{VQ}{Ib} = \frac{\left(16 \times 10^{3}\right)\left(23.55\right)}{\left(310\right)\left(0.295\right)} = 4120\ psi$$

***Answer***

**Part 2**

The maximum shear stress is located **at the neutral axis**, where $Q/b$ is **largest**. Hence, $Q$ is the first moment of the area above (or below) the neutral axis.

The moment of $A'_1$ was calculated in part 1. The moment of $A'_2$ about the neutral axis is where

$$A'_1 \bar{y}' = (8.005 \times 0.515)\frac{11.94 - 0.515}{2} = 23.55 in.^3$$

$$= 23.55 in.^3$$

$$A'_2 = \left(\frac{11.94}{2} - 0.515\right)(0.295) = 1.6092 in.^2$$

$$\bar{y}'_2 = \frac{1}{2}\left(\frac{11.94}{2} - 0.515\right) = 2.7275 in.$$

$$Q = A'_1 \bar{y}'_1 + A'_2 \bar{y}'_2 = 23.55 + (1.6092)(2.7275) = 27.94 in.^3$$

The maximum shear stress in the web becomes

$$\tau_{max} = \frac{VQ}{Ib} = \frac{(16 \times 10^3)(27.94)}{(310)(0.295)} = 4890 \ psi$$

***Answer***

**Part 3**

The distribution of the shear stress in the web is shown in Fig.(c). The shear force carried by the web is

$$V_{\text{web}} = (\text{cross section area of web}) \times (\text{area of shear diagram})$$

The shear stress distribution is parabolic. Recalling that the area of a parabola is (2/3) (base × height).

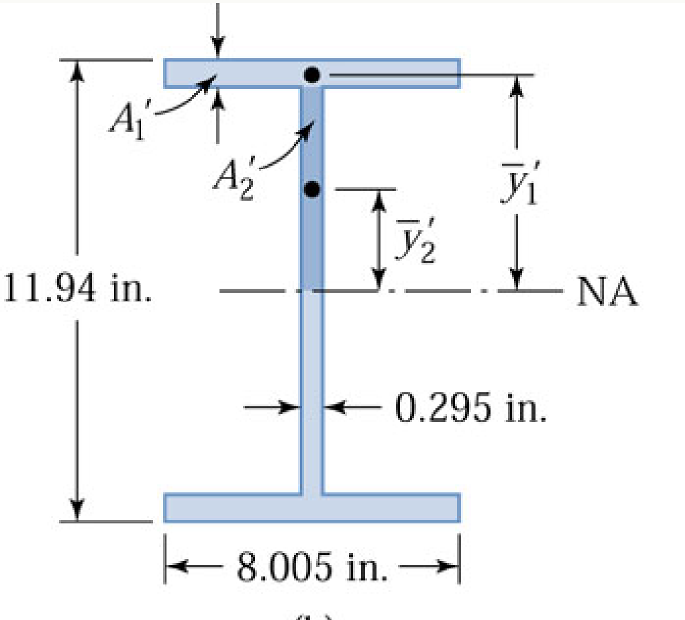
$$V_{web} = (10.91 \times 0.295)\left[4120 + \frac{2}{3}(4890 - 4120)\right] = 14910 lb$$

Therefore the percentage of the shear force carried by the web is

$$\frac{V_{web}}{V}\times 100\% = \frac{14910}{16000}\times 100\% = 93.2\% \qquad \textbf{\textit{Answer}}.$$

The result confirms that the flanges are ineffective in resisting the vertical shear

It was mentioned in Art. 5.5 that we can use $\tau_{max} = V/A_{web}$ as a rough approximation for the maximum shear stress.



(b)

$$\frac{V}{A_{web}} = \frac{16\times 10^{3}}{(10.91)(0.295)} = 4970 \ psi$$

which differs from $\tau_{max} = 4890$ psi computed in Part 2 by less than 2%.

***Sample Problem* 5.8**

The figure shows the cross section of a beam that carries a vertical shear force $V$ = 12 kips. The distance from the bottom of the section to the neutral axis is $d$ = 8.90 in., and the moment of inertia of the cross –sectional area about the neutral axis is $I$ = 547 in.$^4$. Determine the maximum shear stress on this cross section.

***Solution***

The maximum shear stress may occur of the neutral axis (where *Q* is largest) or at level *a-a* in the lower fin (where the width of the cross section is smaller than at the neutral axis).

***Shear Stress at Neutral Axis*** Take *Q* to be the first moment of the rectangular area ***above*** the neutral axis (the area below the neutral axis could also be used).

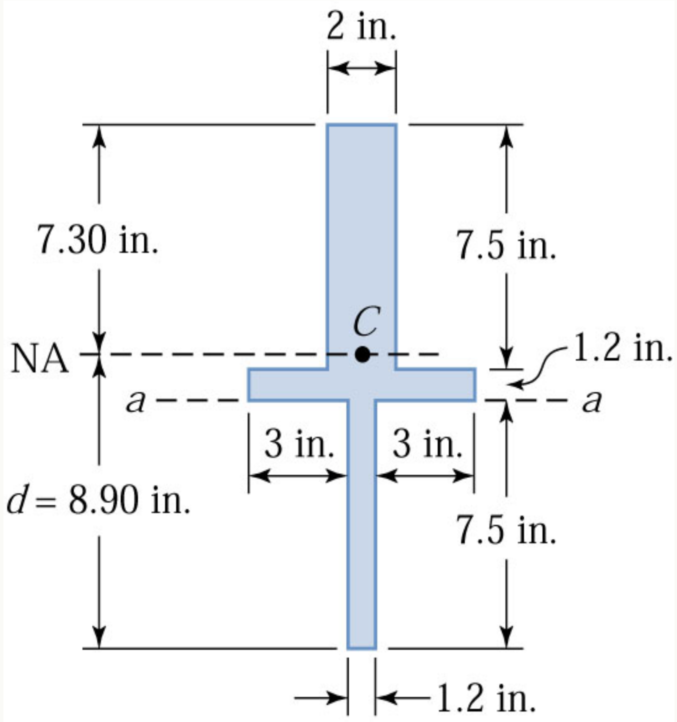
$$Q = A\bar{y} = (2 \times 7.30)\frac{7.30}{2} = 53.29 in.^3$$

and the shear stress at the neutral axis is

$$\tau = \frac{VQ}{Ib} = \frac{(12 \times 10^3)(53.29)}{(547)(2)} = 58.5 psi$$

***Shear Stress at a-a*** It is easier to compute $Q$ by using the area *below* the line *a-a* rather than the area above the line. The dimensions of this area are $b$ =1.2 in. and $h$ =7.5 in. Consequently,

$$Q = A\bar{y} = (1.2\times 7.5)\left(8.90 - \frac{7.5}{2}\right) = 46.35 in.$$




and the shear stress becomes

$$\boxed{\tau = \frac{VQ}{Ib} = \frac{(12\times 10^3)(46.35)}{(547)(1.2)} = 847\,psi}$$

***Shear Stress at Neutral Axis***

$$\boxed{\tau = \frac{VQ}{Ib} = \frac{(12\times 10^3)(53.29)}{(547)(2)} = 58.5\,psi}$$

The maximum shear stress is the largest of the two value;

$\tau_{max}$ = 847 psi (occurring at *a-a*) ***Answer***

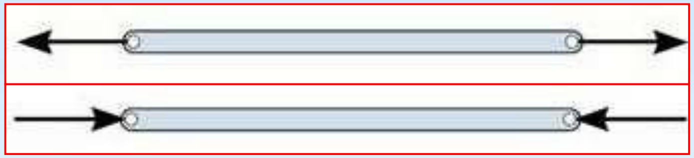
# Trusses

***Truss:*** is a structure composed of slender members (two-force members) joined together at their end points to support stationary or moving load.

❖ Each member of a truss is usually of uniform cross section along its length.

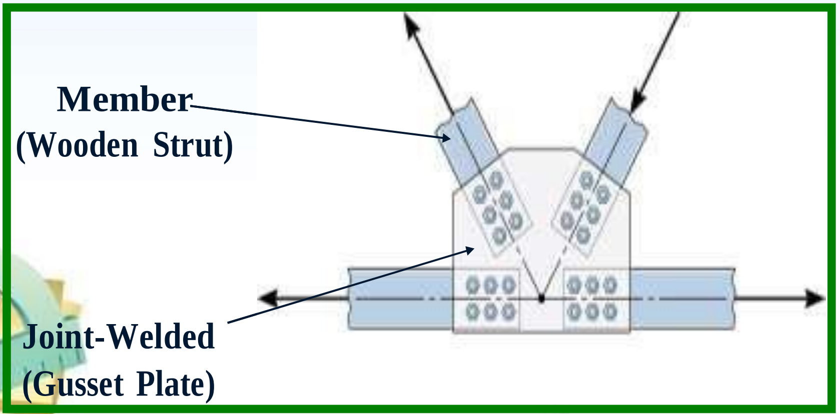
***Calculation are usually based on following assumption:***

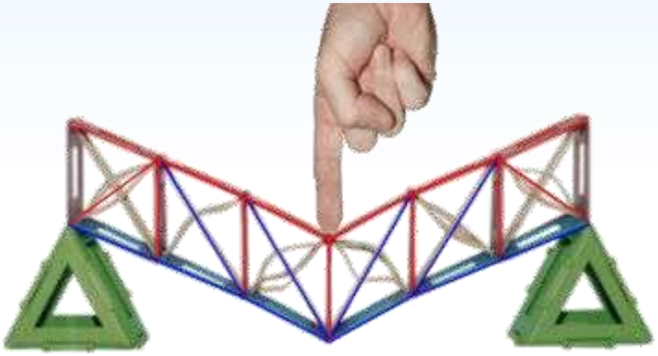
- The loads and reactions act only at the joint.
- Weight of the individual members can be neglected.
- Members are either under tension or compression.



***Joints:*** are usually formed by bolting or welding the members to a common plate, called a gusset plate, or simply passing a large bolt through each member.

- *Joints* are modeled by smooth pin connections.



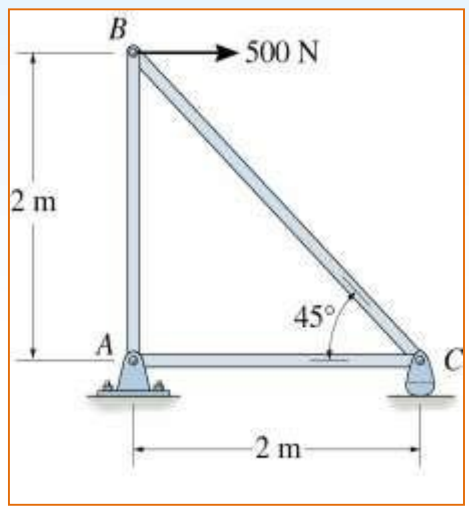
# Analysis of Trusses

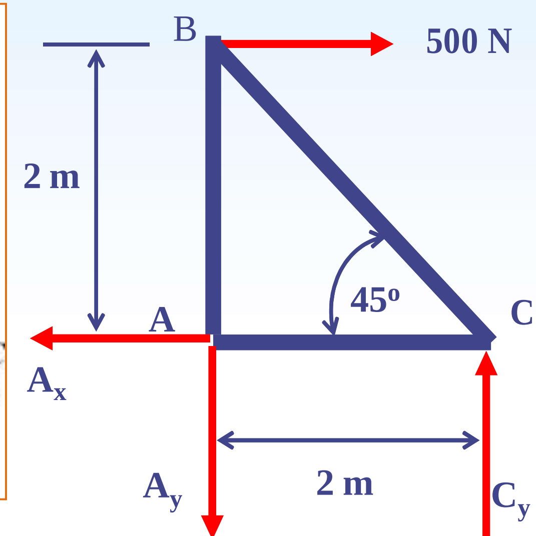
**Truss Analysis**
- **External equilibrium** → **To find the reaction forces**
- **Internal equilibrium** → **To find the force in each member**
  - **Method of joints**
  - **Method of sections**

**External Equilibrium:** to find the ***reaction forces***, follow the below steps:

1. Draw the ***FBD*** for the entire truss system.
2. Determine the ***reactions***. Using the equations of (2 D) which states:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0$$



# Analysis of Trusses

**Method of Joints:** to find the ***forces*** in any *member*, choose a ***joint***, to which that member is connected, and follow the below steps:

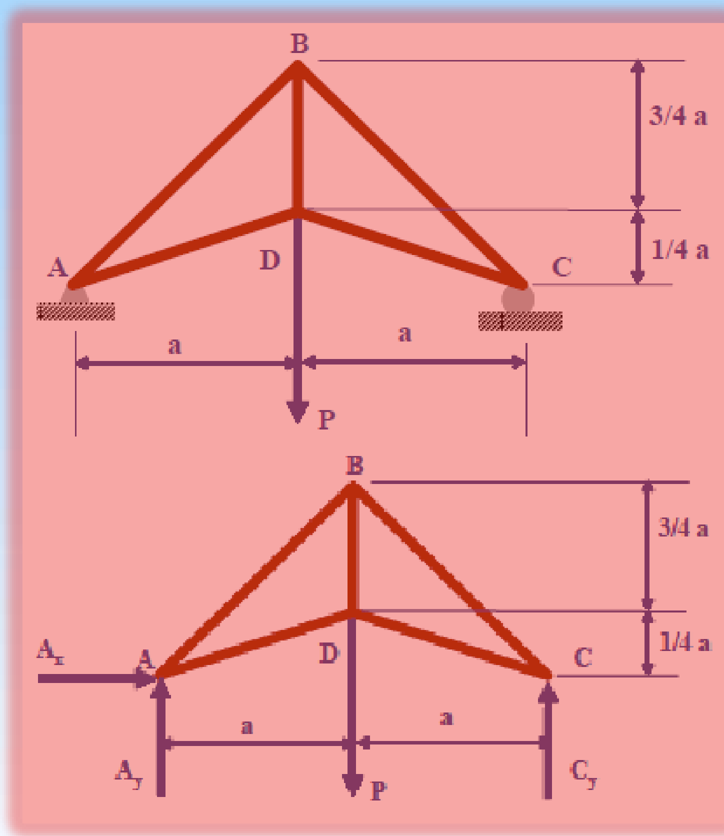
1. Draw the ***FBD*** for the entire truss system.
2. Determine the ***reactions***. Using the equations of (2 D) which states:

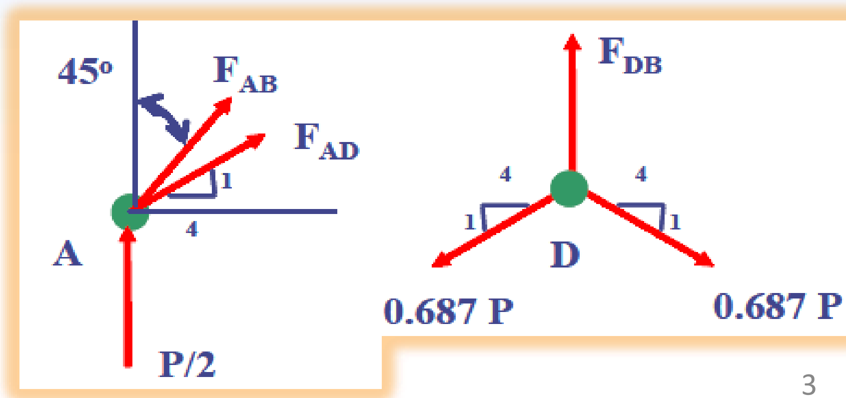
$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0$$

3. Choose the joint, and draw ***FBD*** of a ***joint*** with at least ***one known force*** and at most *two unknown forces*.
4. Using the equation of (**2 D**) which states:

$$\sum F_x = 0, \quad \sum F_y = 0$$

5. The *internal forces* are determined.
6. Choose another ***joint***.



# Analysis of Trusses



**Method of section (Internal equilibrium):** to find the ***forces*** in any *member*, choose a ***section***, to which that *member* is appeared as an internal force, and follow the below steps:

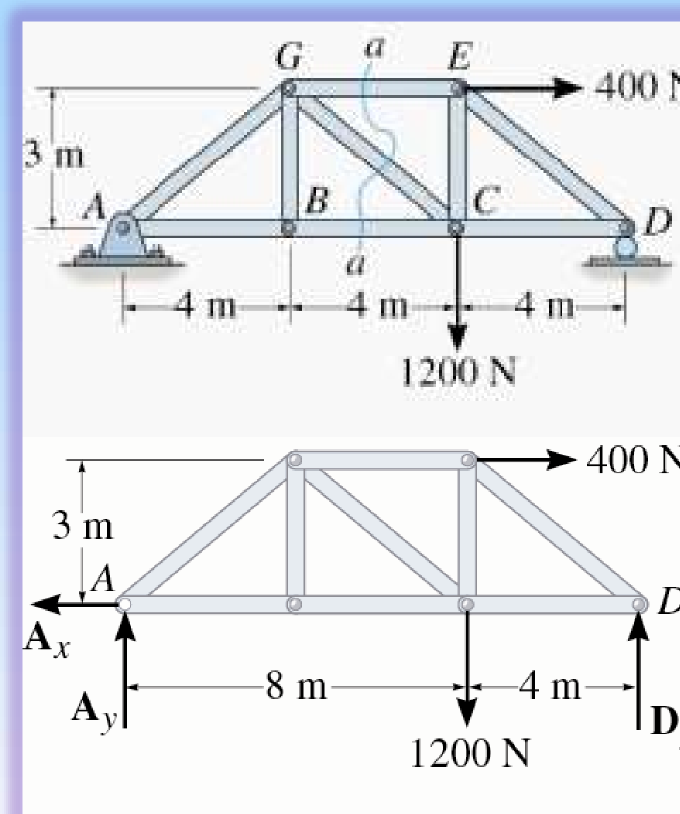
1. Draw the ***FBD*** for the entire truss system.
2. Determine the ***reactions***. Using the equations of (**2 D**) which states:

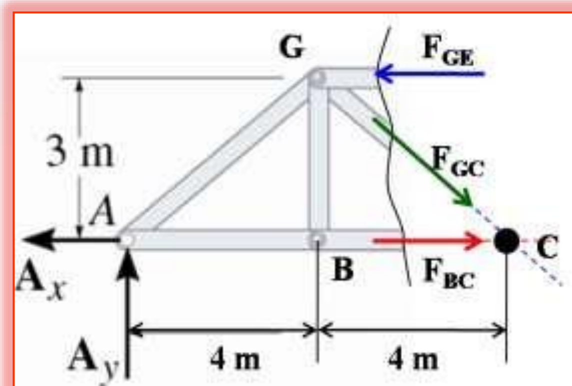
$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0$$

3. Choose the ***section***, and draw ***FBD*** of that *section*, shows how the forces replace the sectioned members.
4. Using the equation of (**2 D**) which states:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0$$

5. The *internal forces* are determined.
6. Choose another *section* or *joint*.



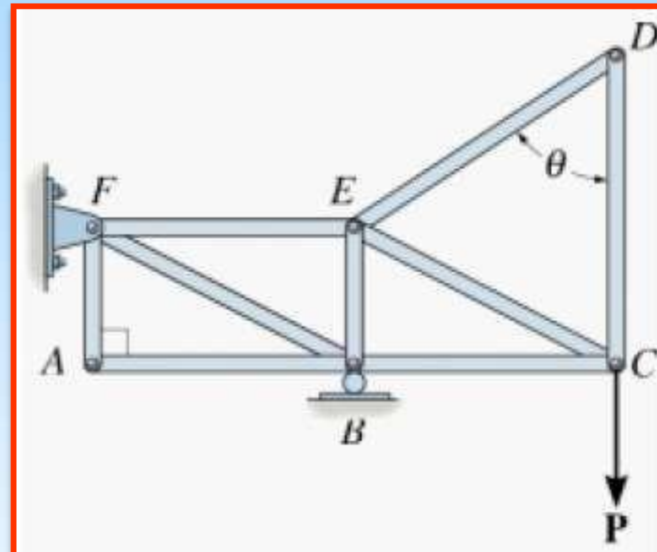



# Analysis of Trusses



## Analysis of trusses (Zero-force members):

Analysis of trusses system is simplified if one can identify those members that support no loads. We call these zero-force members.

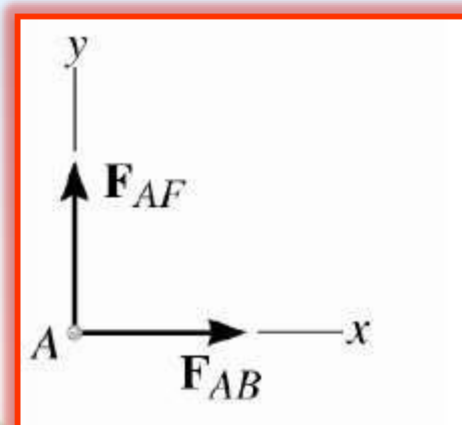



**Examples to follow:**

1. If ***two members*** form a *truss joint* and there is *no external load* or *support reaction* at that joint then *those members* are ***zero-force members***.

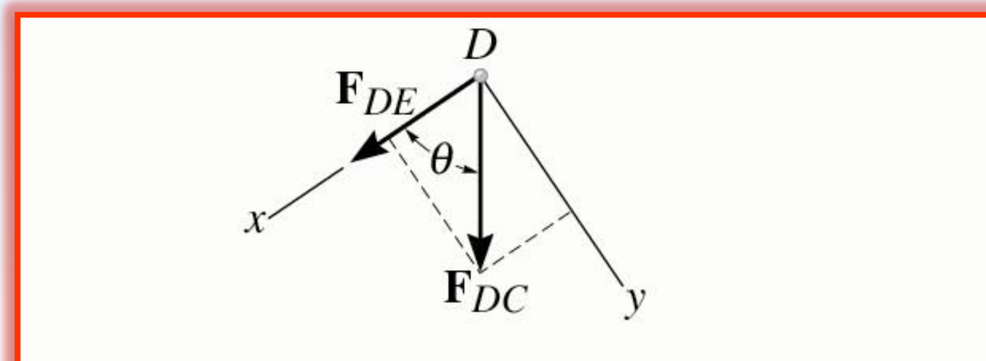
Joints D and A in the following figure are the joints with no external load or support reaction, so:

$$\mathbf{F_{AF} = F_{AB} = F_{DE} = F_{DC} = 0.}$$




$\xrightarrow{+} \Sigma F_x = 0;\quad F_{AB} = 0$

$+\uparrow \Sigma F_y = 0;\quad F_{AF} = 0$




$+\searrow \Sigma F_y = 0;\ F_{DC} \sin\theta = 0;\ \ F_{DC} = 0 \text{ since } \sin\theta \neq 0$

$+\swarrow \Sigma F_x = 0;\ F_{DE} + 0 = 0;\ F_{DE} = 0$

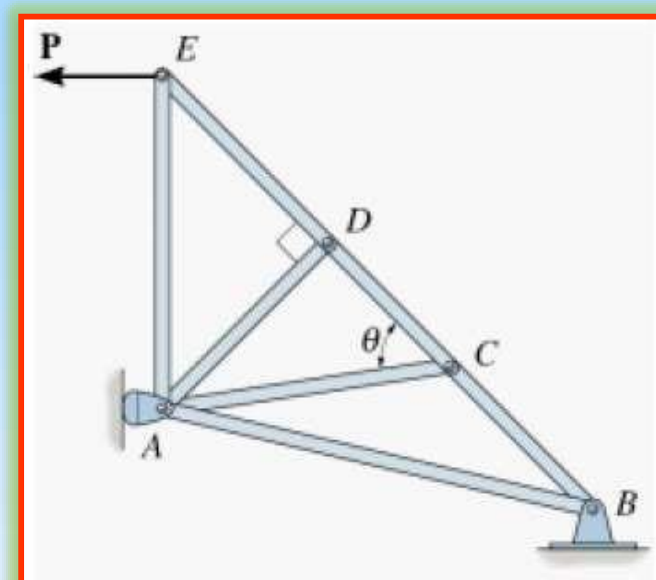
# Analysis of Trusses

## Analysis of trusses (Zero-force members):

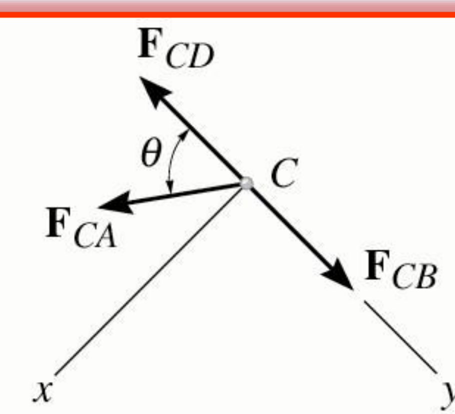
**Examples to follow:**

2. If ***three members*** form a truss **joint** and there is *no external load* or *support reaction* at that *joint* and *two of those members* are ***collinear*** then the *third member* is a ***zero-force member***.

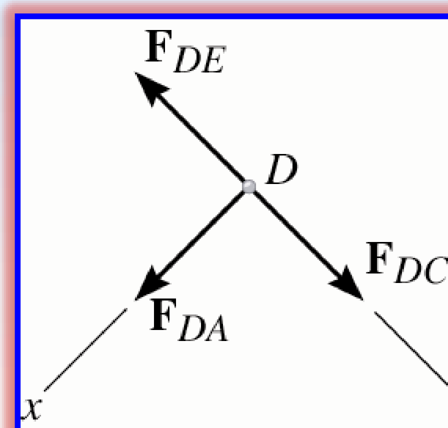
In the following figure, AC and AD are zero-force members, because Joints D and A in the following figure are the joints with three members, there is no external load or support reaction, so:




$$\mathbf{F_{CA} = F_{DA} = 0}$$



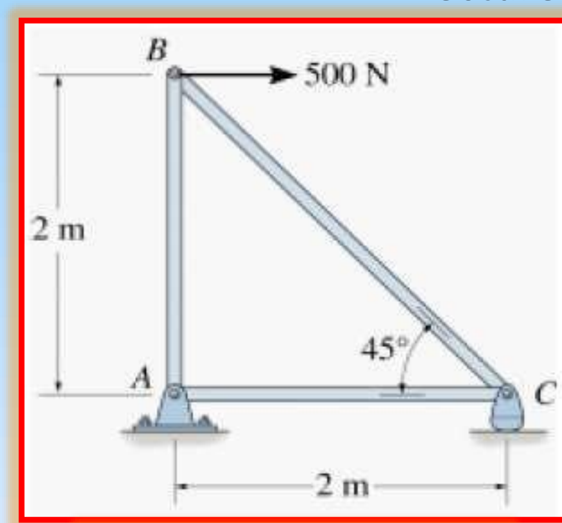

$$+\swarrow \Sigma F_x = 0;\quad F_{CA}\sin\theta = 0;\quad F_{CA} = 0 \text{ since } \sin\theta \neq 0;$$
$$+\searrow \Sigma F_y = 0;\quad F_{CB} = F_{CD}$$




$$+\swarrow \Sigma F_x = 0;\quad F_{DA} = 0$$
$$+\searrow \Sigma F_y = 0;\quad F_{DC} = F_{DE}$$

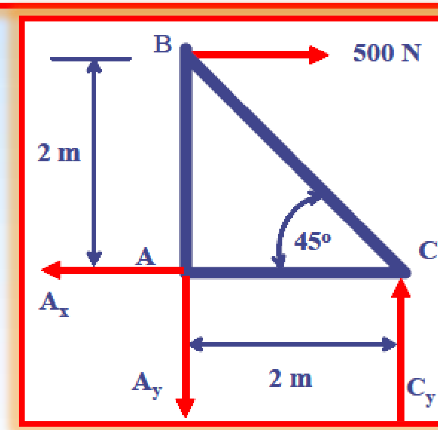
# EXAMPLES of Trusses:

**Example 1:** Determine the support reactions in the joints of the following truss. Calculate the force in member (BA & BC.)




## Solution

1. Draw FBD of entire truss and solve for support reactions:



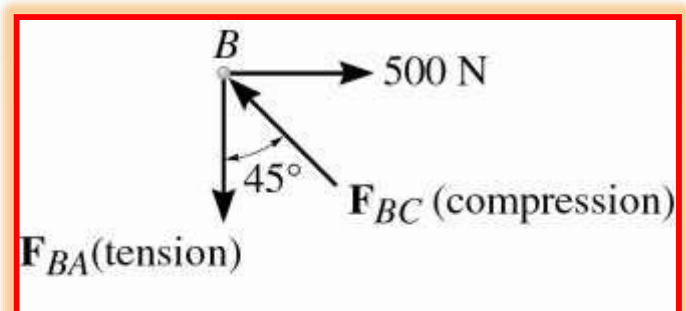

$$\sum F_x = 0 \qquad 500 - A_x = 0 \qquad A_x = 500 \ \text{N}$$

$$\sum F_y = 0 \qquad C_y - A_y = 0 \qquad A_y = C_y$$

$$\sum M_A = 0 \qquad -500(2) + C_y(2) = 0 \qquad A_y = C_y = 500 \ \text{N}$$

2. Draw FBD of a joint with at least one known force and at most two unknown forces. We choose joint B.

- Assume BC is in compression.




$$\sum F_x = 0 \qquad 500 - F_{BC} \ \sin \ 45^\circ = 0 \qquad F_{BC} = 707.1 \ \text{N} \ (C)$$

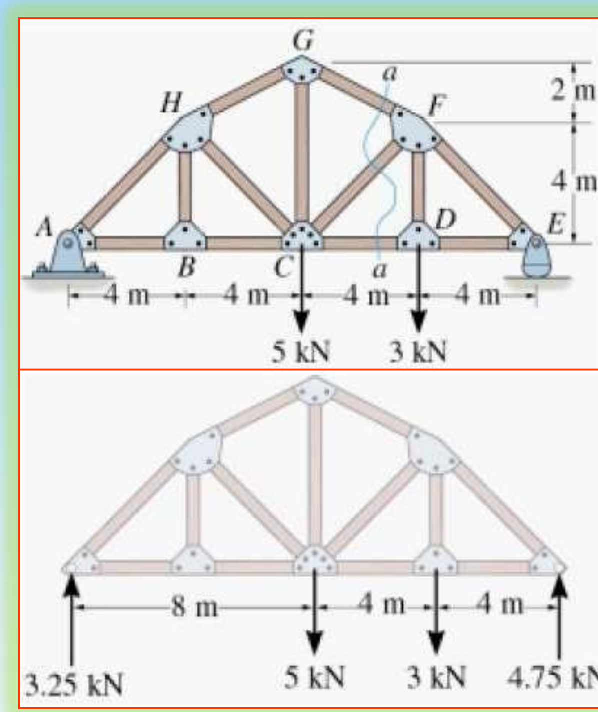
$$\sum F_y = 0 \qquad F_{BC} \ \cos \ 45^\circ - F_{BA} = 0 \qquad F_{BA} = 500 \ \text{N} \ (T)$$

# EXAMPLES of Trusses:

**Example 2:** In the following Bowstring Truss, find the force in member (CF).

**Solution**

draw the FBD and find the support reactions which are shown below




$\sum M_A = 0$

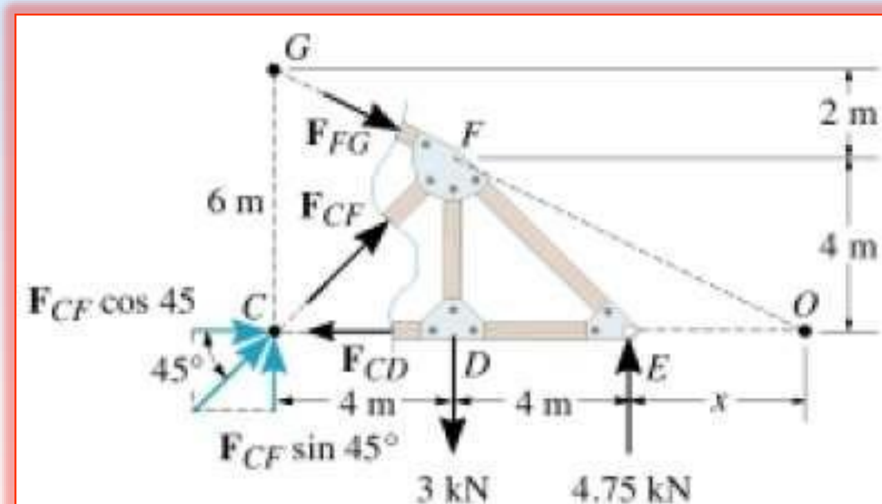
$R_E * 16 - 5 * 8 - 3 * 12 = 0$

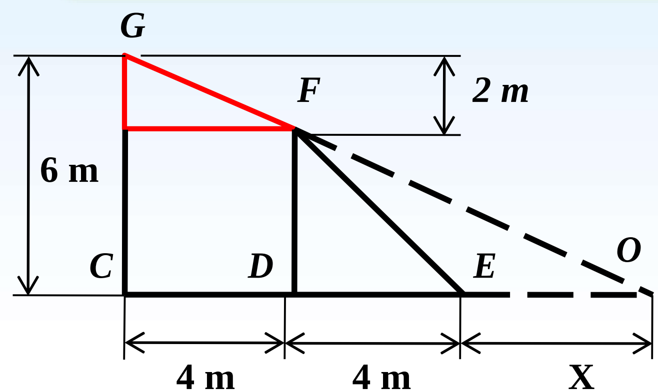
$R_E = 4.75$ kN

$\sum F_y = 0$

$R_E + R_A - 5 - 3 = 0$

$R_A = 3.25$ kN




$$\frac{2}{4} = \frac{6}{(4+4+X)} \rightarrow X = 4\,m$$

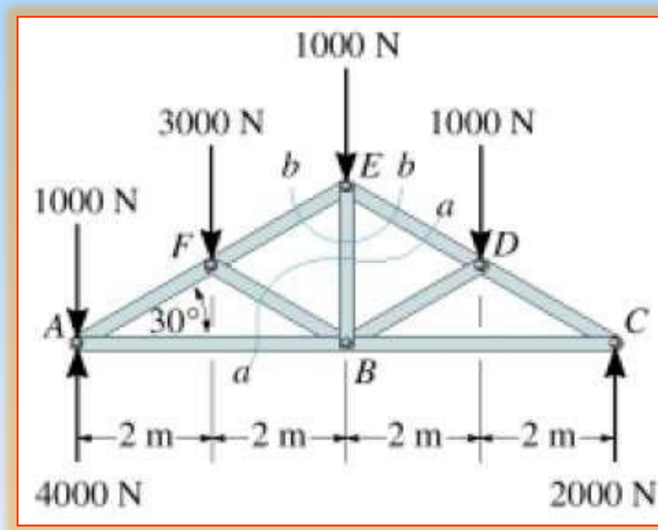
$$\sum M_o = 0$$

$$-F_{CF}\sin 45°(12\text{m}) + (3\text{kN})(8\text{ m}) - (4.75\text{kN})(4\text{m}) = 0$$

$$F_{CF} = 0.589\text{kN}(C)$$

# EXAMPLES of Trusses:

**Example 3:** In the following truss, find the force in member (EB).




**Solution**

Notice that no single cut will provide the answer. Hence, it is best to consider section (a-a and b-b).

$$\sum \mathbf{M_A = 0}$$

$$\mathbf{R_C * 8 - 1000 * 6 - 1000 * 4 - 3000 * 2 = 0}$$

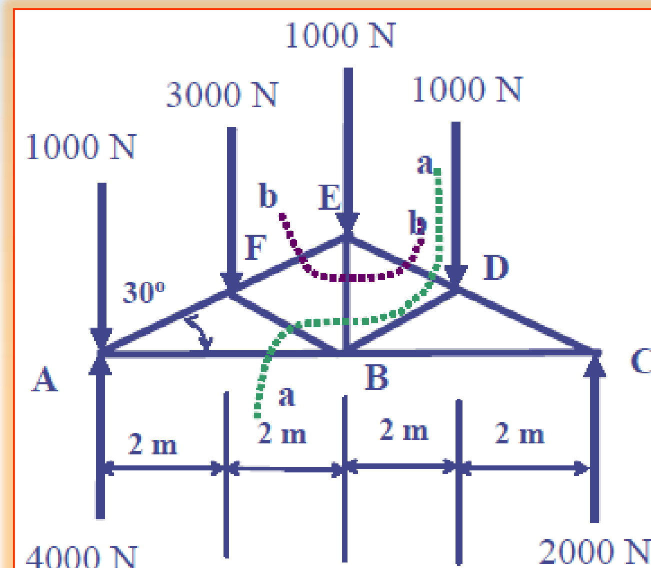
$$\mathbf{R_C = 2000\ N}$$

$$\sum \mathbf{F_y = 0}$$

$$\mathbf{R_A + R_C - 1000 - 1000 - 3000 - 1000 = 0}$$

$$\mathbf{R_A = 4000\ N}$$

Taking the moment about joint (B), to find ($F_{ED}$), as shown in below figure:



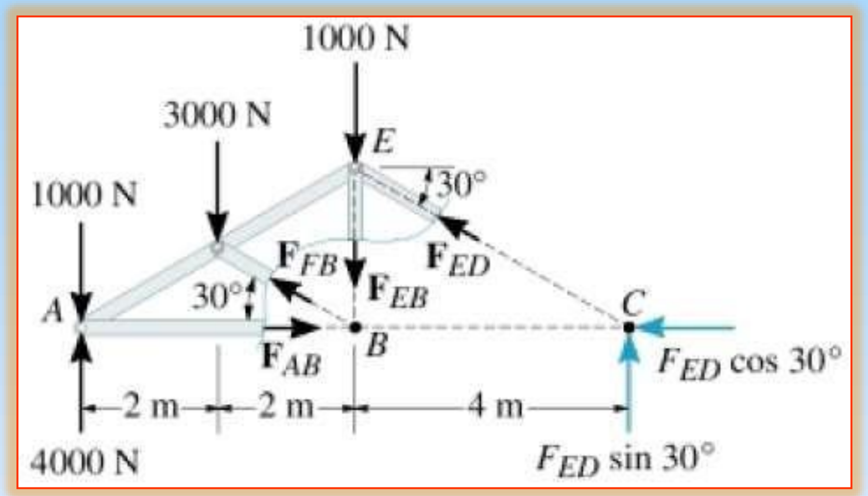

$$\sum \mathbf{M_B = 0}$$

$$\mathbf{1000 * 4 + 3000 * 2 - 4000 * 4 + F_{ED} * \sin 30^o * 4 = 0}$$

$$\mathbf{F_{ED} = 3000\ N} \quad \textbf{(compression)}$$

## Continue Example 3:

From joint (E) to find ($F_{EB}$), as shown in below figure:




$\sum F_x = 0$

$F_{EF} \cdot \cos 30^o - 3000 \cos 30^o = 0$

$F_{EF} = 3000$ N (compression)

$\sum F_y = 0$

$F_{EF} \cdot \sin 30^o + 3000 \cdot \sin 30^o - 1000 - F_{EB} = 0$

$F_{EF} = 2000$ N (Tension)