VEMU INSTITUTE OF TECHNOLOGY

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

PROBABILITY THEORY & STOCHASTIC PROCESSES

LECTURE NOTES

(15A04304)

B.TECH ECE II YEAR I SEMESTER (JNTUA-R15)

SYLLABUS

(15A04304) PROBABILITY THEORY & STOCHASTIC PROCESSES

Course Objectives:

- To understand the concepts of a Random Variable and operations that may be performed on a single Random variable.
- To understand the concepts of Multiple Random Variables and operations that may be performed on Multiple Random variables.
- To understand the concepts of Random Process and Temporal & Spectral characteristics of Random Processes.

Learning Outcomes:

- A student will able to determine the temporal and spectral characteristics of random signal response of a given linear system.

UNIT-I

Probability: Probability introduced through Sets and Relative Frequency: Experiments and Sample Spaces, Discrete and Continuous Sample Spaces, Events, Probability Definitions and Axioms, Mathematical Model of Experiments, Probability as a Relative Frequency, Joint Probability, Conditional Probability, Total Probability, Bays' Theorem, Independent Events:

The Random Variable : Definition of a Random Variable, Conditions for a Function to be a Random Variable, Discrete and Continuous, Mixed Random Variable, Distribution and Density functions, Properties, Binomial, Poisson, Uniform, Gaussian, Exponential, Raleigh, Conditional Distribution, Methods of defining Conditioning Event, Conditional Density, Properties.

UNIT-II

Multiple Random Variables : Vector Random Variables, Joint Distribution Function, Properties of Joint Distribution, Marginal Distribution Functions, Conditional Distribution and Density – Point Conditioning, Conditional Distribution and Density – Interval conditioning, Statistical Independence, Sum of Two Random Variables, Sum of Several Random Variables, Central Limit Theorem, (Proof not expected). Unequal Distribution, Equal Distributions.

Operations on Multiple Random Variables: Expected Value of a Function of Random Variables, Joint Moments about the Origin, Joint Central Moments, Joint Characteristic Functions, Jointly Gaussian Random Variables: Two Random Variables case, N Random Variable case, Properties, Transformations of Multiple Random Variables, Linear Transformations of Gaussian Random Variables.

UNIT-III

Random Processes – Temporal Characteristics: The Random Process Concept, Classification of Processes, Deterministic and Nondeterministic Processes, Distribution and Density Functions, concept of Stationarity and Statistical Independence. First-Order Stationary Processes, Second- Order and Wide-Sense Stationarity, (N-Order) and Strict-Sense Stationarity, Time Averages and Ergodicity, Mean-Ergodic Processes, Correlation-Ergodic Processes, Autocorrelation Function and Its Properties, Cross-Correlation Function and its Properties, Covariance Functions, Gaussian Random Processes, Poisson Random Process.

UNIT-IV

Random Processes – Spectral Characteristics: The Power Spectrum: Properties, Relationship between Power Spectrum and Autocorrelation Function, the Cross-Power Density Spectrum, Properties, Relationship between Cross-Power Spectrum and Cross-Correlation Function.

UNIT-V

Linear Systems with Random Inputs: Random Signal Response of Linear Systems: System Response – Convolution, Mean and Mean-squared Value of System Response, autocorrelation Function of Response, Cross-Correlation Functions of Input and Output, Spectral Characteristics of System Response: Power Density Spectrum of Response, Cross-Power Density Spectrums of Input and Output, Band pass, Band-Limited and Narrowband Processes, Properties.

Text Books:

1. Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", TMH, 4th Edition, 2001.
2. Athanasios Papoulis and S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", PHI, 4th Edition, 2002.

References:

1. R.P. Singh and S.D. Sapre, "Communication Systems Analog & Digital", TMH, 1995.
2. Henry Stark and John W.Woods, "Probability and Random Processes with Application to Signal Processing",Pearson Education, 3rd Edition.
3. George R. Cooper, Clave D. MC Gillem, "Probability Methods of Signal and System Analysis", Oxford, 3rd Edition, 1999.
4. S.P. Eugene Xavier, "Statistical Theory of Communication", New Age Publications, 2003.
5. B.P. Lathi, "Signals, Systems & Communications", B.S. Publications, 2003.

UNIT-1: PROBABILITY AND RANDOM VARIABLES

Introduction: The basic to the study of probability is the idea of a Physical experiment. A single performance of the experiment is called a trial for which there is an outcome. Probability can be defined in three ways. The First one is Classical Definition. Second one is Definition from the knowledge of Sets Theory and Axioms. And the last one is from the concept of relative frequency.

Experiment: Any physical action can be considered as an experiment. Tossing a coin, Throwing or rolling a die or dice and drawing a card from a deck of 52-cards are Examples for the Experiments.

Sample Space: The set of all possible outcomes in any Experiment is called the sample space. And it is represented by the letter s. The sample space is a universal set for the experiment. The sample space can be of 4 types. They are:

1. Discrete and finite sample space.
2. Discrete and infinite sample space.
3. Continuous and finite sample space.
4. Continuous and infinite sample space.

Tossing a coin, throwing a dice are the examples of discrete finite sample space. Choosing randomly a positive integer is an example of discrete infinite sample space. Obtaining a number on a spinning pointer is an example for continuous finite sample space. Prediction or analysis of a random signal is an example for continuous infinite sample space.

Event: An event is defined as a subset of the sample space. The events can be represented with capital letters like A, B, C etc… All the definitions and operations applicable to sets will apply to events also. As with sample space events may be of either discrete or continuous. Again the in discrete and continuous they may be either finite or infinite. If there are N numbers of elements in the sample space of an experiment then there exists 2N number of events.

The event will give the specific characteristic of the experiment whereas the sample space gives all the characteristics of the experiment.

Classical Definition: From the classical way the probability is defined as the ratio of number of favorable outcomes to the total number of possible outcomes from an experiment. i.e. Mathematically, P(A) =F/T.

Where: P(A) is the probability of event A.

F is the number of favorable outcomes and

T is the Total number of possible outcomes.

The classical definition fails when the total number of outcomes becomes infinity.

Definition from Sets and Axioms: In the axiomatic definition, the probability P(A) of an event is always a non negative real number which satisfies the following three Axioms.

Axiom 1: $P(A) \geq 0$.Which means that the probability of event is always a non negative number

Axiom 2: P(S) =1.Which means that the probability of a sample space consisting of all possible outcomes of experiment is always unity or one.

Axiom 3: $P(U_{n=1}^{N})$ or P (A1 A2 . . . AN) = P (A1) + P (A2) + . . . + P (AN)

This means that the probability of Union of N number of events is same as the Sum of the individual probabilities of those N Events.

Probability as a relative frequency: The use of common sense and engineering and scientific observations leads to a definition of probability as a relative frequency of occurrence of some event. Suppose that a random experiment repeated n times and if the event A occurs n(A) times, then the probability of event a is defined as the relative frequency of event a when the number of trials n tends to infinity. Mathematically $P(A) = Lt_{n \to \infty}\ n(A)/n$

Where n (A)/n is called the relative frequency of event, A.

Mathematical Model of Experiments: Mathematical model of experiments can be derived from the axioms of probability introduced. For a given real experiment with a set of possible outcomes, the mathematical model can be derived using the following steps:

1. Define a sample space to represent the physical outcomes.
2. Define events to mathematically represent characteristics of favorable outcomes.
3. Assign probabilities to the defined events such that the axioms are satisfied.

Joint Probability: If a sample space consists of two events A and B which are not mutually exclusive, and then the probability of these events occurring jointly or simultaneously is called the Joint Probability. In other words the joint probability of events A and B is equal to the relative frequency of the joint occurrence. If the experiment repeats n number of times and the joint occurrence of events A and B is n(AB) times, then the joint probability of events A and B is

$$\mathrm{P}\,(\mathrm{A} \cap B) = \lim_{n\to\infty} \frac{n(AB)}{n}$$

$$\mathrm{P}\,(\mathrm{A} \cap B) = P(A) + P\,(B) - P(A \cup B)\ then$$

$$\mathrm{P}\,(\mathrm{A} \cup B) = P(A) + P\,(B) - P(A \cap B)\ \text{also since}$$

$$\mathrm{P}\,(\mathrm{A} \cap B) \neq 0, P(A \cup B) \leq P(A) + \ P(B)$$

Conditional Probability: If an experiment repeats n times and a sample space contains only two events A and B and event A occurs n(A) times, event B occurs n(B) times and the joint event of A and B occurs n(AB) times then the conditional probability of event A given event B is equal to the relative frequency of the joint occurrence n(AB) with respect to n(B) as n tends to infinity.

Mathematically,

$$P\left(\frac{A}{B}\right) = \lim_{n\to\infty} \frac{n(AB)}{n(B)}\ \mathrm{n(B)} > 0$$

$$= \lim_{n\to\infty} \frac{n(AB)/n}{n(B)/n}$$

$$P\left(\frac{A}{B}\right) = \frac{\lim_{n\to\infty} \frac{n(AB)}{n}}{\lim_{n\to\infty} \frac{n(B)}{n}}$$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)},\ \mathrm{P(B)} \neq 0$$

That is the conditional probability P(A/B) is the probability of event A occurring on the condition that the probability of event B is already known. Similarly the conditional probability of occurrence of B when the probability of event A is given can be expressed as

$$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)},\ \mathrm{P(A)} \neq 0 \qquad [\mathrm{P(B} \cap A) = P(A \cap B)]$$

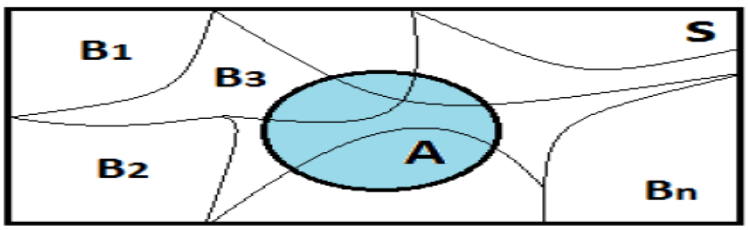
From the conditional probabilities, the joint probabilities of the events A and B can be expressed as

$$\mathrm{P}\,(\mathrm{A} \cap B) = P\left(\frac{A}{B}\right) P(B) = P\left(\frac{B}{A}\right) P(A).$$

Total Probability Theorem: Consider a sample space, s that has n mutually exclusive events Bn, n=1, 2, 3,…,N. such that Bm∩Bn=φ for m =1, 2, 3, ….,N. The probability of any event A, defined on this sample space can be expressed in terms of the Conditional probabilities of events Bn. This probability is known as the total probability of event A. Mathematically,

$$\mathrm{P(A)} = \sum_{n=1}^{N} P\left(\frac{A}{Bn}\right) P(Bn)$$

Proof: The sample space s of N mutually exclusive events, Bn, n=1, 2, 3, …N is shown in the figure.



i.e. $B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_N = S$.

Let an event A be defined on sample space s. Since a is subset of s, then A ∩ S =A or

$\mathrm{A} \cap \mathrm{S} = \mathrm{A} \cap [\cup_{n=1}^{N} B_n] = \mathrm{A}$ or $\mathrm{A} = \cup_{n=1}^{N}(A \cap B_n)$

Applying probability $\mathrm{P(A)} = \mathrm{P}\,[\cup_{n=1}^{N}(A \cap B_n)] = \cup_{n=1}^{N} P(A \cap B_n)$

Since the events $P(A \cap B_n)$ are mutually exclusive, by applying axiom 3 of probability we get,

$$\mathrm{P(A)} = \sum_{n=1}^{N} P\,(A \cap B_n).$$

From the definition of joint probability,

$$\mathrm{P}\,(\mathrm{A} \cap B_n) = P\left(\frac{A}{B_n}\right) P(B_n)$$

Baye's Theorem: It states that if a sample space S has N mutually exclusive events Bn, n=1, 2, 3,…,N. such that Bm∩Bn=φ for m =1, 2, 3, ….,N. and any event A is defined on this sample space then the conditional probability of Bn and A can be Expressed as

$$P(B_n/A) = \frac{P\left(\frac{A}{B_n}\right)P(B_n)}{P\left(\frac{A}{B_1}\right)P(B_1) + P\left(\frac{A}{B_2}\right)P(B_2) + \dots + P\left(\frac{A}{B_n}\right)P(B_n)}$$

Proof: This can be proved from total probability Theorem, and the definition of conditional probabilities.

We know that the conditional probability, $P\left(\frac{B_n}{A}\right) = P(B_n \cap A)/P(A)$, $P(A) \neq 0$ also

$P(B_n \cap A) = P\left(\frac{A}{B_n}\right)P(B_n)$ And from the total probability theorem,

$P(A) = \sum_{n=1}^{N} P(B_n \cap A)$.

Therefore $P(B_n/A) = \frac{P(B_n \cap A)}{\sum_{n=1}^{N} P(B_n \cap A).}$

$$P(B_n/A) = \frac{P\left(\frac{A}{B_n}\right)P(B_n)}{\sum_{n=1}^{N} P\left(\frac{A}{Bn}\right)P(Bn)} \text{ or}$$

$$P(B_n/A) = \frac{P\left(\frac{A}{B_n}\right)P(B_n)}{P\left(\frac{A}{B_1}\right)P(B_1) + P\left(\frac{A}{B_2}\right)P(B_2) + \dots + P\left(\frac{A}{B_n}\right)P(B_n)} \text{ Hence Proved.}$$

Independent events: Consider two events A and B in a sample space S, having non-zero probabilities. If the probability of occurrence of one of the event is not affected by the occurrence of the other event, then the events are said to be Independent events.

$P(A \cap B) = P(A)P(B)$. For $P(A) \neq 0$ and $P(B) \neq 0$.

If A and B are two independent events then the conditional probabilities will become

$P(A/B) = P(A)$ and $P(B/A) = P(B)$. That is the occurrence of an event does not depend on the occurrence of the other event. Similarly the necessary and sufficient conditions for three events A, B and C to be independent are:

$$P(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C) \text{ and}$$
$$P(A \cap B \cap C) = P(A) \cap P(B) \cap P(C).$$

Multiplication Theorem of Probability: Multiplication theorem can be used to find out probability of outcomes when an experiment is performing on more than one event. It states that if there are N events An, n=1,2, . . . N, in a given sample space, then the joint probability of all the events can be expressed as

$$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_N) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \ldots P(A_N/A_1 \cap A_2 \cap \ldots A_{N-1})$$

And if all the events are independent, then

$$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_N) = P(A_1)\, P(A_2)\, P(A_3) \ldots P(A_N).$$

Permutations & Combinations: An ordered arrangement of events is called Permutation. If there are n numbers of events in an experiment, then we can choose and list them in order by two conditions. One is with replacement and another is without replacement. In first condition, the first event is chosen in any of the n ways thereafter the outcome of this event is replaced in the set and another event is chosen from all v events. So the second event can be chosen again in n ways. For choosing r events in succession, the numbers of ways are n^r.

$$N_{P_r} = \frac{N!}{(N-r)!}$$

In the second condition, after choosing the first event, in any of the n ways, the outcome is not replaced in the set so that the second event can be chosen only in (n-1) ways. The third event in (n-2) ways and the rth event in (n-r+1) ways. Thus the total numbers of ways are n(n-1)(n-2) . . . (n-r+1).

$$N_{C_r} = \frac{N!}{(N-r)!r!}$$

RANDOM VARIABLE

Introduction: A random variable is a function of the events of a given sample space, S. Thus for a given experiment, defined by a sample space, S with elements, s the random variable is a function of S. and is represented as X(s) or X(x). A random variable X can be considered to be a function that maps all events of the sample space into points on the real axis.

Typical random variables are the number of hits in a shooting game, the number of heads when tossing coins, temperature/pressure variations of a physical system etc…For example, an experiment consists of tossing two coins. Let the random variable X chosen as the number of heads shown. So X maps the real numbers of the event "showing no head" as zero, the event "any one head" as One and "both heads" as Two. Therefore the random variable is X = {0,1,2}

The elements of the random variable X are $x1=0$, $x2=1$ & $x3=2$.

Conditions for a function to be a Random Variable: The following conditions are required for a function to be a random variable.

1. Every point in the sample space must correspond to only one value of the random variable. i.e. it must be a single valued.
2. The set $\{X \le x\}$ shall be an event for any real number. The probability of this event is equal to the sum of the probabilities of all the elementary events corresponding to $\{X \le x\}$. This is denoted as $P\{X \le x\}$.
3. The probability of events $\{X=\infty\}$ and $\{X=-\infty\}$are zero.

Classification of Random Variables: Random variables are classified into continuous, discrete and mixed random variables.

The values of continuous random variable are continuous in a given continuous sample space. A continuous sample space has infinite range of values. The discrete value of a continuous random variable is a value at one instant of time. For example the Temperature, T at some area is a continuous random variable that always exists in the range say, from T1 and T2. Another example is an experiment where the pointer on a wheel of chance is spun. The events are the continuous range of values from 0 t0 12 marked in the wheel.

The values of a discrete random variable are only the discrete values in a given sample space. The sample space for a discrete random variable can be continuous, discrete or even both continuous and discrete points .They may be also finite or infinite. For example the "Wheel of chance" has the continuous sample space. If we define a discrete random variable n as integer numbers from 0 to 12, then the discrete random variable is X = {0,1,3,4…..12}

The values of mixed random variable are both continuous and discrete in a given sample space. The sample space for a mixed random variable is a continuous sample space. The random variable maps some points as continuous and some points as discrete values. The mixed random variable has least practical significance or importance.

Probability Distribution Function: The probability distribution function (PDF) describes the probabilistic behavior of a random variable. It defines the probability P $\{X \le x\}$ of the event $\{X \le x\}$ for all values of the random variable X up to the value of x. It is also called as the Cumulative Distribution Function of the random variable X and denotes as $F_X(x)$ which is a function of x. Mathematically, $F_X(x) = P\{X \le x\}$. Where x is a real number in the range $-\infty \le x \le \infty$. We can call $F_X(x)$ simply as the distribution function of x. If x is a discrete random variable, the distribution function $F_X(x)$ is a cumulative sum of all probabilities of x up to the value of x. as x is a discrete $F_X(x)$ must have a stair case form with step functions. The amplitude of the step is equal to the probability of X at that value of x. If the values of x are $\{x_i\}$, the distribution function can be written mathematically as

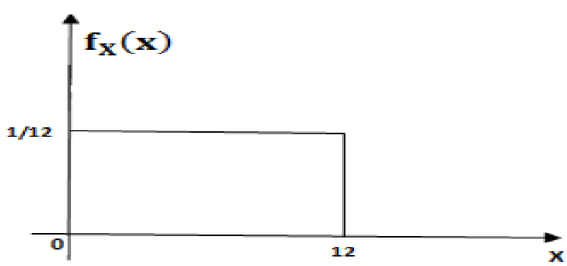
$$\mathbf{F_X(x)} = \sum_{i=1}^{N} P(x_i)\, u(x - x_i).$$

$$\text{Where } \mathbf{u(x)} = \begin{cases} 1 \; for \; x \ge 0. \\ 0 \; for \; x < 0. \end{cases}$$

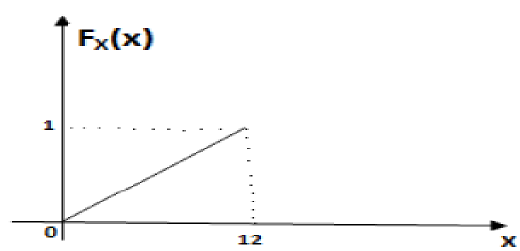
is a unit step function and N is the number of elements in x. N may be infinite.

If x is a continuous random variable, the distribution function $F_X(x)$ is an integration of all continuous probabilities of x up to the value of x. Let $f_X(x)$ be a probability function of x, a continuous random variable. The distribution function for X is given by

$$\mathbf{F_X(x)} = \int_{-\infty}^{x} \mathbf{f_X(x)\,dx}.$$



Density function of x



Distribution function of x

Probability density function: The probability density function (pdf) is a basic mathematical tool to design the probabilistic behavior of a random variable. It is more preferable than PDF. The probability density function of the random variable x is defined as the values of probabilities at a given value of x. It is the derivative of the distribution function $F_X(x)$ and is denoted as $f_X(x)$. Mathematically,

$$f_X(x) = \frac{dFX(x)}{dx}.$$

Where x is a real number in the range -∞≤x≤∞

We can call $f_X(x)$ simply as density function of x. The expression of density function for a discrete random variable is

$$\mathbf{f_X(x)} = \sum_{i=1}^{N} P(x_i)\ \delta(x - x_i).$$

From the definition we know that

$$f_X(x) = \frac{dFX(x)}{dx} = \frac{d[\sum_{i-1}^{N} P(x_i)\ u(x-x_i)]}{dx} = \sum_{i=1}^{N} P(x_i)\ \frac{du(x-x_i)}{dx} = \sum_{i=1}^{N} P(x_i)\ \delta(x - x_i)$$

Since derivative of a unit step function u(x) is the unit impulse function δ(x) . And it is defined as

$$\delta(x) = \begin{cases} 1 & for\ x = 0 \\ 0 & otherwise \end{cases}$$

For continuous random variables, since the distribution function is continuous in the given range, the density function fX(x) can be expressed directly as a derivative of the distribution function. i.e.

$$f_X(x) = \frac{dFX(x)}{dx} \text{ where } -\infty \le x \le \infty.$$

Properties of Probability Distribution Function: If FX(x) is a probability distribution function of a random variable X, then

(i) $F_X(-\infty) = 0.$

(ii) $F_X(\infty) = 1.$

(iii) $0 \le F_{X(x)} \le 1.$

(iv) $F_X(x_1) \le FX(x2)$ if $x_1 < x_2.$

(v) $P\{x_1 \le X \le x_2\} = F_X(x_2) - F_X(x_1).$

(vi) $F_X(x^+) = F_X(x) = F_X(x^-)$

Properties of Probability Density Function: If fX(x) is a probability density function of a random variable X, then

(i) $0 \leq f_X(x)$ for all x.

(ii) $\int_{-\infty}^{\infty} f_{X(x)}\, dx = 1.$

(iii) $F_X(x) = \int_{-\infty}^{x} f_{X(x)}\, dx.$

(iv) $P\{x_1 \leq X \leq x_2\} = \int_{x_1}^{x_2} f_{X(x)}\, dx$

Real Distribution and Density Function: The following are the most generally used distribution and density functions.

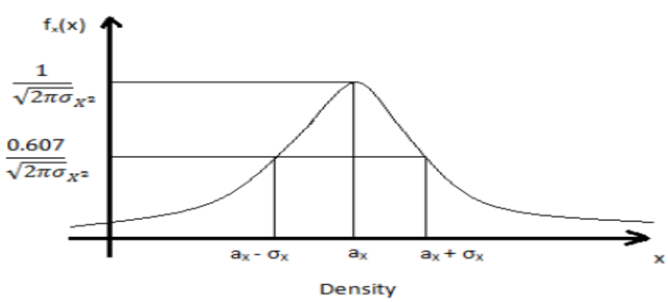
1. Gaussian Function.
2. Uniform Function.
3. Exponential Function.
4. Rayleigh Function.
5. Binomial Function.
6. Poisson's Function.

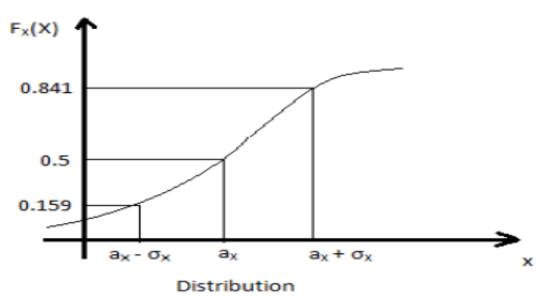
1. Gaussian Function: The Gaussian density and distribution function of a random variable X are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X{}^2}} e^{-(x-a_X)^2} / 2\sigma_X{}^2 \qquad \text{for all x.}$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X{}^2}} \int_{-\infty}^{x} e^{-(x-a_X)^2} / 2\sigma_X{}^2\, dx \quad \text{for all x}$$

Where $\sigma_X > 0$, $-\infty \leq a_X \leq \infty$. Are constants called standard deviation and mean values of X respectively. The Gaussian density function is also called as the normal density function.





The plot of Gaussian density function is bell shaped and symmetrical about its mean value aX. The total area under the density function is one. i.e.

$$\frac{1}{\sqrt{2\pi\sigma_X{}^2}}\int_{-\infty}^{\infty}\frac{e^{-(x-a_X)^2}}{2\sigma_X{}^2}\,dx = 1$$

Applications: The Gaussian probability density function is the most important density function among all density functions in the field of Science and Engineering. It gives accurate descriptions of many practical random quantities. Especially in Electronics & Communication Systems, the distribution of noise signal exactly matches the Gaussian probability function. It is possible to eliminate noise by knowing its behavior using the Gaussian Probability density function.

2. Uniform Function: The uniform probability density function is defined as

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & other\ wise \end{cases}$$

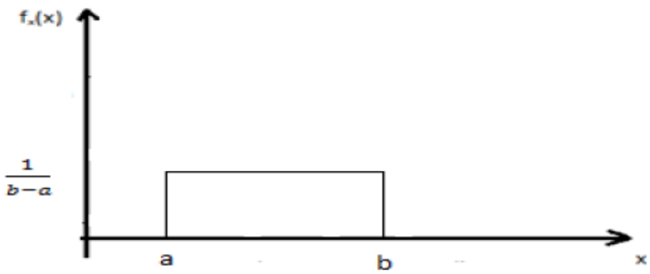
Where 'a' and 'b' are real constants, $-\infty \le a \le \infty$. And b > a. The uniform distribution function is $F_X(x) = \int_a^x f_X(x)dx$.

$F_X(x) = \int_a^x \frac{1}{(b-a)}\,dx = \frac{(x-a)}{(b-a)}$.

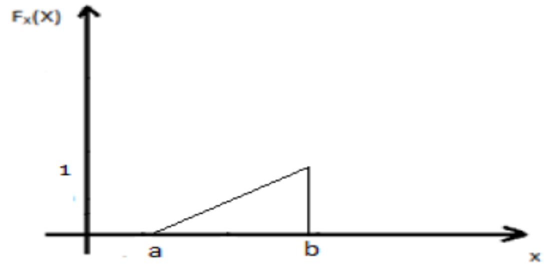
$F_X(a) = 0$.

$F_X(b) = \frac{(b-a)}{(b-a)} = 1$.

$$\text{Therefore } F_X(x) = \begin{cases} \mathbf{0} & \boldsymbol{for\ x} > a \\ (\boldsymbol{x}-\boldsymbol{a})/(\boldsymbol{x}-\boldsymbol{b}) & \boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{b} \\ \mathbf{1} & \boldsymbol{x} > b \end{cases}$$



Density



Distribution

Applications: 1.The random distribution of errors introduced in the round off process is uniformly distributed. 2. In digital communications to round off samples.

3. Exponential function: The exponential probability density function for a continuous random variable, X is defined as

$$f_X(x) = \begin{cases} \frac{1}{b}\, e^{-(x-a)/b} & for\ x > a \\ 0 & for\ x < a \end{cases}$$

Where a and b are real constants , $-\infty \leq a \leq \infty$. And b > 0. The distribution function is

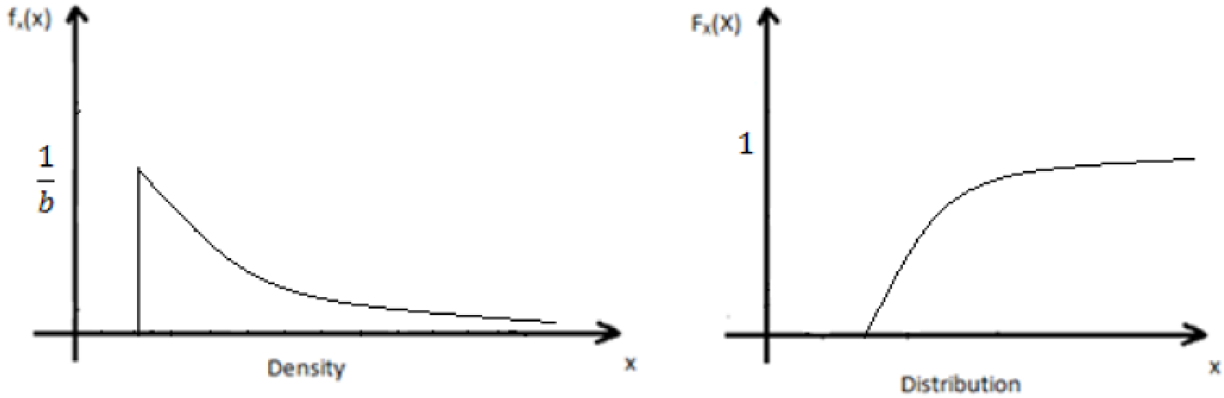
$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx.$$

$$F_X(x) = \int_{a}^{x} \frac{1}{b} e^{\frac{-(x-a)}{b}}\,dx$$

$$F_X(x) = 1 - e^{-(x-a)/b}$$

Therefore

$$F_X(x) = \begin{cases} 0 & for\ x < a \\ 1 - e^{-\frac{x-a}{b}} & for\ x \geq a \\ 1 & for\ x = \infty \end{cases}$$



Applications: 1. The fluctuations in the signal strength received by radar receivers from certain types of targets are exponential. 2. Raindrop sizes, when a large number of rain storm measurements are made, are also exponentially distributed.

4. Rayleigh function: The Rayleigh probability density function of random variable X is defined as

$$f_X(x) = \begin{cases} \frac{2}{b}\ (x-a)e^{-(x-a)^2/b} & for\ x \geq a \\ 0 & for\ x < a \end{cases}$$

Where a and b are real constants

$$F_X(x) = \int_a^x \frac{2}{b}\ (x-a)e^{-(x-a)^2/b} dx$$

Let $\frac{(x-a)^2}{b} = y$

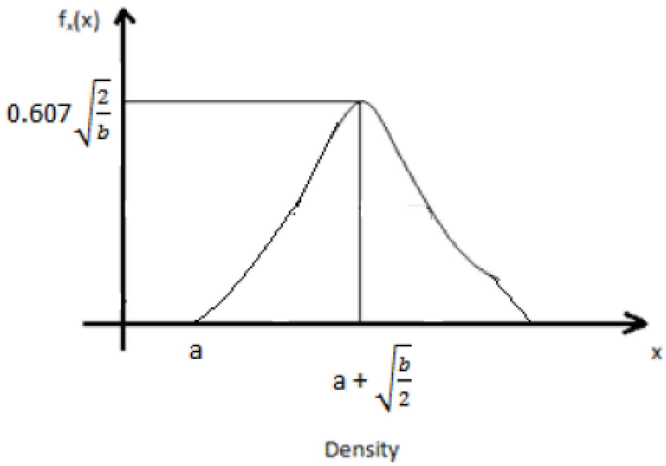
$$\frac{2}{b}\ (x-a)dx = dy$$

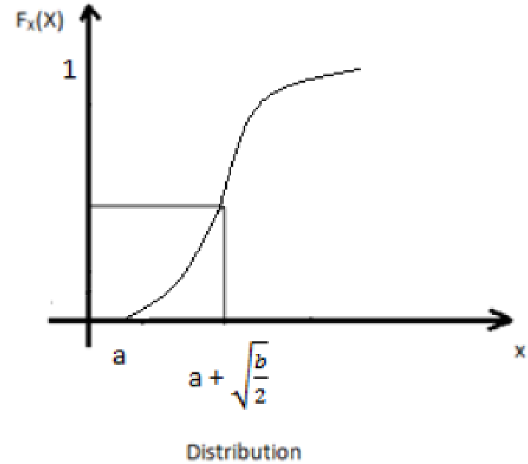
Therefore $F_X(x) = \int_a^x e^{-y} dy = \ -e^{-y}|_a^x$

$F_X(x) = 1 - (e^{-(x-a)^2/b}$

Therefore

$$F_X(x) = \begin{cases} 0 & for\ x < a \\ 1-(e^{-(x-a)^2/b} & for\ x \geq a \\ 1 & for\ x = \infty \end{cases}$$





Applications: 1. It describes the envelope of white noise, when noise is passed through a band pass filter.2. The Rayleigh density function has a relationship with the Gaussian density function.3. Some types of signal fluctuations received by the receiver are modeled as Rayleigh distribution.

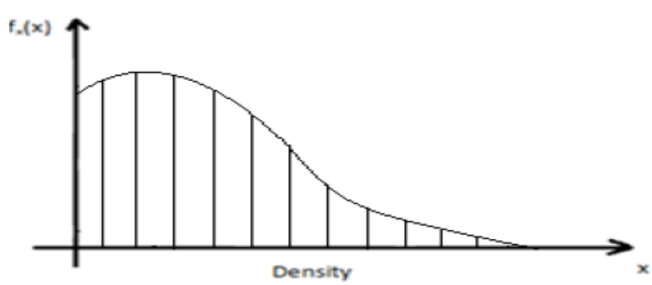
5. Binomial function: Consider an experiment having only two possible outcomes such as one or zero; yes or no: tails or heads etc… If the experiment is repeated for N number of times then the Binomial probability density function of a random variable X is defined as

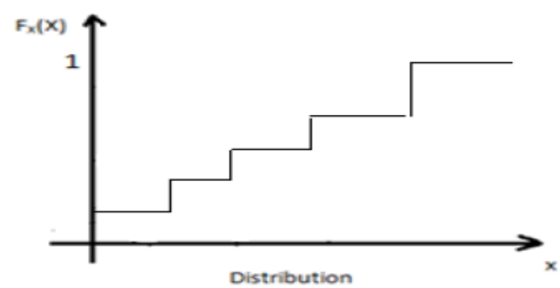
$$f_X(x) = \sum_{K=0}^{N} N_{C_K}\, p^k (1-p)^{N-k} \delta(x-k)$$

$$F_X(x) = \sum_{K=0}^{N} N_{C_K}\, p^k (1-p)^{N-k} u(x-k)$$

Where

$$N_{C_K} = \frac{N!}{(N-k)!k!}$$





Applications: The distribution can be applied to many games of chance, detection problems in radar and sonar and many experiments having only two possible outcomes in any given trial.

6. Poisson's function: Poisson's probability density function of a random variable X is defined as

$$f_X(x) = e^{-b} \sum_{K=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

$$F_X(x) = e^{-b} \sum_{K=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

Poisson's distribution is the approximated function of the Binomial distribution when N→∞ and p→0 . Here the constant b=Np. Poisson's density and distribution plots are similar to Binomial density and distribution plots.

Applications: It is mostly applied to counting type problems. It describes 1. The number of telephone calls made during a period of time. 2. The number of defective elements in a given sample. 3. The number of electrons emitted from a cathode in a time interval.4. The number of items waiting in a queue etc…

Conditional distribution Function: If A and B are two events. If A is an event {X≤x} for random variable X, then the conditional distribution function of X when the event B is known is denoted as FX(x/B) and is defined as

$$F_X(x/B) = P\{X \leq x/B\}.$$

We know that the conditional probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \text{ Then } F_X(x/B) = \frac{p(X \leq x \cap B)}{P(B)}$$

The expression for discrete random variable is

$$F_X(x/B) = \sum_{i=1}^{N} P\left(\frac{x_i}{B}\right) u(x-x_i)$$

The properties of conditional distribution function will be similar to distribution function and are given by

(i) $F_X(-\infty/B) = 0.$

(ii) $F_X(\infty/B) = 1.$

(iii) $0 \leq F_{X(x/B)} \leq 1.$

(iv) $F_X(x_{1/B}) \leq FX(x2/B)$ if $x_1 < x_2.$

(v) $P\{x_1 \leq X \leq x_2/B\} = F_X(x_2/B) - F_X(x_{1/B})$

(vi) $F_X(x^+/B) = F_X(x/B) = F_X(x^-/B)$

Conditional density Function: The conditional density function of a random variable, X is defined as the derivative of the conditional distribution function.

$$f_X(x/B) = \frac{dF_{X}\left(\frac{x}{B}\right)}{dx}$$

For discrete random variable

$$f_X(x/B) = \sum_{i=1}^{N} P\left(\frac{x_i}{B}\right) \delta(x - x_i)$$

The properties of conditional density function are similar to the density function and are given by

(i) $0 \leq f_X(x/B)$ for all x.

(ii) $\int_{-\infty}^{\infty} f_{X(X/B)}\, dx = 1.$

(iii) $F_X(x/B) = \int_{-\infty}^{x} f_{X(X/B)}\, dx.$

(iv) $P\{x_1 \leq X \leq x_2/B\} = \int_{x_1}^{x_2} f_{X(X/B)}\, dx$

UNIT-2: MULTIPLE RANDOM VARIABLES & OPERATIONS

In many practical situations, multiple random variables are required for analysis than a single random variable. The analysis of two random variables especially is very much needed. The theory of two random variables can be extended to multiple random variables.

Joint Probability Distribution Function: Consider two random variables X and Y. And let two events be A{X ≤ x} and B{Y≤y} Then the joint probability distribution function for the joint event {X ≤ x, Y≤y} is defined as FX,Y (x, y) = P{ X ≤ x, Y≤y} = P(A∩B)

For discrete random variables, if X = {x1, x2, x3,…,xn} and Y = {y1, y2, y3,…, ym} with joint probabilities P(xn, ym) = P{X= xn, Y= ym} then the joint probability distribution function is

$$F_{X,Y}(x,y)=\sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)u(x-x_n)\, u(y-y_m)$$

Similarly for N random variables Xn, where n=1, 2, 3 … N the joint distribution function is given as Fx1,x2,x3,…xn (x1,x2,x3,…xn) = P{X1≤ x1, X2≤ x2, X3≤ x3, ... Xn ≤xn}

Properties of Joint Distribution Functions: The properties of a joint distribution function of two random variables X and Y are given as follows.

(1) FX,Y (-∞,-∞) = 0

FX, Y (x,-∞) = 0

FX, Y (-∞, y) = 0

(2) FX,Y (∞,∞) = 1

(3) 0 ≤ FX,Y (x, y) ≤ 1

(4) FX, Y (x, y) is a monotonic non-decreasing function of both x and y.

(5) The probability of the joint event {x1≤ X ≤x2, y1 ≤ Y ≤ y2} is given by

P {x1 ≤ X ≤ x2, y1 ≤ Y ≤ y2} = FX, Y (x2, y2) + FX, Y (x1, y1) - FX, Y (x1, y2) - FX, Y (x2, y1)

(6) The marginal distribution functions are given by FX, Y (x, ∞) = FX (x) and FX, Y (∞, y) = FY (y).

Joint Probability Density Function: The joint probability density function of two random variables X and Y is defined as the second derivative of the joint distribution function. It can be expressed as

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

It is also simply called as joint density function. For discrete random variables X = {x1, x2, x3,…,xn} and Y = {y1, y2, y3,…, ym} the joint density function is

$$f_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\delta(x - x_n)\ \delta(y - y_m)$$

By direct integration, the joint distribution function can be obtained in terms of density as

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\ dx\ dy$$

For N random variables Xn, n=1,2,…N, The joint density function becomes the N-fold partial derivative of the N-dimensional distribution function. That is,

$$f_{X1,X2,X3\ldots\ldots XN}(X_1,X_2,X_3\ldots Xn) = \frac{\partial^2 F_{X1,X2,X3\ldots XN}(x1,X2,X3\ldots\ldots XN)}{\partial x_1 \partial x_2 \partial x_3 \ldots \partial x_n}$$

By direct integration the N-Dimensional distribution function is

$$F_{X1,X2,X3\ldots\ldots XN}(X_1,X_2,X_3\ldots Xn) = \int_{-\infty}^{x1} \int_{-\infty}^{x2} \int_{-\infty}^{x3} \ldots \int_{-\infty}^{xN} f_{X1.X2.X3\ldots XN(x1,x2,x3\ldots xN)} dx_1\ dx_2\ dx_3 \ldots dx_N$$

Properties of Joint Density Function: The properties of a joint density function for two random variables X and Y are given as follows:

(1) $f_{X,Y}(x, y) \geq 0$ A Joint probability density function is always non-negative.

(2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx\ dy = 1$ i.e. the area under the density function curve is always equals to one.

(3) *The joint distribution function is always equals to*

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\ dx\ dy$$

(4) The probability of the joint event $\{x1 \leq X \leq x2,\ y1 \leq Y \leq y2\}$ is given as

$$P\{x_1 \le X \le x_2,\ y_1 \le Y \le y_2\} = \int_{x1}^{x2}\int_{y1}^{y2} f_{X,Y(x,y)}\,dx\,dy$$

(5) The marginal distribution function of X and Y are

$$F_{X(x)} = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\,dx\,dy$$

$$F_{Y(y)} = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y(x,y)}\,dx\,dy$$

(6) The marginal density functions of X and Y are

$$f_{X(x)} = \int_{-\infty}^{\infty} f_{X,Y(x,y)}\,dy$$

$$f_{Y(y)} = \int_{-\infty}^{\infty} f_{X,Y(x,y)}\,dx$$

Conditional Density and Distribution functions:

Point Conditioning: Consider two random variables X and Y. The distribution of random variable X when the distribution function of a random variable Y is known at some value of y is defined as the conditional distribution function of X. It can be expressed as

$$F_X(x/\ Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y(x,y)}\,dx}{f_{Y(y)}}$$

and the conditional density function of X is

$$f_X(x/\ Y=y) = \frac{d}{dx}[F_X(x/\ Y=y)]$$

$$= \frac{\int_{-\infty}^{x}\frac{d}{dx} f_{X,Y(x,y)}}{f_{Y(y)}}$$

$f_X(x/\ Y=y) = \frac{f_{X,Y(x,y)}}{f_{Y(y)}}$ or we can simply write $f_X(x/y) = \frac{f_{X,Y(x,y)}}{f_{Y(y)}}$

Similarly, the conditional density function of Y is

$$f_Y(y/x) = \frac{f_{X,Y(x,y)}}{f_{X(x)}}$$

For discrete random variables, Consider both X and Y are discrete random variables. Then we know that the conditional distribution function of X at a specified value of y_k is given by

$$F_X\left(x/(y-\Delta y<Y<y+\Delta y)\right) = \frac{\sum_{j=y-\Delta y}^{y+\Delta y}\sum_{i=1}^{N} P(x_i,y_j)u(x-x_i)\,u(y-y_j)}{\sum_{j=y-\Delta y}^{y+\Delta y} P(y_j)\,u(y-y_j)}$$

At $y = y_k$, $\Delta y \to 0$

$$F_X(x/Y=y_k) = \sum_{i=1}^{N}\frac{p(x_i,y_j)}{p(y_k)}u(x-x_i)$$

Then the conditional density function of X is

$$f_X(x/Y=y_k) = \sum_{i=1}^{N}\frac{p(x_i,y_j)}{p(y_k)}\delta(x-x_i)$$

Similarly, for random variable Y the conditional distribution function at x = xk is

$$F_Y(y/x_k) = \sum_{i=1}^{N}\frac{p(x_k,y_j)}{p(x_k)}u(y-y_j)$$

And conditional density function is

$$f_Y(y/x_k) = \sum_{i=1}^{N}\frac{p(x_k,y_j)}{p(x_k)}\delta(y-y_j)$$

Interval Conditioning: Consider the event B is defined in the interval y1 ≤ Y ≤ y2 for the random variable Y i.e. B = { y1 ≤ Y ≤ y2}. Assume that P(B) =P(y1 ≤ Y ≤ y2) 0, then the conditional distribution function of x is given by

$$F_X(x/\,y_1 \le Y \le y_2) = \frac{\int_{y1}^{y2}\int_{-\infty}^{x} f_{X,Y}(x,y)\; dx\, dy}{\int_{y1}^{y2} f_Y(y)\, dy}$$

We know that the conditional density function

$$\int_{y1}^{y2} f_Y(y)\, dy = \int_{y1}^{y2}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$$

Or $$F_X(x/\,y_1 \le Y \le y_2) = \frac{\int_{y1}^{y2}\int_{-\infty}^{x} f_{X,Y}(x,y)\; dx\, dy}{\int_{y1}^{y2}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\; dx\, dy}$$

By differentiating we can get the conditional density function of X as

$$f_X(x/\ y_1 \le Y \le y_2) = \frac{\int_{y1}^{y2} f_{Y(y)}\, dy}{\int_{y1}^{y2}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\, dx\, dy}$$

Similarly, the conditional density function of Y for the given interval $x_1 \le X \le x_2$ is

$$f_Y(y/(\ x_1 \le X \le x_2)) = \frac{\int_{x1}^{x2} f_{Y(y)}\, dx}{\int_{x1}^{x2}\int_{-\infty}^{\infty} f_{X,Y(x,y)}\, dx\, dy}$$

Statistical Independence of Random Variables: Consider two random variables X and Y with events A= {X≤ x } and B = {Y ≤ y} for two real numbers x and y. The two random variables are said to be statistically independent if and only if the joint probability is equal to the product of the individual probabilities.

P {X≤x ,Y ≤ y} P {X≤x } P {Y ≤ y} Also the joint distribution function is

$$F_{X,Y(x,y)} = F_{X(x)} F_{Y(y)}$$

And the joint density function is

$$f_{X,Y(x,y)} = f_{X(x)} f_{Y(y)}$$

These functions give the condition for two random variables X and Y to be statistically independent.

The conditional distribution functions for independent random variables are given by

$$F_X(x/\ Y=y) = F_X(x/\ y) = \frac{F_{X,Y(x,y)}}{F_{Y(y)}} = \frac{F_{X(x)} F_{Y(y)}}{F_{Y(y)}}$$

Therefore $F_X(x/\ y) = F_X(x)$

Also $F_Y(y/\ x) = F_Y(\ y)$

Similarly, the conditional density functions for independent random variables are

$$f_X(x/\ y) = f_{X(x)}$$

$$f_Y(y/\ x) = f_{Y(y)}$$

Hence the conditions on density functions do not affect independent random variables.

Sum of two Random Variables: The summation of multiple random variables has much practical importance when information signals are transmitted through channels in a communication system. The resultant signal available at the receiver is the algebraic sum of the information and the noise signals generated by multiple noise sources. The sum of two independent random variables X and Y available at the receiver is W =X+Y

If F_X (x) and F_Y (y) are the distribution functions of X and Y respectively, then the probability distribution function of W is given as F_W (w) =P {W≤w }= P {X+Y≤w }.Then the distribution function is

$$F_{W(w)}=\int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{X,Y(x,y)}\,dx\,dy$$

Since X and Y are independent random variables,

$$f_{X,Y(x,y)}= f_{X(x)} f_{Y(y)}$$

Therefore

$$F_{W(w)}=\int_{-\infty}^{\infty} f_{Y(y)} \int_{-\infty}^{w-y} f_{X(x)}\,dx\,dy$$

Differentiating using Leibniz rule, the density function is

$$f_{W(w)} =\frac{dF_{W(w)}}{dw}=\int_{-\infty}^{\infty} f_{Y(y)}\frac{d}{dw}\int_{-\infty}^{w-y} f_{X(x)}\,dx\,dy$$

$$f_{W(w)} = \int_{-\infty}^{\infty} f_{Y(y)}\ f_{X(w-y)}\ dy$$

Similarly it can be written as

$$f_{W(w)} = \int_{-\infty}^{\infty} f_{X(x)}\ f_{Y(w-x)}\ dx$$

This expression is known as the convolution integral. It can be expressed as

$$f_{W(w)} = f_{X(x)} * f_{Y(y)}$$

Hence the density function of the sum of two statistically independent random variables is equal to the convolution of their individual density functions.

Sum of several Random Variables: Consider that there are N statistically independent random variables then the sum of N random variables is given by W=X1+X2+X3+…+XN.

Then the probability density function of W is equal to the convolution of all the individual density functions. This is given as

$$f_{W}(w) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) * \dots * f_{X_N}(x_N)$$

Central Limit Theorem: It states that the probability function of a sum of N independent random variables approaches the Gaussian density function as N tends to infinity. In practice, whenever an observed random variable is known to be a sum of large number of random variables, according to the central limiting theorem, we can assume that this sum is Gaussian random variable.

Equal Functions: Let N random variables have the same distribution and density functions. And Let Y=X1+X2+X3+…+XN. Also let W be normalized random variable

$$W = \frac{Y-\bar{Y}}{\sigma_Y} \text{ Where } Y=\sum_{n=1}^{N} X_n \text{ , } \bar{Y} = \sum_{n=1}^{N} \overline{X_n} \text{ and } {\sigma_Y}^2 = \sum_{n=1}^{N} {\sigma_{X_n}}^2$$

So

$$W = \frac{\sum_{n=1}^{N} X_n - \sum_{n=1}^{N} \overline{X_n}}{\left[\sum_{n=1}^{N} {\sigma_{X_n}}^2\right]^{1/2}}$$

Since all random variables have same distribution

$${\sigma_{X_n}}^2 = {\sigma_X}^2, \ \left[\sum_{n=1}^{N} {\sigma_{X_n}}^2\right]^{\frac{1}{2}} = \sqrt{{\sigma_X}^2} = \sqrt{N}\ \sigma_X \text{ and } \overline{X_n} = \bar{X}$$

Therefore

$$W = \frac{1}{\sqrt{N}\,\sigma_X} \sum_{n=1}^{N} (X_n - \bar{X})$$

Then W is Gaussian random variable.

Unequal Functions: Let N random variables have probability density functions, with mean and variance. The central limit theorem states that the sum of the random variables W=X1+X2+X3+…+XN have a probability distribution function which approaches a Gaussian distribution as N tends to infinity.

Introduction: In this Part of Unit we will see the concepts of expectation such as mean, variance, moments, characteristic function, Moment generating function on Multiple Random variables. We are already familiar with same operations on Single Random variable. This can be used as basic for our topics we are going to see on multiple random variables.

Function of joint random variables: If g(x,y) is a function of two random variables X and Y with joint density function fx,y(x,y) then the expected value of the function g(x,y) is given as

$$\overline{g} = E[g(x,y)]$$

$$\overline{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

Similarly, for N Random variables X1, X2, . . . XN With joint density function fx1,x2, . . . Xn(x1,x2, . . . xn), the expected value of the function g(x1,x2, . . . xn) is given as

$$\overline{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} . . . \int_{-\infty}^{\infty} g(x_1, x_2, . . . x_N)\, f_{x_1, x_2, . . . x_N}(x_1, x_2, . . . x_N)\; dx_1 dx_2 . . . dx_N$$

Joint Moments about Origin: The joint moments about the origin for two random variables, X, Y is the expected value of the function g(X,Y) =E(X^n,Y^k) and is denoted as mnk.. Mathematically,

$$m_{nk} = E[X^n\, Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\,dx\,dy$$

Where n and k are positive integers. The sum n+k is called the order of the moments. If k=0, then

$$m_{10} = E[X] = \bar{X} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy$$

$$m_{01} = E[Y] = \bar{Y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy$$

The second order moments are m20= E[X2] ,m02= E[Y2] and m11 = E[XY]

For N random variables X1, X2, . . . XN, the joint moments about the origin is defined as

$$m_{n1,n2, . . . nN} = E[X_1^{\,n_1}, X_2^{\,n_2}, . . . X_N^{\,n_N}]$$
$$\int_{-\infty}^{\infty} . . . \int_{-\infty}^{\infty} X_1^{\,n_1}, X_2^{\,n_2}, . . . X_N^{\,n_N} f_{x_1, x_2, . . . x_N}(x_1, x_2, . . . x_N)\; dx_1 dx_2 . . . dx_N$$

Where n1,n2, . . . nN are all positive integers.

Correlation: Consider the two random variables X and Y, the second order joint moment m11 is called the Correlation of X and Y. I t is denoted as RXY. RXY = m11 = E [XY] =

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y^{(x,y)dx\,dy}}$$

For discrete random variables

$$\mathrm{R_{XY}} = \sum_{n=1}^{N}\sum_{m=1}^{M} x_n\ y_m\ P_{XY}{}^{(x_n,y_m)}$$

Properties of Correlation:

1. If two random variables X and Y are statistically independent then X and Y are said to be uncorrelated. That is RXY = E[XY]= E[X] E[Y].

Proof: Consider two random variables, X and Y with joint density function fx,y(x,y)and marginal density functions fx(x) and fy(y). If X and Y are statistically independent, then we know that fx,y(x,y) = fx(x) fy(y).

The correlation is

$$\mathrm{R_{XY}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{x,y^{(X,Y)}}\ \mathrm{dx\ dy.}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{x^{(X)}} f_{y^{(Y)}}\ \mathrm{dx\ dy.}$$

$$=\int_{-\infty}^{\infty} x\ f_{X^{(X)}}\ \mathrm{dx}\ \ \int_{-\infty}^{\infty} y\ f_{y^{(Y)}}\ \mathrm{dy.}$$

$$\mathrm{R_{XY} = E[XY]= E[X]\ E[Y].}$$

2. If the Random variables X and Y are orthogonal then their correlation is zero. i.e. RXY = 0.

Proof: Consider two Random variables X and Y with density functions fx(x) and fy(y). If X and Y are said to be orthogonal, their joint occurrence is zero. That is fx,y(x,y)=0. Therefore the correlation is

$$\mathrm{R_{XY}} = \mathrm{E[XY]} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{x,y^{(X,Y)}}\ \mathrm{dx\ dy} = 0.$$

Joint central moments: Consider two random variables X and Y. Then the expected values of the function g(x,y)=$(x-\overline{X})^n(y-\overline{Y})^k$ are called joint central moments. Mathematically μ_{nk} =E[$(x-\overline{X})^n(y-\overline{Y})^k$]

= $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\overline{X})^n(y-\overline{Y})^k f_{x,y(X,Y)}$ dx dy =0. Where n, k are positive integers0,1,2,… The order of the central moment is n+k. The 0th Order central moment is μ_{00} = E[1]=1. The first order central moments are μ_{10} =E[x-$\bar{X}$] =E[$\bar{X}$] -E[$\bar{X}$]=0and μ_{01} =E[y-$\bar{Y}$] =E[$\bar{Y}$] -E[$\bar{Y}$]=0. The second order central moments are

μ_{20} =E[$(x-\overline{X})^2$]=σ_{X^2}, μ_{02} =E[$(y-\overline{Y})^2$]=σ_{Y^2} and μ_{11} =E[$(x-\overline{X})^1(y-\overline{Y})^1$]= σ_{XY}

For N random Variables X_1, X_2, . . . X_N , the joint central moments are defined as $\mu_{n1,n2,\ldots nN} = E[(x_1-\overline{X_1})^{n_1}(x_2-\overline{X_2})^{n_2}\ldots(x_N-\overline{X_N})^{n_N}]$

$$= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}(x_1-\overline{X_1})^{n_1}(x_2-\overline{X_2})^{n_2}\ldots(x_N- XN)\, nN f x1,x2,\ldots xN(x1,x2,\ldots xN)\, dx1dx2\ldots dxN$$

The order of the joint central moment $n_1+n_2+\ldots+n_N$.

Covariance: Consider the random variables X and Y. The second order joint central moment μ_{11} is called the covariance of X and Y. It is expressed as C_{XY}= σ_{XY} = μ_{11}=E[x-$\bar{X}$] E[y-$\bar{Y}$]

C_{XY} = $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\overline{X})^1(y-\overline{Y})^1 f_{x,y(X,Y)}$ dx dy

For discrete random variables X and Y, C_{XY} = $\sum_{n=1}^{N}\sum_{k=1}^{K}(x_n-\overline{X_n})^1(y_k-\overline{Y_k})^1 P(x_n,y_k)$

Correlation coefficient: For the random variables X and Y, the normalized second order Central moment is called the correlation coefficient It is denoted as ρ and is given by

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{XY}}{\sqrt{\sigma_X{}^2\sigma_Y{}^2}} = \frac{C_{XY}}{\sigma_X\sigma_Y} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{E[x-\bar{X}]\, E[y-\bar{Y}]}{\sigma_X\sigma_Y}.$$

Properties of ρ: 1. The range of correlation coefficient is $-1 \leq \rho \leq 1$.

2. If X and Y are independent then ρ=0.

3. If the correlation between X and Y is perfect then $\rho \pm 1$.

4. If X=Y, then ρ=1.

Properties of Covariance:

1. If X and Y are two random variables, then the covariance is

$$C_{XY} = R_{XY} - \bar{X}\,\bar{Y}$$

Proof: If X and Y are two random variables, We know that

$$C_{XY} = E[x-\bar{X}]\, E[y-\bar{Y}]$$

$$= E[XY - \bar{X}Y - \bar{Y}X - \bar{X}\bar{Y}]$$

$$= E[XY] - E[\bar{X}Y] - E[\bar{Y}X] - E[\bar{X}\bar{Y}]$$

$$= E[XY] - \bar{X}E[Y] - \bar{Y}E[X] - \bar{X}\bar{Y}E[1]$$

$$= E[XY] - \bar{X}\,\bar{Y} - \bar{Y}\,\bar{X} + \bar{X}\,\bar{Y}$$

$$= E[XY] - \bar{X}\,\bar{Y}$$

2. If two random variables X and Y are independent, then the covariance is zero. i.e. CXY = 0. But the converse is not true.

Proof: Consider two random variables X and Y. If X and Y are independent, We know that E[XY]=E[X]E[Y] and the covariance of X and Y is

$C_{XY}=R_{XY}-\bar{X}\,\bar{Y}$

$= E[XY]-\bar{X}\,\bar{Y}$

$= E[X]\,E[Y]-\bar{X}\,\bar{Y}$

$= C_{XY}=\bar{X}\,\bar{Y}-\bar{X}\,\bar{Y}=0.$

3. If X and Y are two random variables, Var(X+Y) = Var(X) + Var(Y) + 2 CXY.

Proof: If X and Y are two random variables, We know that $Var(X)=\sigma_X{}^2 = E[X^2]-E[X]^2$

Then $Var(X+Y) = E[(X+Y)^2]-(E[X+Y])^2$

$= E[X^2+Y^2+2XY]-(E[X]+E[Y])^2$

$= E[X^2]+E[Y^2]+2E[XY]-E[X]^2-E[Y]^2-2E[X]E[Y]$

$= E[X^2]-E[X]^2+E[Y^2]-E[Y]^2+2(E[XY]-E[X]E[Y])$

$= \sigma_X{}^2+\sigma_Y{}^2+2\,C_{XY}.$

Therefore $Var(X+Y) = Var(X) + Var(Y) + 2\,C_{XY}$. hence proved.

4. If X and Y are two random variables, then the covariance of X+a,Y+b, Where ‘a’and ‘b’ are constants is Cov (X+a,Y+b) = Cov (X,Y) = CXY.

Proof: If X and Y are two random variables, Then

$Cov(X+a,Y+b)=E[((X+a)-(\overline{X+a})\,(Y+b)-\overline{Y+b})]$

$= E[(X+a-\bar{X}-a)(Y+b-\bar{Y}-b)]$

$= E[(X-\bar{X})(Y-\bar{Y})]$

Therefore $Cov(X+a,Y+b) = Cov(X,Y) = C_{XY}$. hence proved.

5. If X and Y are two random variables, then the covariance of aX,bY, Where 'a'and 'b' are constants is Cov (aX,bY) = abCov (X,Y) = abCXY.

<u>Proof:</u> Proof: If X and Y are two random variables, Then

Cov(aX,bY)=E[((aX)-($\overline{aX}$) (bY-$\overline{bY}$)]

= E[a(X -$\bar{X}$)b(Y-$\bar{Y}$)]

= E[ab(X -$\bar{X}$)(Y-$\bar{Y}$)]

Therefore Cov (aX,bY) = abCov (X,Y) = abC_{XY}. hence proved.

6. If X, Y and Z are three random variables, then Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z).

<u>Proof:</u> We know that Cov(X+Y,Z)=E[((X+Y)-($\overline{X+Y}$) (Z-$\bar{Z}$)]

= E[X+Y- $\bar{X}-\bar{Y}$ (Z-$\bar{Z}$)]

= E[((X $-\overline{X}$) + (Y $-\overline{Y}$)) (Z-$\bar{Z}$)]

= E[(X $-\overline{X}$)(Z $-\bar{Z}$)] + E[(Y $-\overline{Y}$) (Z-$\bar{Z}$)]

Therefore Cov (X+Y,Z) = Cov (X,Z) + Cov (Y,Z). hence proved.

<u>Joint characteristic Function:</u> The joint characteristic function of two random variables X and Y is defined as the expected value of the joint function g(x,y)=$e^{j\omega 1X}e^{j\omega 2Y}$. It can be expressed as $\Phi_{X,Y(\omega 1,\omega 2)}$ = E[$e^{j\omega 1X}e^{j\omega 2Y}$.]= $e^{j\omega 1X+j\omega 2Y}$. Where $\omega 1$ *and* $\omega 2$ *are* real variables.

Therefore $\Phi_{X,Y(\omega 1,\omega 2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega 1X+j\omega 2Y} f_{x,y(X,Y)}$ dx dy.

This is known as the two dimensional Fourier transform with signs of $\omega 1$ *and* $\omega 2$ are reversed for the joint density function. So the inverse Fourier transform of the joint characteristic function gives the joint density function again the signs of $\omega 1$ *and* $\omega 2$ are reversed. i.e. The joint density function is $f_{x,y(X,Y)}= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{X,Y(\omega 1,\omega 2)} e^{-(j\omega 1X+j\omega 2Y)}$ dω1 dω2.

Joint Moment Generating Function: the joint moment generating function of two random variables X and Y is defined as the expected value of the joint function g(x,y)=$e^{\vartheta 1X}e^{\vartheta 2Y}$.It can be expressed as

$M_{X,Y(\vartheta 1,\vartheta 2)}$ = E[$e^{\vartheta 1X}e^{\vartheta 2Y}$]= $e^{\vartheta 1X+\vartheta 2Y}$. Where $\vartheta 1$ *and* $\vartheta 2$ *are* real variables.

Therefore $M_{X,Y(\vartheta 1,\vartheta 2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\vartheta 1X+\vartheta 2Y} f_{x,y(X,Y)}$ dx dy.

And the joint density function is

$f_{x,y(X,Y)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M_{X,Y(\vartheta 1,\vartheta 2)} e^{-(\vartheta 1X+\vartheta 2Y)}\, d\vartheta 1\, d\vartheta 2$.

Gaussian Random Variables:

(2 Random variables): If two random variables X and Y are said to be jointly Gaussian, then the joint density function is given as

$$f_{x,y(X,Y)}=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\{\frac{-1}{2(1-\rho^2)}[\frac{(x-\overline{X})^2}{\sigma_X{}^2}-\frac{2\rho((X-\bar{X})(Y-\bar{Y})}{\sigma_X\sigma_Y}+\frac{(y-\overline{Y})^2}{\sigma_Y{}^2}]\}$$

This is also called as bivariate Gaussian density function.

N Random variables: Consider N random variables X_n, n=1,2, . . . N. They are said to be jointly Gaussian if their joint density function(N variate density function) is given by

$$f_{x_1,x_2,..x_N}(x_1,x_2,..x_N) = \frac{1}{(2\pi)^{N/2}|C_X|^{1/2}}\ \exp\ \{\frac{-[X-\bar{X}]^t[C_X]^{-1}[X-\bar{X}]}{2}\}$$

Where the covariance matrix of N random variables is

$$[C_X] = \begin{bmatrix} C_{11} & C_{12}\cdots & C_{1N} \\ C_{21} & C_{22}\cdots & C_{2N} \\ \vdots & & \vdots \\ C_{N1} & C_{N2}\cdots & C_{NN} \end{bmatrix},\ [X-\bar{X}] = \begin{bmatrix} X_1-\bar{X}_1 \\ X_2-\bar{X}_2 \\ \vdots \\ X_N-\bar{X}_{1N} \end{bmatrix}$$

$[X-\bar{X}]^t$ = transpose of $[X-\bar{X}]$

$|C_X|$= determinant of $[C_X]$

And $[[C_X]^{-1}]$ = inverse of $[C_X]$.

The joint density function for two Gaussian random variables X_1 and X_2 can be derived by substituting N=2 in the formula of N Random variables case.

Properties of Gaussian Random Variables:

1. The Gaussian random variables are completely defined by their means, variances and covariances.
2. If the Gaussian random variables are uncorrelated, then they are statistically independent.
3. All marginal density functions derived from N-variate Gaussian density functions are Gaussian.
4. All conditional density functions are also Gaussian.
5. All linear transformations of Gaussian random variables are also Gaussian.

Linear Transformations of Gaussian Random variables: Consider N Gaussian random variables Yn, n=1,2, . . .N. having a linear transformation with set of N Gaussian random variables Xn, n=1,2, . . .N. The linear transformations can be written as

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1N} \\ a_{21} & a_{22} \cdots & a_{2N} \\ \vdots & & \vdots \\ a_{N1} & a_{N2} \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

The transformation T is

$$[T] = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1N} \\ a_{21} & a_{22} \cdots & a_{2N} \\ \vdots & & \vdots \\ a_{N1} & a_{N2} \cdots & a_{NN} \end{bmatrix}$$

Therefore [Y]=[T] [X]. Also with mean values of X and Y. $[Y-\bar{Y}] = [T]\,[X-\bar{X}]$.

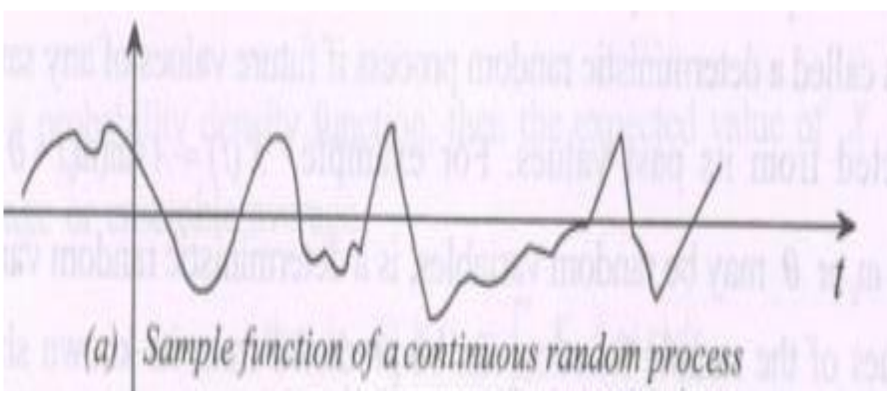
And $[X-\bar{X}]=[T]^{-1}\,[Y-\bar{Y}]$.

UNIT-3: RANDOM PROCESSES: TEMPORAL CHARACTERISTICS

The random processes are also called as stochastic processes which deal with randomly varying time wave forms such as any message signals and noise. They are described statistically since the complete knowledge about their origin is not known. So statistical measures are used. Probability distribution and probability density functions give the complete statistical characteristics of random signals. A random process is a function of both sample space and time variables. And can be represented as {X x(s,t)}.

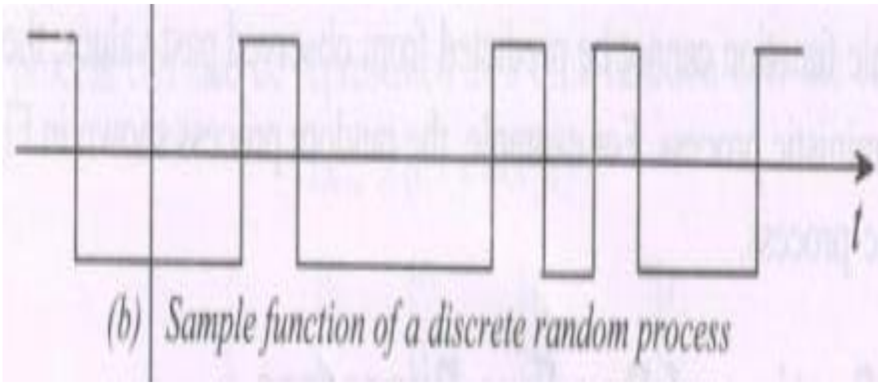
Deterministic and Non-deterministic processes: In general a random process may be deterministic or non deterministic. A process is called as deterministic random process if future values of any sample function can be predicted from its past values. For example, X(t) = A sin (ω0t+Θ), where the parameters A, ω0 and Θ may be random variables, is deterministic random process because the future values of the sample function can be detected from its known shape. If future values of a sample function cannot be detected from observed past values, the process is called non-deterministic process.

Classification of random process: Random processes are mainly classified into four types based on the time and random variable X as follows. 1. Continuous Random Process: A random process is said to be continuous if both the random variable X and time t are continuous. The below figure shows a continuous random process. The fluctuations of noise voltage in any network is a continuous random process.

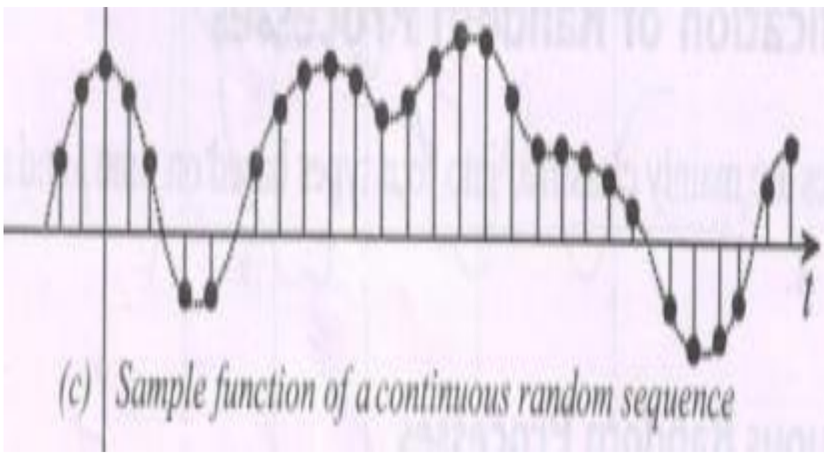


(a) Sample function of a continuous random process

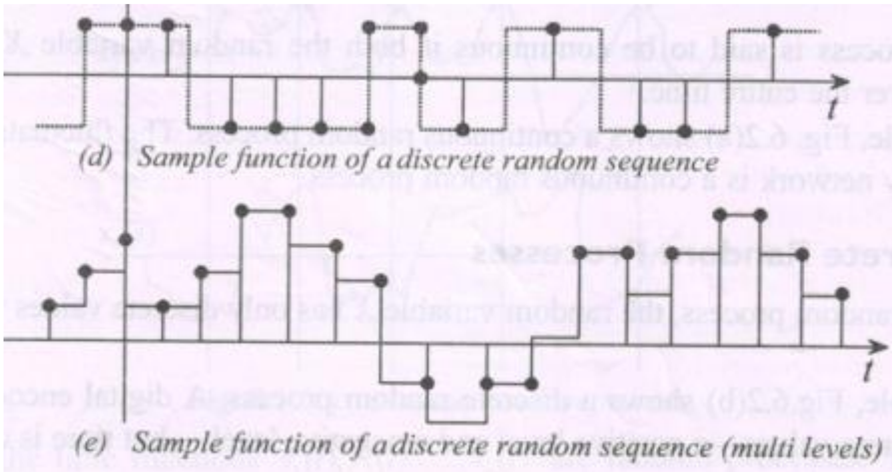
2. Discrete Random Process: In discrete random process, the random variable X has only discrete values while time, t is continuous. The below figure shows a discrete random process. A digital encoded signal has only two discrete values a positive level and a negative level but time is continuous. So it is a discrete random process.



3. Continuous Random Sequence: A random process for which the random variable X is continuous but t has discrete values is called continuous random sequence. A continuous random signal is defined only at discrete (sample) time intervals. It is also called as a discrete time random process and can be represented as a set of random variables {X(t)} for samples tk, k=0, 1, 2,….



4. Discrete Random Sequence: In discrete random sequence both random variable X and time t are discrete. It can be obtained by sampling and quantizing a random signal. This is called the random process and is mostly used in digital signal processing applications. The amplitude of the sequence can be quantized into two levels or multi levels as shown in below figure s (d) and (e).



(d) Sample function of a discrete random sequence

(e) Sample function of a discrete random sequence (multi levels)

Joint distribution functions of random process: Consider a random process X(t). For a single random variable at time t1, X1=X(t1), The cumulative distribution function is defined as FX(x1;t1) = P {(X(t1) x1} where x1 is any real number. The function FX(x1;t1) is known as the first order distribution function of X(t). For two random variables at time instants t1 and t2 X(t1) = X1 and X(t2) = X2, the joint distribution is called the second order joint distribution function of the random process X(t) and is given by FX(x1, x2 ; t1, t2) = P {(X(t1)≤ x1, X(t2)≤ x2}. In general for N random variables at N time intervals X(ti) = Xi i=1,2,…N, the Nth order joint distribution function of X(t) is defined as FX(x1, x2…… xN ; t1, t2,….. tN) = P {(X(t1)≤ x1, X(t2) ≤x2,…. X(tN)≤ xN}.

Joint density functions of random process:: Joint density functions of a random process can be obtained from the derivatives of the distribution functions.

1. First order density function: $f_X(x_1;t_1) = \frac{dF_X(x1;t1)}{dx_1}$
2. Second order density function: $f_X(x_{1,}\ x_2\ ;\ t_{1,}\ t_2) = \frac{\partial^2 F_X(x1,x2\ ;\ t1,t2)}{\partial x_1 \partial x_2}$
3. N^{th} order density function: $f_X(x_1,\ x_2 \ldots\ x_N\ ;\ t_1,\ t_2 \ldots\ t_N) = \frac{\partial^2 F_X(x1,x2 \ldots\ xN\ ;\ t1,t2,\ldots tN)}{\partial x_1 \partial x_2 \ldots \partial x_N}$

Independent random processes: Consider a random process X(t). Let X(ti) = xi, i= 1,2,…N be N Random variables defined at time constants t1,t2, … t N with density functions fX(x1;t1), fX(x2;t2), … fX(xN ; tN). If the random process X(t) is statistically independent, then the Nth order joint density function is equal to the product of individual joint functions of X(t) i.e. fX(x1, x2…… xN ; t1, t2,….. tN)

= fX(x1;t1) fX(x2;t2). . . . fX(xN ; tN). Similarly the joint distribution will be the product of the individual distribution functions.

Statistical properties of Random Processes: The following are the statistical properties of random processes.

1. **Mean:** The mean value of a random process X(t) is equal to the expected value of the random process i.e. $\bar{X}(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_x(x;t)dx$
2. **Autocorrelation:** Consider random process X(t). Let X_1 and X_2 be two random variables defined at times t_1 and t_2 respectively with joint density function $f_X(x_1, x_2 ; t_1, t_2)$. The correlation of X_1 and X_2, $E[X_1 X_2] = E[X(t_1) X(t_2)]$ is called the autocorrelation function of the random process X(t) defined as
   $R_{XX}(t_1,t_2) = E[X_1 X_2] = E[X(t_1) X(t_2)]$ or

   $$R_{XX}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_x(x1, x2 ;\ t1, t2)\, dx_1 dx_2$$

3. **Cross correlation:** Consider two random processes X(t) and Y(t) defined with random variables X and Y at time instants t_1 and t_2 respectively. The joint density function is $f_{xy}(x,y ; t_1,t_2)$.Then the correlation of X and Y, $E[XY] = E[X(t_1) Y(t_2)]$ is called the cross correlation function of the random processes X(t) and Y(t) which is defined as
   $R_{XY}(t_1,t_2) = E[X Y] = E[X(t_1) Y(t_2)]$ or

   $$R_{XY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{x,y}(x, y ;\ t1, t2)\, dxdy$$

Stationary Processes: A random process is said to be stationary if all its statistical properties such as mean, moments, variances etc… do not change with time. The stationarity which depends on the density functions has different levels or orders.

1. **First order stationary process:** A random process is said to be stationary to order one or first order stationary if its first order density function does not change with time or shift in time value. If X(t) is a first order stationary process then fX(x1;t1) = fX(x1;t1+Δt) for any time t1. Where Δt is shift in time value. Therefore the condition for a process to be a first order stationary random process is that its mean value must be constant at any time instant. i.e. E[X(t)] = constant.
2. **Second order stationary process:** A random process is said to be stationary to order two or second order stationary if its second order joint density function does not change with time or shift in

time value i.e. $f_X(x_1, x_2 ; t_1, t_2) = f_X(x_1, x_2; t_1+\Delta t, t_2+\Delta t)$ for all t1,t2 and Δt. It is a function of time difference (t2, t1) and not absolute time t. Note that a second order stationary process is also a first order stationary process. The condition for a process to be a second order stationary is that its autocorrelation should depend only on time differences and not on absolute time. i.e. If $R_{XX}(t_1,t_2) = E[X(t_1)\ X(t_2)]$ is autocorrelation function and $\tau = t_2 - t_1$ then $R_{XX}(t_1,t_1+\tau) = E[X(t_1)\ X(t_1+\tau)] = R_{XX}(\tau)$. $R_{XX}(\tau)$ should be independent of time t.

3. **Wide sense stationary (WSS) process:** If a random process X(t) is a second order stationary process, then it is called a wide sense stationary (WSS) or a weak sense stationary process. However the converse is not true. The condition for a wide sense stationary process are 1. E[X(t)] = constant. 2. $E[X(t)\ X(t+\tau)] = R_{XX}(\tau)$ is independent of absolute time t. Joint wide sense stationary process: Consider two random processes X(t) and Y(t). If they are jointly WSS, then the cross correlation function of X(t) and Y(t) is a function of time difference $\tau = t_2 - t_1$only and not absolute time. i.e. $R_{XY}(t_1,t_2) = E[X(t_1)\ Y(t_2)]$. If $\tau = t_2 - t_1$ then $R_{XY}(t,t+\tau) = E[X(t)\ Y(t+\tau)] = R_{XY}(\tau)$. Therefore the conditions for a process to be joint wide sense stationary are 1. E[X(t)] = Constant. 2. E[Y(t)] = Constant 3. $E[X(t)\ Y(t+\tau)] = R_{XY}(\tau)$ is independent of time t.

4. **Strict sense stationary (SSS) processes:** A random process X(t) is said to be strict Sense stationary if its Nth order joint density function does not change with time or shift in time value. i.e. $f_X(x_1, x_2 \ldots\ldots x_N ; t_1, t_2,\ldots.. t_N) = f_X(x_1, x_2\ldots\ldots x_N ; t_1+\Delta t, t_2+\Delta t, . . . t_N+\Delta t)$ for all t1, t2 . . . tN and Δt. A process that is stationary to all orders n=1,2,. . . N is called strict sense stationary process. Note that SSS process is also a WSS process. But the reverse is not true.

Time Average Function: Consider a random process X(t). Let x(t) be a sample function which exists for all time at a fixed value in the given sample space S. The average value of x(t) taken over all times is called the time average of x(t). It is also called mean value of x(t). It can be expressed as $\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$.

Time autocorrelation function: Consider a random process X(t). The time average of the product X(t) and X(t+ τ) is called time average autocorrelation function of x(t) and is denoted as $\mathcal{R}_{xx}(\tau) = A[X(t)\ X(t+\tau)]$ or $\mathcal{R}_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$.

Time mean square function: If τ = 0, the time average of $x^2(t)$ is called time mean square value of x(t) defined as $= A[X^2(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^2(t)dt$.

Time cross correlation function: Let X(t) and Y(t) be two random processes with sample functions x(t) and y(t) respectively. The time average of the product of x(t) y(t+ τ) is called time cross correlation function of x(t) and y(t). Denoted as

$\mathcal{R}_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt$.

Ergodic Theorem and Ergodic Process: The Ergodic theorem states that for any random process $X(t)$, all time averages of sample functions of $x(t)$ are equal to the corresponding statistical or ensemble averages of $X(t)$. i.e. $\overline{x} = \overline{X}$ or $R_{xx}(\tau) = R_{XX}(\tau)$. Random processes that satisfy the Ergodic theorem are called Ergodic processes.

Joint Ergodic Process: Let $X(t)$ and $Y(t)$ be two random processes with sample functions $x(t)$ and $y(t)$ respectively. The two random processes are said to be jointly Ergodic if they are individually Ergodic and their time cross correlation functions are equal to their respective statistical cross correlation functions. i.e. $\overline{x}=\overline{X}$ $\overline{y}=\overline{Y}$ 2. $R_{xx}(\tau) = R_{XX}(\tau)$, $R_{xy}(\tau) = R_{XY}(\tau)$ and $R_{yy}(\tau) = R_{YY}(\tau)$.

Mean Ergodic Random Process: A random process $X(t)$ is said to be mean Ergodic if time average of any sample function $x(t)$ is equal to its statistical average, which is constant and the probability of all other sample functions is equal to one. i.e. $E[X(t)] = \overline{X} = A[x(t)] = \overline{x}$ with probability one for all $x(t)$.

Autocorrelation Ergodic Process: A stationary random process $X(t)$ is said to be Autocorrelation Ergodic if and only if the time autocorrelation function of any sample function $x(t)$ is equal to the statistical autocorrelation function of $X(t)$. i.e. $A[x(t)\,x(t+\tau)] = E[X(t)\,X(t+\tau)]$ or $R_{xx}(\tau) = R_{XX}(\tau)$.

Cross Correlation Ergodic Process: Two stationary random processes $X(t)$ and $Y(t)$ are said to be cross correlation Ergodic if and only if its time cross correlation function of sample functions $x(t)$ and $y(t)$ is equal to the statistical cross correlation function of $X(t)$ and $Y(t)$. i.e. $A[x(t)\,y(t+\tau)] = E[X(t)\,Y(t+\tau)]$ or $R_{xy}(\tau) = R_{XY}(\tau)$.

Properties of Autocorrelation function: Consider that a random process $X(t)$ is at least WSS and is a function of time difference $\tau = t_2 - t_1$. Then the following are the properties of the autocorrelation function of $X(t)$.

1. Mean square value of X(t) is $E[X^2(t)] = R_{XX}(0)$. It is equal to the power (average) of the process, X(t).
   Proof: We know that for X(t), $R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$. If $\tau = 0$, then $R_{XX}(0) = E[X(t)\, X(t)] = E[X^2(t)]$ hence proved.
2. Autocorrelation function is maximum at the origin i.e. $|R_{XX}(\tau)| \leq R_{XX}(0)$.
   Proof: Consider two random variables $X(t_1)$ and $X(t_2)$ of X(t) defined at time intervals t_1 and t_2 respectively. Consider a positive quantity $[X(t_1) \pm X(t_2)]^2 \geq 0$
   Taking Expectation on both sides, we get $E[X(t_1) \pm X(t_2)]^2 \geq 0$
   $E[X^2(t_1)+ X^2(t_2) \pm 2X(t_1)\, X(t_2)] \geq 0$
   $E[X^2(t_1)]+ E[X^2(t_2) \pm 2E[X(t_1)\, X(t_2)] \geq 0$
   $R_{XX}(0)+ R_{XX}(0)\pm 2\, R_{XX}(t_1,t_2) \geq 0$ [Since $E[X^2(t)] = R_{XX}(0)$]
   Given X(t) is WSS and $\tau = t_2-t_1$.
   Therefore $2\, R_{XX}(0\pm 2\, R_{XX}(\tau) \geq 0$
   $R_{XX}(0\pm R_{XX}(\tau) \geq 0$ or
   $|R_{XX}(\tau)| \leq R_{XX}(0)$ hence proved.

3. $R_{XX}(\tau)$ is an even function of τ i.e. $R_{XX}(-\tau) = R_{XX}(\tau)$.
   Proof: We know that $R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$
   Let $\tau = -\tau$ then
   $R_{XX}(-\tau) = E[X(t)\, X(t-\tau)]$
   Let $u=t-\tau$ or $t= u+\tau$
   Therefore $R_{XX}(-\tau) = E[X(u+\tau)\, X(u)] = E[X(u)\, X(u+\tau)]$
4. If a random process X(t) has a non zero mean value, $E[X(t)] \neq 0$ and Ergodic with no periodic components, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$.
   Proof : Consider a random variable X(t)with random variables $X(t_1)$ and $X(t_2)$. Given the mean value is $E[X(t)] = \bar{X} \neq 0$. We know that
   $R_{XX}(\tau) = E[X(t)X(t+\tau)] = E[X(t_1)\ X(t_2)]$. Since the process has no periodic components, as $|\tau| \to \infty$, the random variable becomes independent, i.e.
   $\lim_{|\tau|\to\infty} R_{XX}(\tau) = E[X(t_1)\, X(t_2)] = E[X(t_1)]\, E[X(t_2)]$
   Since X(t) is Ergodic $E[X(t_1)] = E[X(t_2)] = \bar{X}$
   Therefore $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$ hence proved.
5. If X(t) is periodic then its autocorrelation function is also periodic.
   Proof: Consider a Random process X(t) which is periodic with period T_0
   Then $X(t) = X(t\pm T_0)$ or
   $X(t+\tau) = X(t+\tau \pm T_0)$. Now we have $R_{XX}(\tau) = E[X(t)X(t+\tau)]$ then
   $R_{XX}(\tau\pm T_0) = E[X(t)X(t+\tau\pm T_0)]$
   Given X(t) is WSS, $R_{XX}(\tau\pm T_0) = E[X(t)X(t+\tau)]$
   $R_{XX}(\tau\pm T_0) = R_{XX}(\tau)$
   Therefore $R_{XX}(\tau)$ is periodic hence proved.

6. If X(t) is Ergodic has zero mean, and no periodic components then
$\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$.
Proof: It is already proved that $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$. Where $\bar{X}$ is the mean value of X(t) which is given as zero.
Therefore $\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$ hence proved.
7. The autocorrelation function of random process $R_{XX}(\tau)$ cannot have any arbitrary shape.
Proof: The autocorrelation function $R_{XX}(\tau)$ is an even function of τ and has maximum value at the origin. Hence the autocorrelation function cannot have an arbitrary shape hence proved.
8. If a random process X(t) with zero mean has the DC component A as Y(t) =A + X(t), Then $R_{YY}(\tau) = A^2 + R_{XX}(\tau)$.
Proof: Given a random process Y(t) =A + X(t).
We know that $R_{YY}(\tau) = E[Y(t)Y(t+\tau)] = E[(A + X(t)) (A + X(t+ \tau))]$
$= E[(A^2 + AX(t) + AX(t+ \tau)+ X(t) X(t+ \tau)]$
$= E[(A^2] + AE[X(t)] + E[AX(t+ \tau)]+ E[X(t) X(t+ \tau)]$
$=A^2+0+0+ R_{XX}(\tau)$.
Therefore $R_{YY}(\tau) = A^2+ R_{XX}(\tau)$ hence proved.

9. If a random process Z(t) is sum of two random processes X(t) and Y(t) i.e, Z(t) = X(t) + Y(t). Then $R_{ZZ}(\tau) = R_{XX}(\tau)+ R_{XY}(\tau)+ R_{YX}(\tau)+ R_{YY}(\tau)$
Proof: Given Z(t) = X(t) + Y(t).
We know that $R_{ZZ}(\tau) = E[Z(t)Z(t+\tau)]$
$= E[(X(t)+Y(t)) (X(t+\tau)Y(t+\tau))]$
$= E[(X(t) X(t+\tau)+ X(t) Y(t+\tau) +Y(t) X(t+\tau) +Y(t) Y(t+\tau))]$
$= E[(X(t) X(t+\tau)]+ E[X(t) Y(t+\tau)] +E[Y(t) X(t+\tau)] +E[Y(t) Y(t+\tau))]$
Therefore $R_{ZZ}(\tau) = R_{XX}(\tau)+ R_{XY}(\tau)+ R_{YX}(\tau)+ R_{YY}(\tau)$ hence proved.

Properties of Cross Correlation Function: Consider two random processes X(t) and Y(t) are at least jointly WSS. And the cross correlation function is a function of the time difference τ = t2-t1. Then the following are the properties of cross correlation function.

1. RXY(τ) = RYX(-τ) is a Symmetrical property.

Proof: We know that RXY(τ) = E[X(t) Y(t+ τ)] also RYX(τ) = E[Y(t) X(t+ τ)] Let τ = - τ then RYX(-τ) = E[Y(t) X(t- τ)] Let u=t- τ or t= u+ τ. then RYX(-τ) = E[Y(u+ τ) X(u)] = E[X(u) Y(u+ τ)] Therefore RYX(-τ) = RXY(τ) hence proved.

2. If RXX(τ) and RYY(τ) are the autocorrelation functions of X(t) and Y(t) respectively then the cross correlation satisfies the inequality

$$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}.$$

$$E[\frac{X(t)}{\sqrt{R_{XX}(0)}} \pm \frac{Y(t+\tau)}{\sqrt{R_{YY}(0)}}]^2 = \geq 0$$

$$E[\frac{X^2(t)}{\sqrt{R_{XX}(0)}} + \frac{Y^2((t+\tau)}{\sqrt{R_{YY}(0)}}] \pm 2 \frac{X(t)Y(t+\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}}] \geq 0$$

$$E[\frac{X^2(t)}{\sqrt{R_{XX}(0)}}] + E[\frac{Y^2((t+\tau)}{\sqrt{R_{YY}(0)}}] \pm 2\, E[\frac{X(t)Y(t+\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}}] \geq 0$$

We know that $E[X^2(t)] = R_{XX}(0)$ and $E[Y^2(t)] = R_{YY}(0)$ and $E[X(t)\, X(t+\tau)] = R_{XY}(\tau)$

Therefore $\frac{R_{XX}(0)}{R_{XX}(0)} + \frac{R_{YY}(0)}{R_{YY}(0)} \pm 2\frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}} \geq 0$

$$2 \pm 2\frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}} \geq 0$$

$$1 \pm \frac{R_{XY}(\tau)}{\sqrt{R_{XX}(0)R_{YY}(0)}} \geq 0$$

$\sqrt{R_{XX}(0)R_{YY}(0)}. \geq |R_{XY}(\tau)|$ Or

3. If RXX(τ) and RYY(τ) are the autocorrelation functions of X(t) and Y(t) respectively then the cross correlation satisfies the inequality:

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0)+ R_{YY}(0)].$$

Proof: We know that the geometric mean of any two positive numbers cannot exceed their arithmetic mean that is if RXX(τ) and RYY(τ) are two positive quantities then at τ=0,

$\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0)+ R_{YY}(0)]$.We know that $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$

4.If two random processes X(t) and Y(t) are statistically independent and are at least WSS, then RXY(τ) $=\overline{X}\,\overline{Y}$. Proof: Let two random processes X(t) and Y(t) be jointly WSS, then we know that RXY(τ) =E[X(t) Y(t+ τ)] Since X(t) and Y(t) are independent RXY(τ) =E[X(t)]E[Y(t+ τ)]

Proof: We know that $R_{XY}(\tau)$ =E[X(t) Y(t+ τ)]. Taking the limits on both sides

$\lim_{|\tau|\to\infty} R_{XY}(\tau) = \lim_{|\tau|\to\infty} E[X(t)\, Y(t+\tau)]$.

As $|\tau| \to \infty$, the random processes X(t) and Y(t) can be considered as independent processes therefore

$\lim_{|\tau|\to\infty} R_{XY}(\tau) = E[X(t)]E[Y(t+\tau)] = \bar{X}\,\bar{Y}$

Given $\bar{X} = \bar{Y} = 0$

Therefore $\lim_{|\tau|\to\infty} R_{XY}(\tau) = 0$. Similarly $\lim_{|\tau|\to\infty} R_{YX}(\tau) = 0$. Hence proved.

Covariance functions for random processes: Auto Covariance function: Consider two random processes X(t) and X(t+ τ) at two time intervals t and t+ τ. The auto covariance function can be expressed as

$C_{XX}(t, t+\tau) = E[(X(t)-E[X(t)]) ((X(t+\tau) - E[X(t+\tau)])]$ or

$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - E[(X(t) E[X(t+\tau)]$

If X(t) is WSS, then $C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$. At $\tau = 0$, $C_{XX}(0) = R_{XX}(0) - \bar{X}^2 = E[X^2] - \bar{X}^2 = \sigma X^2$

Therefore at τ = 0, the auto covariance function becomes the Variance of the random process. The autocorrelation coefficient of the random process, X(t) is defined as

$$\rho_{XX}(t, t+\tau) = \frac{C_{XX}(t,t+\tau)}{\sqrt{C_{XX}(t,t)C_{XX}(t+\tau,t+\tau)}} \text{ if } \tau = 0,$$

$$\rho_{XX}(0) = \frac{C_{XX}(t,t)}{C_{XX}(t,t)} = 1. \text{ Also } |\rho_{XX}(t, t+\tau)| \leq 1$$

Cross Covariance Function: If two random processes X(t) and Y(t) have random variables X(t) and Y(t+ τ), then the cross covariance function can be defined as

CXY(t, t+τ) = E[(X(t)-E[X(t)]) ((Y(t+τ) – E[Y(t+τ)])] or CXY(t, t+τ) = RXY(t, t+τ) - E[(X(t) E[Y(t+τ)].
If X(t) and Y(t) are jointly WSS, then CXY(τ) = RXY(τ) -X ̅Y ̅. If X(t) and Y(t) are Uncorrelated then CXY(t, t+τ) =0.

The cross correlation coefficient of random processes X(t) and Y(t) is defined as

$$\rho_{XY}(t, t+\tau) = \frac{C_{XY}(t,t+\tau)}{\sqrt{C_{XX}(t,t)C_{YY}(t+\tau,t+\tau)}} \text{ if } \tau = 0,$$

$$\rho_{XY}(0) = \frac{C_{XY}(0)}{\sqrt{C_{XX}(0)C_{YY}(0)}} = \frac{C_{XY}(0)}{\sigma_X\sigma_Y}.$$

Gaussian Random Process: Consider a continuous random process X(t). Let N random variables X1=X(t1),X2=X(t2), . . . ,XN =X(tN) be defined at time intervals t1, t2, . . . tN respectively. If random variables are jointly Gaussian for any N=1,2,…. And at any time instants t1,t2,. . . tN. Then the random process X(t) is called Gaussian random process. The Gaussian density function is given as

$$f_X(x_1, x_2 \ldots\ldots x_N ; t_1, t_2, \ldots\ldots t_N) = \frac{1}{(2\pi)^{N/2}|[C_{XX}]|^{1/2}} exp(-[X-\overline{X}]^T[C_{XX}]^{-1}[X-\overline{X}])/2$$

Poisson's random process: The Poisson process X(t) is a discrete random process which represents the number of times that some event has occurred as a function of time. If the number of occurrences of an event in any finite time interval is described by a Poisson distribution with the average rate of occurrence is λ, then the probability of exactly occurrences over a time interval (0,t) is

$$P[X(t)=K] = \frac{(\lambda t)^K e^{-\lambda t}}{k!}, K=0,1,2, \ldots$$

And the probability density function is

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^K e^{-\lambda t}}{k!} \delta(x-k).$$

UNIT-4: RANDOM PROCESSES: SPECTRAL CHARACTERISTICS

In this unit we will study the characteristics of random processes regarding correlation and covariance functions which are defined in time domain. This unit explores the important concept of characterizing random processes in the frequency domain. These characteristics are called spectral characteristics. All the concepts in this unit can be easily learnt from the theory of Fourier transforms.

Consider a random process X (t). The amplitude of the random process, when it varies randomly with time, does not satisfy Dirichlet's conditions. Therefore it is not possible to apply the Fourier transform directly on the random process for a frequency domain analysis. Thus the autocorrelation function of a WSS random process is used to study spectral characteristics such as power density spectrum or power spectral density (psd).

Power Density Spectrum: The power spectrum of a WSS random process X (t) is defined as the Fourier transform of the autocorrelation function RXX (τ) of X (t). It can be expressed as

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

We can obtain the autocorrelation function from the power spectral density by taking the inverse Fourier transform i.e.

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

Therefore, the power density spectrum SXX(ω) and the autocorrelation function RXX (τ) are Fourier transform pairs.

The power spectral density can also be defined as

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[\left|X_T(\omega)\right|^2]}{2T}$$

Where XT(ω) is a Fourier transform of X(t) in interval [-T,T]

Average power of the random process: The average power PXX of a WSS random process X(t) is defined as the time average of its second order moment or autocorrelation function at τ =0.

Mathematically

$$P_{XX} = A\ \{E[X^2(t)]\}$$

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X^2(t)]dt$$

$$\text{Or } P_{XX} = R_{XX}(\tau)|\tau = 0$$

We know that from the power density spectrum

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\text{At } \tau = 0\ P_{XX} = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

Therefore average power of X(t) is

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

Properties of power density spectrum: The properties of the power density spectrum SXX(ω) for a WSS random process X(t) are given as

1. $S_{XX}(\omega) \geq 0$

Proof: From the definition, the expected value of a non negative function

2. The power spectral density at zero frequency is equal to the area under the curve of the autocorrelation $R_{XX}(\tau)$ i.e.

$$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

Proof: From the definition we know that

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau \text{ at } \omega = 0,$$

$$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

3. The power density spectrum of a real process X(t) is an even function i.e.

SXX(-ω)= SXX(ω)

Proof: Consider a WSS real process X(t). then

$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$ also $S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{j\omega\tau}\, d\tau$

Substitute τ = -τ then

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(-\tau)\, e^{-j\omega\tau}\, d\tau$$

Since X (t) is real, from the properties of autocorrelation we know that, RXX (-τ) = RXX (τ)

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{j\omega\tau}\, d\tau$$

4. SXX(ω) is always a real function

5. If SXX(ω) is a psd of the WSS random process X(t), then

6. If X(t) is a WSS random process with psd SXX(ω), then the psd of the derivative of X(t) is equal to ω^2 times the psd SXX(ω).

$$S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$$

Cross power density spectrum: Consider two real random processes X(t) and Y(t). which are jointly WSS random processes, then the cross power density spectrum is defined as the Fourier transform of the cross correlation function of X(t) and Y(t).and is expressed as

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$ and $S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau}\, d\tau$ by inverse Fourier transformation, we can obtain the cross correlation functions as

$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega$ and $R_{YX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX}(\omega)\, e^{j\omega\tau}\, d\omega$

Th $\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = A\,\{E[X^2(t)]\} = R_{XX}(0)$ forms a Fourier transform pair.

If XT(ω) and YT(ω) are Fourier transforms of X(t) and Y(t) respectively in interval [-T,T], Then the cross power density spectrum is defined as

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{E[\left|X_T(\omega) Y_T(\omega)\right|^1]}{2T} \text{ and } S_{YX}(\omega) = \lim_{T\to\infty} \frac{E[\left|Y_T(\omega) X_T(\omega)\right|^1]}{2T}$$

Average cross power: The average cross power PXY of the WSS random processes X(t) and Y(t) is defined as the cross correlation function at τ =0. That is

$P_{XY} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t,t)dt$ or

$P_{XY} = R_{XY}(\tau)|\tau = 0 = R_{XY}(0)$ Also $P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY(\omega)}\, d\omega$ and $P_{YX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YX(\omega)}\, d\omega$

Properties of cross power density spectrum: The properties of the cross power for real random processes X(t) and Y(t) are given by

(1)SXY(-ω)= SXY(ω) and SYX(-ω)= SYX(ω)

Proof: Consider the cross correlation function $R_{XY}(\tau)$. The cross power density spectrum is

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY(\tau)}\, e^{-j\omega\tau}\, d\tau$

Let τ = - τ Then

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY(-\tau)}\, e^{j\omega\tau}\, d\tau$ Since $R_{XY}(-\tau) = R_{XY}(\tau)$

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY(\tau)}\, e^{-j-\omega\tau}\, d\tau$

Therefore $S_{XY}(-\omega) = S_{XY}(\omega)$ Similarly $S_{YX}(-\omega) = S_{YX}(\omega)$ hence proved.

(2) The real part of SXY(ω) and real part SYX(ω) are even functions of ω i.e.
Re [SXY(ω)] and Re [SYX(ω)] are even functions.

Proof: We know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY(\tau)}\, e^{-j\omega\tau}\, d\tau$ and also we know that

$e^{-j\omega\tau}$=cosωt-jsinωt, $\text{Re}\,[S_{XY}(\omega)] = \int_{-\infty}^{\infty} R_{XY(-\tau)} \cos\omega t\, d\tau$

Since cos ωt is an even function i.e. cos ωt = cos (-ωt)

$\text{Re}\,[S_{XY}(\omega)] = \int_{-\infty}^{\infty} R_{XY(\tau)} \cos\omega t\, d\tau = \int_{-\infty}^{\infty} R_{XY(\tau)} \cos(-\omega t)\, d\tau$

Therefore $S_{XY}(\omega) = S_{XY}(-\omega)$ Similarly $S_{YX}(\omega) = S_{YX}(-\omega)$ hence proved.

(3) The imaginary part of SXY(ω) and imaginary part SYX(ω) are odd functions of ω i.e.

Im [SXY(ω)] and Im [SYX(ω)] are odd functions.

Proof: We know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$ and also we know that

$e^{-j\omega\tau}$=cosωt-jsinωt,

$\text{Im}\,[S_{XY}(\omega)] = \int_{-\infty}^{\infty} R_{XY}(\tau)(-\sin\omega t)\, d\tau = -\int_{-\infty}^{\infty} R_{XY}(\tau)\sin\omega t\, d\tau = -\text{Im}\,[S_{XY}(\omega)]$

Therefore $\text{Im}\,[S_{XY}(\omega)] = -\text{Im}\,[S_{XY}(\omega)]$ Similarly $\text{Im}\,[S_{YX}(\omega)] = -\text{Im}\,[S_{YX}(\omega)]$ hence proved.

(4) SXY(ω)=0 and SYX(ω)=0 if X(t) and Y(t) are Orthogonal.

Proof: From the properties of cross correlation function, We know that the random processes X(t) and Y(t) are said to be orthogonal if their cross correlation function is zero.

i.e. $R_{XY}(\tau) = R_{YX}(\tau) = 0$.

We know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$

Therefore $S_{XY}(\omega)=0$. Similarly $S_{YX}(\omega)=0$ hence proved.

(5) If X(t) and Y(t) are uncorrelated and have mean values and , then

$$S_{XY}(\omega)=2\pi\bar{X}\,\bar{Y}\,\delta(\omega).$$

Proof: We know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$

$= S_{XY}(\omega) = \int_{-\infty}^{\infty} E[X(t)Y(t+\tau)]e^{-j\omega\tau}\, d\tau$

Since X(t) and Y(t) are uncorrelated, we know that

$E[X(t)Y(t+\tau) = E[X(t)]E[Y(t+\tau)]$

Therefore $S_{XY}(\omega) = \int_{-\infty}^{\infty} E[X(t)]E[Y(t+\tau)]e^{-j\omega\tau}\, d\tau$

$S_{XY}(\omega) = \int_{-\infty}^{\infty} \bar{X}\bar{Y}e^{-j\omega\tau}\, d\tau$

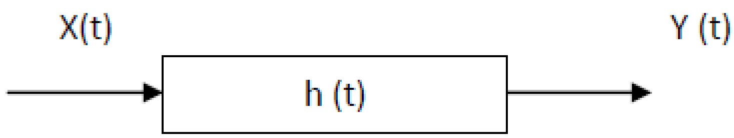
$S_{XY}(\omega) = \bar{X}\,\bar{Y}\int_{-\infty}^{\infty} e^{-j\omega\tau}\, d\tau$

Therefore $S_{XY}(\omega)=2\pi\bar{X}\,\bar{Y}\,\delta(\omega)$. hence proved.

UNIT-5: LINEAR SYSTEMS RESPONSE TO RANDOM INPUTS

Consider a continuous LTI system with impulse response h (t). Assume that the system is always causal and stable. When a continuous time Random process X (t) is applied on this system, the output response is also a continuous time random process Y (t). If the random processes X and Y are discrete time signals, then the linear system is called a discrete time system. In this unit we concentrate on the statistical and spectral characteristics of the output random process Y (t).

System Response: Let a random process X (t) be applied to a continuous linear time invariant system whose impulse response is h(t) as shown in below figure. Then the output response Y (t) is also a random process. It can be expressed by the convolution integral, Y (t) = h (t) *X (t)



Y (t) = $\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau.$

Mean Value of Output Response: Consider that the random process X (t) is wide sense stationary process.

Mean value of output response=E[Y (t)], Then

E[Y (t)] = E [h (t) * X (t)]

=E [$\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$]

=$\int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]d\tau$

But $E[X(t-\tau)]$ = $\bar{X}$ =constant, since X (t) is WSS.

Then E[Y (t)] = $\bar{Y}$ = $\bar{X}\int_{-\infty}^{\infty} h(\tau)\, d\tau$. Also if H (ω) is the Fourier transform of h (t), then

H (ω) = $\int_{-\infty}^{\infty} h(\tau)e^{-j\omega t}\, dt$. At ω = 0, H (0) = $\int_{-\infty}^{\infty} h(t)\, dt$ is called the zero frequency response of the system. Substituting this we get E[Y (t)] = $\bar{Y}$ = $\bar{X}$ H (0) is constant. Thus the mean value of the output response Y (t) of a WSS random process is equal to the product of the mean value of the input process and the zero frequency response of the system.

Mean square value of output response is

$E[Y^2(t)] = E[(h(t) * X(t))^2]$

$= E[(h(t) * X(t))(h(t) * X(t))]$

$= E[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t-\tau_2)d\tau_2]$

$= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t-\tau_1)X(t-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2]$

$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\tau_1)X(t-\tau_2)]h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$

Where τ_1 and τ_2 are shifts in time intervals. If input X(t) is a WSS random process then

$E[X(t-\tau_1)X(t-\tau_2)] = R_{XX}(\tau_1 - \tau_2)$

Therefore $E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau_1-\tau_2)\, h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$

This expression is independent of time t. And it represents the Output power.

Autocorrelation Function of Output Response: The autocorrelation of Y (t) is

$R_{YY}(\tau_1, \tau_2) = E[Y(t_1)\, Y(t_2)]$

$= E[(h(t_1) * X(t_1))(h(t_2) * X(t_2))]$

$= E[\int_{-\infty}^{\infty} h(\tau_1)X(t_1-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t_2-\tau_2)d\tau_2]$

$= E[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t_1-\tau_1)X(t_2-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t_1-\tau_1)X(t_2-\tau_2)]h(\tau_1)h(\tau_2)d\tau_1 d\tau_2$

We know that E $[X(t_1 - \tau_1)X(t_2 - \tau_2)] = R_{XX}(t_2 - t_1 + \tau_1 - \tau_2)$.

If input X (t) is a WSS random process, Let the time difference $\tau = t_1 - t_2$ and $t=t_1$ Then

E $[X(t - \tau_1)X(t + \tau - \tau_2)] = R_{XX}(\tau + \tau_1 - \tau_2)$. Then

$R_{YY}(t, t + \tau) = R_{YY}(t, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau + \tau_1 - \tau_2)\, h(\tau_1)h(\tau_2)d\,\tau_1 d\tau_2$

If = $R_{XX}(\tau)$ is the autocorrelation function of X (t), then $R_{YY}(\tau) = R_{XX}(\tau) * h(\tau)\, h(-\tau)$

It is observed that the output autocorrelation function is a function of only τ. Hence the output random process Y(t) is also WSS random process.

If the input X (t) is WSS random process, then the cross correlation function of input X (t) and output Y(t) is

$R_{XY}(t, t + \tau)$ = E [X (t) Y (t + τ)]

$R_{XY}(\tau)$ = E [X (t) $\int_{-\infty}^{\infty} h(\tau_1)$ X (t + τ - τ_1) dτ_1]

$R_{XY}(\tau) = \int_{-\infty}^{\infty}$ E [X (t) X (t + τ - τ_1)] $h(\tau_1)$dτ_1]

$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_{XX}(\tau - \tau_1)]\, h(\tau_1)d\tau_1$ which is the convolution of $R_{XX}(\tau)$ and h (τ).

Therefore $R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$ similarly we can show that $R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$

This shows that X (t) and Y(t) are jointly WSS. And we can also relate the autocorrelation functions and the cross correlation functions as

$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$

$R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$

Spectral Characteristics of a System Response: Consider that the random process X (t) is a WSS random process with the autocorrelation function Rxx(τ) applied through an LTI system. It is noted that the output response Y (t) is also a WSS and the processes X (t) and Y (t) are jointly WSS. We can obtain power spectral characteristics of the output process Y(t) by taking the Fourier transform of the correlation functions.

Power Density Spectrum of Response: Consider that a random process X (t) is applied on an LTI system having a transfer function H(ω). The output response is Y (t). If the power spectrum of the input process is SXX (ω), then the power spectrum of the output response is given by SYY (ω) =

$|H(\omega)|^2$ $S_{XX}(\omega)$.

Proof: Let $R_{YY}(\tau)$ be the autocorrelation of the output response Y (t). Then the power spectrum of the response is the Fourier transform of $R_{YY}(\tau)$.

Therefore $S_{YY}(\omega)$ = F [$S_{YY}(\omega)$]

$= \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-j\omega\tau}\, d\tau$

We know that $R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1)h(\tau_2)d\,\tau_1 d\tau_2$

Then $S_{YY}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1)h(\tau_2)d\,\tau_1 d\tau_2\, e^{-j\omega\tau}d\tau$

$= \int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) \int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega\tau}d\tau\, d\tau_2\, d\tau_1$

$= \int_{-\infty}^{\infty} h(\tau_1)e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)e^{j\omega\tau_2} \int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega\tau}e^{j\omega\tau_1}e^{j\omega\tau_2}d\tau\, d\tau_2\, d\tau_1$

Let $\tau+\tau_1-\tau_2=t$, $d\tau = dt$

Therefore $S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\tau_1)e^{j\omega\tau_1}d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)e^{j\omega\tau_2}d\tau_2 \int_{-\infty}^{\infty} R_{XX}(t)\, e^{-j\omega t}dt$

We know that $H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega t}\, dt$.

Therefore $S_{YY}(\omega) = H^*(\omega)\, H(\omega)\, S_{XX}(\omega) = H(-\omega)H(\omega)\, S_{XX}(\omega)$

Therefore $S_{YY}(\omega) = |H(\omega)|^2\, S_{XX}(\omega)$. Hence proved.

Similarly, we can prove that the cross power spectral density function is

$S_{XY}(\omega) = S_{XX}(\omega)\, H(\omega)$ and $S_{YX}(\omega) = S_{XX}(\omega)\, H(-\omega)$

Spectrum Bandwidth: The spectral density is mostly concentrated at a certain frequency value. It decreases at other frequencies. The bandwidth of the spectrum is the range of frequencies having significant values. It is defined as “the measure of spread of spectral density” and is also called rms bandwidth or normalized bandwidth. It is given by

$$W_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega}$$

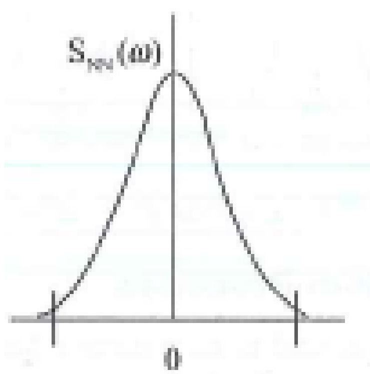
Types of Random Processes: In practical situations, random process can be categorized into different types depending on their frequency components. For example information bearing signals such as audio, video and modulated waveforms etc., carry the information within a specified frequency band.

The Important types of Random processes are;

1. Low pass random processes
2. Band pass random processes
3. Band limited random processes
4. Narrow band random processes

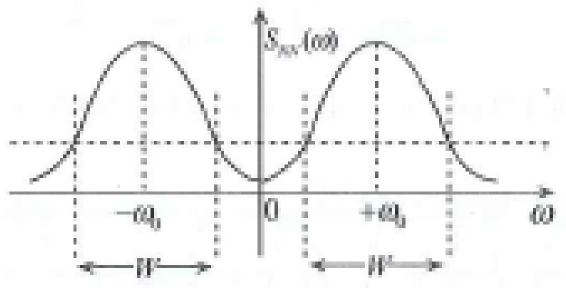
(1).Low pass random processes:

A random process is defined as a low pass random process X (t) if its power spectral density SXX (ω) has significant components within the frequency band as shown in below figure. For example baseband signals such as speech, image and video are low pass random processes.

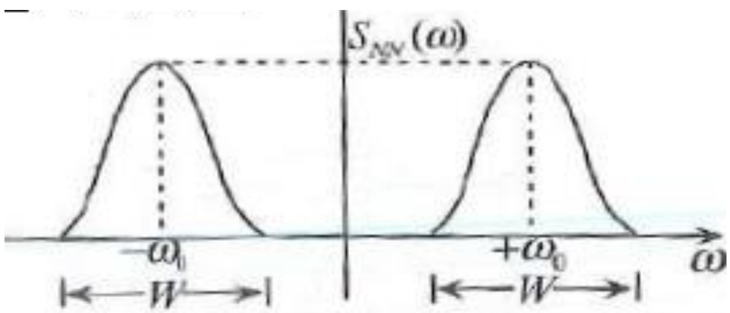


(2).Band pass random processes: A random process X (t) is called a band pass process if its power spectral density SXX (ω) has significant components within a band width W that does not include ω =0. But in practice, the spectrum may have a small amount of power spectrum at ω =0, as shown in the below figure. The spectral components outside the band W are very small and can be neglected.

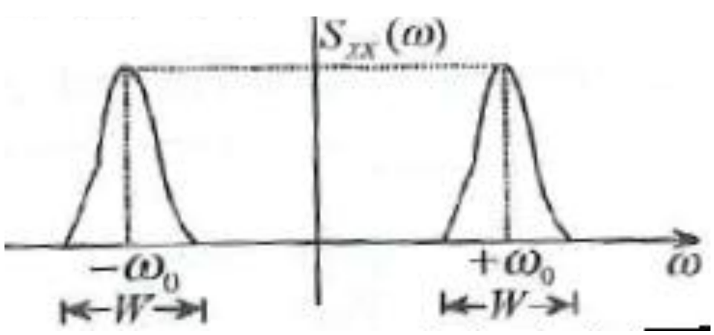
For example, modulated signals with carrier frequency ω0 and band width W are band pass random processes. The noise transmitting over a communication channel can be modelled as a band pass process.



(3).Band Limited random processes: A random process is said to be band limited if its power spectrum components are zero outside the frequency band of width W that does not include ω =0. The power density spectrum of the band limited band pass process is shown in below figure.



(4).Narrow band random processes: A band limited random process is said to be a narrow band process if the band width W is very small compared to the band centre frequency, i.e. W<< ω0, where W=band width and ω0 is the frequency at which the power spectrum is maximum. The power density spectrum of a narrow band process N(t) is shown in below figure.



Representation of a narrow band process: For any arbitrary WSS random processes N(t), The quadrature form of narrow band process can be represented as N(t) = X(t) Cos ω0t – Y(t)Sin ω0t

Where X(t) and Y(t) are respectively called the in-phase and quadrature phase components of N(t). They can be expressed as

X (t) = A (t) Cos[Θ(t)]

Y (t) = A (t) Sin[Θ(t)] and the relationship between the processes A(t) and Θ(t) are given by

A (t) = $\sqrt{X^{2(t)} + Y^{2(t)}}$ and Θ (t) = $\tan^{-1}(\frac{Y(t)}{X(t)})$

Properties of Band Limited Random Processes: Let N (t) be any band limited WSS random process with zero mean value and a power spectral density, SNN(ω). If the random process is represented by N (t) = X (t) Cos ω0t – Y(t)Sin ω0t then some important properties of X (t) and Y (t) are given below

1. If N (t) is WSS, then X (t) and Y (t) are jointly WSS.
2. If N (t) has zero mean i.e. E [N(t)] = 0, then E [x (t)] = E [Y (t)] = 0
3. The mean square values of the processes are equal i.e. E [N^2(t)] = E [X^2(t)] = E [Y^2(t)].
4. Both processes X (t) and Y (t) have the same autocorrelation functions i.e. $R_{XX}(\tau) = R_{YY}(\tau)$.
5. The cross correlation functions of X (t) and Y (t) are given by $R_{YX}(\tau) = -R_{YX}(\tau)$. If the processes are orthogonal, then $R_{YX}(\tau) = R_{YX}(\tau) = 0$.
6. Both X (t) and Y (t) have the same power spectral densities

   $S_{YY}(\omega) = S_{XX}(\omega) = \begin{cases} S_{N^{(\omega-\omega_0)}} + S_{N(\omega+\omega_0)} & for\ |W| \le \omega_0 \\ 0 & \end{cases}$
7. The cross power spectrums are $S_{XY}(\omega) = -S_{YX}(\omega)$.
8. If N (t) is a Gaussian random process, then X (t) and Y (t) are jointly Gaussian.
9. The relationship between autocorrelation and power spectrum $S_{NN}(\omega)$ is

   $R_{XX}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN(\omega)\cos[(\omega-\omega_0})\tau]d\omega$ and

   $R_{YY}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN(\omega)\cos[(\omega-\omega_0})\tau]d\omega$

10. If N (t) is zero mean Gaussian and its psd, $S_N(\omega)$ is symmetric about $\pm\omega_0$, then X (t) and Y (t) are statistically independent.