PRINCIPLES OF DIGITAL SIGNAL PROCESSING(19A04704b)

LECTURE NOTES

IV -B.TECH & I-SEM

Prepared by:

Mr. M. SRINIVASULU, Assistant Professor

Department of Electronics & Communications Engineering

VEMU INSTITUTE OF TECHNOLOGY

(Approved By AICTE, New Delhi and Affiliated to JNTUA, Ananthapuramu)

Accredited By NAAC, NBA(EEE, ECE & CSE) & ISO: 9001-2015 Certified Institution

Near Pakala, P.Kothakota, Chittoor- Tirupathi Highway

Chittoor, Andhra Pradesh-517 112

Web Site: www.vemu.org

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY ANANTAPUR
(Established by Govt. of A.P., ACT No.30 of 2008)
ANANTHAPURAMU – 515 002 (A.P) INDIA

Electrical & Electronics Engineering

Course Code	Principles of Digital Signal Processing	L	T	P	C
19A04704b		3	0	0	3
Pre-requisite	Analysis of Digital Signal Processing	Semester			VII

Course Objectives:

- **To explain about signals and perform various operations on it.**
- **To understand discrete time signals and systems.**
- **To solve Laplace transforms and z-transforms for various signals.**
- **To find Discrete Fourier Transform of a sequence by using Fast Fourier Transform.**
- **To design and realize IIR and FIR filters.**

Course Outcomes (CO): After completion of the course, the student can able to

CO-1: Define basic signals and its operations, Classify discrete time signals and systems.

CO-2:Solve Laplace Transform and z-Transform for various signals, Calculate DFT of a given sequence by using Fast Fourier Transform.

CO-3: Analyze the continuous and discrete signals and systems

CO-4: Design and realize IIR and FIR filters from the given specifications.

Unit – I: **INTRODUCTION TO SIGNALS**

Classification of Signals: Analog, Discrete, Digital, Deterministic & Random, Periodic &Aperiodic, Even & Odd, Energy & Power signals. Basic operations on signals: Time shifting,Time scaling, Time reversal, Amplitude scaling and Signal addition. Elementary Signals: Unitstep, Unit ramp, Unit parabolic, Impulse, Sinusoidal function,Exponential function, Gate function, Triangular function, Sinc function and Signum function.

Unit – II: **DISCRETE TIME SIGNALS AND SYSTEMS**

Discrete Time Signals: Elementary discrete time signals, Classification of discrete time signals:power and energy signals, even and odd signals. Simple manipulations of discrete time signals:Shifting and scaling of discrete-time signals.
Discrete Time Systems: Input-Output description of systems, Block diagram representation ofdiscrete time systems, Linear Constant Coefficient Difference Equations, Classification ofdiscrete time systems: linear and nonlinear, time-invariant and variant systems, causal and non causal, stable and unstable systems.

Unit – III: **LAPLACE TRANSFORMS AND Z- TRANSFORMS**

Laplace Transforms: Laplace transforms, Partial fraction expansion, Inverse Laplace transform,Concept of Region of Convergence (ROC), Constraints on ROC for various classes of signals,Properties of Laplace transforms.
Z-Transforms: Concept of Z-transform of a discrete sequence, Region of convergence in Z Transform,constraints on ROC for various classes of signals, inverse Z-transform,properties of Z-Transforms.

Unit – IV: **FAST FOURIER TRANSFORMS**

Discrete Time Fourier Transform (DTFT), Discrete Fourier Transform (DFT), Radix-2 Fast Fourier Transforms (FFT), Decimation in Time and Decimation in Frequency FFTAlgorithms: radix-2 DIT-FFT, DIF-FFT, and Inverse FFT: IDFT-FFT.

Unit – V: **IIR AND FIR DIGITAL FILTERS**

IIR DIGITAL FILTERS: **Analog filters approximations: Butterworth and Chebyshev, Designof IIR digital filters from analog filters. Realization of IIR filters: Direct form-I, Direct form-II,cascade form and parallel form.**

FIR DIGITAL FILTERS: **Characteristics of FIR digital filters, frequency response. Design ofFIR digital filters using window techniques: Rectangular window, Triangular or Bartlett window,Hamming window, Hanning window, Blackman window. Realization of FIR filters: Linearphase and Lattice structures.**

Textbooks:

1. **B. P. Lathi, "Signals, Systems and Communications", BS Publications, 2008.**
2. **John G. Proakis, Dimitris G. Manolakis, "Digital signal processing, principles, Algorithms and applications" , 4th edition , Pearson Education/PHI, 2007.**
3. **A.V. Oppenheim and R.W. Schaffer, "Discrete Time Signal Processing", 2nd edition., PHI**

Reference Books:

1. **A. Anand Kumar, "Signals and Systems", PHI Publications, Third Edition, 2013**
2. **P. Ramesh Babu. "Digital Signal Processing".**

UNIT-I:INTRODUCTION TO SIGNALS

Lecture notes

UNIT-II: DISCRETE TIME SIGNALS AND SYSTEMS

Lecture notes

UNIT-III:LAPLACE TRANSFORMS AND Z- TRANSFORMS

Lecture notes

UNIT-IV :FAST FOURIER TRANSFORMS

Lecture notes

UNIT-V:IIR AND FIR DIGITAL FILTERS

Lecture notes

UNIT-I

INTRODUCTION TO SIGNALS

Introduction

Definition of Signal:

In a communication system, the word 'signal' is commonly used. Therefore we must know its exact meaning.

- Mathematically, signal is described as a function of one or more independent variables.
- Basically it is a physical quantity. It varies with some dependent or independent variables.
- So the term signal is defined as "A physical quantity which contains some information and which is a function of one or more independent variables."

Examples: speech signal, ECG signal, radio signal, TV signal.

These signals can be one dimensional or multidimensional.

One dimensional signal: when the function depends on a single variable, the signal is said to be one dimensional.

Example: speech signal

Multidimensional signal: When the function depends on two or more variables, the signal is said to be multi dimensional.

Example: image

Definition of system:

A system is a physical device (or an algorithm) which performs required operation on a signal.

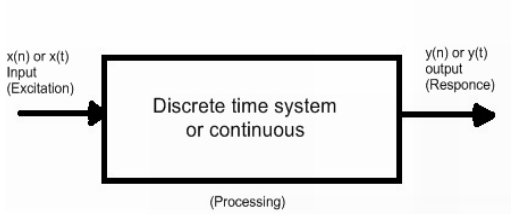


Fig. 1.1 Functional block diagram of system

A system is a set of elements or functional blocks that are connected together & produces an output in response to an input signal. The response of the system depends on transfer function of the system.

The functional relationship between input & output is y(t)= T [x(t)]

Where x (t) is the input to the system or excitation.

Y (t) is the output or response of the system.

T is the transfer function of the system.

Examples: communication filters, amplifiers, TV, audio amplifiers, transmitters, receivers etc.

Classification of signals

There are various types of signals. Every signal has its own characteristics. The processing of signals mainly depends on the characteristic of that particular signal. So classification of signal is necessary. Broadly the signals are classified as below:

- Continuous and discrete time signals
- Periodic and non-periodic signals
- Even and odd signals
- Energy and power signals
- Deterministic and random signals

Continuous Time and Discrete Time Signals

Continuous Time signal: A signal is said to be continuous when it is defined for all instants of time. Or a signal of continuous amplitude & time is known as Continuous Time signal.

Examples: electrical signals derived in proportion with the physical quantities such as temperature, pressure, sound etc.

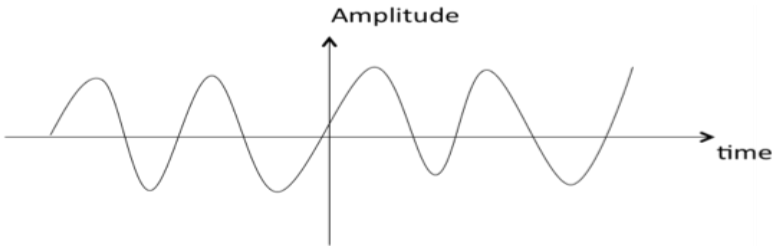


Fig. 1.2 Continous time sine signal

Discrete Time Signals: A signal is said to be discrete when it is defined at only discrete instants of time.

Example: if we take blood pressure readings of a patient after every one hour & plot the graph then it is discrete signal.

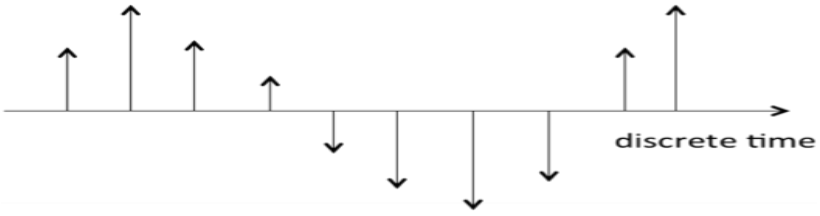


Fig. 1.3. Discrete time signal

Periodic and Aperiodic Signals

Periodic signal: A signal which repeats itself after a fixed time period is called as Periodic signal. Alternatively a signal is said to be periodic if it satisfies the condition

$x(t) = x(t + T_0)$ for CT signal

$x(n) = x(n + N)$ for DT signal

Where T_0 = fundamental time period. The minimum possible interval over which a function repeats is called fundamental period T_0.

$1/ T_0$= F= fundamental frequency.

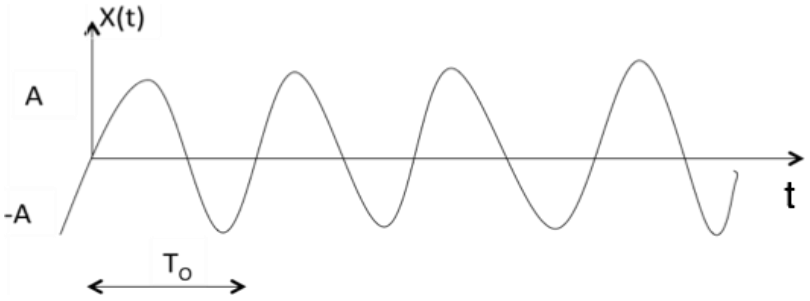


Fig 1.4 Periodic signal

The above signal will repeat for every time interval T_0 hence it is periodic with period T_0.

Examples: sine, cosine waves, square waves etc..

Aperiodic Signals: A signal which does not repeat at regular interval is called as Aperiodic Signals. Alternatively a signal is said to be non periodic if it satisfies the condition

$x(t) \neq x(t + T_0)$ for CT signal

$x(n) \neq x(n + N)$ for DT signal

Examples : exponential signal, rectangular pulse, noise etc.

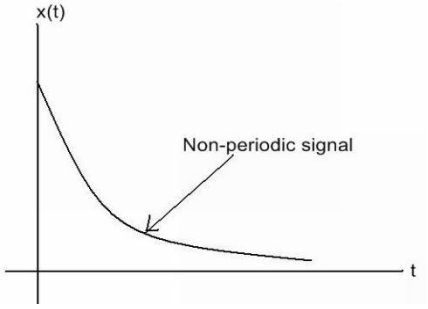


Fig 1.4 Aperiodic Signal

The above signal is a exponential signal which does not repeat for a particular time interval. So it is a non periodic signal.

Condition for periodicity of DT signal

For DT signal, the condition of periodicity is x(n) = x(n + N)

$$\text{Let } x(n) = A\cos[2\pi f_0 n + \theta]$$

$$\text{So } x(n+N) = A\cos[2\pi f_0 (n+N) + \theta]$$

According to the condition of periodicity $A\cos[2\pi f_0 n + \theta] = A\cos[2\pi f_0 (n+N) + \theta]$

$$= A\cos[2\pi f_0 n + 2\pi f_0 N + \theta]$$

To satisfy this condition, $2\pi f_0 N = 2\pi k$

$$f_0 = k/N$$

Here k,N are integers. Thus DT signal is periodic if its frequency f_0 is rational.

Periodicity condition for $x(n) = x_1(n)+x_2(n)$

According to the Periodicity $f_1 = k_1/N_1 \quad f_2 = k_2/N_2$

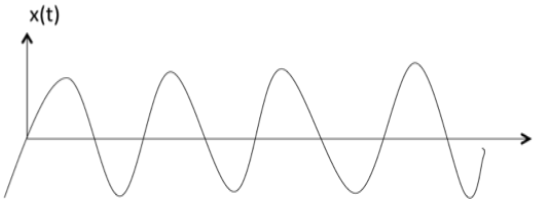
The resultant signal is periodic if N_1/N_2 is the ratio of two integers. The period of x(n) will be least common multiple of N_1, N_2.

Deterministic and random Signals

Deterministic signal: A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time. Or, signals which can be defined exactly by a mathematical formula, look up table or some well defined rule are known as deterministic signals. Example: sine, cosine wave.

Random Signals: A signal is said to be non-deterministic if there is uncertainty with respect to its value at some instant of time. Non-deterministic signals are random in nature hence they are called random signals. Random signals cannot be described by a mathematical equation. They are modeled in probabilistic terms.

Example: noise generated in electronic components, transmission channels etc



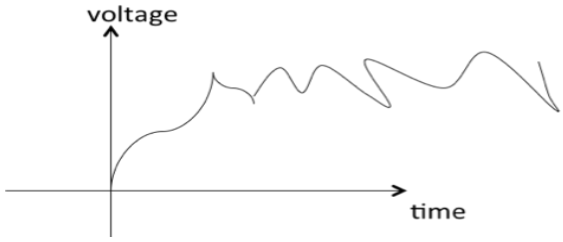


Fig 1.5 Deterministic signal **Fig 1.6 Random signal**

Deterministic signal	**Random Signal**
1. A deterministic signal is one which can be completely represented by Mathematical equation at any time.	A random signal is one which cannot be represented by any mathematical equation.
2.eg. Sine wave, cosine wave.	Noise generated in electronic components, transmission channels, cables.
3. These signals can be periodic or non periodic.	These signals are non periodic.

Even and odd signal

Even or symmetrical signal

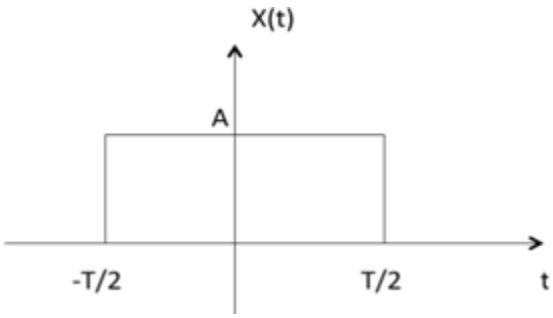
A signal x(t) is said to be symmetrical (or) even if it satisfies the following condition:

X(t) = x(-t) for CT signal

X(n) = x(-n) for DT signal

Example: cosine waveform, rectangular pulse.

As shown in the following diagram, rectangular function satisfies the condition x(t) = x(-t) so it is also even function.



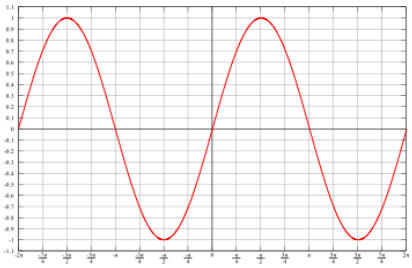


Fig 1.7(a) Example of Even signal **Fig 1.7(b) Example of odd signal**

Odd (or) Anti symmetrical signal

A signal x(t) is said to be anti symmetrical (or) odd if it satisfies the following condition:

X(t) = -x(-t) for CT signal

X(n)= -x(-n) for DT signal

Example: sine waveform

Decomposing a CT signal into even & odd parts:

Any CT signal can be expressed as the summation of even part & odd part.

$X(t)= x_e(t)+ x_o(t)$

Even part of x(t) is expressed as, $x_{ev}(t) = \frac{x(t) + x(-t)}{2}$

Odd part of x(t) is expressed as, $x_{od}(t) = \frac{x(t) - x(-t)}{2}$

The even part of a signal is an even signal, since

$$x_{ev}(-t) = \frac{x(-t) + x(t)}{2} = x_{ev}(t)$$

Similarly any DT signal can be expressed as the summation of even part & odd part.

$X(n)= x_e(n)+ x_o(n)$

Even part of x(t) is expressed as, $x_e(n) = [x(n)+ x(-n] / 2$

Odd part of x(t) is expressed as, $x_o(n) = [x(n)- x(-n)] / 2.$

Steps to be followed to find even and odd component of the given signal is:

1) Draw the signal x (t).
2) Draw its folded version x (-t)
3) Add x (t) and x (-t) or subtract x (-t) from x (t)
4) Divide the addition by 2 to get $x_e(t)$& subtraction by 2 to get $x_o(t)$.

Energy and power signal

Power signal: A signal x (t) is said to be power signal, if and only if the normalized average power p is finite and non-zero. ($0 < P < \infty$, $E \to \infty$).

Almost all the periodic signals are power signals.

The average normalized power is given by

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

,

For periodic signals, the power *P* can be computed using a simpler form based on the periodicity of the signal as

$$\text{Power } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

,

where *T* is the period of the signal.

Energy signal

A signal x (t) is said to be energy signal if and only if the total normalized energy is finite and non-zero. ($0 \le E < \infty$, $P = 0$)

The total energy contained in and average power provided by a signal x(t) (which is a function of time) are defined as

$$\text{Energy } E = \int_{-\infty}^{\infty} x^2(t)dt$$

Comments:

1. The square root of the average power $\sqrt{P}$ of a power signal is what is usually defined as the RMS value of that signal.
2. If a signal approaches zero as t approaches ∞ then the signal is an energy signal. This is in most cases true but not always.
3. All periodic signals are power signals (but not all non–periodic signals are energy signals).
4. Any signal f that has limited amplitude ($|f| < \infty$) and is time limited

   ($f = 0$ for $|t| > t_0$ for some $t_0 > 0$) is an energy signal .

Energy and Power: The *total energy* of a discrete-time signal is defined by

$$E_\infty = \sum_{n=-\infty}^{\infty} x^2[n] = \lim_{N\to\infty} \sum_{n=-N}^{N} x^2[n]$$

The time-average power is

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

Comparison of energy and power signals

Power signals	Energy signals
The normalized average power is finite	Total normalized energy is finite and

and non zero.	non- zero.
Almost all periodic signals are power signals.	Almost all non periodic signals are energy signals.
Energy of the power signal is infinite.	Power of the energy signal is zero.
Power signals can exist over an infinite time. They are not time limited.	Energy signals exist over a short period of time. They are time limited.

PROBLEMS BASED ON CLASSIFICATION OF SIGNALS

1. Prove that the sine wave is a periodic signal.

The sinewave is mathematically expressed as, $x(t) = A \sin w_0 t$

If it is a periodic signal it should satisfy the condition $x(t) = x(t+T_0)$

Now let us calculate $x(t+T_0) = A \sin w_0 (t + T_0)$

$= A \sin (w_0 t + w_0 T_0)$

$= A \sin (w_0 t + 2 \pi T_0 / T_0)$

$= A \sin (w_0 t + 2 \pi)$

$= A [\sin w_0 t \cos 2 \pi + \cos w_0 t \sin 2 \pi)$

$= A [\sin w_0 t .1 + \cos w_0 t .0]$

$= A \sin w_0 t$

$= x(t)$

Therefore $x(t+T_0) = x(t)$. The sine wave is a periodic signal.

2. Prove that the exponential signal is non periodic.

The exponential signal is expressed as, $x(t) = e^{-\alpha t}$

Put $t= t+T_0$ in above equation, we get

$$x(t+T_0) = e^{-\alpha(t+T_0)}$$

$$= e^{-\alpha t} e^{-\alpha T_0}$$

But for exponential signal $T_0=\infty$.

$$\text{So } e^{-\alpha T_0} = 0$$

$$x (t+T_0) = e^{-\alpha.t} .0=0$$

$$x(t) \neq x(t+T_0).$$

Hence the exponential signal is non periodic signal.

3. State whether the following signals x(t) is periodic or not. If it is periodic find the corresponding period.

(a) x(t)= 3 sin 4t

The given equation is a sine wave. So it is a periodic signal.

To find time period, compare the given equation with standard sine wave x(t)= A sin wt

$$w =4$$

$$f = 4 / 2\pi = 2 / \pi.$$

So time period $T= \pi / 2$.

(b) x(t)= 2 cos 100πt + 5 sin 50t

The given signal x(t) is the addition of two signals $x_1(t)$ & $x_2(t)$.

Let $x_1(t) = 2 \cos 100\pi t$ & $x_2(t)= 5 \sin 50t$

If x(t) is a periodic, the ratio T_1/T_2 is the ratio of two integers.(rational no)

So calculate T_1 & T_2 .

Compare $x_1(t)= 2 \cos 100\pi t$ with the standard cosine wave equation $x_1(t)= A \cos w_1 t$

$$w_1 = 100\pi$$

$2\pi f_1 = 100\pi$

$2\pi/T_1 = 100\pi$

$T_1 = 1/50$

Similarly Compare $x_2(t)= 5 \sin 50t$ with the standard sine wave equation $x_2(t)= A \sin w_2t$,

$W_2 = 50$

$2\pi f_2 = 50$

$2\pi/T_2 = 50$

$T_2 = 2\pi /50$

Therefore the ratio $T_1/ T_2 = 1/2\pi$.

It is not the ratio of two integers. Thus x (t) is non periodic signal.

4. Find the fundamental period and frequency of the following signals

(a) $x(t) = 5 \sin 24\pi t + 7 \sin 36\pi t$

(b) $x(t) = 5 \cos \pi t \sin 3\pi t$

(a)

$$x(t) = 5 \sin 24\pi t + 7 \sin 36\pi t$$
$$= x_1(t) + x_2(t)$$

where

$$x_1(t) = 5 \sin 24\pi t$$
$$x_2(t) = 7 \sin 36\pi t$$

Let T_1 &T_2 be the fundamental periods of $x_1(t)$ $x_2(t)$ respectively

$$\omega_1 = 24\pi$$
$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{24\pi} = \frac{1}{12} \text{ (rational)}$$
$$\omega_2 = 36\pi$$
$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{36\pi} = \frac{1}{18} \text{ (rational)}$$
$$\frac{T_1}{T_2} = \frac{1}{12} \times 18 = \frac{3}{2} \text{ (rational)}$$

The composite signal is periodic signal. Since T_1 &T_2 are rational,x(t) is periodic. The fundamental period is the LCM of T_1 &T_2.

In this case, T_1 &T_2 are fractions; they are made integers by multiplying by a least number. for T_1 &T_2 thus obtained,LCM is found.T_0 is obtained by dividing by the same number which was chosen to make T_1 &T_2 as integers.

(1)

$$T_1 = \frac{1}{12} \quad \text{and} \quad T_2 = \frac{1}{18}$$

(2) By multiplying T_1 &T_2 by 36, T_1 = 3 T_2 = 2.

(3) The LCM for the new T_1 &T_2 is easily obtained as 6.

(4)T_0 is obtained by dividing LCM by 36.

$$T_0 = \frac{\text{LCM}}{36} = \frac{6}{36} = \frac{1}{6} \text{ sec.}$$

$$\boxed{\begin{aligned} T_0 &= \frac{1}{6} \text{ sec.} \\ f_0 &= 6 \text{ Hz.} \end{aligned}}$$

(b)

$$x(t) = 5 \cos \pi t \sin 3\pi t$$
$$= x_1(t) x_2(t)$$

Where

$$x_1(t) = 5\cos \pi t$$
$$x_2(t) = \sin \pi t$$

Let T_1 &T_2 be the fundamental periods of $x_1(t)$ $x_2(t)$ respectively.

$$\omega_1 = \pi$$
$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\pi} = 2 \text{ sec. (rational)}$$
$$\omega_2 = 3\pi$$
$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{3\pi} = \frac{2}{3} \text{ sec. (rational).}$$
$$\frac{T_1}{T_2} = 2 \times \frac{3}{2} = 3 \text{ (rational).}$$

The composite signal is periodic with fundamental period is

$$T_0 = T_1 = 3T_2 = 2 \text{ sec.}$$

$$\boxed{\begin{aligned} T_0 &= 2 \text{ sec.} \\ f_0 &= 0.5 \text{ Hz} \end{aligned}}$$

5. Find whether the following signals are even or odd. Find the even and odd components.

(a) $\boldsymbol{x(t) = t^2 - 5t + 10}$
Put $t = -t$

$$x(-t) = t^2 + 5t + 10$$
$$\neq x(t)$$
$$\neq -x(t)$$

The function is neither even nor odd.

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$
$$= \frac{1}{2}[t^2 - 5t + 10 + t^2 + 5t + 10]$$

$$\boxed{x_e(t) = (t^2 + 10)}$$

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$
$$= \frac{1}{2}[t^2 - 5t + 10 - t^2 - 5t - 10]$$

$$\boxed{x_0(t) = -5t}$$

(b) $x(t) = t^4 + 4t^2 + 6$

Put $t = -t$

$$x(-t) = t^4 + 4t^2 + 6 = x(t)$$
$$x(t) = x(-t)$$

The function is even. The odd part should be zero which can be verified as

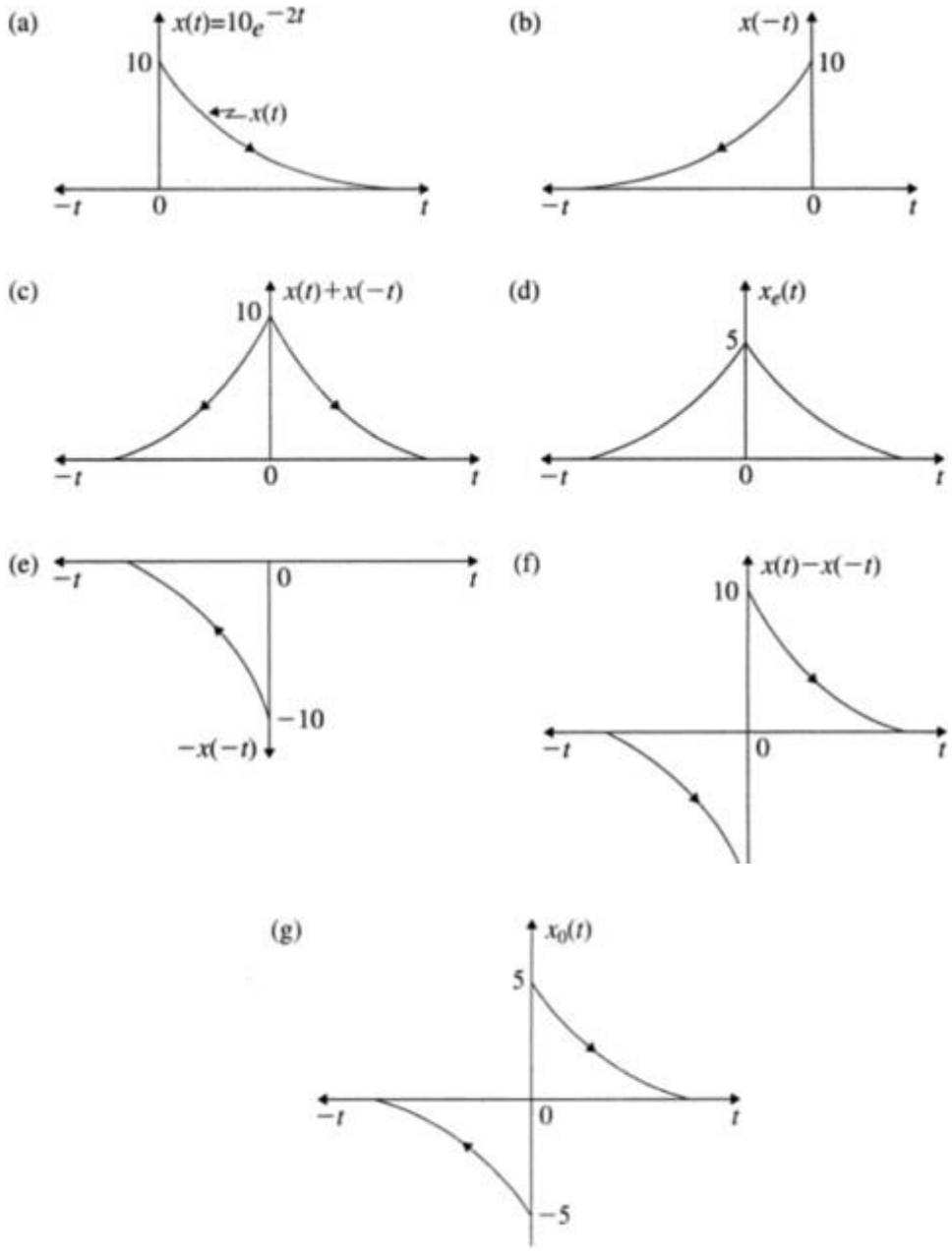
$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$
$$= \frac{1}{2}[t^4 + 4t^2 + 6 - t^4 - 4t^2 - 6]$$
$$= 0$$

$$\boxed{x_e(t) = x(t) = t^4 + 4t^2 + 6}$$

6. Sketch the even and odd components of exponential signal x(t)=10 e^{-2t}

1) Draw the signal x (t).
2) Draw its folded version x (-t).
3) Add x (t) and x (-t) or subtract x (-t) from x (t).
4) Even & odd component is calculated by,

$$x_{ev}(t) = \frac{x(t) + x(-t)}{2} \qquad x_{od}(t) = \frac{x(t) - x(-t)}{2}$$


(a)
$x(t)=10e^{-2t}$
10
$x(t)$
$-t$
0
t
(b)
$x(-t)$
10
$-t$
0
t
(c)
$x(t)+x(-t)$
10
$-t$
0
t
(d)
$x_e(t)$
5
$-t$
0
t
(e)
$-t$
0
t
-10
$-x(-t)$
(f)
$x(t)-x(-t)$
10
$-t$
0
t
(g)
$x_0(t)$
5
$-t$
0
t
-5

8. Determine the following signals are energy or power signal?

(a) x(t)= u(t)

The unit-step function, defined by

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

is a power signal, since

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u^2(t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} 1\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

P=1/2 watts

Elementary signals

In the analysis of communication system, standard test signals play very important role. Such signals are used to check the performance of the system. Applying such signals at the system; the output is checked. Now depending on the input-output characteristic of that particular system study of different properties of a system can be done. Some standard test signals are as follows:

1) DC signal 2) Unit step signal 3) Delta or unit impulse function

4) Unit ramp signal 5) Sinusoidal signal 6) Rectangular pulse

7) Exponential signal 8) Signum function 9) Sinc function

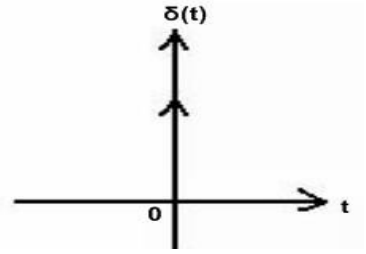
1. Unit delta function δ(t)

CT unit impulse δ(t) : A continuous time delta function is denoted by δ(t). Mathematically it is expressed as follows:

δ(t)=1 for t=0

δ(t)=0 for t≠0

The graphical representation of delta function for C.T. signal is shown in figure



CT Unit impulse

The delta function is an extremely important function used for the analysis of communication systems.The unit impulse, $\delta(t)$, is a function that is zero for all $t \neq 0$ and for which

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

The area under the pulse is unity. Due to its unity area it is called as a unit impulse function.

Importance of impulse function:

1. By applying impulse signal to a system one can get the impulse response of the system. From the impulse response it is possible to get the transfer function of the system.

2. For a LTI system if the area under the impulse response curve is finite, then the system is said to be stable.

3. From the impulse response of the system, one can easily get the step response by integrating it once &twice respectively.

4. It is easy to generate &apply to any system.

Properties of impulse response:

1. $\delta(at) = 1/a\,\delta(t)$
2. $\delta(-t) = \delta(t)$
3. $\int_{-\infty}^{\infty} \delta(t)dt = 1$
4. $X(t) * \delta(t) = X(t)$
5. $X(t)) * \delta(t-t_0) = X(t_0)$

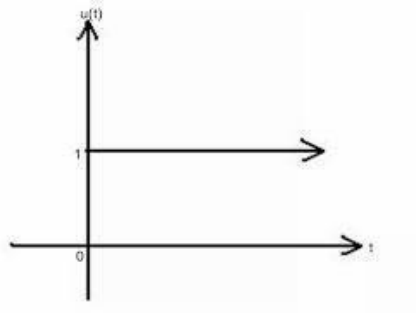
s

3. Unit step function

CT unit step : A continuous time unit step signal is denoted by u(t). Its value is unity (1) for all positive values of t. that means its value is one for t ≥ 0. While for other values of t, its value is zero. Unit step function is defined as U(t)= 1 for t ≥ 0

0 Otherwise

The graphical representation of CT unit step function is as shown in figure below:



Unit step function

The unit step and the impulse functions are related to one another by

$$u(t) = \int_{-\infty}^{t} \delta(t)dt$$

$$\delta(t) = \frac{du(t)}{dt}$$

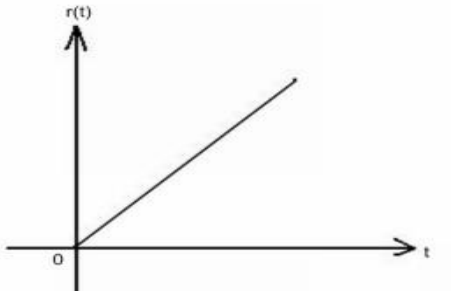
4. Unit Ramp signal

CT unit Ramp signal: A continuous time unit ramp signal is denoted by r(t). Its value is t for all positive values of t. While for other values of t, its value is zero.

Unit ramp function is defined as r(t) = t for t ≥ 0

0 otherwise

The graphical representation of CT unit ramp signal is as shown in figure below:



CT unit Ramp signal

Relationship between unit impulse, step and ramp signal

1. Integrating the unit step signal we get,

$$\int u(t)dt = \int dt = t$$

By integrating the unit step function, unit ramp function is obtained. In the reverse process, by differentiating unit ramp function, the unit step function is obtained.

2. The continuous time unit step function is the running integral of unit impulse function which is expressed as,

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$\boxed{\frac{du(t)}{dt} = \delta(t)}$$

3. By differentiating the ramp function twice, the impulse function is obtained.

$$r(t) = t$$

$$\frac{dr(t)}{dt} = 1 = u(t)$$

$$\boxed{\frac{d^2r(t)}{dt^2} = \frac{du(t)}{dt} = \delta(t)}$$

Thus impulse function is obtained by differentiating the ramp function twice. In reverse process, By integrating unit impulse function twice, the ramp function is obtained which is mathematically expressed as follows:

$$\boxed{r(t) = \iint \delta(t)\, dt}$$

The relationship between unit step, impulse and ramp signals are represented below

$$\delta(t) \xrightarrow{\text{integrate}} u(t) \xrightarrow{\text{integrate}} r(t)$$

$$r(t) \xrightarrow{\text{differentiate}} u(t) \xrightarrow{\text{differentiate}} \delta(t)$$

4 . Signum function (sgn(t))

The CT signum function is defined as

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

The graphical representation of CT signum function is as shown in figure below:

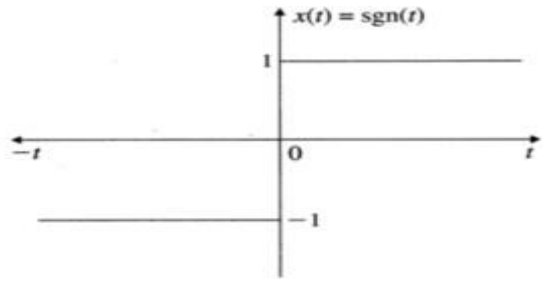


Fig : Representation of unit Signum function

Signum function is an odd function.

The relation between sgn(t) and u(t) as follows: Sgn(t) = -1+ 2 u(t)

Sgn(t) = u(t)- u(-t)

5. Sinusoidal signal

Continuous-Time Sinusoidal Signals: A simple harmonic oscillation is mathematically described by the following continuous-time sinusoidal signal as shown in figure.

$$x_a(t) = A\cos(\Omega t + \theta), -\infty < t < \infty$$

The subscript *a* used with x (t) denotes an analog signal. This signal is completely characterized by three parameters:

A is the *amplitude* of the sinusoid.

Ω is the *frequency* in radian s per second (rad/s),

Θ is the *phase* in radians.

Instead of Ω, we often use the frequency *F* in cycles per second or hertz (Hz), where

$$\Omega = 2\pi F$$

$X_a(t)$ can be written as

$$x_a(t) = A\cos(2\pi Ft + \theta), -\infty < t < \infty$$

We will use both forms for representing sinusoidal signals.

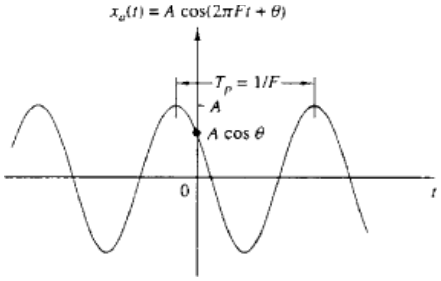


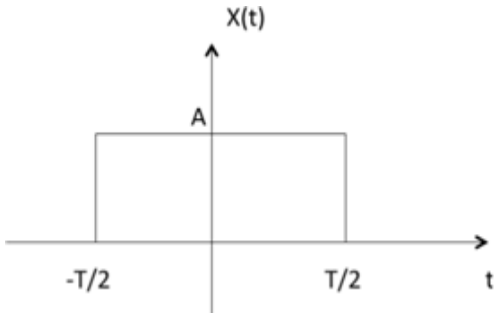
Fig : Analog sinusoidal signal

6. Rectangular pulse

The Rectangular signal is having constant amplitude A for the time interval between –T/2 to T/2. Mathematically it is defined as , A rect(t/T) = A for –T/2 <t< T/2

= 0 Otherwise

The graphical representation of CT Rectangular function is as shown in figure below:



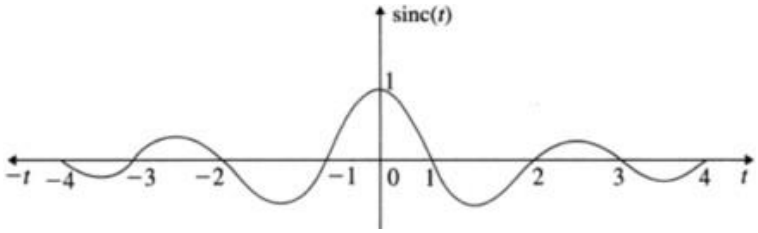
CT rectangular pulse

7. SINC Function

Sinc function is defined as follows:

$$\text{sinc}(t) = \frac{\sin \pi t}{\pi t} \qquad -\infty < t < \infty.$$

Sinc(t) = 1 at t=0 and Sinc(t) = 0 at t =1,-1,2,-2,3,-3 …..



Representation of SINC Function

8 . CT complex exponential signal

The CT complex exponential signal is of the following form $x(t) = c\, e^{\alpha t}$

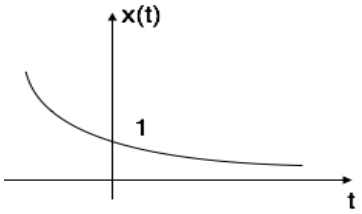
Where c,α are parameters.

Types of CT complex exponential signal:

Depending on the type of c,α, the complex exponential signal can be classified as,

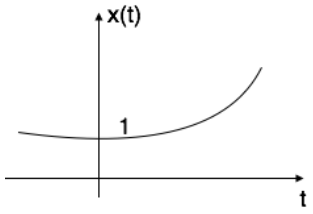
1. Real exponential signal: If c,α are real.

a) Decaying exponential $x(t) = e^{-\alpha t}$



Decaying signal

b) Rising exponential $x(t) = e^{\alpha t}$



exponential Rising

Basic operations on the signal

The basic operations performed on the signal are

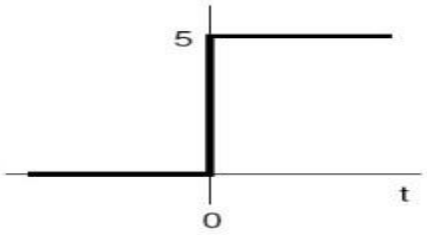
1) Amplitude scaling

Amplitude scaling means changing an amplitude of given continuous time signal. We will denote continuous time signal by x(t). If it is multiplied by some constant 'B' then resulting signal is,

y(t)= B x(t)

Example: Sketch y(t) = 5u(t)

Solution: we know that u(t) is unit step function. So if we multiply it with 5, its amplitude will become 5 and it shown as follow



2)Sum and difference of two signals:

Consider two signals $x_1(t)$ and $x_2(t)$. Then addition of these signals is denoted by $y(t)=x_1(t)+x_2(t)$. Similarly subtraction is given by $y(t)=x_1(t)-x_2(t)$.

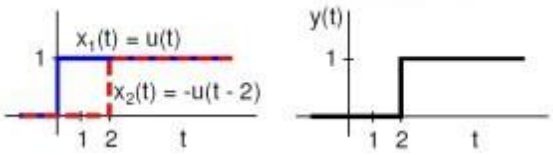
Example: Sketch y(t) = u(t) – u(t – 2)

Solution: First, plot each of the portions of this signal separately

$x_1(t) = u(t)$ …….Simply a step signal

$x_2(t) = –u(t-2)$ ……. Delayed step signal by 2 units and multiplied by -1.

Then, move from one side to the other, and add their instantaneous values:



2) Product of two signals:

If $x_1(t)$ and $x_2(t)$ are two continuous signals then the product of $x_1(t)$ and $x_2(t)$ is,**$Y(t) = x_1(t)\, x_2(t)$.**

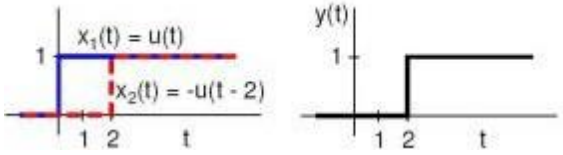
Example: Sketch $y(t) = u(t) \cdot u(t - 2)$

Solution: First, plot each of the portions of this signal separately

$x_1(t) = u(t)$ Simply a step signal

$x_2(t) = u(t-2)$ Delayed step signal

Then, move from one side to the other, and multiply instantaneous values:

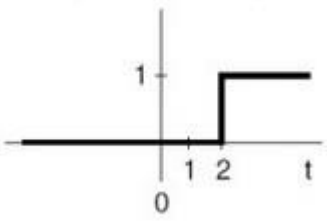


Operations performed on independent variables:

1) Time shifting:

A signal x(t) is said to be 'shifted in time' if we replace t by (t-T). thus x(t-T) represents the time shifted version of x(t) and the amount of time shift is 'T' sec. if T is positive then the shift is to right (delay) and if T is negative then the shift is to the left (advance).

Example: Sketch $y(t) = u(t - 2)$



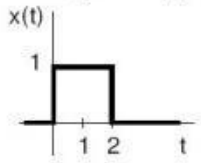
2) Time scaling:

The compression or expression of a signal in time is known as the time scaling. If x(t) is the original signal then x(at) represents its time scaled version. Where a is constant.

- If a> 1 then x(at) will be a compressed version of x(t) and
- if a< 1 then it will be a expanded version of x(t).

Example: Let x(t) = u(t) – u(t – 2). Sketch y(t) = x(t/2)

First, plot x(t)

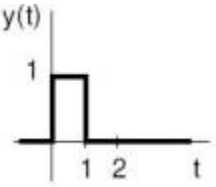


Replace all t's with 2t

$$y(t) = x(2t) = u(2t) - u(2t - 2)$$

Turns on at $2t \geq 0$, $t \geq 0$, No change

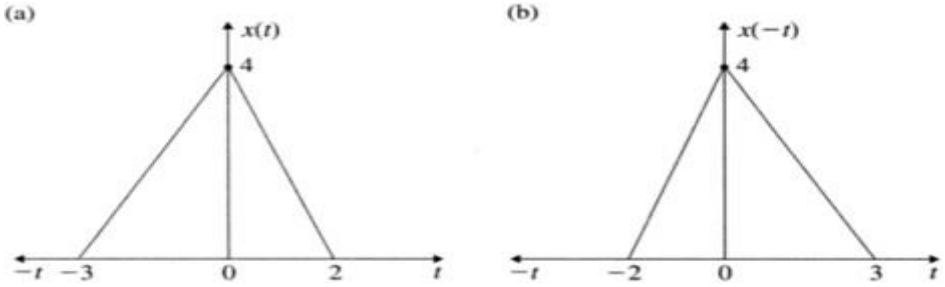
Turns on at $2t - 2 \geq 0$, $t \geq 1$



3) Time reversal (Time inversion):

Flips the signal about the y axis. y (t) = x(-t) .

Consider the signal x (t) shown in figure (a). The signal x (-t) is obtained by putting a mirror along the vertical axis. The signal to the right of the vertical axis gets reflected to the left and vice versa.Alternatively, if we make a folding across the vertical axis, the signal in the right of the axis is printed in left and vice versa. The signal so obtained is x (-t).

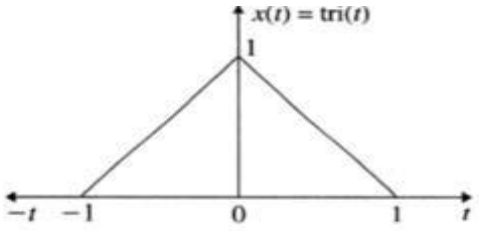


Original signal folded version x(-t)

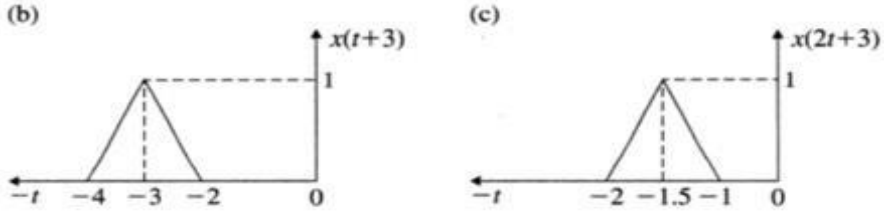
Q.Consider the triangular waveform x (t) shown in figure. Sketch the following waveforms. (a) x(2t+3) (b) x [(t+3)/2] (c) x (t/2-3) (d) x(-2t+3) (e) x (-2t-3)

To sketch x(2t+3)

Figure (a) shows x (t) = tri(t).by shifting by t=3 towards left,x(t+3) is obtained and this is sketched in figure (b).x(t+3) is time compressed by a factor to get x(2t+3).this is sketched in figure(c).



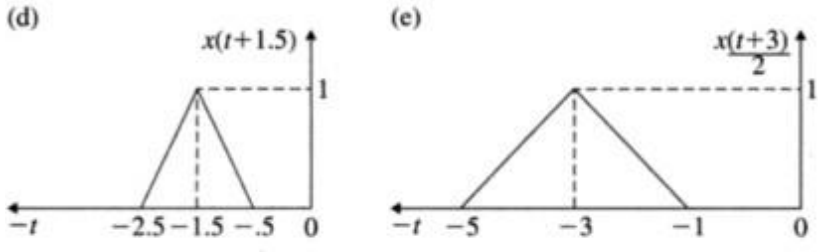
(a) $x(t) = \text{tri}(t)$.



(b) $x(t + 3)$; (c) $x(2t + 3)$.

(a)To sketch x [(t+3)/2]

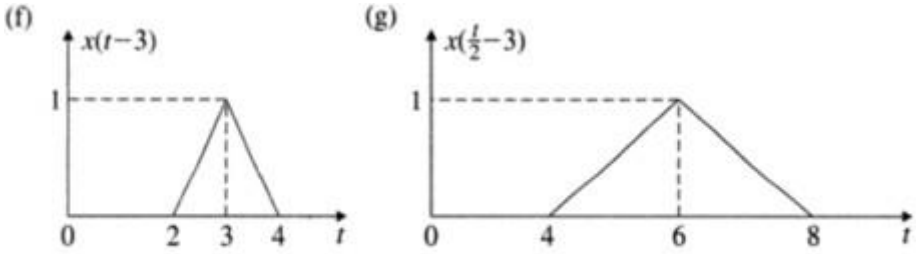
The signal x [(t+3)/2] is written as x(t/2+1.5). The signal x (t) is time shifted to the left by 1.5 unit to get x(t+1.5) which is nothing but x[(t+3)/2].This is sketched in figure(e).



(d) $x(t + 1.5)$; (e) $x\left(\frac{t+3}{2}\right)$.

(b) To sketch x (t/2-3)

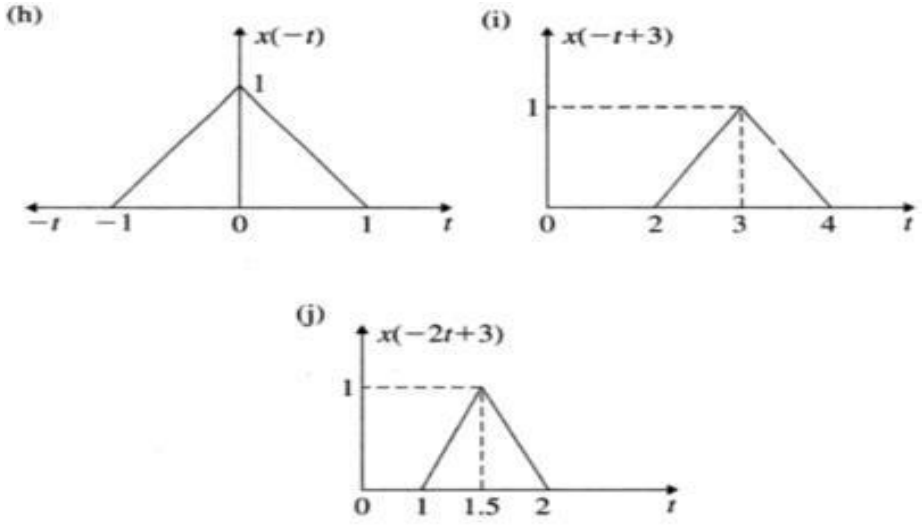
X(t-3) is obtained from x(t) by shifting the signal x(t) to the right by 3 unit and is shown in figure(f).by time expansion of x(t-3) by a factor 2, x(t/2-3) is obtained and sketched as shown in figure (g).



(f) $x(t-3)$; (g) $x\left(\frac{t}{2}-3\right)$.

(c) To sketch x(-2t+3):

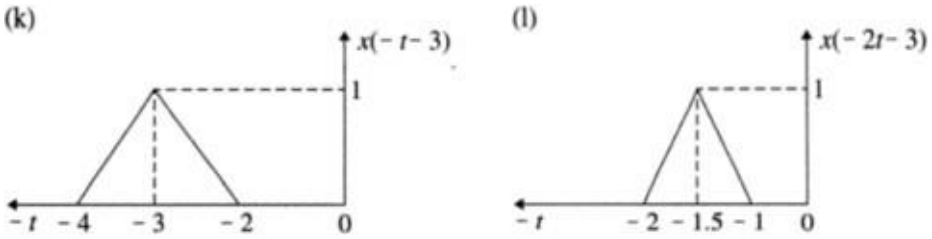
Signal x(-t) is obtained by folding and it is shown in figure (h).x(-t) is time shifted to the right by 3 unit to get x(-t+3).this is shown in figure(i).the signal x(-t+3) is time compressed by a factor 2 to get x(-2t+3).



(h) $x(-t)$; (i) $x(-t+3)$; (j) $x(-2t+3)$.

(d) To sketch x (-2t-3)

X(-t) is time shifted towards left by 3 units to get x(-t-3).this is shown in figure(k).x(t-3) is time compressed by a factor 2 to get x(-2t-3).this is sketched in figure (l)



(k) $x(-t-3)$; (l) $x(-2t-3)$.

UNIT-II
DISCRETE TIME SIGNALS AND SYSTEMS

2.1 DISCRETE TIME SIGNALS

A discrete time signal is defined as the one that is defined at distinct time intervals

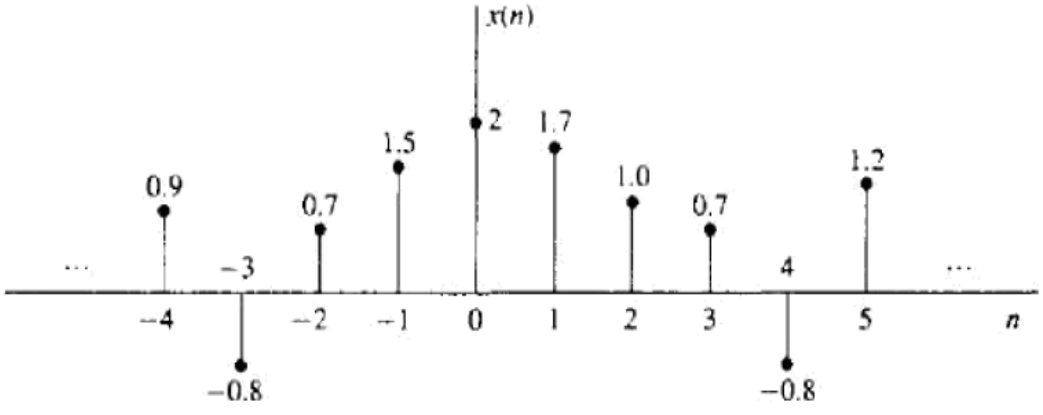


Figure 2.1 Graphical representation of a discrete-time signal.

two successive samples. Also, it is incorrect to think that $x(n)$ is equal to zero if n is not an integer. Simply, the signal $x(n)$ is not defined for noninteger values of n.

In the sequel we will assume that a discrete-time signal is defined for every integer value n for $-\infty < n < \infty$. By tradition, we refer to $x(n)$ as the "nth sample" of the signal even if the signal $x(n)$ is inherently discrete time (i.e., not obtained by sampling an analog signal). If, indeed, $x(n)$ was obtained from sampling an analog signal $x_a(t)$, then $x(n) \equiv x_a(nT)$, where T is the sampling period (i.e., the

by sampling an analog signal). If, indeed, $x(n)$ was obtained from sampling an analog signal $x_a(t)$, then $x(n) \equiv x_a(nT)$, where T is the sampling period (i.e., the time between successive samples).

Besides the graphical representation of a discrete-time signal or sequence as illustrated in Fig. 2.1, there are some alternative representations that are often more convenient to use. These are:

1. Functional representation, such as

$$x(n) = \begin{cases} 1, & \text{for } n = 1, 3 \\ 4, & \text{for } n = 2 \\ 0, & \text{elsewhere} \end{cases} \qquad (2.1.1)$$

2. Tabular representation, such as

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\cdots$

3. Sequence representation

An infinite-duration signal or sequence with the time origin ($n = 0$) indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{\ldots 0, 0, 1, 4, 1, 0, 0, \ldots\} \qquad (2.1.2)$$
$$\uparrow$$

A sequence $x(n)$, which is zero for $n < 0$, can be represented as

$$x(n) = \underset{\uparrow}{\{0, 1, 4, 1, 0, 0, \ldots\}} \tag{2.1.3}$$

The time origin for a sequence $x(n)$, which is zero for $n < 0$, is understood to be the first (leftmost) point in the sequence.

A finite-duration sequence can be represented as

$$x(n) = \{3, -1, -2, 5, 0, 4, -1\} \tag{2.1.4}$$
$$\uparrow$$

whereas a finite-duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

$$x(n) = \{0, 1, 4, 1\} \tag{2.1.5}$$
$$\uparrow$$

The signal in (2.1.4) consists of seven samples or points (in time), so it is called or identified as a seven-point sequence. Similarly, the sequence given by (2.1.5) is a four-point sequence.

2.1.1 Some Elementary Discrete-Time Signals

In our study of discrete-time signals and systems there are a number of basic signals that appear often and play an important role. These signals are defined below.

1. The *unit sample sequence* is denoted as $\delta(n)$ and is defined as

$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \tag{2.1.6}$$

   In words, the unit sample sequence is a signal that is zero everywhere, except at $n = 0$ where its value is unity. This signal is sometimes referred to as a *unit impulse*. In contrast to the analog signal $\delta(t)$, which is also called a unit impulse and is defined to be zero everywhere except $t = 0$, and has unit area, the unit sample sequence is much less mathematically complicated. The graphical representation of $\delta(n)$ is shown in Fig. 2.2.

2. The *unit step signal* is denoted as $u(n)$ and is defined as

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.7}$$

   Figure 2.3 illustrates the unit step signal.

3. The *unit ramp signal* is denoted as $u_r(n)$ and is defined as

$$u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \tag{2.1.8}$$

   This signal is illustrated in Fig. 2.4.

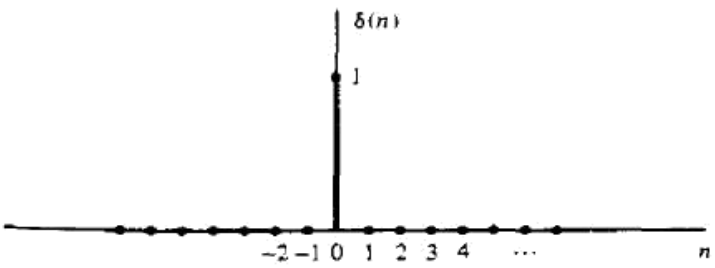


Figure 2.2 Graphical representation of the unit sample signal.

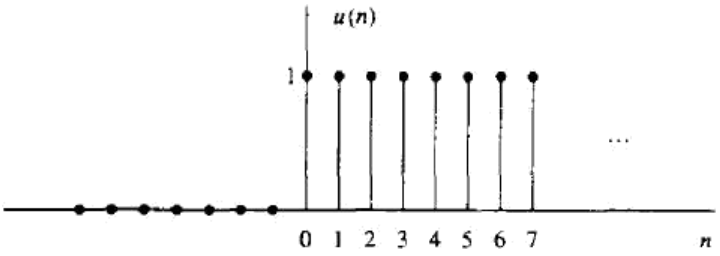


Figure 2.3 Graphical representation of the unit step signal.

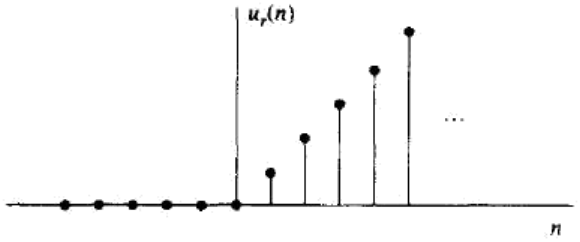


Figure 2.4 Graphical representation of the unit ramp signal.

4. The *exponential signal* is a sequence of the form

$$x(n) = a^n \qquad \text{for all } n \tag{2.1.9}$$

If the parameter a is real, then $x(n)$ is a real signal. Figure 2.5 illustrates $x(n)$ for various values of the parameter a.

When the parameter a is complex valued, it can be expressed as

$$a \equiv re^{j\theta}$$

where r and θ are now the parameters. Hence we can express $x(n)$ as

$$\begin{aligned} x(n) &= r^n e^{j\theta n} \\ &= r^n(\cos\theta n + j\sin\theta n) \end{aligned} \tag{2.1.10}$$

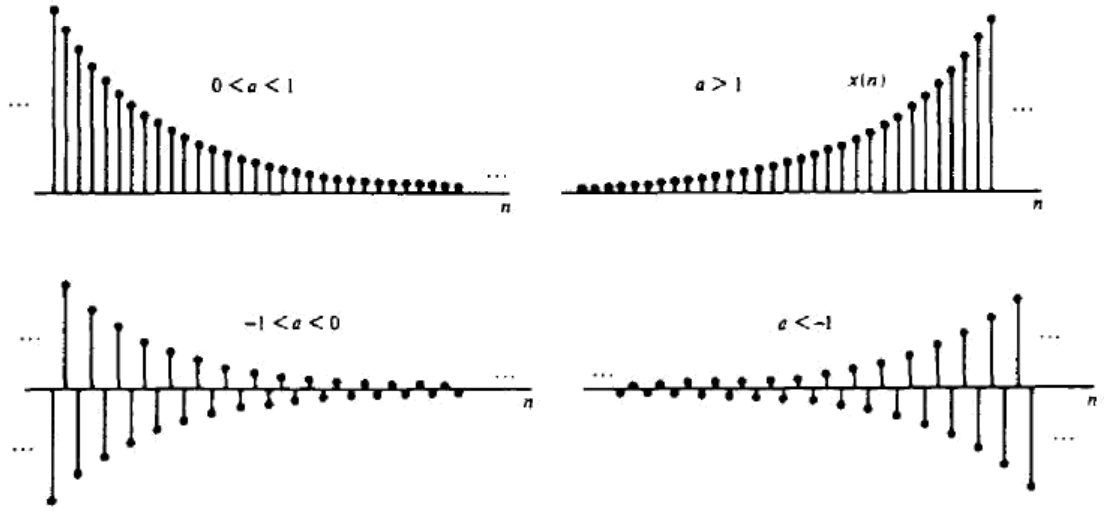


Figure 2.5 Graphical representation of exponential signals.

Since $x(n)$ is now complex valued, it can be represented graphically by plotting the real part

$$x_R(n) \equiv r^n \cos \theta n \tag{2.1.11}$$

as a function of n, and separately plotting the imaginary part

$$x_I(n) \equiv r^n \sin \theta n \tag{2.1.12}$$

as a function of n. Figure 2.6 illustrates the graphs of $x_R(n)$ and $x_I(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the signals $x_R(n)$ and $x_I(n)$ are a damped (decaying exponential) cosine function and a damped sine function. The angle variable θ is simply the frequency of the sinusoid, previously denoted by the (normalized) frequency variable ω. Clearly, if $r = 1$, the damping disappears and $x_R(n)$, $x_I(n)$, and $x(n)$ have a fixed amplitude, which is unity.

Alternatively, the signal $x(n)$ given by (2.1.10) can be represented graphically by the amplitude function

$$|x(n)| = A(n) \equiv r^n \tag{2.1.13}$$

and the phase function

$$\angle x(n) = \phi(n) \equiv \theta n \tag{2.1.14}$$

Figure 2.7 illustrates $A(n)$ and $\phi(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the phase function is linear with n. However, the phase is defined only over the interval $-\pi < \theta \leq \pi$ or, equivalently, over the interval $0 \leq \theta < 2\pi$. Consequently, by convention $\phi(n)$ is plotted over the finite interval $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. In other words, we subtract multiples of 2π from $\phi(n)$ before plotting. In one case, $\phi(n)$ is constrained to the range $-\pi < \theta \leq \pi$ and in the other case $\phi(n)$ is constrained to the range $0 \leq \theta < 2\pi$. The subtraction of multiples of 2π from $\phi(n)$ is equivalent to interpreting the function $\phi(n)$ as $\phi(n)$, modulo 2π. The graph for $\phi(n)$, modulo 2π, is shown in Fig. 2.7b.

2.1.2 Classification of Discrete-Time Signals

The mathematical methods employed in the analysis of discrete-time signals and systems depend on the characteristics of the signals. In this section we classify discrete-time signals according to a number of different characteristics.

Energy signals and power signals. The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.1.15}$$

We have used the magnitude-squared values of $x(n)$, so that our definition applies to complex-valued signals as well as real-valued signals. The energy of a signal can be finite or infinite. If E is finite (i.e., $0 < E < \infty$), then $x(n)$ is called an *energy*

Since $x(n)$ is now complex valued, it can be represented graphically by plotting the real part

$$x_R(n) \equiv r^n \cos \theta n \tag{2.1.11}$$

as a function of n, and separately plotting the imaginary part

$$x_I(n) \equiv r^n \sin \theta n \tag{2.1.12}$$

as a function of n. Figure 2.6 illustrates the graphs of $x_R(n)$ and $x_I(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the signals $x_R(n)$ and $x_I(n)$ are a damped (decaying exponential) cosine function and a damped sine function. The angle variable θ is simply the frequency of the sinusoid, previously denoted by the (normalized) frequency variable ω. Clearly, if $r = 1$, the damping disappears and $x_R(n)$, $x_I(n)$, and $x(n)$ have a fixed amplitude, which is unity.

Alternatively, the signal $x(n)$ given by (2.1.10) can be represented graphically by the amplitude function

$$|x(n)| = A(n) \equiv r^n \tag{2.1.13}$$

and the phase function

$$\angle x(n) = \phi(n) \equiv \theta n \tag{2.1.14}$$

Figure 2.7 illustrates $A(n)$ and $\phi(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the phase function is linear with n. However, the phase is defined only over the interval $-\pi < \theta \leq \pi$ or, equivalently, over the interval $0 \leq \theta < 2\pi$. Consequently, by convention $\phi(n)$ is plotted over the finite interval $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. In other words, we subtract multiplies of 2π from $\phi(n)$ before plotting. In one case, $\phi(n)$ is constrained to the range $-\pi < \theta \leq \pi$ and in the other case $\phi(n)$ is constrained to the range $0 \leq \theta < 2\pi$. The subtraction of multiples of 2π from $\phi(n)$ is equivalent to interpreting the function $\phi(n)$ as $\phi(n)$, modulo 2π. The graph for $\phi(n)$, modulo 2π, is shown in Fig. 2.7b.

2.1.2 Classification of Discrete-Time Signals

The mathematical methods employed in the analysis of discrete-time signals and systems depend on the characteristics of the signals. In this section we classify discrete-time signals according to a number of different characteristics.

Energy signals and power signals. The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.1.15}$$

We have used the magnitude-squared values of $x(n)$, so that our definition applies to complex-valued signals as well as real-valued signals. The energy of a signal can be finite or infinite. If E is finite (i.e., $0 < E < \infty$), then $x(n)$ is called an *energy*

Since $x(n)$ is now complex valued, it can be represented graphically by plotting the real part

$$x_R(n) \equiv r^n \cos \theta n \tag{2.1.11}$$

as a function of n, and separately plotting the imaginary part

$$x_I(n) \equiv r^n \sin \theta n \tag{2.1.12}$$

as a function of n. Figure 2.6 illustrates the graphs of $x_R(n)$ and $x_I(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the signals $x_R(n)$ and $x_I(n)$ are a damped (decaying exponential) cosine function and a damped sine function. The angle variable θ is simply the frequency of the sinusoid, previously denoted by the (normalized) frequency variable ω. Clearly, if $r = 1$, the damping disappears and $x_R(n)$, $x_I(n)$, and $x(n)$ have a fixed amplitude, which is unity.

Alternatively, the signal $x(n)$ given by (2.1.10) can be represented graphically by the amplitude function

$$|x(n)| = A(n) \equiv r^n \tag{2.1.13}$$

and the phase function

$$\angle x(n) = \phi(n) \equiv \theta n \tag{2.1.14}$$

Figure 2.7 illustrates $A(n)$ and $\phi(n)$ for $r = 0.9$ and $\theta = \pi/10$. We observe that the phase function is linear with n. However, the phase is defined only over the interval $-\pi < \theta \leq \pi$ or, equivalently, over the interval $0 \leq \theta < 2\pi$. Consequently, by convention $\phi(n)$ is plotted over the finite interval $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$. In other words, we subtract multiples of 2π from $\phi(n)$ before plotting. In one case, $\phi(n)$ is constrained to the range $-\pi < \theta \leq \pi$ and in the other case $\phi(n)$ is constrained to the range $0 \leq \theta < 2\pi$. The subtraction of multiples of 2π from $\phi(n)$ is equivalent to interpreting the function $\phi(n)$ as $\phi(n)$, modulo 2π. The graph for $\phi(n)$, modulo 2π, is shown in Fig. 2.7b.

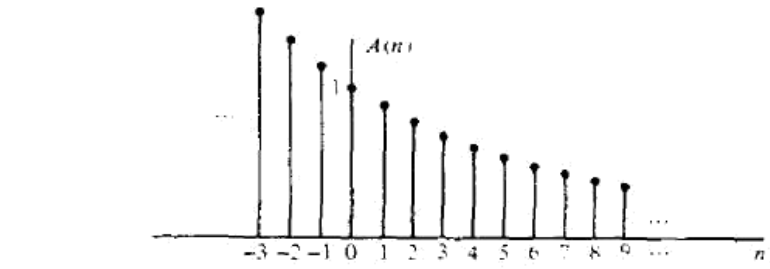
2.1.2 Classification of Discrete-Time Signals

The mathematical methods employed in the analysis of discrete-time signals and systems depend on the characteristics of the signals. In this section we classify discrete-time signals according to a number of different characteristics.

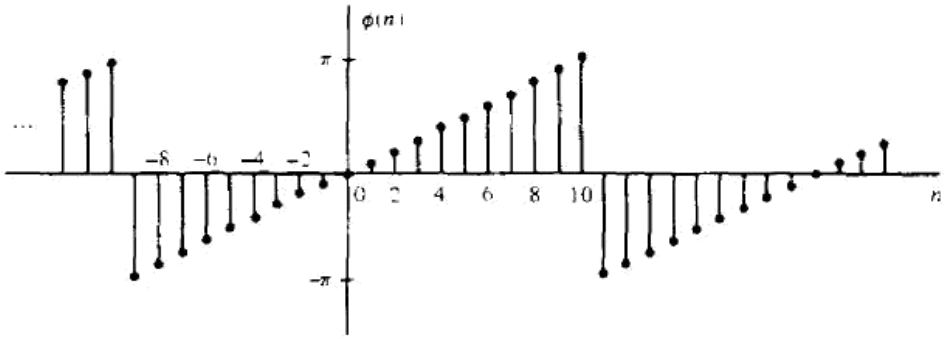
Energy signals and power signals. The energy E of a signal $x(n)$ is defined as

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 \tag{2.1.15}$$

We have used the magnitude-squared values of $x(n)$, so that our definition applies to complex-valued signals as well as real-valued signals. The energy of a signal can be finite or infinite. If E is finite (i.e., $0 < E < \infty$), then $x(n)$ is called an *energy*



(a) Graph of $A(n) = r^n$, $r = 0.9$



(b) Graph of $\phi(n) = \frac{\pi}{10} n$, modulo 2π plotted in the range $(-\pi, \pi)$

Figure 2.7 Graph of amplitude and phase function of a complex-valued exponential signal: (a) graph of $A(n) = r^n$, $A = 0.9$; (b) graph of $\phi(n) = (\pi/10)n$, modulo 2π plotted in the range $(-\pi, \pi]$.

signal. Sometimes we add a subscript x to E and write E_x to emphasize that E_x is the energy of the signal $x(n)$.

Many signals that possess infinite energy, have a finite average power. The average power of a discrete-time signal $x(n)$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \tag{2.1.16}$$

If we define the signal energy of $x(n)$ over the finite interval $-N \leq n \leq N$ as

$$E_N \equiv \sum_{n=-N}^{N} |x(n)|^2 \tag{2.1.17}$$

then we can express the signal energy E as

$$E \equiv \lim_{N \to \infty} E_N \tag{2.1.18}$$

and the average power of the signal $x(n)$ as

$$P \equiv \lim_{N \to \infty} \frac{1}{2N+1} E_N \tag{2.1.19}$$

Clearly, if E is finite, $P = 0$. On the other hand, if E is infinite, the average power P may be either finite or infinite. If P is finite (and nonzero), the signal is called a *power signal*. The following example illustrates such a signal.

Example 2.1.1

Determine the power and energy of the unit step sequence. The average power of the unit step signal is

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} u^2(n)$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1} = \lim_{N\to\infty} \frac{1+1/N}{2+1/N} = \frac{1}{2}$$

Consequently, the unit step sequence is a power signal. Its energy is infinite.

Similarly, it can be shown that the complex exponential sequence $x(n) = Ae^{j(\omega_0 n}$ has average power A^2, so it is a power signal. On the other hand, the unit ramp sequence is neither a power signal nor an energy signal.

Periodic signals and aperiodic signals. As defined on Section 1.3, a signal $x(n)$ is periodic with period $N (N > 0)$ if and only if

$$x(n + N) = x(n) \text{ for all } n \tag{2.1.20}$$

The smallest value of N for which (2.1.20) holds is called the (fundamental) period. If there is no value of N that satisfies (2.1.20), the signal is called *nonperiodic* or *aperiodic*.

We have already observed that the sinusoidal signal of the form

$$x(n) = A \sin 2\pi f_0 n \tag{2.1.21}$$

is periodic when f_0 is a rational number, that is, if f_0 can be expressed as

$$f_0 = \frac{k}{N} \tag{2.1.22}$$

where k and N are integers.

The energy of a periodic signal $x(n)$ over a single period, say, over the interval $0 \le n \le N-1$, is finite if $x(n)$ takes on finite values over the period. However, the energy of the periodic signal for $-\infty \le n \le \infty$ is infinite. On the other hand, the average power of the periodic signal is finite and it is equal to the average power over a single period. Thus if $x(n)$ is a periodic signal with fundamental period N and takes on finite values, its power is given by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \tag{2.1.23}$$

Consequently, periodic signals are power signals.

Symmetric (even) and antisymmetric (odd) signals. A real-valued signal $x(n)$ is called symmetric (even) if

$$x(-n) = x(n) \tag{2.1.24}$$

On the other hand, a signal $x(n)$ is called antisymmetric (odd) if

$$x(-n) = -x(n) \tag{2.1.25}$$

We note that if $x(n)$ is odd, then $x(0) = 0$. Examples of signals with even and odd symmetry are illustrated in Fig. 2.8.

We wish to illustrate that any arbitrary signal can be expressed as the sum of two signal components, one of which is even and the other odd. The even signal component is formed by adding $x(n)$ to $x(-n)$ and dividing by 2, that is,

$$x_e(n) = \tfrac{1}{2}[x(n) + x(-n)] \tag{2.1.26}$$

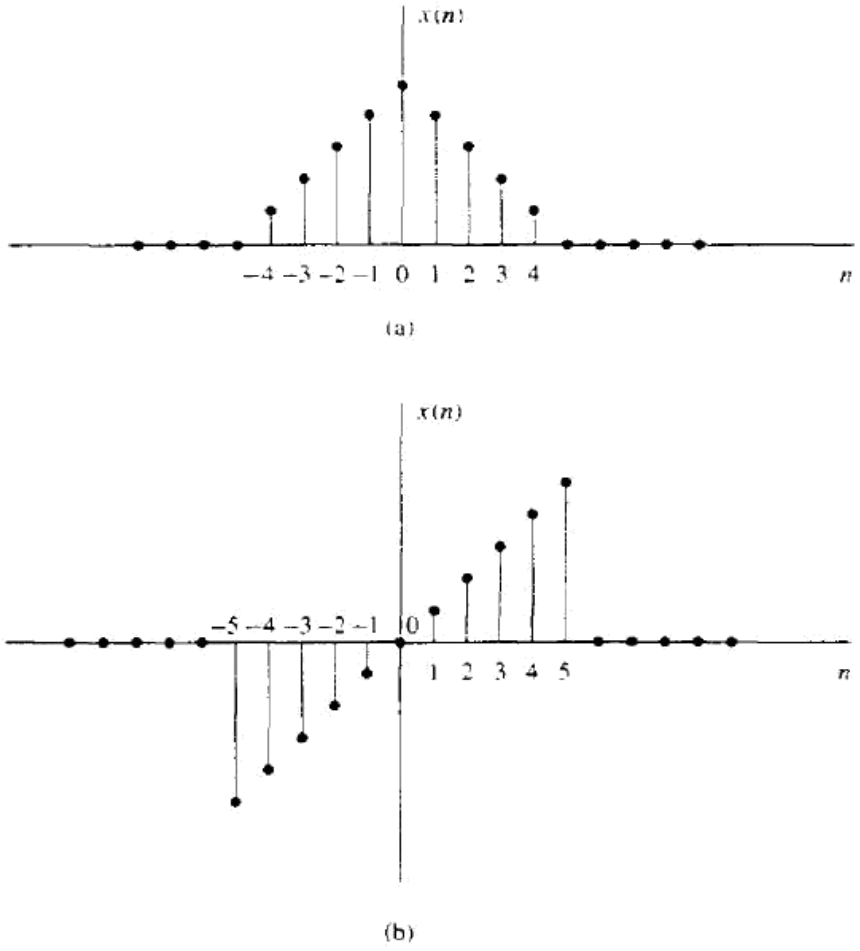


Figure 2.8 Example of even (a) and odd (b) signals.

DISCRETE-TIME SYSTEMS

In many applications of digital signal processing we wish to design a device or an algorithm that performs some prescribed operation on a discrete-time signal. Such a device or algorithm is called a discrete-time system. More specifically, a *discrete-time system* is a device or algorithm that operates on a discrete-time signal, called the *input* or *excitation*, according to some well-defined rule, to produce another discrete-time signal called the *output* or *response* of the system. In general, we view a system as an operation or a set of operations performed on the input signal $x(n)$ to produce the output signal $y(n)$. We say that the input signal $x(n)$ is *transformed* by the system into a signal $y(n)$, and express the general relationship between $x(n)$ and $y(n)$ as

$$y(n) \equiv \mathcal{T}[x(n)] \tag{2.2.1}$$

where the symbol $\mathcal{T}$ denotes the transformation (also called an operator), or processing performed by the system on $x(n)$ to produce $y(n)$. The mathematical relationship in (2.2.1) is depicted graphically in Fig. 2.12.

There are various ways to describe the characteristics of the system and the operation it performs on $x(n)$ to produce $y(n)$. In this chapter we shall be concerned with the time-domain characterization of systems. We shall begin with an input–output description of the system. The input–output description focuses on the behavior at the terminals of the system and ignores the detailed internal construction or realization of the system. Later, in Section 7.5, we introduce the state-space description of a system. In this description we develop mathematical equations that not only describe the input–output behavior of the system but specify its internal behavior and structure.

2.2.1 Input–Output Description of Systems

The input–output description of a discrete-time system consists of a mathematical expression or a rule, which explicitly defines the relation between the input and output signals (*input–output relationship*). The exact internal structure of the system is either unknown or ignored. Thus the only way to interact with the system is by using its input and output terminals (i.e., the system is assumed to be a "black box" to the user). To reflect this philosophy, we use the graphical representa-

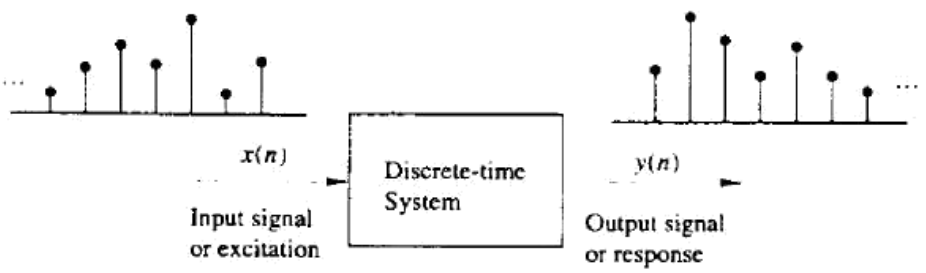


Figure 2.12 Block diagram representation of a discrete-time system.

internal description of the system, we know exactly how the system building blocks are configured. In terms of such a realization, we can see that a system is *relaxed* at time $n = n_0$ if the outputs of all the *delays* existing in the system are zero at $n = n_0$ (i.e., all memory is *filled* with zeros).

2.2.3 Classification of Discrete-Time Systems

In the analysis as well as in the design of systems, it is desirable to classify the systems according to the general properties that they satisfy. In fact, the mathematical techniques that we develop in this and in subsequent chapters for analyzing and designing discrete-time systems depend heavily on the general characteristics of the systems that are being considered. For this reason it is necessary for us to develop a number of properties or categories that can be used to describe the general characteristics of systems.

We stress the point that for a system to possess a given property, the property must hold for every possible input signal to the system. If a property holds for some input signals but not for others, the system does not possess that property. Thus a counterexample is sufficient to prove that a system does not possess a property. However, to prove that the system has some property, we must prove that this property holds for every possible input signal.

Static versus dynamic systems. A discrete-time system is called *static* or memoryless if its output at any instant n depends at most on the input sample at the same time, but not on past or future samples of the input. In any other case, the system is said to be *dynamic* or to have memory. If the output of a system at time n is completely determined by the input samples in the interval from $n - N$ to $n (N \geq 0)$, the system is said to have *memory* of duration N. If $N = 0$, the system is static. If $0 < N < \infty$, the system is said to have *finite memory*, whereas if $N = \infty$, the system is said to have *infinite memory*.

The systems described by the following input–output equations

$$y(n) = ax(n) \tag{2.2.7}$$

$$y(n) = nx(n) + bx^3(n) \tag{2.2.8}$$

are both static or memoryless. Note that there is no need to store any of the past inputs or outputs in order to compute the present output. On the other hand, the systems described by the following input–output relations

$$y(n) = x(n) + 3x(n-1) \tag{2.2.9}$$

$$y(n) = \sum_{k=0}^{n} x(n-k) \tag{2.2.10}$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \tag{2.2.11}$$

are dynamic systems or systems with memory. The systems described by (2.2.9)

(b) The input–output equation for this system is

$$y(n) = \mathcal{T}[x(n)] = nx(n) \tag{2.2.18}$$

The response of this system to $x(n-k)$ is

$$y(n, k) = nx(n-k) \tag{2.2.19}$$

Now if we delay $y(n)$ in (2.2.18) by k units in time, we obtain

$$\begin{aligned} y(n-k) &= (n-k)x(n-k) \\ &= nx(n-k) - kx(n-k) \end{aligned} \tag{2.2.20}$$

This system is time variant, since $y(n, k) \neq y(n-k)$.

(c) This system is described by the input–output relation

$$y(n) = \mathcal{T}[x(n)] = x(-n) \tag{2.2.21}$$

The response of this system to $x(n-k)$ is

$$y(n, k) = \mathcal{T}[x(n-k)] = x(-n-k) \tag{2.2.22}$$

Now, if we delay the output $y(n)$, as given by (2.2.21), by k units in time, the result will be

$$y(n-k) = x(-n+k) \tag{2.2.23}$$

Since $y(n, k) \neq y(n-k)$, the system is time variant.

(d) The input–output equation for this system is

$$y(n) = x(n) \cos \omega_0 n \tag{2.2.24}$$

The response of this system to $x(n-k)$ is

$$y(n, k) = x(n-k) \cos \omega_0 n \tag{2.2.25}$$

If the expression in (2.2.24) is delayed by k units and the result is compared to (2.2.25), it is evident that the system is time variant.

Linear versus nonlinear systems. The general class of systems can also be subdivided into linear systems and nonlinear systems. A linear system is one that satisfies the *superposition principle*. Simply stated, the principle of superposition requires that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual input signals. Hence we have the following definition of linearity.

Definition. A relaxed $\mathcal{T}$ system is linear if and only if

$$\mathcal{T}[a_1 x_1(n) + a_2 x_2(n)] = a_1 \mathcal{T}[x_1(n)] + a_2 \mathcal{T}[x_2(n)] \tag{2.2.26}$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

Figure 2.20 gives a pictorial illustration of the superposition principle.

and (2.2.10) have finite memory, whereas the system described by (2.2.11) has infinite memory.

We observe that static or memoryless systems are described in general by input–output equations of the form

$$y(n) = \mathcal{T}[x(n), n] \tag{2.2.12}$$

and they do not include delay elements (memory).

Time-invariant versus time-variant systems. We can subdivide the general class of systems into the two broad categories, time-invariant systems and time-variant systems. A system is called time-invariant if its input–output characteristics do not change with time. To elaborate, suppose that we have a system $\mathcal{T}$ in a relaxed state which, when excited by an input signal $x(n)$, produces an output signal $y(n)$. Thus we write

$$y(n) = \mathcal{T}[x(n)] \tag{2.2.13}$$

Now suppose that the same input signal is delayed by k units of time to yield $x(n-k)$, and again applied to the same system. If the characteristics of the system do not change with time, the output of the relaxed system will be $y(n-k)$. That is, the output will be the same as the response to $x(n)$, except that it will be delayed by the same k units in time that the input was delayed. This leads us to define a time-invariant or shift-invariant system as follows.

Definition. A relaxed system $\mathcal{T}$ is *time invariant* or *shift invariant* if and only if

$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

implies that

$$x(n-k) \xrightarrow{\mathcal{T}} y(n-k) \tag{2.2.14}$$

for every input signal $x(n)$ and every time shift k.

To determine if any given system is time invariant, we need to perform the test specified by the preceding definition. Basically, we excite the system with an arbitrary input sequence $x(n)$, which produces an output denoted as $y(n)$. Next we delay the input sequence by same amount k and recompute the output. In general, we can write the output as

$$y(n, k) = \mathcal{T}[x(n-k)]$$

Now if this output $y(n, k) = y(n-k)$, for all possible values of k, the system is time invariant. On the other hand, if the output $y(n, k) \neq y(n-k)$, even for one value of k, the system is time variant.

and the output of the second system is

$$\begin{aligned} y(n) &= \mathcal{T}_2[y_1(n)] \\ &= \mathcal{T}_2[\mathcal{T}_1[x(n)]] \end{aligned} \tag{2.2.47}$$

We observe that systems $\mathcal{T}_1$ and $\mathcal{T}_2$ can be combined or consolidated into a single overall system

$$\mathcal{T}_c \equiv \mathcal{T}_2\mathcal{T}_1 \tag{2.2.48}$$

Consequently, we can express the output of the combined system as

$$y(n) = \mathcal{T}_c[x(n)]$$

In general, the order in which the operations $\mathcal{T}_1$ and $\mathcal{T}_2$ are performed is important. That is,

$$\mathcal{T}_2\mathcal{T}_1 \neq \mathcal{T}_1\mathcal{T}_2$$

for arbitrary systems. However, if the systems $\mathcal{T}_1$ and $\mathcal{T}_2$ are linear and time invariant, then (a) $\mathcal{T}_c$ is time invariant and (b) $\mathcal{T}_2\mathcal{T}_1 = \mathcal{T}_1\mathcal{T}_2$, that is, the order in which the systems process the signal is not important. $\mathcal{T}_2\mathcal{T}_1$ and $\mathcal{T}_1\mathcal{T}_2$ yield identical output sequences.

The proof of (a) follows. The proof of (b) is given in Section 2.3.4. To prove time invariance, suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are time invariant; then

$$x(n-k) \xrightarrow{\mathcal{T}_1} y_1(n-k)$$

and

$$y_1(n-k) \xrightarrow{\mathcal{T}_2} y(n-k)$$

Thus

$$x(n-k) \xrightarrow{\mathcal{T}_c=\mathcal{T}_2\mathcal{T}_1} y(n-k)$$

and therefore, $\mathcal{T}_c$ is time invariant.

In the parallel interconnection, the output of the system $\mathcal{T}_1$ is $y_1(n)$ and the output of the system $\mathcal{T}_2$ is $y_2(n)$. Hence the output of the parallel interconnection is

$$\begin{aligned} y_3(n) &= y_1(n) + y_2(n) \\ &= \mathcal{T}_1[x(n)] + \mathcal{T}_2[x(n)] \\ &= (\mathcal{T}_1 + \mathcal{T}_2)[x(n)] \\ &= \mathcal{T}_p[x(n)] \end{aligned}$$

where $\mathcal{T}_p = \mathcal{T}_1 + \mathcal{T}_2$.

In general, we can use parallel and cascade interconnection of systems to construct larger, more complex systems. Conversely, we can take a larger system and break it down into smaller subsystems for purposes of analysis and implementation. We shall use these notions later, in the design and implementation of digital filters.

Definition. A system is said to be *causal* if the output of the system at any time n [i.e., $y(n)$] depends only on present and past inputs [i.e., $x(n)$, $x(n-1)$, $x(n-2), \ldots$], but does not depend on future inputs [i.e., $x(n+1)$, $x(n+2), \ldots$]. In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots] \tag{2.2.44}$$

where $F[\cdot]$ is some arbitrary function.

If a system does not satisfy this definition, it is called *noncausal*. Such a system has an output that depends not only on present and past inputs but also on future inputs.

It is apparent that in real-time signal processing applications we cannot observe future values of the signal, and hence a noncausal system is physically unrealizable (i.e., it cannot be implemented). On the other hand, if the signal is recorded so that the processing is done off-line (nonreal time), it is possible to implement a noncausal system, since all values of the signal are available at the time of processing. This is often the case in the processing of geophysical signals and images.

Example 2.2.6

Determine if the systems described by the following input–output equations are causal or noncausal.

(a) $y(n) = x(n) - x(n-1)$ **(b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$ **(c)** $y(n) = ax(n)$
(d) $y(n) = x(n) + 3x(n+4)$ **(e)** $y(n) = x(n^2)$ **(f)** $y(n) = x(2n)$
(g) $y(n) = x(-n)$

Solution The systems described in parts (a), (b), and (c) are clearly causal, since the output depends only on the present and past inputs. On the other hand, the systems in parts (d), (e), and (f) are clearly noncausal, since the output depends on future values of the input. The system in (g) is also noncausal, as we note by selecting, for example, $n = -1$, which yields $y(-1) = x(1)$. Thus the output at $n = -1$ depends on the input at $n = 1$, which is two units of time into the future.

Stable versus unstable systems. Stability is an important property that must be considered in any practical application of a system. Unstable systems usually exhibit erratic and extreme behavior and cause overflow in any practical implementation. Here, we define mathematically what we mean by a stable system, and later, in Section 2.3.6, we explore the implications of this definition for linear, time-invariant systems.

Definition. An arbitrary relaxed system is said to be bounded input–bounded output (BIBO) stable if and only if every bounded input produces a bounded output.

The conditions that the input sequence $x(n)$ and the output sequence $y(n)$ are bounded is translated mathematically to mean that there exist some finite numbers,

UNIT - III
Laplace and Z-transforms

Introduction

We know that Fourier transform exists if the signals have finite energy. But for the signals such as ramp, rising exponents etc. This condition of finite energy is not satisfied. Thus FT does not exist for such signals. By the use of Laplace transformation this limitation can be avoided.WKT in FT the variable s = jw.

But in Laplace transform variable s can be expressed as s = σ+jώ

σ - real part of which represents the attenuation factor.

Jώ- imaginary part, ώ- angular frequency.

Laplace transform exists for almost all signals of practical interest. Some of the advantages of Laplace transform are as follows: Laplace transform can be used for the analysis of unstable systems.

There are two types of Laplace transform:

1) Bilateral (or) two sided Laplace transform

2) Unilateral (or) one sided Laplace transform

1) Bilateral Laplace Transform

The Laplace transform can be alternatively defined as the bilateral Laplace transform or two-sided Laplace transform by extending the limits of integration to be the entire real axis.

The bilateral Laplace transform is defined as follows:

$$L\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

The f(t) and F(s) are Laplace transform pairs. It is written as,

$$f(t) \leftrightarrow F(s)$$

Where‘s’ is complex frequency, it is given as

$$s = \sigma + j\omega$$

Here σ is the attenuation constant or damping factor and w is the angular frequency. With above value of 's' we can write F(s) equation as,

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[f(t)\, e^{-\sigma t} \right] e^{-j\omega t}\, dt$$

The above equation shows that F(s) is basically a Fourier transform of f(t) $e^{-\sigma t}$.This is the relationship between Fourier transform and Laplace transform. The Fourier transform given by the above equation must exist, which is actually Laplace transform. Hence sufficient condition of f(t) to be Laplace transformable is that

$$\int_0^{\infty} |f(t)|\, e^{-\sigma t}\, dt < \infty$$

For real and positive values of σ.

2) Unilateral Laplace Transform

The unilateral Laplace transform is the special case of Laplace transform and is defined as,

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt.$$

The unilateral Laplace transform has the following features:

1. The unilateral Laplace transform simplifies the system analysis considerably.
2. The signals are restricted to casual signals.
3. There is one to one correspondence between LT and ILT.
4. In view of above advantages Laplace transform means unilateral Laplace transform.

3) Inverse Laplace Transform

The inverse Laplace transform is given as

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)\, e^{st}\, ds$$

The above formula of inverse Laplace transform involves a complex integration. In this chapter we will use a partial fraction expansion method to evaluate inverse Laplace transforms.

Region of convergence

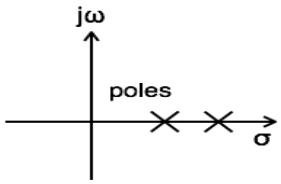
The range variation of σ for which the Laplace transform converges is called region of convergence.

Properties of ROC

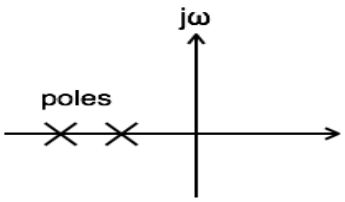
- ROC contains strip lines parallel to jω axis in s-plane.
- If x (t) is absolutely integral and it is of finite duration, then ROC is entire s-plane.
- If x (t) is a right sided sequence then ROC: Re{s} > σ_o.
- If x (t) is a left sided sequence then ROC: Re{s} < σ_o.
- If x (t) is a two sided sequence then ROC is the combination of two regions.

2.4.3 Causality and Stability

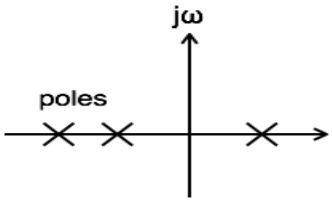
1. For a system to be causal, all poles of its transfer function must be right half of s-plane.



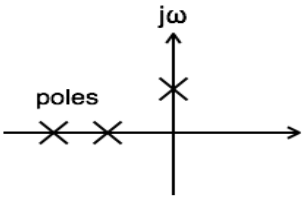
2. A system is said to be stable when all poles of its transfer function lay on the left half of s-plane.



3. A system is said to be unstable when at least one pole of its transfer function is shifted to the right half of s-plane.



4. A system is said to be marginally stable when at least one pole of its transfer function lies on the jω axis of s-plane.



Properties of Laplace transform

1. Linearity

Let $x_1(t)$ be the two Laplace transform pairs. Then linearity property states that,

$$\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

Here a_1 and a_2 are constants.

Proof: let us find the Laplace transform of $a_1f_1(t) + a_2 f_2(t)$ by applying definition.ie

$$x_1(t) \xleftrightarrow{L} X_1(s)$$

$$x_2(t) \xleftrightarrow{L} X_2(s)$$

$$[a_1x_1(t) + a_2x_2(t)] \xleftrightarrow{L} [a_1X_1(s) + a_2X_2(s)]$$

2. Time scaling property

It states that

$$x(at) \xleftrightarrow{L} \frac{1}{a}X\left(\frac{s}{a}\right)$$

3. Scaling in s domain

It states that

$$\boxed{\frac{1}{a}x\left(\frac{t}{a}\right) \xleftrightarrow{L} X(as)}$$

Proof: According to time scaling property**,**

$$x(at) \xleftrightarrow{L} \frac{1}{a}X\left(\frac{s}{a}\right)$$

Let

$$b = \frac{1}{a}$$

$$x\left(\frac{t}{b}\right) \xleftrightarrow{L} bX(bs)$$

Replacing b by a we get

$$\boxed{\frac{1}{a}x\left(\frac{t}{a}\right) \overset{L}{\longleftrightarrow} X(as)}$$

3. Time differentiation

It states that

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$
$$\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s) - x(0^-)$$
$$\frac{d^2x(t)}{dt^2} \overset{L}{\longleftrightarrow} s^2X(s) - sx(0^-) - \frac{d}{dt}x(0^-)$$

Proof: According to the definition of Laplace transform

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

The above integral is evaluated by parts using

$$\int udv = uv - \int vdu$$

Let $u = x(t)$ and $dv = e^{-st}dt$; $du = \frac{d}{dt}x(t)dt$ and $v = -\frac{1}{s}e^{-st}$

$$\int_0^\infty x(t)e^{-st}dt = \left[\frac{-1}{s}x(t)e^{-st}\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\frac{d}{dt}x(t)dt$$

or

$$X(s) = \frac{1}{s}x(0) + \frac{1}{s}\int_0^\infty e^{-st}\frac{d}{dt}x(t)dt.$$

But

$$L\left[\frac{d}{dt}(x(t))\right] = \int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt$$

$$\therefore \quad L\frac{d}{dt}(x(t)) \overset{L}{\longleftrightarrow} sX(s) - x(0^-)$$

The time differentiation twice is proved as follows:

$$\frac{d^2}{dt^2}(x(t)) = \frac{d}{dt}\left(\frac{d}{dt}(x(t))\right)$$

Using the property

$$\frac{d}{dt}(x(t)) \overset{L}{\longleftrightarrow} sX(s) - x(0^-)$$

We get

$$L\left[\frac{d^2(x(t))}{dt^2}\right] = sL\left[\frac{d}{dt}(x(t))\right] - \frac{d}{dt}(x(0^-))\Big|_{t=0}$$

$$\frac{d^2(x(t))}{dt^2} \overset{L}{\longleftrightarrow} s^2X(s) - sx(0^-) - \frac{d}{dt}(x(0^-))$$

In general

$$\frac{d^n x(t)}{dt^n} \overset{L}{\longleftrightarrow} s^n X(s) - s^{n-1}x(0^-) - s^{n-2}x(0^-) \cdots x^{n-1}(0^-)$$

OR

$$\frac{d^n x(t)}{dt^n} \overset{L}{\longleftrightarrow} s^n X(s) - \sum_{k=1}^{n} s^n x^{k-1}(0^-)$$

5. Time integration

The time integration property states that if

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$

$$\int_0^t x(\tau)d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s}$$

Proof: We define

$$f(t) = \int_0^t x(\tau)d\tau.$$

Differentiating the above equation we get

$$\frac{df(t)}{dt} = x(t) \quad \text{and} \quad x(0^-) = 0$$

if

$$f(t) \overset{L}{\longleftrightarrow} F(s)$$

$$X(s) = L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0^-) = sF(s) \quad \text{if} \quad f(0^-) = 0$$

$$F(s) = \frac{X(s)}{s}$$

$$\boxed{\int_0^t x(\tau)d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s}}$$

6. Time convolution

The time convolution property states that if

$$x_1(t) \xleftrightarrow{L} X_1(s)$$
$$x_2(t) \xleftrightarrow{L} X_2(s)$$
$$x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s)X_2(s)$$

Proof:

$$L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} e^{-st} \left[\int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau \right] dt$$
$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} e^{-st} x_2(t-\tau) dt \right] d\tau$$

The inner integral is the LT of $x_2(t-\varsigma)$ with a time delay ς. substituting

$$\int_{-\infty}^{\infty} e^{-st} x_2(t-\tau) dt = X_2(s) e^{-\tau s}$$

In the above equation, we get

$$L[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) X_2(s) e^{-\tau s} d\tau$$
$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau) e^{-\tau s} d\tau$$
$$= X_2(s) X_1(s)$$

$$\boxed{[x_1(t) * x_2(t)] \xleftrightarrow{L} X_1(s)X_2(s)}$$

7. Complex frequency differentiation

According to this property,

$$-tx(t) \xleftrightarrow{L} \frac{d}{ds}(X(s))$$

Proof: By definition of LT,

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

Differentiating both sided with respect to s,

$$\begin{aligned}\frac{d}{ds}(X(s)) &= \frac{d}{ds}\int_0^\infty x(t)e^{-st}dt \\ &= -\int_0^\infty tx(t)e^{-st}dt \\ &= -L[tx(t)]\end{aligned}$$

$$\therefore \quad \boxed{-tx(t) \xleftrightarrow{L} \frac{d}{ds}(X(s))}$$

8. Frequency shifting

According to this property,

$$[e^{s_0 t}x(t)] \xleftrightarrow{L} X(s - s_0)$$

$$L[e^{s_0 t}x(t)] = \int_0^\infty e^{s_0 t}x(t)e^{-st}dt \qquad \text{where } s_0 \text{ is a constant}$$

$$= \int_0^\infty x(t)e^{-(s-s_0 t)}dt = X(s - s_0)$$

$$\boxed{[e^{s_0 t}x(t)] \xleftrightarrow{L} X(s - s_0)}$$

9. Conjugation property

According to this property,

$$x(t) \overset{L}{\longleftrightarrow} X(s) \text{ then}$$

$$x^*(t) \overset{L}{\longleftrightarrow} X^*(-s)$$

Proof: By definition of LT

$$\begin{aligned} L[x^*(t)] &= \int_0^\infty x^*(t)e^{-st}dt \\ &= \int_0^\infty [x(t)e^{-(-s)t}dt]^* \\ &= X^*(-s) \end{aligned}$$

$$\boxed{x^*(t) \overset{L}{\longleftrightarrow} X^*(-s)}$$

10. Initial value theorem

According to this theorem,

$$\lim_{t \to 0} x(t) = \lim_{s \to \infty} sX(s)$$

Proof:

$$L\left[\frac{d}{dt}x(t)\right] = \int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = sX(s) - x(0)$$

Let $s \to \infty$; then

$$\lim_{s\to\infty}\int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = \lim_{s\to\infty}[sX(s) - x(0)]$$

$$0 = \lim_{s\to\infty}[sX(s) - x(0)]$$

Since $x(0) = \lim_{t\to 0} x(t)$

$$\boxed{\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s)}$$

11. Final value theorem

According to this theorem,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Proof: The LT of d/dt(x (t)) could be written as

$$\int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = [sX(s) - x(0)]$$

Taking s-> 0 on both sides of the above equation, we get

$$\int_0^\infty \frac{d}{dt}(x(t))dt = \lim_{s\to 0}[sX(s) - x(0)]$$

$$\lim_{t\to\infty}[x(t) - x(0)] = \lim_{s\to 0}[sX(s) - x(0)]$$

$$\boxed{\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)}$$

The above theorem is valid if X(s) has no poles in RHP of s-plane.

1. Find the Laplace transform of ramp signal.

The ramp signal is given as,

$$r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{otherwise.}$$

or $$r(t) = t\,u(t)$$

By the definition of Laplace transform,

$$\mathcal{L}[r(t)] = \int_{0-}^{\infty} t\, e^{-st}\, dt$$

Integrating the above equation by parts we get,

$$\mathcal{L}[r(t)] = \left[t \cdot \frac{e^{-st}}{-s}\right]_{0-}^{\infty} - \int_{0-}^{\infty} 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= \left[\frac{t\,e^{-st}}{-s}\right]_{0-}^{\infty} - \left[\frac{e^{-st}}{-s^2}\right]_{0-}^{\infty}$$

$$= \frac{1}{s^2}$$

Thus $$\boxed{\mathcal{L}\, t\,u(t) = \frac{1}{s^2}}$$

2. Find the Laplace transform of delayed ramp signal.

Solution:

If unit ramp function is delayed by time t_0, it is given as,

$$r(t - t_0) = t - t_0 \quad \text{for } t \geq t_0$$
$$= 0 \quad \text{otherwise.}$$

By the shifting property of Laplace transform

$$\mathcal{L}\left[r(t-t_0)\right] = \frac{e^{-st_0}}{s^2}$$

Similarly

$$\mathcal{L}\left[Kr(t-t_0)\right] = \frac{Ke^{-st_0}}{s^2}$$

3. Find out the Laplace transform of impulse function.

We have evaluated the relationship between unit impulse function and step function. The differentiation of unit step function gives unit impulse function i.e.,

$$\delta(t) = \frac{d}{dt}u(t)$$

Taking Laplace transform on both sides,

$$\mathcal{L}\left[\delta(t)\right] = \mathcal{L}\left[\frac{d}{dt}u(t)\right]$$

By differentiation property, the RHS of above equation will be,

$$\mathcal{L}\left[\delta(t)\right] = \mathcal{L}\left[\frac{d}{dt}u(t)\right]$$
$$= sF(s) - f(0-)$$

In the above equation

$$F(s) = \mathcal{L}\,u(t) = \frac{1}{s}.$$

$$f(0-) = u(t)\big|_{t=0} = 0$$

Therefore

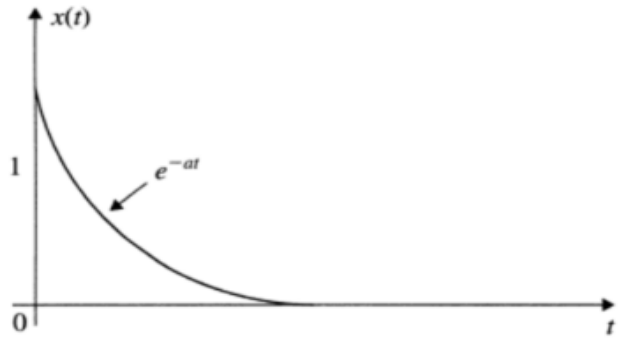
$$\mathcal{L}\left[\delta(t)\right] = s\cdot\frac{1}{s} - 0 = 1$$

Thus $\boxed{\mathcal{L}\left[\delta(t)\right] = 1}$

If the impulse function is delayed by t_0, then its Laplace transform can be obtained by shifting property as,

$$\mathcal{L}\left[\delta(t-t_0)\right] = e^{-st_0}$$

4. Determine the LT of an exponential decay which is shown in figure.



Solution: The exponential decay is represented by

$$x(t) = e^{-at}u(t) \quad t \geq 0.$$

Taking LT for the above function we get

$$L[e^{-at}u(t)] = \int_0^\infty e^{-at}e^{-st}dt$$
$$= \int_0^\infty e^{-(s+a)t}dt$$

$$L[e^{-at}u(t)] = -\frac{1}{(s+a)}\left[e^{-(s+a)t}\right]_0^{\infty}$$

$$= \frac{1}{(s+a)} \quad \text{with ROC: Re } s > -a$$

$$\boxed{L[e^{-at}u(t)] = \frac{1}{(s+a)}}$$

5. Find out the Laplace transform of sine wave.

Solution:

A sine wave is given as,

$$f(t) = A \sin \omega_0 t$$

We know that sin w_0t can be represented using Euler's identity as,

$$\sin \omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

So f(t) becomes

$$f(t) = \frac{A}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

Taking Laplace transform on both sides,

$$\mathcal{L}[f(t)] = \frac{A}{2j}\left\{\mathcal{L}\left[e^{j\omega_0 t}\right] - \mathcal{L}\left[e^{-j\omega_0 t}\right]\right\}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\therefore \quad \mathcal{L}\left[e^{j\omega_0 t}\right] = \frac{1}{s-j\omega_0}$$

$$\text{and} \quad \mathcal{L}\left[e^{-j\omega_0 t}\right] = \frac{1}{s+j\omega_0}$$

Putting these values in L{f(t)} we get,

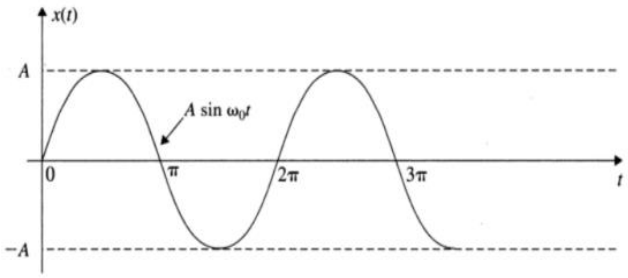
$$\mathcal{L}[f(t)] = \frac{A}{2j}\left\{\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right\}$$

$$= \frac{A}{2j}\cdot\frac{2j\omega_0}{s^2+\omega_0^2}$$

$$= \frac{A\omega_0}{s^2+\omega_0^2}$$

Thus, $\boxed{\mathcal{L}[A\sin\omega_0 t] = \frac{A\omega_0}{s^2+\omega_0^2}}$

6.Determine the LT of a sine function which is shown in figure



Solution: A sinusoidal function shown in figure is mathematically expressed as follows:

$$x(t) = A\sin\omega_0 t\, u(t) \quad t \geq 0$$

The given sinusoidal function is written as follows using Euler's identity.

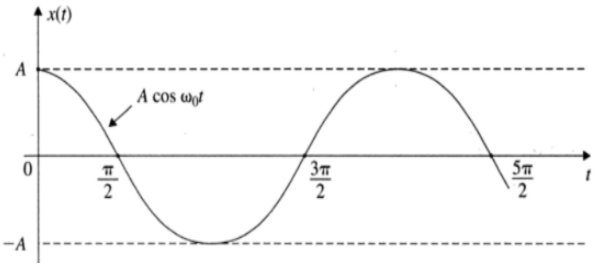
$$\sin\omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$L[A\sin\omega_0 t] = \frac{A}{2j}[L(e^{j\omega_0 t}) - Le^{-j\omega_0 t}]$$

$$L[A \sin \omega_0 t] = \frac{A}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right]$$
$$= \frac{A}{2j}\frac{2j\omega_0}{(s^2 + \omega_0^2)}$$

$$\boxed{L[A \sin \omega_0 t] = \frac{A\omega_0}{(s^2 + \omega_0^2)}} \quad \text{ROC: Re } s > 0.$$

7. Determine the LT of a cosine function which is shown in figure



Solution: A cosine function shown in figure is mathematically expressed as follows:

$$x(t) = A \cos \omega_0 t u(t) \quad t \leq 0.$$

The given sinusoidal function is written as follows using Euler's identity

$$A \cos \omega_0 t = \frac{A}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Taking LT for x (t), the following equation is written

$$L[A \cos \omega_0 t u(t)] = \frac{A}{2}[Le^{j\omega_0 t}u(t) + Le^{-j\omega_0 t}u(t)]$$

$$L[A\cos\omega_0 tu(t)] = \frac{A}{2}\left[\frac{1}{(s+j\omega_0)} + \frac{1}{(s-j\omega_0)}\right]$$

$$= \frac{As}{(s^2+\omega_0^2)}$$

$$\boxed{L[A\cos\omega_0 tu(t)] = \frac{As}{(s^2+\omega_0^2)}} \quad \text{ROC: Re}s > 0.$$

8. Find out the Laplace transform of sine wave.

Solution: A sine wave is given as,

$$f(t) = A\cos\omega_0 t$$

We know that sin w_0t can be represented using Euler's identity as,

$$\cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

So f(t) becomes

$$f(t) = \frac{A}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

Taking Laplace transform on both sides,

$$\mathcal{L}[f(t)] = \frac{A}{2}\mathcal{L}\left\{e^{j\omega_0 t} + e^{-j\omega_0 t}\right\}$$

$$\mathcal{L}\left[e^{j\omega_0 t}\right] = \frac{1}{s-j\omega_0}$$

and $$\mathcal{L}\left[e^{-j\omega_0 t}\right] = \frac{1}{s+j\omega_0}$$

$$\mathcal{L}[f(t)] = \frac{A}{2}\left\{\frac{1}{s-j\omega_0}+\frac{1}{s+j\omega_0}\right\}$$

$$= \frac{A}{2}\cdot\frac{2s}{s^2+\omega_0^2}$$

$$= \frac{A\cdot s}{s^2+\omega_0^2}$$

Thus,

$$\boxed{\mathcal{L}[A\cos\omega_0 t] = \frac{A\cdot s}{s^2+\omega_0^2}}$$

9. Find the Laplace transform of damped sine wave.

$$x(t) = e^{-at}\sin\omega_0 t.$$

We know that,

$$L[\sin\omega_0 t] = \frac{\omega_0}{(s^2+\omega_0^2)}$$

Using the complex shifting property,

$$L[e^{-at}x(t)] = X(s+a)$$

Applying the above property we get,

$$\boxed{L[e^{-at}\sin\omega_0 t] = \frac{\omega_0}{(s+a)^2+\omega_0^2}}$$

ROC: Re $s > -a$.

(Or)

$$f(t) = e^{-at}\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]$$

$$= \frac{1}{2j}\left\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right\}$$

Taking Laplace transform on both sides,

$$\mathcal{L}\, f(t) = \frac{1}{2j}\,\mathcal{L}\left\{e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2j}\left\{\frac{1}{s+(a-j\omega)} - \frac{1}{s+(a+j\omega)}\right\}$$

$$= \frac{1}{2j}\cdot\frac{2j\omega}{(s+a)^2+\omega^2}$$

$$= \frac{\omega}{(s+a)^2+\omega^2}$$

Thus, $\boxed{\mathcal{L}\left[e^{-at}\sin\omega t\right] = \frac{\omega}{(s+a)^2+\omega^2}}$

10. Find the Laplace transform of damped cosine wave.

$$f(t) = e^{-at}\cos\omega t$$

Solution: With the help of Euler's identity,

$$f(t) = e^{-at}\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right]$$

$$= \frac{1}{2}\cdot\left[e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right]$$

Taking Laplace transform on both sides,

$$\mathcal{L}\, f(t) = \frac{1}{2}\,L\left\{e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right\}$$

$$\mathcal{L}\, f(t) = \frac{1}{2}\, \mathcal{L}\left\{e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s+(a-j\omega)} + \frac{1}{s+(a+j\omega)}\right\}$$

$$= \frac{1}{2} \cdot \frac{2\,(s+a)}{(s+a)^2 + \omega^2}$$

$$= \frac{(s+a)}{(s+a)^2 + \omega^2}$$

11. By applying the complex differentiation property, determine the LT of

$$x(t) = t \sin \omega_0 t.$$

Solution: We know that

$$L[\sin \omega_0 t] = \frac{\omega_0}{(s^2 + \omega_0^2)}$$

According to the complex differentiation property,

$$L[-tx(t)] = \frac{d}{ds} X(s)$$

$$\therefore \quad L[\sin \omega_0 t] = \frac{d}{ds} \frac{\omega_0}{(s^2 + \omega_0^2)}$$

$$\boxed{L[t \sin \omega_0 t] = \frac{2\omega_0 s}{(s^2 + \omega_0^2)^2}}$$

12. Determine the LT of

$$x(t) = \cos at \sin bt.$$

Solution: The given signal x(t) is written in the following form.

$$x(t) = \frac{1}{2}[\sin(a+b)t - \sin(a-b)t]$$

$$L[\cos at \sin bt] = \frac{1}{2}[L\sin(a+b)t - L\sin(a-b)t]$$

$$\boxed{L[\cos at \sin bt] = \frac{1}{2}\left[\frac{(a+b)}{s^2+(a+b)^2} - \frac{(a-b)}{s^2+(a-b)^2}\right]}$$

13. Find the Laplace transform of $x(t) = t^n u(t)$.

Solution: Using the definition of LT for the given function we get

$$L[x(t)] = \int_0^\infty t^n e^{-st} dt$$

Let

$$u = t^n \quad \text{and} \quad du = nt^{n-1}dt$$

$$dv = \int e^{-st}dt \quad \text{and} \quad v = \frac{e^{-st}}{(-s)}$$

Using the property

$$\int udv = uv - \int vdu$$

we get

$$L[t^n] = \left[t^n \frac{e^{-st}}{(-s)}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{(-s)} nt^{n-1}dt$$

$$= 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt.$$

It can be shown that

$$\int_0^\infty t^{n-1}e^{-st}dt = \frac{(n-1)}{s}\int_0^\infty t^{n-2}e^{-st}dt.$$

Thus $L[t^n]$ is written as

$$L[t^n] = \frac{n}{s}\frac{(n-1)}{s}\frac{(n-2)}{s}\cdots\frac{2}{s}\frac{1}{s}$$
$$= \frac{n(n-1)(n-2)\ldots 2}{s^n}\frac{1}{s}$$
$$= \frac{\angle n}{s^{n+1}}$$

$$\boxed{L[t^n] = \frac{\angle n}{s^{n+1}}} \quad \text{ROC: Re } s > 0.$$

14. Determine the Laplace transform and sketch the ROC in the s- plane.

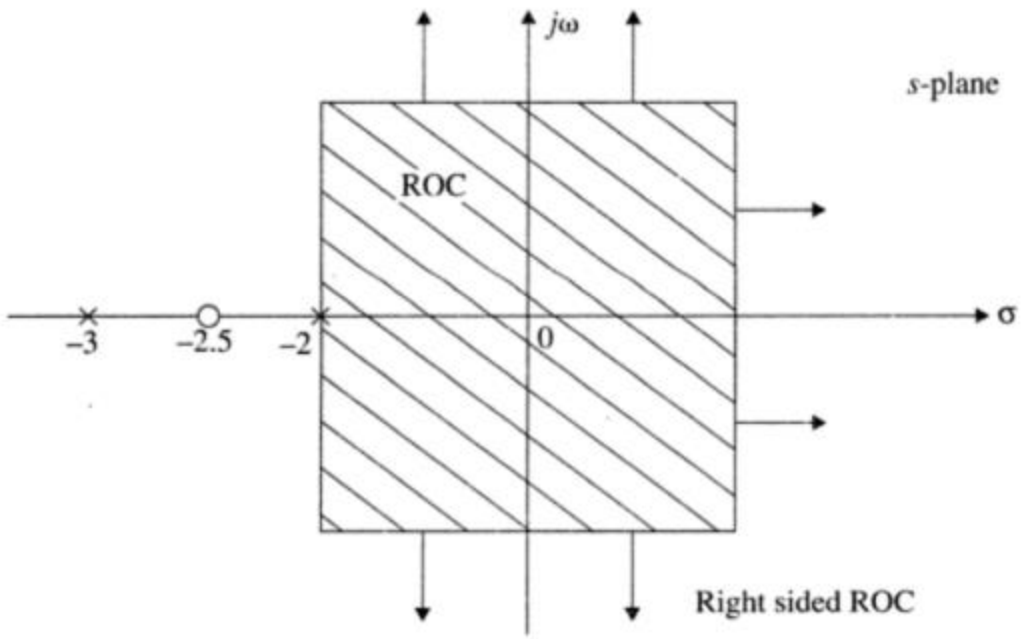
$$x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

Solution:

1. X (t) is completely a right sided signal and hence the limit of the integration is from t=0 to t=∞.Thus the following equation is written for X(s).

$$X(s) = \int_0^\infty e^{-2t}e^{-st}dt + \int_0^\infty e^{-3t}e^{-st}dt$$
$$= \int_0^\infty e^{-(s+2)t}dt + \int_0^\infty e^{-(s+3)t}dt$$
$$= \frac{1}{(s+2)} + \frac{1}{(s+3)}$$
$$= \frac{(2s+5)}{(s+2)(s+3)}$$

$$\boxed{X(s) = \frac{2(s+2.5)}{(s+2)(s+3)}}$$



Poles and zeros and ROC of $X(s) = \dfrac{2(s+2.5)}{(s+2)(s+3)}$.

2. The poles are at s= -2 and s= -3 and a zero is at s= -2.5 and are marked in figure.

3. For the pole 1/(s+2), the ROC is right sided to the vertical line passing through σ = -2.For the pole 1/(s+3), the ROC is also right sided passing through σ = -3.If ROC where σ > -2 is satisfied the ROC where σ > -3 is automatically satisfied. Further no pole of X(s) will be inside the ROC.

4. A strip to the right of σ =-2 is created and shaded. The strip is enlarged to ∞ in the direction of real and imaginary axis.

5. Thus, the ROC of a casual signal is to the right of the right most pole of X(s).

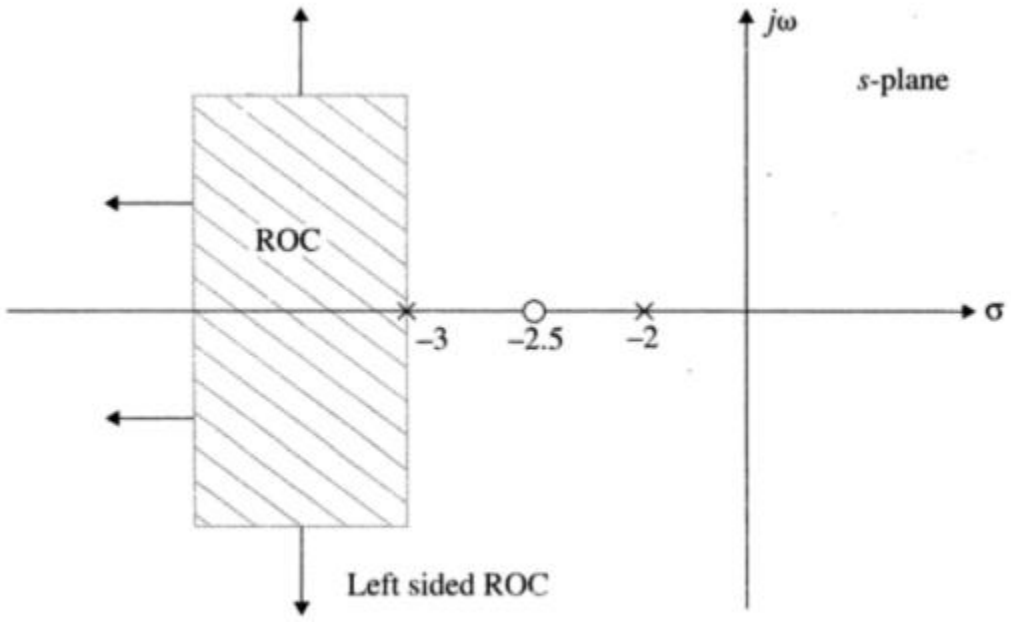
15. Determine the Laplace transform and locate the poles and zeros of X(s) and also the ROC in the s- plane.

$$x(t) = e^{-2t}u(-t) + e^{-3t}u(-t)$$

Solution:

The given signal is fully a left sided signal and hence the limit of LT integration is from -∞ to 0. The LT of x (t) is obtained as follows:

$$\begin{aligned} X(s) &= \int_{-\infty}^{0} e^{-2t}e^{-st}dt + \int_{-\infty}^{0} e^{-3t}e^{-st}dt \\ &= \int_{-\infty}^{0} e^{-(s+2)t}dt + \int_{-\infty}^{0} e^{-(s+3)t}dt \\ &= \frac{-1}{(s+2)}\left[e^{-(s+2)t}\right]_{-\infty}^{0} - \frac{1}{(s+3)}\left[e^{-(s+3)t}\right]_{-\infty}^{0} \end{aligned}$$



Poles and zeros and ROC of $X(s) = \dfrac{-2(s+2.5)}{(s+2)(s+3)}$.

16. Determine the Laplace transform and locate the poles and zeros and ROC in the s- Plane for the following signal.

$$x(t) = Au(t)$$

Solution:

1. The given signal is right- sided signal. And hence the limit of LT integration is from 0 to ∞. The LT of x (t) is obtained as follows

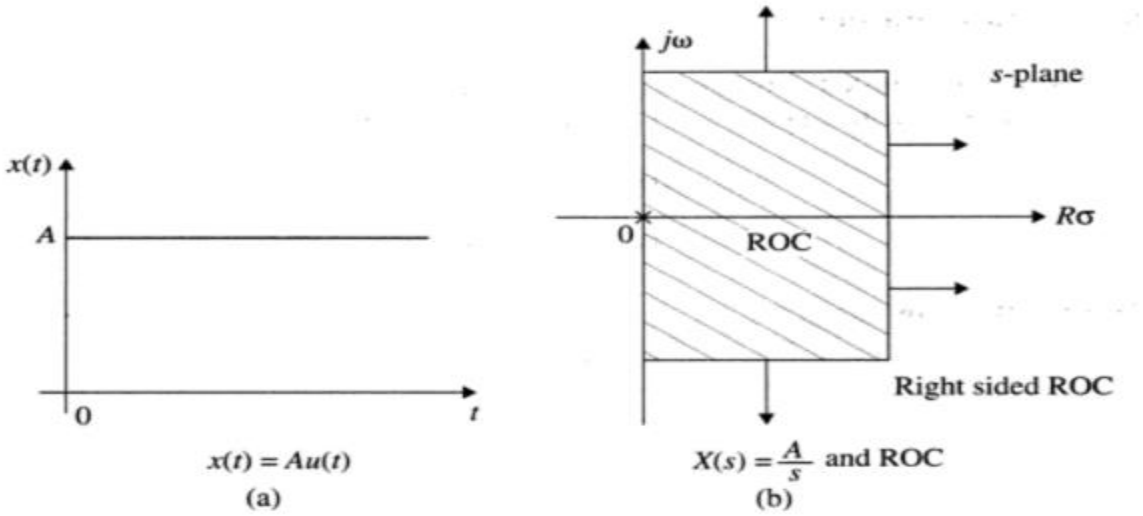
$$X(s) = \int_0^\infty Ae^{-st}dt$$

$$= \frac{-A}{s}\left[e^{-st}\right]_0^\infty$$

$$\boxed{X(s) = \frac{A}{s}} \quad \text{ROC Re } s > 0.$$

2. For the given signal, a pole at the origin exists and it is marked in figure (b).

3. The LT converges only σ >0.Thus the ROC is the entire right half of s-plane.



Representation of $x(t)$ and ROC.

17. Find the Laplace transform and ROC of

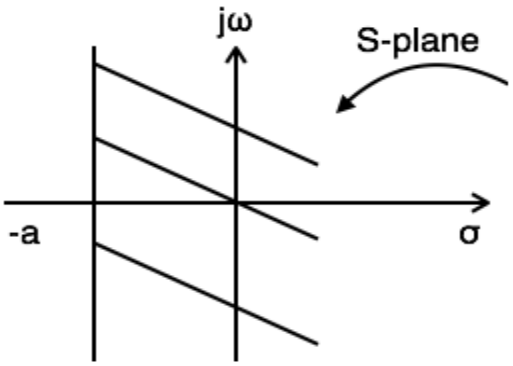
$$x(t) = e^{-at}\, u(t)$$

Solution:

$$L.T[x(t)] = L.T[e^{-at}\, u(t)] = \frac{1}{S+a}$$

$$Re > -a$$

$$ROC : Res >> -a$$



18. Find the Laplace transform and ROC of

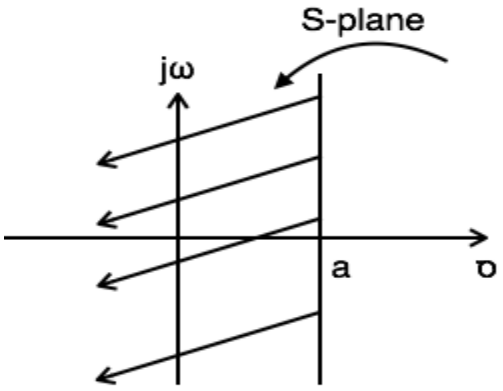
$$x(t) = e^{at}u(-t)$$

Solution:

$$L.T[x(t)] = L.T[e^{at}u(t)] = \frac{1}{S-a}$$

$$Res < a$$

$$ROC : Res < a$$



19. Find the Laplace transform and ROC of

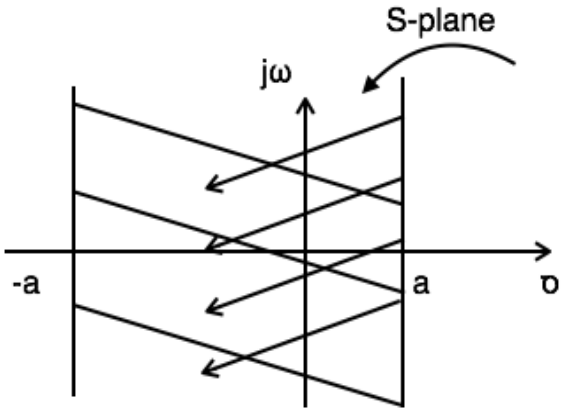
$$x(t) = e^{-at}u(t) + e^{at}u(-t)$$

Solution:

$$L.T[x(t)] = L.T[e^{-at}u(t) + e^{at}u(-t)] = \frac{1}{S+a} + \frac{1}{S-a}$$

For $\frac{1}{S+a}\ Re\{s\} > -a$

For $\frac{1}{S-a}\ Re\{s\} < a$



Referring to the above diagram, combination region lies from –a to a. Hence,

$$ROC: -a < Res < a$$

2.4.5 Laplace transform of periodic signal

If a signal x (t) is a periodic signal with period T, then the LT of X(s) is given as

$$X(s) = X_1(s)\left[1 + e^{-Ts} + e^{-2Ts} + \ldots\right]$$
$$= \frac{X_1(s)}{(1 - e^{-Ts})}$$

Here $x_1(t)$ is the signal which is repeated for every T.

2.4.6 Inverse Laplace transform

The time signal x (t) is the Inverse LT of X(s).This is represented by the following mathematical equation.

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds \qquad \ldots\ldots \textbf{(1)}$$

Use of the above equation to obtain x (t) from X(s) is really a tedious process. The alternative is to express X(s) in polynomial form both in the numerator and the denominator. Both these polynomials are factorized as

$$X(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \quad \text{……(2)}$$

The points in the s plane at which X(s) = 0 are called zeros. Thus $(s+z_1)$, $(s+z_2)$, $(s+z_3)$….. $(s+z_m)$ are the zeros of X(s). Similarly, the points in the s-plane at which X(s) = ∞ are called poles of X(s).

The zeros are identified by a small circle 0 and the poles by a small cross x in the s- plane. For m<n the degree of the numerator polynomial is less than the degree of the denominator polynomial. Under this condition X(s) in equation (2) is written in the following partial fraction form.

$$X(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3} + \cdots + \frac{A_n}{s+p_n} \quad \textbf{…… (3)}$$

In equation (3) $A_1, A_2, \dots A_n$ are called the residues and are determined by any convenient method. Once the residues are determined, one can easily obtain x(t) which is the required inverse LT of X(s).

1. Find the inverse LT of

$$X(s) = \frac{10e^{-3s}}{(s-2)(s+2)}.$$

Solution: Consider the function

$$X_1(s) = \frac{10}{(s-2)(s+2)}$$

Putting this into partial fraction we get

$$X_1(s) = \frac{A_1}{(s-2)} + \frac{A_2}{(s+2)}$$
$$= \frac{A_1(s+2) + A_2(s-2)}{(s-2)(s+2)}$$
$$10 = A_1(s+2) + A_2(s-2)$$

Substitute $s = -2$

$$10 = 0 + A_2(-2-2)$$
$$A_2 = -2.5$$

Substitute $s = 2$

$$10 = A_1(2+2) + 0$$
$$A_1 = 2.5$$
$$X_1(s) = 2.5\left[\frac{1}{s-2} - \frac{1}{s+2}\right]$$

Taking inverse LT we get

$$x_1(t) = 2.5[e^{+2t} - e^{-2t}]u(t)$$

According to time shifting property of LT

$$X(s) = X_1(s)e^{-3s}$$

$$\boxed{x(t) = 2.5[e^{2(t-3)} - e^{-2(t-3)}]u(t-3)}$$

2. Find the inverse LT of

$$X(s) = \frac{(s+1) + 3e^{-4s}}{(s+2)(s+3)}.$$

Solution: The given function is written in the following form:

$$X(s) = \frac{(s+1)}{(s+2)(s+3)} + \frac{3e^{-4s}}{(s+2)(s+3)}$$
$$= X_1(s) + X_2(s)$$
$$X_1(s) = \frac{(s+1)}{(s+2)(s+3)}$$
$$= \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)}$$
$$= \frac{A_1(s+3) + A_2(s+2)}{(s+2)(s+3)}$$
$$(s+1) = A_1(s+3) + A_2(s+2)$$

Put $s = -3$

$$(-3+1) = 0 + A_2(-3+2)$$
$$A_2 = 2$$

Put $s = -2$

$$(-2+1) = A_1(-2+3) + 0$$
$$A_1 = -1$$
$$X_1(s) = -\frac{1}{s+2} + \frac{2}{s+3}$$
$$x_1(t) = (-e^{-2t} + 2e^{-3t})u(t).$$

Now consider $X_2(s)$ without delay as $X_3(s)$

$$X_3(s) = \frac{3}{(s+2)(s+3)}$$

$$= \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)}$$

$$3 = A_1(s+3) + A_2(s+2)$$

Put $s = -2$

$$3 = A_1$$

Put $s = -3$

$$3 = A_2(-3+2)$$

$$A_2 = -3$$

$$X_3(s) = 3\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$$

$$X_2(s) = X_3(s)e^{-4s}$$

$$x_3(t) = 3[e^{-2t} - e^{-3t}]u(t)$$

$$x_2(t) = 3[e^{-2(t-4)} - e^{-3(t-4)}]u(t-4)$$

$$x(t) = x_1(t) + x_2(t)$$

$$\boxed{x(t) = \left[-e^{-2t} + 2e^{-3t}\right]u(t) + 3\left[e^{-2(t-4)} - e^{-3(t-4)}\right]u(t-4)}$$

Z-transform

In mathematics and signal processing, the Z-transform converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency domain representation.

Definition

The Z-transform can be defined as either a one-sided or two-sided transform.

(i) Bilateral Z-transform

The bilateral or two-sided Z-transform of a discrete-time signal x[n] is the formal power series X(z) defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where *n* is an integer and *z* is, in general, a complex number:

$$z = Ae^{j\phi} = A(\cos\phi + j\sin\phi)$$

where *A* is the magnitude of *z*, *j* is the imaginary unit, and Φ is the complex argument (also referred to as angle or phase) in radians.

(ii) Unilateral Z-transform

Alternatively, in cases where *x[n]* is defined only for $n \geq 0$, the single-sided or unilateral Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}.$$

Inverse Z-transform

The inverse Z-transform is calculated by using the formula,

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

where *C* is a counter clockwise closed path encircling the origin and entirely in the region of convergence (ROC).

A special case of this contour integral occurs when *C* is the unit circle (and can be used when the ROC includes the unit circle which is always guaranteed when *X(z)* is stable, (i.e. all the poles are within the unit circle). The inverse Z-transform simplifies to the inverse discrete-time Fourier transform:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}d\omega.$$

The Z-transform with a finite range of *n* and a finite number of uniformly spaced *z* values can be computed efficiently via Bluestein's FFT algorithm.

Region of convergence

The region of convergence (ROC) is the set of points in the complex plane for which the Z-transform summation converges.

$$ROC = \left\{z : \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| < \infty\right\}$$

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

The stability of a system can also be determined by knowing the ROC alone. If the ROC contains the unit circle (i.e., |*z*| = 1) then the system is stable.

If you are provided a Z-transform of a system without an ROC .you can determine a unique x[n] provided you desire the following:

- Stability
- Causality

If you need stability then the ROC must contain the unit circle.

If you need a causal system then the ROC must contain infinity and the system function will be a right-sided sequence.

If you need an anti-causal system then the ROC must contain the origin and the system function will be a left-sided sequence.

If you need both, stability and causality, all the poles of the system function must be inside the unit circle.

4.3.4 Properties of ROC of Z-Transform

- ROC of z-transform is indicated with circle in z-plane.
- Roc is a ring, whose center is at origin.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

4.3.5 Causality and Stability

Causality condition for discrete time LTI systems is as follows:

A discrete time LTI system is causal when

- ROC is outside the outermost pole.

- In The transfer function H[Z], the order of numerator cannot be greater than the order of denominator.

Stability Condition for Discrete Time LTI Systems

A discrete time LTI system is stable when

- Its system function H[Z] include unit circle |z|=1.
- All poles of the transfer function lay inside the unit circle |z|=1.

Properties of the *z*-Transform

The z-transform has a few very useful properties, and its definition extends to infinite signals/impulse responses.

(1)Linearity

This property states that

$$\mathcal{Z}[ax[n] + by[n]] = aX(z) + bY(z), \quad ROC \supseteq (R_x \cap R_y)$$

While it is obvious that the ROC of the linear combination of x(n) and y(n) should be the intersection of the their individual ROCs in which both X(Z) and Y(Z) exist.

(2)Time Shifting

This property states that

$$\mathcal{Z}[x[n - n_0]] = z^{-n_0}X(z), \quad ROC = R_x$$

Proof:

$$\mathcal{Z}[x[n - n_0]] = \sum_{n=-\infty}^{\infty} x[n - n_0]z^{-n}$$

Let $m = n - n_0$, we have $n = m + n_0$ and

$$\sum_{m=-\infty}^{\infty} x[m]z^{-m}z^{-n_0} = z^{-n_0}X(z)$$

The new ROC is the same as the old one except the possible addition/deletion of the origin or infinity as the shift may change the duration of the signal.

(3)Time Expansion (Scaling)

This property states that

$$\mathcal{Z}[x[n/k]] = X(z^k), \quad ROC = R_x^{1/k}$$

Proof: The z-transform of such an expanded signal is

$$\mathcal{Z}[x[n/k]] = \sum_{n=-\infty}^{\infty} x[n/k]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-km} = X(z^k)$$

Note that the change of the summation index from n to m has no effect as the terms skipped are all zeros.

(4)Convolution

This property states that

$$\mathcal{Z}[x[n] * y[n]] = X(z)Y(z), \quad ROC \supseteq (R_x \cap R_y)$$

The ROC of the convolution could be larger than the intersection of R_X and R_Y, due to the possible pole-zero cancellation caused by the convolution.

(5)Time Difference

This property states that

$$\mathcal{Z}[x[n]-x[n-1]]=(1-z^{-1})X(z), \qquad ROC=R_x$$

Proof:

$$\mathcal{Z}[x[n]-x[n-1]]=X(z)-z^{-1}X(z)=(1-z^{-1})X(z)=\frac{z-1}{z}X(z)$$

Note that due to the additional zero z=1 and pole z=0, the resulting ROC is the same as R_X except the possible deletion of z=0 caused by the added pole and/or addition of z = 1 caused by the added zero which may cancel an existing pole.

(6) Time Accumulation

This property states that

$$\mathcal{Z}[\sum_{k=-\infty}^{n} x[k]]=\frac{1}{1-z^{-1}}X(z), \qquad ROC \supseteq [R_x \cap (|z|>1)]$$

Proof: The accumulation of x[n] can be written as its convolution with u[n]:

$$u[n]*x[n]=\sum_{k=-\infty}^{\infty} u[n-k]x[k]=\sum_{k=-\infty}^{n} x[k]$$

Applying the convolution property, we get

$$\mathcal{Z}[\sum_{k=-\infty}^{n} x[k]]=\mathcal{Z}[u[n]*x[n]]=\frac{1}{1-z^{-1}}X(z)$$

$$\mathcal{Z}[u[n]] = 1/(1 - z^{-1})$$

.

(7) Time Reversal

This property states that

$$\mathcal{Z}[x[-n]] = X(1/z) \quad ROC = 1/R_x$$

Proof:

$$\mathcal{Z}[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{m=-\infty}^{\infty} x[m](\frac{1}{z})^{-m} = X(1/z)$$

where m=-n.

(8) Scaling in Z-domain

This property states that,

$$\mathcal{Z}[a^n x[n]] = X\left(\frac{z}{a}\right), \quad ROC = |a|R_x$$

Proof:

$$\mathcal{Z}[a^n x[n]] = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{a}\right)^{-n} = X\left(\frac{z}{a}\right)$$

In particular, if $a=e^{jw0}$, the above becomes

$$\mathcal{Z}[e^{jn\omega_0} x[n]] = X(e^{-j\omega_0} z) \quad ROC = R_x$$

This property is essentially the same as the frequency shifting property of discrete Fourier transform.

(9) Conjugation

$$\mathcal{Z}[x^*[n]] = X^*(z^*), \qquad ROC = R_x$$

Proof: Complex conjugate of the z-transform of $x[n]$ is

$$X^*(z) = [\sum_{n=-\infty}^{\infty} x[n]z^{-n}]^* = \sum_{n=-\infty}^{\infty} x^*[n](z^*)^{-n}$$

Replacing z by z^*, we get the desired result.

(10) Differentiation in z-Domain

$$\mathcal{Z}[nx[n]] = -\frac{d}{dz}X(z), \qquad ROC = R_x$$

Proof:

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n}) = \sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1} = \frac{-1}{z}\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

i.e.,

$$\mathcal{Z}[nx[n]] = -z\frac{d}{dz}X(z)$$

SOLVED PROBLEMS

1) Find the z transform of following:

$$x[n] = \delta[n] + \tfrac{1}{2}\delta(n+1) + \delta(n-3)$$

$$\delta[n] \overset{Z}{\longleftrightarrow} 1$$

$$\frac{1}{2}\delta[n+1] \overset{Z}{\longleftrightarrow} \frac{1}{2}z$$

$$\delta[n-3] \overset{Z}{\longleftrightarrow} z^{-3}$$

$$\boxed{X[z] = 1 + \frac{1}{2}z + z^{-3}}$$

$$x[n] = nu[n-1]$$

Method 1

$$u[n-1] \overset{Z}{\longleftrightarrow} \frac{1}{(z-1)}$$

Using differential property we get

$$nu[n-1] \overset{Z}{\longleftrightarrow} -z\frac{d}{dz}\frac{1}{(z-1)}$$

$$\boxed{z[nu[n-1]] = \frac{z}{(z-1)^2}}$$

Method 2

$$nu[n-1] = (n-1)u[n-1] + u[n-1]$$

$$(n-1)u[n-1] \overset{Z}{\longleftrightarrow} \frac{zz^{-1}}{(z-1)^2} = \frac{1}{(z-1)^2}$$

$$u[n-1] \overset{Z}{\longleftrightarrow} \frac{1}{(z-1)}$$

$$nu[n-1] \overset{Z}{\longleftrightarrow} \frac{1}{(z-1)^2} + \frac{1}{(z-1)}$$

$$\boxed{Z[nu[n-1]] = \frac{z}{(z-1)^2}}$$

$$\boldsymbol{x[n] = 4^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u[-n-1]}$$

$$\cos\left(\frac{2\pi n}{6}\right) = \cos\frac{\pi n}{3}$$

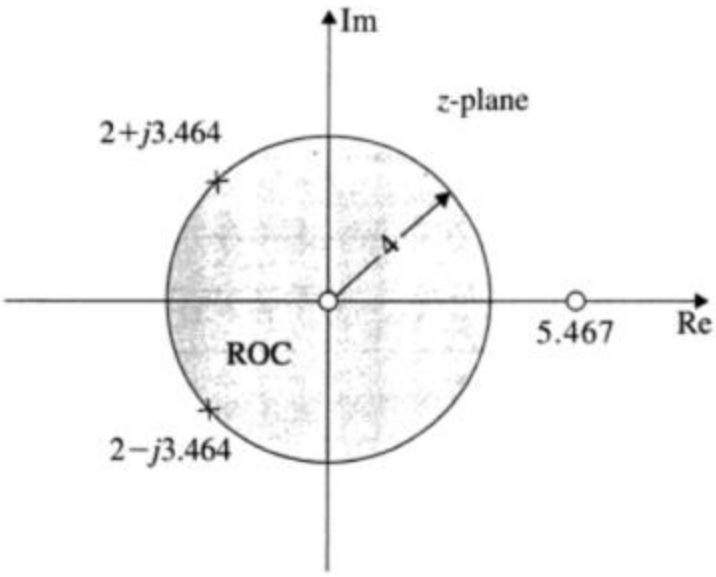
$$\cos\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) = \frac{e^{j\left(\frac{\pi}{4}+\frac{\pi n}{3}\right)} + e^{-j\left(\frac{\pi}{4}+\frac{\pi n}{3}\right)}}{2}$$

$$= \frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}e^{j\left(\frac{\pi n}{3}\right)} + \frac{1}{2}e^{-j\left(\frac{\pi}{4}\right)}e^{-j\left(\frac{\pi n}{3}\right)}$$

$$4^n \cos\left(\frac{2\pi n}{6} + \frac{\pi}{n}\right) = \frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}\left(4e^{j\frac{\pi}{3}}\right)^n + \frac{1}{2}e^{-j\frac{\pi}{4}}\left(4e^{-j\frac{\pi}{3}}\right)^n$$

$$\left(4e^{j\frac{\pi}{3}}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{(z-4e^{j\frac{\pi}{3}})} \qquad \text{ROC: } |z| < 4$$

$$\left(4e^{-j\frac{\pi}{3}}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{(z-4e^{-j\frac{\pi}{3}})} \qquad \text{ROC: } |z| < 4$$



$$X[z] = -\frac{1}{2}z\left[\frac{e^{j\frac{\pi}{4}}}{(z-4e^{j\frac{\pi}{3}})} + \frac{e^{-j\frac{\pi}{4}}}{(z-4e^{-j\frac{\pi}{3}})}\right]$$

$$= -\frac{1}{2}z\left[\frac{ze^{j\frac{\pi}{4}} - 4e^{-j\frac{\pi}{12}} - 4e^{j\frac{\pi}{12}} + ze^{-j\frac{\pi}{4}}}{z^2 - z4(e^{j\frac{\pi}{3}} + e^{-j\frac{\pi}{3}}) + 16}\right]$$

$$= \frac{-\frac{1}{2}z[\sqrt{2}z - 7.73]}{(z^2 - 4z + 16)}$$

$$\boxed{X[z] = \frac{-0.707z[z-5.467]}{(z-2+j3.464)(z-2-j3.464)}} \quad \text{ROC: } |z| < 4$$

The pole zero diagram is shown in above figure. The ROC is the interior of the circle.

2.Find the inverse z transform of following:

Solution:

(a) $X[z] = \frac{4z}{(z^2-3z+2)}$; **ROC: $|z| > 2$**

$$X[z] = \frac{4z}{(z^2 - 3z + 2)}$$
$$= \frac{4z}{(z-1)(z-2)}$$

For ROC: IzI > 2, x[n] is a right-sided sequence where n ≥ 0.Hence , the long division is done in such a way that X[z] is expressed in power of z^{-1}.

$$\begin{array}{r} 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots \\ z^2 - 3z + 2 \overline{)\,4z} \\ \underline{4z - 12 + 8z^{-1}} \\ 12 - 8z^{-1} \\ \underline{12 - 36z^{-1} + 24z^{-2}} \\ 28z^{-1} - 24z^{-2} \\ \underline{28z^{-1} - 84z^{-2} + 56z^{-3}} \end{array}$$

$$X[z] = 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots$$

$$x[n] = \{0,\ 4,\ 12,\ 28, \ldots\}$$
$$\uparrow$$

(b) $X[z] = \frac{4z}{(z^2-3z+2)}$; **ROC: $|z| < 1$**

For ROC: IzI < 1, x[n] sequence is negative where n≤0.Hence , the long division is done in such a way that X[z] is expressed in power of z.

$$
\begin{array}{r}
2z + 3z^2 + \frac{7}{2}z^3 \\
2 - 3z + z^2 \overline{)\,4z} \\
\underline{4z - 6z^2 + 2z^3} \\
6z^2 - 2z^3 \\
\underline{6z^2 - 9z^3 + 3z^4} \\
7z^3 - 3z^4 \\
\underline{7z^3 - \frac{21}{2}z^4 + \frac{7}{2}z^5} \\
\end{array}
$$

$$X[z] = 2z + 3z^2 + \frac{7}{2}z^3 + \cdots$$

$$\boxed{x[n] = \left\{ \cdots \frac{7}{2},\ 3,\ 2, 0 \right\}}$$
$$\uparrow$$

3. Find the inverse z transform of following:

(a)

$$X[z] = \frac{z(z+1)}{(z+3)(z+5)}$$

$$x[n] = \sum \text{Residue of } \frac{z(z+1)}{(z+3)(z+5)} z^{n-1}$$

$$= \text{Residue of } (z+3)\frac{z(z+1)}{(z+3)(z+5)} z^{n-1}\Big|_{z=-3}$$

$$+ \text{Residue of } (z+5)\frac{z(z+1)z^{n-1}}{(z+3)(z+5)}\Big|_{z=-5}$$

$$\boxed{x[n] = -(-3)^n + 2(-5)^n}$$

(b) $X[z] = \frac{z^{-1}}{(1-10z^{-1}+24z^{-2})}$; $4 < |z| < 6$

$$X[z] = \frac{z}{(z^2 - 10z + 24)} = \frac{z}{(z-4)(z-6)}$$

For $n \geq 0$

$$\begin{aligned} x[n] &= \text{ Residue of } X[z]z^{n-1}\Big|_{z=4} \\ &= (z-4)\frac{z(z^{n-1})}{(z-4)(z-6)}\Big|_{z=4} \\ &= -\frac{1}{2}(4)^n u[n] \end{aligned}$$

For $n < 0$

$$\begin{aligned} x[n] &= -\left[(z-6)\frac{zz^{n-1}}{(z-4)(z-6)}\right]_{z=6} \\ &= -\frac{1}{2}(6)^n u(-n-1) \end{aligned}$$

$$\boxed{x[n] = -\frac{1}{2}[(4)^n u[n] + (6)^n u(-n-1)]}$$

(c) $X[z] = \frac{z}{(z-\frac{1}{2})^2}$

$$\begin{aligned} x[n] &= \frac{d}{dz}\left[\left(z-\frac{1}{2}\right)^2 \frac{zz^{n-1}}{\left(z-\frac{1}{2}\right)}\right]_{z=\frac{1}{2}} \\ &= \frac{d}{dz}z^n\Big|_{z=1/2} = nz^{n-1}\Big|_{z=1/2} \end{aligned}$$

$$\boxed{x[n] = 2n\left(\frac{1}{2}\right)^n u[n]}$$

PART-A QUESTIONS

1. Define ROC.

2. Write any 2 properties of Laplace Transform.

3. Describe the relation between Laplace & Z-Transform.

4. Mention any 2 properties of Laplace Transform.

5. Name the signal which has ROC in entire z- plane and justify your answer.

6. Find the Laplace transform of $f(t) = e^{-j2t}\, u(t)$

7. What are the properties of ROC?

8. Define Unilateral and Bilateral Laplace Transform.

9. State Initial and final theorems of Laplace transform.

10. State Initial and final theorems of Z transform.

PART-B QUESTIONS

(1) Determine the LT and associated ROC and pole-zero plot for the following function of time x(t) = $e_{-2t}\,u(t) + e_{-3t}\,u(t)$.

(2)Find the signal corresponding to X(s) = $\frac{3s^2 + 22S+27}{(S+1)(S+2)(s^2+2S+5)}$.

(3)(a) Explain the properties of Z-T.

Find the Z-T of $a^n \sin(n\omega)\,u(n)$.

Find the inverse Z-T of X(Z) = $\frac{3}{Z-2}$ $|Z| > 2$.

(4)(a) Find Laplace transforms and sketch their ROC of:

x(t) = u(t - 5)

x(t) = $e^{j\omega(t)}\,u(t)$

(5) Find z transform, ROC and pole zero locations of: X(n) = $a^n\,u(n)$.

(6)State and prove z – transform time reversal property.

(7)Find the inverse z – transform of X(z) = (1/1 + z) (2z/z – 0.2).

(8) Find Z transforms of the following by using properties

(i) $x(n) = n^2\left(\frac{1}{3}\right)^n . u(n-2)$ (ii) $x(n) = n\left(\frac{1}{2}\right)^n u(n) * [\delta(n) - \frac{1}{2}\delta(n-1)]$

(9) A causal LTI system described by the differential equation $\frac{d^2y(t)}{dt^2} + \frac{3d(t)}{dt} + 2y(t) = x(t)$with input

x (t) =2u (t) and with initial conditions$\frac{d}{dt}y(0^-) = 3; y(0^-) = -5$ & $x(t) = 2u(t)$ find system

Response y(t)

(10)Describe the ROC of the signal:

x(t) = $e^{-a|t|}$

for a > 0 and a $\leq$ 0.

(11) Determine z transform, pole zero locations and sketch of ROC of following signal:
x(n) $= (1/2)^n\,u(n) + (-1/3)_n\,u(n)$.

(12) Find the Laplace transform of: $x(t) = e^{-(t-2)}\,(t-2)\,u(t-2)$

(13)Find the inverse z – transform of: $X(z) = (1/1 + 2z) + (2z/z - 0.25)$

(14)Prove that the Laplace transform of even and odd functions is even and odd functions respectively.

(15)Find the Laplace transform of $x(t) = e^{-t}\,Cos(w_0 t + \varphi)u(t)$ and its ROC.

(16)Find the Z-Transform of x[n] = {1, 2, $-\underset{\uparrow}{1}$, 3, 2, 1}.

(17)Find the Inverse Z-Transform of $X(z) = \dfrac{8z - 9}{z^2 - 5z + 6}$, $ROC\ |z| > 3$.

(18)By using Laplace Transform, solve the differential equation

$$\frac{d^3y(t)}{dt^3} + 7.\frac{d^2y(t)}{dt^2} + 16.\frac{dy(t)}{dt} + 12\,y(t) = x(t) \text{ if } x(t) = \delta(t),\ y(0^-) = 0,\ y^1(0^-) = 0 \text{ and } y^{11}(0^-) = 0$$

(19)Find the Z Transform of the signal along with ROC

(i) $x(n) = \sin(w_0 n).u(n)$ (ii) $x(n) = n.\sin(\pi n/2).u(n)$

(20) Find the inverse Laplace Tansform of X(s) = (-5s + 7)/(s+1) (s-1) (s-2).

APPENDIX

	Signal, $x[n]$	Z-transform, $X(z)$	ROC
1	$\delta[n]$	1	all *z*
2	$\delta[n-n_0]$	z^{-n_0}	$z \neq 0$
3	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
4	$e^{-\alpha n}u[n]$	$\frac{1}{1-e^{-\alpha}z^{-1}}$	$\|z\| > \|e^{-\alpha}\|$
5	$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
6	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$
7	$-nu[-n-1]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\|z\| < 1$
8	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| > 1$
9	$-n^2u[-n-1]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| < 1$
10	$n^3u[n]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\|z\| > 1$
11	$-n^3u[-n-1]$	$\frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$	$\|z\| < 1$
12	$a^nu[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$

13	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
14	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
15	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
16	$n^2 a^n u[n]$	$\dfrac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert > \lvert a\rvert$
17	$-n^2 a^n u[-n-1]$	$\dfrac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\lvert z\rvert < \lvert a\rvert$
18	$\cos(\omega_0 n)u[n]$	$\dfrac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
19	$\sin(\omega_0 n)u[n]$	$\dfrac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
20	$a^n\cos(\omega_0 n)u[n]$	$\dfrac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
21	$a^n\sin(\omega_0 n)u[n]$	$\dfrac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

UNIT -IV
FAST FOURIER TRANSFORM

Introduction to discrete time signals ans systems.

Anything that carries some information can be called as signals. Some examples are ECG, EEG, ac power, seismic, speech, interest rates of a bank, unemployment rate of a country, temperature, pressure etc.

A **signal** is also defined as any physical quantity that varies with one or more independent variables.

A **discrete time signal** is the one which is not defined at intervals between two successive samples of a signal. It is represented as graphical, functional, tabular representation and sequence.

Some of the elementary discrete time signals are unit step, unit impulse, unit ramp, exponential and sinusoidal signals (as you read in signals and systems).

Classification of discrete time signals

Energy and Power signals

If the value of E is finite, then the signal x(n) is called energy signal.

If the value of the P is finite, then the signal x(n) is called Power signal.

Periodic and Non periodic signals

A discrete time signal is said to be periodic if and only if it satisfies the condition X (N+n) =x (n), otherwise non periodic

Symmetric (even) and Anti-symmetric (odd) signals

The signal is said to be even if x(-n)=x(n)

The signal is said to be odd if x(-n)= - x(n)

Causal and non causal signal

The signal is said to be causal if its value is zero for negative values of 'n'.

Some of the **operations** on discrete time signals are shifting, time reversal, time scaling, signal multiplier, scalar multiplication and signal addition or multiplication.

Discrete time systems

A discrete time signal is a device or algorithm that operates on discrete time signals and produces another discrete time output.

Classification of discrete time systems

Static and dynamic systems

A system is said to be static if its output at present time depend on the input at present time only.

Causal and non causal systems
A system is said to be causal if the response of the system depends on present and past values of the input but not on the future inputs.

Linear and non linear systems
A system is said to be linear if the response of the system to the weighted sum of inputs should be equal to the corresponding weighted sum of outputs of the systems. This principle is called superposition principle.

Time invariant and time variant systems
A system is said to be time invariant if the characteristics of the systems do not change with time.

Stable and unstable systems
A system is said to be stable if bounded input produces bounded output only.

TIME DOMAIN ANALYSIS OF DISCRETE TIME SIGNALS AND SYSTEMS
Representation of an arbitrary sequence
Any signal x(n) can be represented as weighted sum of impulses as given below

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

The response of the system for unit sample input is called impulse response of the system h(n)

$$y(n) = \mathcal{T}[x(n)] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n,k)$$

By time invariant property, we have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The above equation is called **convolution sum**.
Some of the properties of convolution are commutative law, associative law and distributive law.

Correlation of two sequences
It is basically used to compare two signals. It is the measure of similarity between two signals. Some of the applications are communication systems, radar, sonar etc.
The cross correlation of two sequences x(n) and y(n) is given by

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

One of the important properties of cross correlation is given by

$$r_{xy}(l) = r_{yx}(-l)$$

The auto correlation of the signal x(n) is given by

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

Linear time invariant systems characterized by constant coefficient difference equation

The response of the first order difference equation is given by

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k) \qquad n \geq 0$$

The first part contain initial condition y(-1) of the system, the second part contains input x(n) of the system.

The response of the system when it is in relaxed state at n=0 or y(-1)=0 is called **zero state response** of the system or **forced response**.

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k) \qquad n \geq 0$$

The output of the system at zero input condition x(n)=0 is called **zero input response** of the system or **natural response**.

The impulse response of the system is given by zero state response of the system

$$y_{zs}(n) = \sum_{k=0}^{n} a^k \delta(n-k)$$

$$= a^n \qquad n \geq 0$$

The total response of the system is equal to sum of natural response and forced responses.

FREQUENCY DOMAIN ANALYSIS OF DISCRETE TIME SIGNALS AND SYSTEMS

A s we have observed from the discussion o f Section 4.1, the Fourier series representation o f a continuous-time periodic signal can consist of an infinite number of frequency components, where the frequency spacing between two successive harmonically related frequencies is 1 / *T p*, and where *Tp* is the fundamental period.

Since the frequency range for continuous-time signals extends infinity on both sides it is possible to have signals that contain an infinite number of frequency components.

In contrast, the frequency range for discrete-time signals is unique over the interval. A discrete-time signal of fundamental period N can consist of frequency components separated by $2n / N$ radians.
Consequently, the Fourier series representation o f the discrete-time periodic signal will contain at most N frequency components. This is the basic difference between the Fourier series representations for continuous-time and discrete-time periodic signals.

4.2.1 The Fourier Series for Discrete-Time Periodic Signals

Suppose that we are given a periodic sequence $x(n)$ with period N, that is, $x(n) = x(n + N)$ for all n. The Fourier series representation for $x(n)$ consists of N harmonically related exponential functions

$$e^{j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1$$

and is expressed as

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \tag{4.2.1}$$

where the $\{c_k\}$ are the coefficients in the series representation.

To derive the expression for the Fourier coefficients, we use the following formula:

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.2.2}$$

Note the similarity of (4.2.2) with the continuous-time counterpart in (4.1.3). The proof of (4.2.2) follows immediately from the application of the geometric summation formula

$$\sum_{n=0}^{N-1} a^n = \begin{cases} N, & a = 1 \\ \dfrac{1 - a^N}{1 - a}, & a \neq 1 \end{cases} \tag{4.2.3}$$

The expression for the Fourier coefficients c_k can be obtained by multiplying both sides of (4.2.1) by the exponential $e^{-j2\pi ln/N}$ and summing the product from $n = 0$ to $n = N - 1$. Thus

$$\sum_{n=0}^{N-1} x(n) e^{-j2\pi ln/N} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k e^{j2\pi (k-l)n/N} \tag{4.2.4}$$

If we perform the summation over n first, in the right-hand side of (4.2.4), we obtain

$$\sum_{n=0}^{N-1} e^{j2\pi (k-l)n/N} = \begin{cases} N, & k - l = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases} \tag{4.2.5}$$

where we have made use of (4.2.2). Therefore, the right-hand side of (4.2.4) reduces to Nc_l and hence

$$c_l = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi ln/N} \qquad l = 0, 1, \ldots, N-1 \tag{4.2.6}$$

Thus we have the desired expression for the Fourier coefficients in terms of the signal $x(n)$.

4.2.3 The Fourier Transform of Discrete-Time Aperiodic Signals

Just as in the case of continuous-time aperiodic energy signals, the frequency analysis of discrete-time aperiodic finite-energy signals involves a Fourier transform of the time-domain signal. Consequently, the development in this section parallels to a large extent, that given in Section 4.1.3.

The Fourier transform of a finite-energy discrete-time signal $x(n)$ is defined as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{4.2.23}$$

Physically, $X(\omega)$ represents the frequency content of the signal $x(n)$. In other words, $X(\omega)$ is a decomposition of $x(n)$ into its frequency components.

We observe two basic differences between the Fourier transform of a discrete-time finite-energy signal and the Fourier transform of a finite-energy analog signal. First, for continuous-time signals, the Fourier transform, and hence the spectrum of the signal, have a frequency range of $(-\infty, \infty)$. In contrast, the frequency range for a discrete-time signal is unique over the frequency interval of $(-\pi, \pi)$ or, equivalently, $(0, 2\pi)$. This property is reflected in the Fourier transform of the signal. Indeed, $X(\omega)$ is periodic with period 2π, that is,

$$\begin{aligned} X(\omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}e^{-j2\pi kn} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega) \end{aligned} \tag{4.2.24}$$

Hence $X(\omega)$ is periodic with period 2π. But this property is just a consequence of the fact that the frequency range for any discrete-time signal is limited to $(-\pi, \pi)$ or $(0, 2\pi)$, and any frequency outside this interval is equivalent to a frequency within the interval.

The second basic difference is also a consequence of the discrete-time nature of the signal. Since the signal is discrete in time, the Fourier transform of the signal involves a summation of terms instead of an integral, as in the case of continuous-time signals.

Since $X(\omega)$ is a periodic function of the frequency variable ω, it has a Fourier series expansion, provided that the conditions for the existence of the Fourier series, described previously, are satisfied. In fact, from the definition of the Fourier transform $X(\omega)$ of the sequence $x(n)$, given by (4.2.23), we observe that $X(\omega)$ has the form of a Fourier series. The Fourier coefficients in this series expansion are the values of the sequence $x(n)$.

To demonstrate this point, let us evaluate the sequence $x(n)$ from $X(\omega)$. First, we multiply both sides (4.2.23) by $e^{j\omega m}$ and integrate over the interval $(-\pi, \pi)$. Thus we have

$$\int_{-\pi}^{\pi} X(\omega)e^{j\omega m}d\omega = \int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right]e^{j\omega m}d\omega \tag{4.2.25}$$

The integral on the right-hand side of (4.2.25) can be evaluated if we can interchange the order of summation and integration. This interchange can be made if the series

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

converges uniformly to $X(\omega)$ as $N \to \infty$. Uniform convergence means that, for every ω, $X_N(\omega) \to X(\omega)$, as $N \to \infty$. The convergence of the Fourier transform is discussed in more detail in the following section. For the moment, let us assume that the series converges uniformly, so that we can interchange the order of summation and integration in (4.2.25). Then

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)}d\omega = \begin{cases} 2\pi, & m = n \\ 0, & m \neq n \end{cases}$$

Consequently,

$$\sum_{n=-\infty}^{\infty} x(n)\int_{-\pi}^{\pi} e^{j\omega(m-n)}d\omega = \begin{cases} 2\pi x(m), & m = n \\ 0, & m \neq n \end{cases} \tag{4.2.26}$$

By combining (4.2.25) and (4.2.26), we obtain the desired result that

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \tag{4.2.27}$$

If we compare the integral in (4.2.27) with (4.1.9), we note that this is just the expression for the Fourier series coefficient for a function that is periodic with period 2π. The only difference between (4.1.9) and (4.2.27) is the sign on the exponent in the integrand, which is a consequence of our definition of the Fourier transform as given by (4.2.23). Therefore, the Fourier transform of the sequence $x(n)$, defined by (4.2.23), has the form of a Fourier series expansion.

FREQUENCY DOMAIN SAMPLING: THE DISCRETE FOURIER TRANSFORM

Before we introduce the DFT, we consider the sampling of the Fourier transform of an aperiodic discrete-time sequence. Thus, we establish the relationship between the sampled Fourier transform and the DFT.

5.1.1 Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

We recall that aperiodic finite-energy signals have continuous spectra. Let us consider such an aperiodic discrete-time signal $x(n)$ with Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.1.1}$$

Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. Since $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are necessary. For convenience, we take N equidistant samples in the interval $0 \le \omega < 2\pi$ with spacing $\delta\omega = 2\pi/N$, as shown in Fig. 5.1. First, we consider the selection of N, the number of samples in the frequency domain.

If we evaluate (5.1.1) at $\omega = 2\pi k/N$, we obtain

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.2}$$

The summation in (5.1.2) can be subdivided into an infinite number of summations, where each sum contains N terms. Thus

$$\begin{aligned} X\left(\frac{2\pi}{N}k\right) = & \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ & + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots \\ = & \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N} \end{aligned}$$

If we change the index in the inner summation from n to $n - lN$ and interchange the order of the summation, we obtain the result

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right]e^{-j2\pi kn/N} \tag{5.1.3}$$

for $k = 0, 1, 2, \ldots, N-1$.

The signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \tag{5.1.4}$$

obtained by the periodic repetition of $x(n)$ every N samples, is clearly periodic with fundamental period N. Consequently, it can be expanded in a Fourier

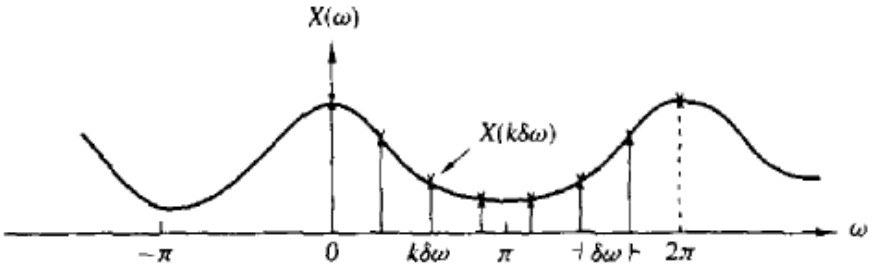


Figure 5.1 Frequency-domain sampling of the Fourier transform.

series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.5}$$

with Fourier coefficients

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.6}$$

Upon comparing (5.1.3) with (5.1.6), we conclude that

$$c_k = \frac{1}{N}X\left(\frac{2\pi}{N}k\right) \qquad k = 0, 1, \ldots, N-1 \tag{5.1.7}$$

Therefore,

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.8}$$

5.1.2 The Discrete Fourier Transform (DFT)

The development in the preceding section is concerned with the frequency-domain sampling of an aperiodic finite-energy sequence $x(n)$. In general, the equally spaced frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, do not uniquely represent the original sequence $x(n)$ when $x(n)$ has infinite duration. Instead, the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, correspond to a periodic sequence $x_p(n)$ of period N, where $x_p(n)$ is an aliased version of $x(n)$, as indicated by the relation in (5.1.4), that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.1.15}$$

When the sequence $x(n)$ has a finite duration of length $L \leq N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is given as

$$x_p(n) = \begin{cases} x(n), & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases} \tag{5.1.16}$$

Consequently, the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, uniquely represent the finite-duration sequence $x(n)$. Since $x(n) \equiv x_p(n)$ over a single period (padded by $N - L$ zeros), the original finite-duration sequence $x(n)$ can be obtained from the frequency samples $\{X(2\pi k/N)\}$ by means of the formula (5.1.8).

It is important to note that *zero padding* does not provide any additional information about the spectrum $X(\omega)$ of the sequence $\{x(n)\}$. The L equidistant samples of $X(\omega)$ are sufficient to reconstruct $X(\omega)$ using the reconstruction formula (5.1.13). However, padding the sequence $\{x(n)\}$ with $N - L$ zeros and computing an N-point DFT results in a "better display" of the Fourier transform $X(\omega)$.

In summary, a finite-duration sequence $x(n)$ of length L [i.e., $x(n) = 0$ for $n < 0$ and $n \geq L$] has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \qquad 0 \leq \omega \leq 2\pi \tag{5.1.17}$$

where the upper and lower indices in the summation reflect the fact that $x(n) = 0$ outside the range $0 \leq n \leq L-1$. When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N-1$, where $N \geq L$, the resultant samples are

$$\begin{aligned} X(k) \equiv X\left(\frac{2\pi k}{N}\right) &= \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N} \\ X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \end{aligned} \tag{5.1.18}$$

where for convenience, the upper index in the sum has been increased from $L-1$ to $N-1$ since $x(n) = 0$ for $n \geq L$.

The relation in (5.1.18) is a formula for transforming a sequence $\{x(n)\}$ of length $L \leq N$ into a sequence of frequency samples $\{X(k)\}$ of length N. Since the frequency samples are obtained by evaluating the Fourier transform $X(\omega)$ at a set of N (equally spaced) discrete frequencies, the relation in (5.1.18) is called the *discrete Fourier transform* (DFT) of $x(n)$. In turn, the relation given by (5.1.10), which allows us to recover the sequence $x(n)$ from the frequency samples

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.19}$$

is called the *inverse DFT* (IDFT). Clearly, when $x(n)$ has length $L < N$, the N-point IDFT yields $x(n) = 0$ for $L \leq n \leq N-1$. To summarize, the formulas for the DFT and IDFT are

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N-1 \tag{5.1.18}$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \qquad n = 0, 1, 2, \ldots, N-1 \tag{5.1.19}$$

5.1.3 The DFT as a Linear Transformation

The formulas for the DFT and IDFT given by (5.1.18) and (5.1.19) may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.20}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.21}$$

where, by definition,

$$W_N = e^{-j2\pi/N} \tag{5.1.22}$$

which is an Nth root of unity.

With these definitions, the N-point DFT may be expressed in matrix form as

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \tag{5.1.24}$$

where $\mathbf{W}_N$ is the matrix of the linear transformation. We observe that $\mathbf{W}_N$ is a symmetric matrix. If we assume that the inverse of $\mathbf{W}_N$ exists, then (5.1.24) can be inverted by premultiplying both sides by $\mathbf{W}_N^{-1}$. Thus we obtain

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N \tag{5.1.25}$$

Relationship to the Fourier series coefficients of a periodic sequence. A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier series of the form

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N} \qquad -\infty < n < \infty \tag{5.1.29}$$

where the Fourier series coefficients are given by the expression

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.30}$$

If we compare (5.1.29) and (5.1.30) with (5.1.18) and (5.1.19), we observe that the formula for the Fourier series coefficients has the form of a DFT. In fact, if we define a sequence $x(n) = x_p(n)$, $0 \le n \le N-1$, the DFT of this sequence is simply

$$X(k) = Nc_k \tag{5.1.31}$$

Furthermore, (5.1.29) has the form of an IDFT. Thus the N-point DFT provides the exact line spectrum of a periodic sequence with fundamental period N.

Relationship to the Fourier transform of an aperiodic sequence. We have already shown that if $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$, which is sampled at N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, the spectral components

$$X(k) = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.32}$$

are the DFT coefficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.1.33}$$

Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \le n \le N-1$. The finite-duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1.34}$$

bears no resemblance to the original sequence $\{x(n)\}$, unless $x(n)$ is of finite duration and length $L \leq N$, in which case

$$x(n) = \hat{x}(n) \qquad 0 \leq n \leq N-1 \tag{5.1.35}$$

Only in this case will the IDFT of $\{X(k)\}$ yield the original sequence $\{x(n)\}$.

Relationship to the *z*-transform. Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{5.1.36}$$

with a ROC that includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, $0, 1, 2, \ldots, N-1$, we obtain

$$\begin{aligned} X(k) &\equiv X(z)|_{z=e^{j2\pi k/N}} \qquad k = 0, 1, \ldots, N-1 \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nk/N} \end{aligned} \tag{5.1.37}$$

The expression in (5.1.37) is identical to the Fourier transform $X(\omega)$ evaluated at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, which is the topic treated in Section 5.1.1.

If the sequence $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Hence its z-transform is uniquely determined by its N-point DFT. Consequently, $X(z)$ can be expressed as a function of the DFT $\{X(k)\}$ as follows

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n)z^{-n} \\ X(z) &= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \right] z^{-n} \\ X(z) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1}\right)^n \\ X(z) &= \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j2\pi k/N}z^{-1}} \end{aligned} \tag{5.1.38}$$

When evaluated on the unit circle, (5.1.38) yields the Fourier transform of the finite-duration sequence in terms of its DFT, in the form

$$X(\omega) = \frac{1-e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{-j(\omega-2\pi k/N)}} \tag{5.1.39}$$

Relationship to the Fourier series coefficients of a continuous-time signal. Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0} \tag{5.1.40}$$

where $\{c_k\}$ are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain the discrete-time sequence

$$\begin{aligned} x(n) \equiv x_a(nT) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 nT} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N} \\ &= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N} \end{aligned} \tag{5.1.41}$$

It is clear that (5.1.41) is in the form of an IDFT formula. where

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k \tag{5.1.42}$$

and

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN} \tag{5.1.43}$$

Thus the $\{\tilde{c}_k\}$ sequence is an aliased version of the sequence $\{c_k\}$.

PROPERTIES OF DFT:

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n) \textcircled{N} x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n) \textcircled{N} y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \textcircled{N} X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

LINEAR FILTERING METHODS BASED ON THE DFT

Since the D F T provides a discrete frequency representation o f a finite-duration Sequence in the frequency domain, it is interesting to exp lore its use as a computational tool for linear system analysis and, especially, for linear filtering. We have already established that a system with frequency response $H(\omega)$ y w hen excited with an input signal that has a spectrum possesses an output spectrum.

The output sequence *y(n)* is determined from its spectrum via the inverse Fourier transform. Computationally, the problem with this frequency domain approach is

that are functions o f the continuous variable. As a consequence, the computations cannot be done on a digital computer, since the computer can only store and perform computations on quantities at discrete frequencies.
On the other hand, the DFT does lend itself to computation on a digital computer. In the discussion that follows, we describe how the DFT can be used to perform linear filtering in the frequency domain. In particular, we present a computational procedure that serves as an alternative to time-domain convolution.
In fact, the frequency-domain approach based on the DFT, is computationally m ore efficient than time-domain convolution due to the existence of efficient algorithms for computing the DFT . These algorithms, which are described in Chapter 6, are collectively called fast Fourier transform (FFT) algorithms.

5.3.1 Use of the DFT in Linear Filtering

In the preceding section it was demonstrated that the product of two DFTs is equivalent to the circular convolution of the corresponding time-domain sequences. Unfortunately, circular convolution is of no use to us if our objective is to determine the output of a linear filter to a given input sequence. In this case we seek a frequency-domain methodology equivalent to linear convolution.

Suppose that we have a finite-duration sequence $x(n)$ of length L which excites an FIR filter of length M. Without loss of generality, let

$$x(n) = 0, \qquad n < 0 \text{ and } n \geq L$$

$$h(n) = 0, \qquad n < 0 \text{ and } n \geq M$$

where $h(n)$ is the impulse response of the FIR filter.

The output sequence $y(n)$ of the FIR filter can be expressed in the time domain as the convolution of $x(n)$ and $h(n)$, that is

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{5.3.1}$$

Since $h(n)$ and $x(n)$ are finite-duration sequences, their convolution is also finite in duration. In fact, the duration of $y(n)$ is $L + M - 1$.

The frequency-domain equivalent to (5.3.1) is

$$Y(\omega) = X(\omega)H(\omega) \tag{5.3.2}$$

If the sequence $y(n)$ is to be represented uniquely in the frequency domain by samples of its spectrum $Y(\omega)$ at a set of discrete frequencies, the number of distinct samples must equal or exceed $L+M-1$. Therefore, a DFT of size $N \geq L+M-1$, is required to represent $\{y(n)\}$ in the frequency domain.

Now if

$$\begin{aligned} Y(k) &\equiv Y(\omega)|_{\omega=2\pi k/N} \qquad k = 0, 1, \ldots, N-1 \\ &= X(\omega)H(\omega)|_{\omega=2\pi k/N} \qquad k = 0, 1, \ldots, N-1 \end{aligned}$$

then

$$Y(k) = X(k)H(k) \qquad k = 0, 1, \ldots, N-1 \tag{5.3.3}$$

where $\{X(k)\}$ and $\{H(k)\}$ are the N-point DFTs of the corresponding sequences $x(n)$ and $h(n)$, respectively. Since the sequences $x(n)$ and $h(n)$ have a duration less than N, we simply pad these sequences with zeros to increase their length to N. This increase in the size of the sequences does not alter their spectra $X(\omega)$ and $H(\omega)$, which are continuous spectra, since the sequences are aperiodic. However, by sampling their spectra at N equally spaced points in frequency (computing the N-point DFTs), we have increased the number of samples that represent these sequences in the frequency domain beyond the minimum number (L or M, respectively).

Since the $N = L+M-1$-point DFT of the output sequence $y(n)$ is sufficient to represent $y(n)$ in the frequency domain. it follows that the multiplication of the N-point DFTs $X(k)$ and $H(k)$, according to (5.3.3), followed by the computation of the N-point IDFT, must yield the sequence $\{y(n)\}$. In turn, this implies that the N-point circular convolution of $x(n)$ with $h(n)$ must be equivalent to the linear convolution of $x(n)$ with $h(n)$. In other words, by increasing the length of the sequences $x(n)$ and $h(n)$ to N points (by appending zeros), and then circularly convolving the resulting sequences, we obtain the same result as would have been obtained with linear convolution. Thus with zero padding, the DFT can be used to perform linear filtering.

****

FAST FOURIER TRANSFORM

EFFICIENT COMPUTATION OF DFT:

In this section we represent several methods for computing dft efficiently.In view of the importance of the DFT in various digital signal processing applications such as linear filtering, correlation analysis and spectrum analysis, its efficient computation is a topic that has received considerably attention by many mathematicians, engineers and scientists. Basically the computation is done using the formula method.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \le k \le N-1$$

where

$$W_N = e^{-j2\pi/N}$$

In general, the data sequence $x(n)$ is also assumed to be complex valued.

Similarly, the IDFT becomes

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} \qquad 0 \le n \le N-1$$

We observe that for each value of k, direct computation of $X(k)$ involves N complex multiplications ($4N$ real multiplications) and $N-1$ complex additions ($4N-2$ real additions). Consequently, to compute all N values of the DFT requires N^2 complex multiplications and $N^2 - N$ complex additions.

6.1.1 Direct Computation of the DFT

For a complex-valued sequence $x(n)$ of N points, the DFT may be expressed as

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right] \tag{6.1.6}$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right] \tag{6.1.7}$$

The direct computation of (6.1.6) and (6.1.7) requires:

1. $2N^2$ evaluations of trigonometric functions.
2. $4N^2$ real multiplications.

3. $4N(N-1)$ real additions.
4. A number of indexing and addressing operations.

These operations are typical of DFT computational algorithms. The operations in items 2 and 3 result in the DFT values $X_R(k)$ and $X_I(k)$. The indexing and addressing operations are necessary to fetch the data $x(n)$, $0 \leq n \leq N-1$, and the phase factors and to store the results. The variety of DFT algorithms optimize each of these computational processes in a different way.

Divide-and-Conquer Approach to Computation of the DFT

The development of computationally efficient algorithms for the DFT is made possible if we adopt a divide-and-conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFT. This basic approach leads to a family o f computationally efficient algorithm s know n collectively as FFT algorithms.
T o illustrate the basic notions, let us consider the computation of an N point DFT , where N can be factored as a product of two integers, that is, $N = L\,M$

Algorithm 1

1. Store the signal column-wise.
2. Compute the M-point DFT of each row.
3. Multiply the resulting array by the phase factors W_N^{lq}.
4. Compute the L-point DFT of each column
5. Read the resulting array row-wise.

Algorithm 2

1. Store the signal row-wise.
2. Compute the L-point DFT at each column.
3. Multiply the resulting array by the factors W_N^{pm}.
4. Compute the M-point DFT of each row.
5. Read the resulting array column-wise.

6.1.3 Radix-2 FFT Algorithms

Let us consider the computation of the $N = 2^{\nu}$ point DFT by the divide-and-conquer approach specified by (6.1.16) through (6.1.18). We select $M = N/2$ and $L = 2$. This selection results in a split of the N-point data sequence into two $N/2$-point data sequences $f_1(n)$ and $f_2(n)$, corresponding to the even-numbered and odd-numbered samples of $x(n)$, respectively, that is,

$$\begin{aligned} f_1(n) &= x(2n) \\ f_2(n) &= x(2n+1), \qquad n = 0, 1, \ldots, \frac{N}{2} - 1 \end{aligned} \tag{6.1.23}$$

Thus $f_1(n)$ and $f_2(n)$ are obtained by decimating $x(n)$ by a factor of 2, and hence the resulting FFT algorithm is called a decimation-in-time algorithm.

Now the N-point DFT can be expressed in terms of the DFTs of the decimated sequences as follows:

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \\ &= \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn} \\ &= \sum_{m=0}^{(N/2)-1} x(2m) W_N^{2mk} + \sum_{m=0}^{(N/2)-1} x(2m+1) W_N^{k(2m+1)} \end{aligned} \tag{6.1.24}$$

But $W_N^2 = W_{N/2}$. With this substitution, (6.1.24) can be expressed as

$$\begin{aligned} X(k) &= \sum_{m=0}^{(N/2)-1} f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{(N/2)-1} f_2(m) W_{N/2}^{km} \\ &= F_1(k) + W_N^k F_2(k) \qquad k = 0, 1, \ldots, N-1 \end{aligned} \tag{6.1.25}$$

where $F_1(k)$ and $F_2(k)$ are the $N/2$-point DFTs of the sequences $f_1(m)$ and $f_2(m)$, respectively.

Since $F_1(k)$ and $F_2(k)$ are periodic, with period $N/2$, we have $F_1(k + N/2) = F_1(k)$ and $F_2(k + N/2) = F_2(k)$. In addition, the factor $W_N^{k+N/2} = -W_N^k$. Hence (6.1.25) can be expressed as

$$X(k) = F_1(k) + W_N^k F_2(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \tag{6.1.26}$$

$$X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \tag{6.1.27}$$

To be consistent with our previous notation, we may define

$$G_1(k) = F_1(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

$$G_2(k) = W_N^k F_2(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

Then the DFT $X(k)$ may be expressed as

$$\begin{aligned} X(k) &= G_1(k) + G_2(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \\ X(k + \frac{N}{2}) &= G_1(k) - G_2(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \end{aligned} \tag{6.1.28}$$

$N/4$-point sequences

$$\begin{aligned} v_{11}(n) &= f_1(2n) \qquad n = 0, 1, \ldots, \frac{N}{4} - 1 \\ v_{12}(n) &= f_1(2n+1) \qquad n = 0, 1, \ldots, \frac{N}{4} - 1 \end{aligned} \tag{6.1.29}$$

and $f_2(n)$ would yield

$$\begin{aligned} v_{21}(n) &= f_2(2n) \qquad n = 0, 1, \ldots, \frac{N}{4} - 1 \\ v_{22}(n) &= f_2(2n+1) \qquad n = 0, 1, \ldots, \frac{N}{4} - 1 \end{aligned} \tag{6.1.30}$$

By computing $N/4$-point DFTs, we would obtain the $N/2$-point DFTs $F_1(k)$ and $F_2(k)$ from the relations

$$\begin{aligned} F_1(k) &= V_{11}(k) + W_{N/2}^k V_{12}(k) \qquad k = 0, 1, \ldots, \frac{N}{4} - 1 \\ F_1\left(k + \frac{N}{4}\right) &= V_{11}(k) - W_{N/2}^k V_{12}(k) \qquad k = 0, 1, \ldots, \frac{N}{4} - 1 \end{aligned} \tag{6.1.31}$$

$$\begin{aligned} F_2(k) &= V_{21}(k) + W_{N/2}^k V_{22}(k) \qquad k = 0, 1, \ldots, \frac{N}{4} - 1 \\ F_2\left(k + \frac{N}{4}\right) &= V_{21}(k) - W_{N/2}^k V_{22}(k) \qquad k = 0, \ldots, \frac{N}{4} - 1 \end{aligned} \tag{6.1.32}$$

where the $\{V_{ij}(k)\}$ are the $N/4$-point DFTs of the sequences $\{v_{ij}(n)\}$.

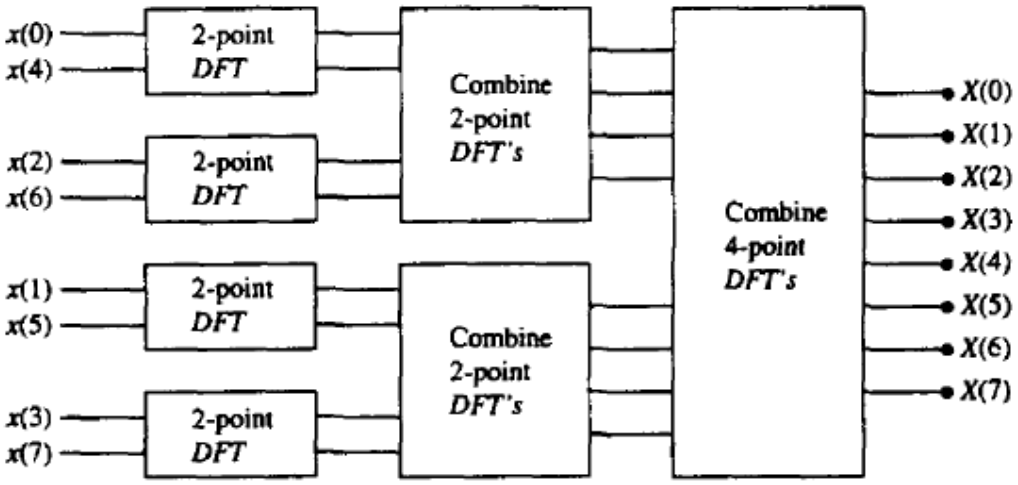


Figure 6.5 Three stages in the computation of an $N = 8$-point DFT.

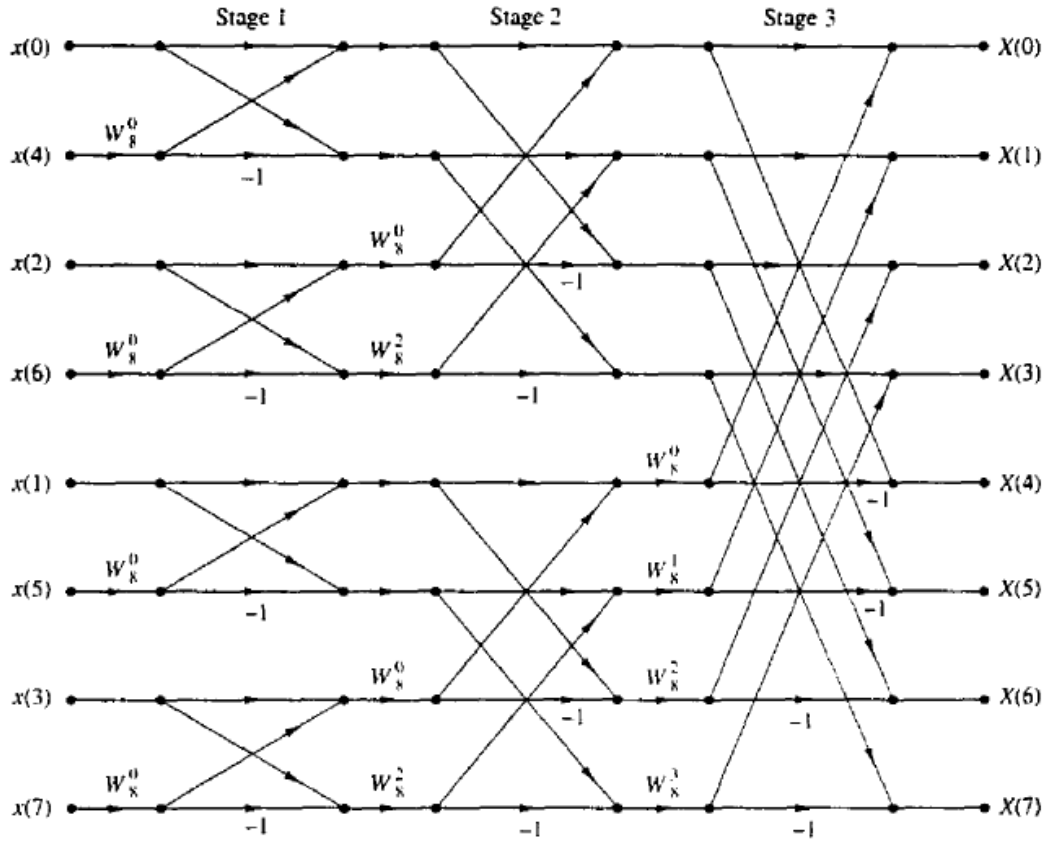


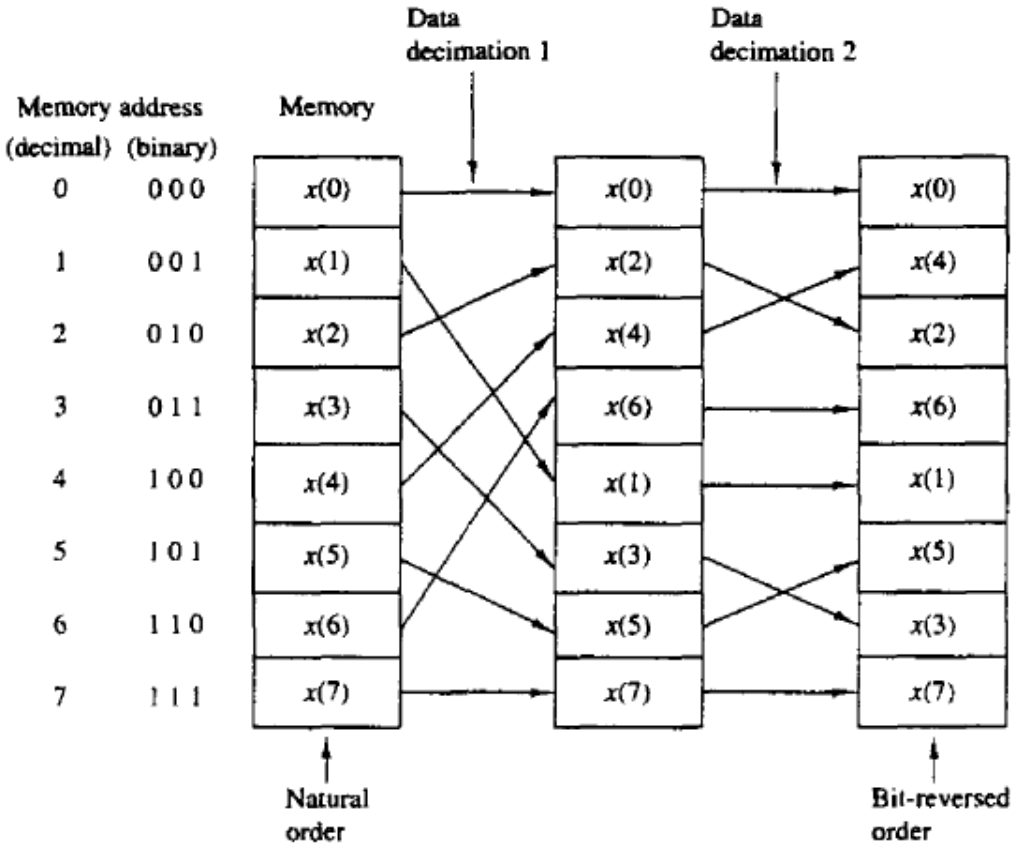
Figure 6.6 Eight-point decimation-in-time FFT algorithm.

Another important radix-2 FFT algorithm, called the decimation-in-frequency algorithm, is obtained by using the divide-and-conquer approach described in Section 6.1.2 with the choice of $M = 2$ and $L = N/2$. This choice of parameters implies a column-wise storage of the input data sequence. To derive the algorithm, we begin by splitting the DFT formula into two summations, one of which involves the sum over the first $N/2$ data points and the second sum involves the last $N/2$ data points. Thus we obtain

$$\begin{aligned} X(k) &= \sum_{n=0}^{(N/2)-1} x(n)W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)W_N^{kn} \\ &= \sum_{n=0}^{(N/2)-1} x(n)W_N^{kn} + W_N^{Nk/2} \sum_{n=0}^{(N/2)-1} x\left(n+\frac{N}{2}\right)W_N^{kn} \end{aligned} \tag{6.1.33}$$

Since $W_N^{kN/2} = (-1)^k$, the expression (6.1.33) can be rewritten as

$$X(k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right)\right] W_N^{kn} \tag{6.1.34}$$



(a)

Now, let us split (decimate) $X(k)$ into the even- and odd-numbered samples. Thus we obtain

$$X(2k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_{N/2}^{kn} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \tag{6.1.35}$$

and

$$X(2k+1) = \sum_{n=0}^{(N/2)-1} \left\{ \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n \right\} W_{N/2}^{kn} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \tag{6.1.36}$$

where we have used the fact that $W_N^2 = W_{N/2}$.

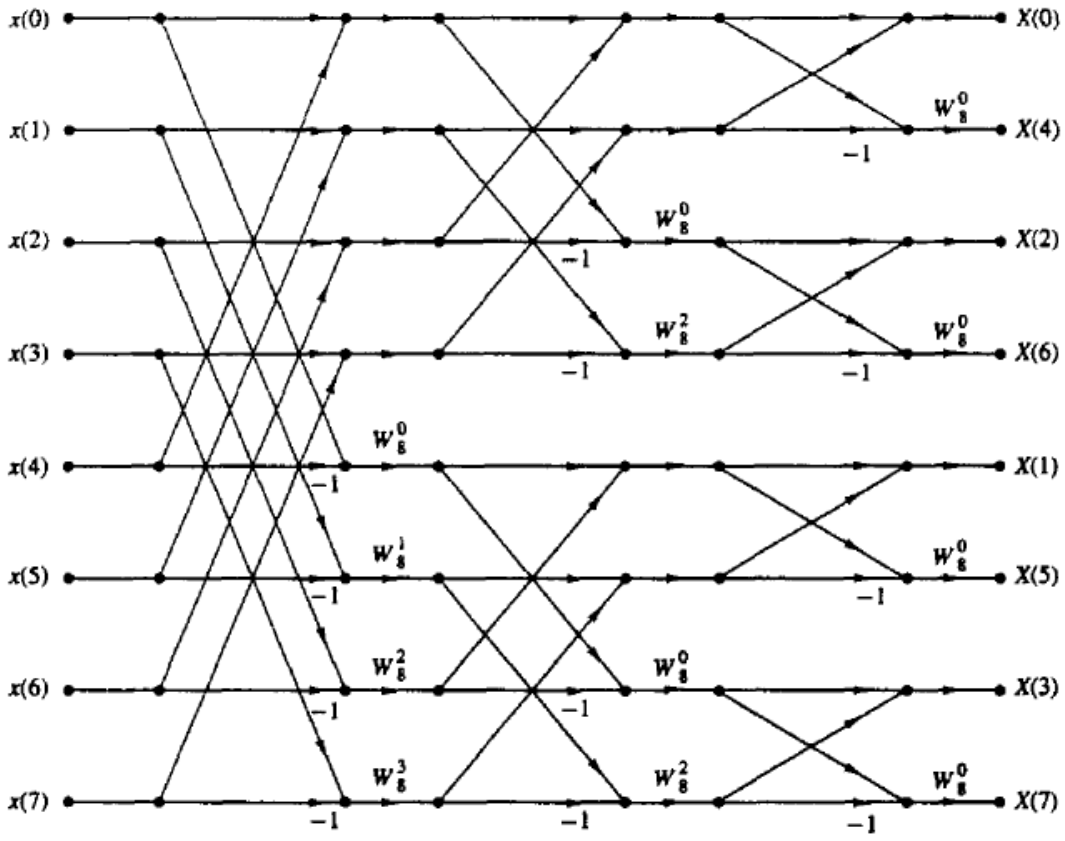
If we define the $N/2$-point sequences $g_1(n)$ and $g_2(n)$ as

$$\begin{aligned} g_1(n) &= x(n) + x\left(n + \frac{N}{2}\right) \\ g_2(n) &= \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^n \qquad n = 0, 1, 2, \ldots, \frac{N}{2} - 1 \end{aligned} \tag{6.1.37}$$

then

$$\begin{aligned} X(2k) &= \sum_{n=0}^{(N/2)-1} g_1(n) W_{N/2}^{kn} \\ X(2k+1) &= \sum_{n=0}^{(N/2)-1} g_2(n) W_{N/2}^{kn} \end{aligned} \tag{6.1.38}$$

We observe from Fig. 6.11, that the input data $x(n)$ occurs in natural order, but the output DFT occurs in bit-reversed order. We also note that the computations are performed in place. However, it is possible to reconfigure the decimation-in-frequency algorithm so that the input sequence occurs in bit-reversed order while the output DFT occurs in normal order. Furthermore, if we abandon the requirement that the computations be done in place, it is also possible to have both the input data and the output DFT in normal order.


x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)
W_8^0
W_8^1
W_8^2
W_8^3
−1

6.1.4 Radix-4 FFT Algorithms

When the number of data points N in the DFT is a power of 4 (i.e., $N = 4^v$), we can, of course, always use a radix-2 algorithm for the computation. However, for this case, it is more efficient computationally to employ a radix-4 FFT algorithm.

Let us begin by describing a radix-4 decimation-in-time FFT algorithm, which is obtained by selecting $L = 4$ and $M = N/4$ in the divide-and-conquer approach described in Section 6.1.2. For this choice of L and M, we have l, $p = 0, 1, 2, 3$; m, $q = 0, 1, \ldots, N/4 - 1$; $n = 4m + l$; and $k = (N/4)p + q$. Thus we split or decimate the N-point input sequence into four subsequences, $x(4n)$, $x(4n + 1)$, $x(4n + 2)$, $x(4n + 3)$, $n = 0, 1, \ldots, N/4 - 1$.

By applying (6.1.15) we obtain

$$X(p, q) = \sum_{l=0}^{3} \left[W_N^{lq} F(l, q) \right] W_4^{lp} \qquad p = 0, 1, 2, 3 \tag{6.1.39}$$

where $F(l, q)$ is given by (6.1.16), that is,

$$F(l, q) = \sum_{m=0}^{(N/4)-1} x(l, m) W_{N/4}^{mq} \qquad \begin{matrix} l = 0, 1, 2, 3, \\ q = 0, 1, 2, \ldots, \dfrac{N}{4} - 1 \end{matrix} \tag{6.1.40}$$

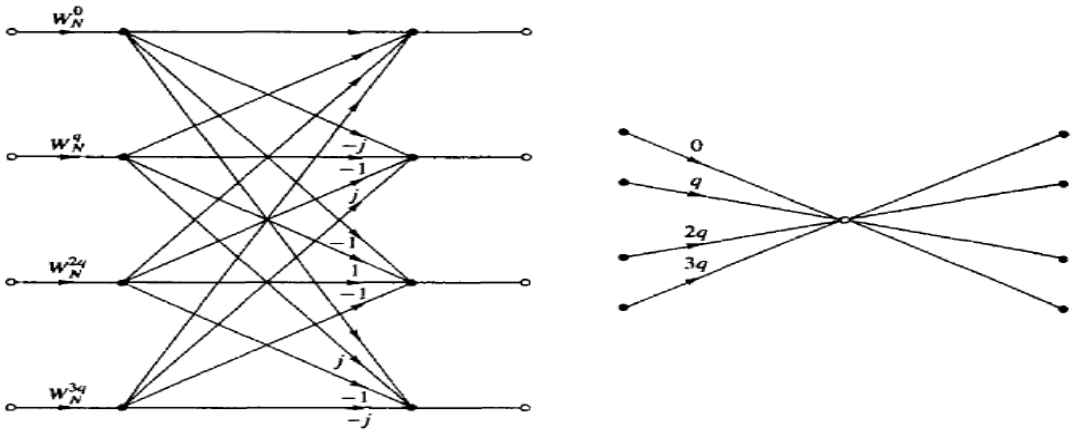
and

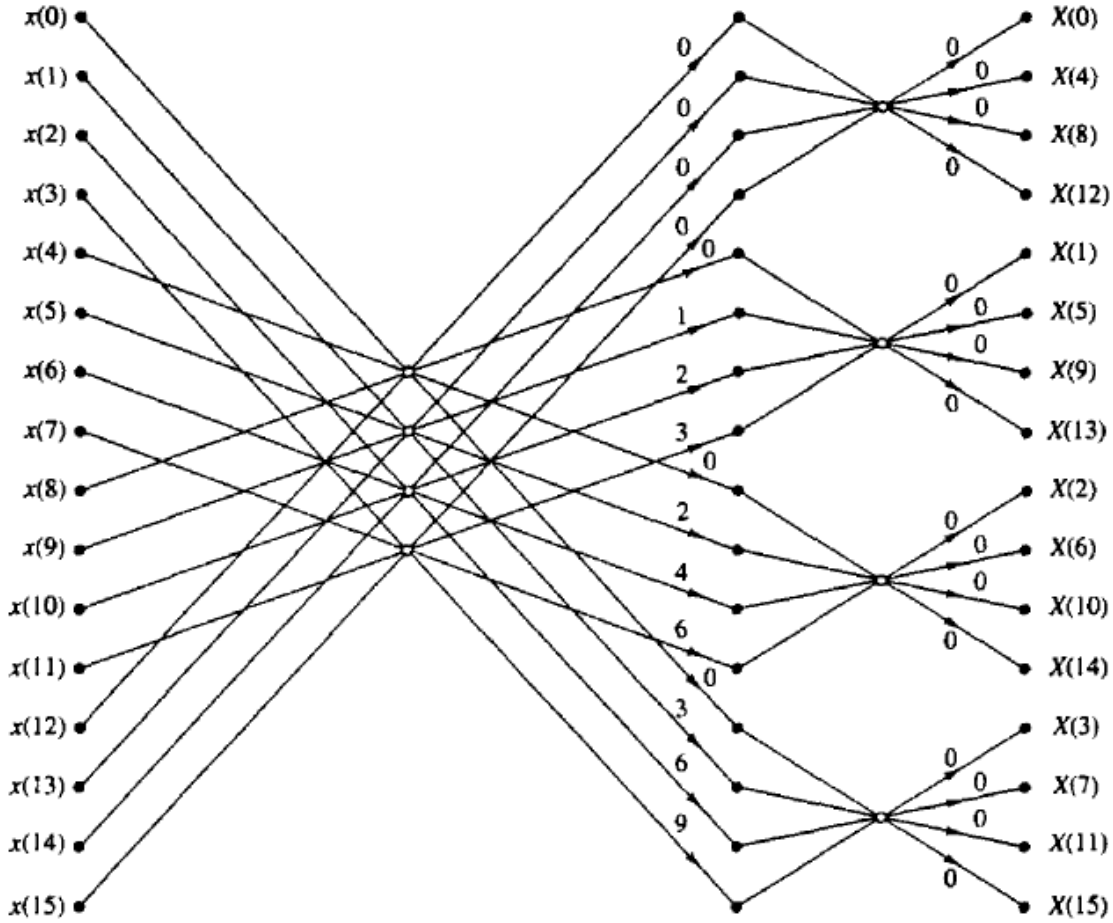
$$x(l, m) = x(4m + l) \tag{6.1.41}$$

$$X(p, q) = X\left(\frac{N}{4}p + q\right) \tag{6.1.42}$$

Thus, the four $N/4$-point DFTs obtained from (6.1.40) are combined according to (6.1.39) to yield the N-point DFT. The expression in (6.1.39) for combining the $N/4$-point DFTs defines a radix-4 decimation-in-time butterfly, which can be expressed in matrix form as

$$\begin{bmatrix} X(0, q) \\ X(1, q) \\ X(2, q) \\ X(3, q) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} W_N^0 F(0, q) \\ W_N^q F(1, q) \\ W_N^{2q} F(2, q) \\ W_N^{3q} F(3, q) \end{bmatrix} \tag{6.1.43}$$




x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
x(8)
x(9)
x(10)
x(11)
x(12)
x(13)
x(14)
x(15)
0
0
0
0
0
1
2
3
0
2
4
6
0
3
6
9
0
0
0
0
0
0
0
0
0
0
0
0
0
0
0
0
X(0)
X(4)
X(8)
X(12)
X(1)
X(5)
X(9)
X(13)
X(2)
X(6)
X(10)
X(14)
X(3)
X(7)
X(11)
X(15)

$$X(4k) = \sum_{n=0}^{N/4-1} \left[x(n) + x\left(n + \frac{N}{4}\right) + x\left(n + \frac{N}{2}\right) + x\left(n + \frac{3N}{4}\right) \right] W_N^0 W_{N/4}^{kn}$$

$$X(4k+1) = \sum_{n=0}^{N/4-1} \left[x(n) - jx\left(n + \frac{N}{4}\right) - x\left(n + \frac{N}{2}\right) + jx\left(n + \frac{3N}{4}\right) \right] W_N^n W_{N/4}^{kn}$$

$$X(4k+2) = \sum_{n=0}^{N/4-1} \left[x(n) - x\left(n + \frac{N}{4}\right) + x\left(n + \frac{N}{2}\right) - x\left(n + \frac{3N}{4}\right) \right] W_N^{2n} W_{N/4}^{kn}$$

$$X(4k+3) = \sum_{n=0}^{N/4-1} \left[x(n) + jx\left(n + \frac{N}{4}\right) - x\left(n + \frac{N}{2}\right) - jx\left(n + \frac{3N}{4}\right) \right] W_N^{3n} W_{N/4}^{kn}$$

6.1.5 Split-Radix FFT Algorithms

An inspection of the radix-2 decimation-in-frequency flowgraph shown in Fig. 6.11 indicates that the even-numbered points of the DFT can be computed independently of the odd-numbered points. This suggests the possibility of using different computational methods for independent parts of the algorithm with the objective of reducing the number of computations. The split-radix FFT (SRFFT) algorithms exploit this idea by using both a radix-2 and a radix-4 decomposition in the same FFT algorithm.

We illustrate this approach with a decimation-in-frequency SRFFT algorithm due to Duhamel (1986). First, we recall that in the radix-2 decimation-in-frequency FFT algorithm, the even-numbered samples of the N-point DFT are given as

$$X(2k) = \sum_{n=0}^{N/2-1} \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_{N/2}^{nk} \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \tag{6.1.55}$$

Note that these DFT points can be obtained from an $N/2$-point DFT without any additional multiplications. Consequently, a radix-2 suffices for this computation.

The odd-numbered samples $\{X(2k+1)\}$ of the DFT require the premultiplication of the input sequence with the twiddle factors W_N^n. For these samples a radix-4 decomposition produces some computational efficiency because the four-point DFT has the largest multiplication-free butterfly. Indeed, it can be shown that using a radix greater than 4, does not result in a significant reduction in computational complexity.

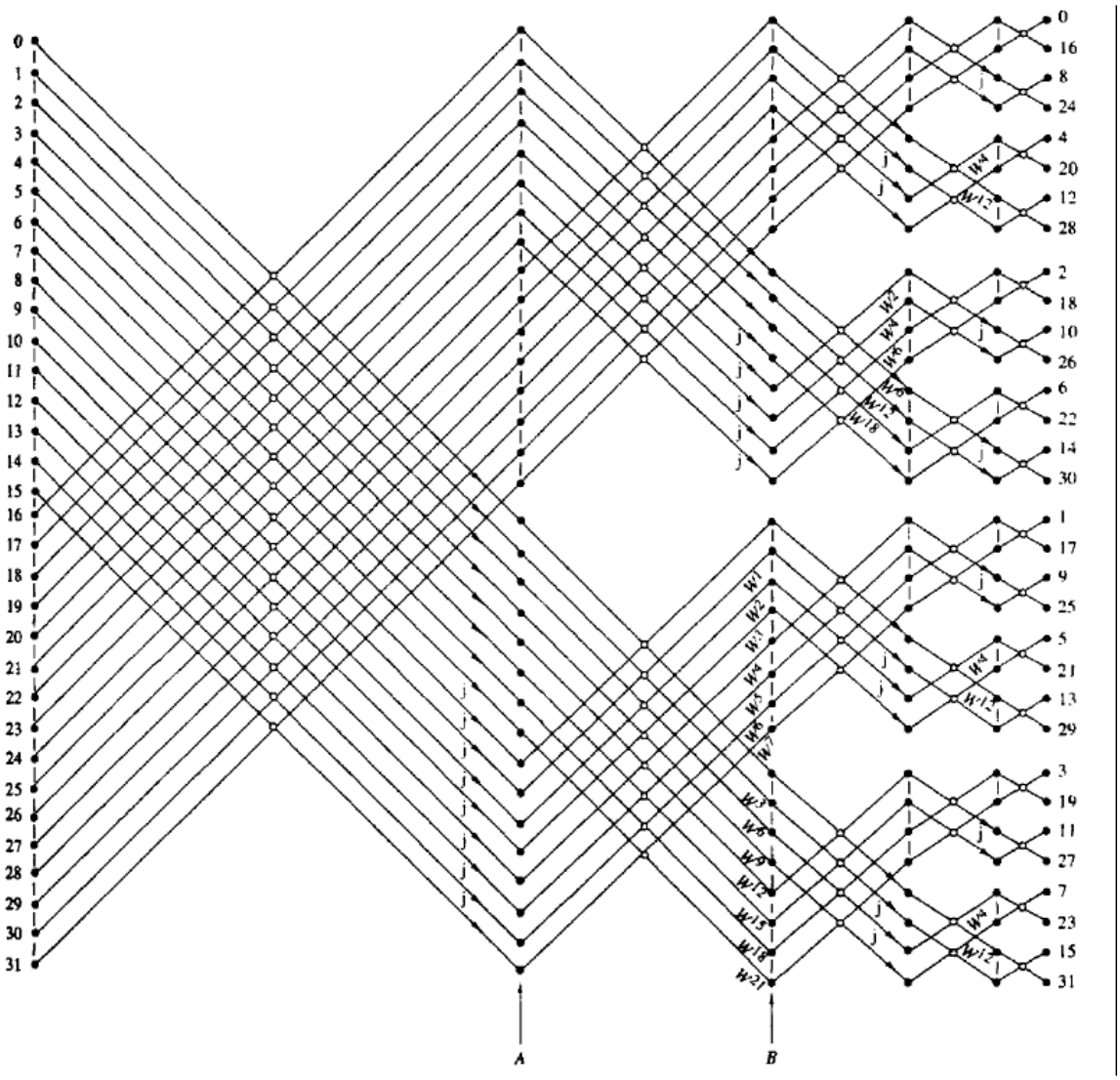
If we use a radix-4 decimation-in-frequency FFT algorithm for the odd-numbered samples of the N-point DFT, we obtain the following $N/4$-point DFTs:

$$\begin{aligned} X(4k+1) = \sum_{n=0}^{N/4-1} & \{[x(n) - x(n+N/2)] \\ & - j[x(n+N/4) - x(n+3N/4)]\} W_N^n W_{N/4}^{kn} \end{aligned} \tag{6.1.56}$$

$$\begin{aligned} X(4k+3) = \sum_{n=0}^{N/4-1} & \{[x(n) - x(n+N/2)] \\ & + j[x(n+N/4) - x(n+3N/4)]\} W_N^{3n} W_{N/4}^{kn} \end{aligned} \tag{6.1.57}$$

Thus the N-point DFT is decomposed into one $N/2$-point DFT without additional twiddle factors and two $N/4$-point DFTs with twiddle factors. The N-point DFT is obtained by successive use of these decompositions up to the last stage. Thus we obtain a decimation-in-frequency SRFFT algorithm.

Figure 6.15 shows the flow graph for an in-place 32-point decimation-in-frequency SRFFT algorithm. At stage A of the computation for $N = 32$, the



An additional factor of 2 savings in storage of twiddle factors can be obtained by introducing a 90° phase offset at the mid point of each twiddle array , which can be removed if necessary at the ouput of the SRFFT computation. The incorporation of this improvement into the SRFFT results in an other algorithm also due to price called the PFFT algorithm.

Implementation of FFT Algorithms

Now that w e has described the basic radix-2 and radix -4 F FT algorithm s, let us consider some of the implementation issues. Our remarks apply directly to

radix-2 algorithms, although similar comments may be made about radix-4 and higher-radix algorithms.

Basically, the radix-2 FFT algorithm consists of taking two data points at a time from memory, performing the butterfly computations and returning the resulting numbers to memory. This procedure is repeated many times ($(N \log_2 N)/2$ times) in the computation of an N-point DFT.

The butterfly computations require the twiddle factors $\{W_N^k\}$ at various stages in either natural or bit-reversed order. In an efficient implementation of the algorithm, the phase factors are computed once and stored in a table, either in normal order or in bit-reversed order, depending on the specific implementation of the algorithm.

Memory requirement is another factor that must be considered. If the computations are performed in place, the number of memory locations required is $2N$ since the numbers are complex. However, we can instead double the memory to 4N, thus simplifying the indexing and control operations in the FFT algorithms. In this case we simply alternate in the use of the two sets of memory locations from one stage of the FFT algorithm to the other. Doubling of the memory also allows us to have both the input sequence and the output sequence in normal order.

Finally, we note that the emphasis in our discussion of FFT algorithms was on radix-2, radix-4, and split-radix algorithms. These are by far the most widely used in practice. When the number of data points is not a power of 2 or 4, it is a simple matter to pad the sequence $x(n)$ with zeros such that $N = 2^v$ or $N = 4^v$.

The measure of complexity for FFT algorithms that we have emphasized is the required number of arithmetic operations (multiplications and additions). Although this is a very important benchmark for computational complexity, there are other issues to be considered in practical implementation of FFT algorithms. These include the architecture of the processor, the available instruction set, the data structures for storing twiddle factors, and other considerations.

For general-purpose computers, where the cost of the numerical operations dominate, radix-2, radix-4, and split-radix FFT algorithms are good candidates. However, in the case of special-purpose digital signal processors, featuring single-cycle multiply-and-accumulate operation, bit-reversed addressing, and a high degree of instruction parallelism, the structural regularity of the algorithm is equally important as arithmetic complexity. Hence for DSP processors, radix-2 or radix-4 decimation-in-frequency FFT algorithms are preferable in terms of speed and accuracy. The irregular structure of the SRFFT may render it less suitable for implementation on digital signal processors. Structural regularity is also important in the implementation of FFT algorithms on vector processors, multiprocessors, and in VLSI. Interprocessor communication is an important consideration in such implementations on parallel processors.

In conclusion, we have presented several important considerations in the implementation of FFT algorithms. Advances in digital signal processing technology, in hardware and software, will continue to influence the choice among FFT algorithms for various practical applications.

APPLICATIONS OF FFT ALGORITHMS

The FFT algorithms described in the preceding section find application in a variety of areas, including linear filtering, correlation, and spectrum analysis. Basically, the FFT algorithm is used as an efficient means to compute the DFT and the IDFT.

In this section we consider the use of the FFT algorithm in linear filtering and in the computation of the crosscorrelation of two sequences. The use of the FFT in spectrum analysis is considered in Chapter 12. In addition we illustrate how to enhance the efficiency of the FFT algorithm by forming complex-valued sequences from real-valued sequences prior to the computation of the DFT.

6.2.1 Efficient Computation of the DFT of Two Real Sequences

The FFT algorithm is designed to perform complex multiplications and additions, even though the input data may be real valued. The basic reason for this situation is

Suppose that $x_1(n)$ and $x_2(n)$ are two real-valued sequences of length N, and let $x(n)$ be a complex-valued sequence defined as

$$x(n) = x_1(n) + jx_2(n) \qquad 0 \le n \le N-1 \tag{6.2.1}$$

The DFT operation is linear and hence the DFT of $x(n)$ can be expressed as

$$X(k) = X_1(k) + jX_2(k) \tag{6.2.2}$$

The sequences $x_1(n)$ and $x_2(n)$ can be expressed in terms of $x(n)$ as follows:

$$x_1(n) = \frac{x(n) + x^*(n)}{2} \tag{6.2.3}$$

$$x_2(n) = \frac{x(n) - x^*(n)}{2j} \tag{6.2.4}$$

Hence the DFTs of $x_1(n)$ and $x_2(n)$ are

$$X_1(k) = \frac{1}{2}\{DFT[x(n)] + DFT[x^*(n)]\} \tag{6.2.5}$$

$$X_2(k) = \frac{1}{2j}\{DFT[x(n)] - DFT[x^*(n)]\} \tag{6.2.6}$$

Recall that the DFT of $x^*(n)$ is $X^*(N-k)$. Therefore,

$$X_1(k) = \frac{1}{2}[X(k) + X^*(N-k)] \tag{6.2.7}$$

$$X_2(k) = \frac{1}{j2}[X(k) - X^*(N-k)] \tag{6.2.8}$$

6.2.2 Efficient Computation of the DFT of a $2N$-Point Real Sequence

Suppose that $g(n)$ is a real-valued sequence of $2N$ points. We now demonstrate how to obtain the $2N$-point DFT of $g(n)$ from computation of one N-point DFT involving complex-valued data. First, we define

$$\begin{aligned} x_1(n) &= g(2n) \\ x_2(n) &= g(2n+1) \end{aligned} \tag{6.2.9}$$

Let $x(n)$ be the N-point complex-valued sequence

$$x(n) = x_1(n) + jx_2(n) \tag{6.2.10}$$

From the results of the preceding section, we have

$$\begin{aligned} X_1(k) &= \frac{1}{2}[X(k) + X^*(N-k)] \\ X_2(k) &= \frac{1}{2j}[X(k) - X^*(N-k)] \end{aligned} \tag{6.2.11}$$

Finally, we must express the $2N$-point DFT in terms of the two N-point DFTs, $X_1(k)$ and $X_2(k)$. To accomplish this, we proceed as in the decimation-in-time FFT algorithm, namely,

$$\begin{aligned} G(k) &= \sum_{n=0}^{N-1} g(2n) W_{2N}^{2nk} + \sum_{n=0}^{N-1} g(2n+1) W_{2N}^{(2n+1)k} \\ &= \sum_{n=0}^{N-1} x_1(n) W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} x_2(n) W_N^{nk} \end{aligned}$$

Consequently,

$$\begin{aligned} G(k) &= X_1(k) + W_2^k N X_2(k) \qquad k = 0, 1, \ldots, N-1 \\ G(k+N) &= X_1(k) - W_2^k N X_2(k) \qquad k = 0, 1, \ldots, N-1 \end{aligned} \tag{6.2.12}$$

Thus we have computed the DFT of a $2N$-point real sequence from one N-point DFT and some additional computation as indicated by (6.2.11) and (6.2.12).

6.2.3 Use of the FFT Algorithm in Linear Filtering and Correlation

An important application of the FFT algorithm is in FIR linear filtering of long data sequences. In Chapter 5 we described two methods, the overlap-add and the overlap-save methods for filtering a long data sequence with an FIR filter, based on the use of the DFT. In this section we consider the use of these two methods in conjunction with the FFT algorithm for computing the DFT and the IDFT.

Let $h(n)$, $0 \le n \le M-1$, be the unit sample response of the FIR filter and let $x(n)$ denote the input data sequence. The block size of the FFT algorithm is N, where $N = L + M - 1$ and L is the number of new data samples being processed by the filter. We assume that for any given value of M, the number L of data samples is selected so that N is a power of 2. For purposes of this discussion, we consider only radix-2 FFT algorithms.

The N-point DFT of $h(n)$, which is padded by $L-1$ zeros, is denoted as $H(k)$. This computation is performed once via the FFT and the resulting N complex numbers are stored. To be specific we assume that the decimation-in-frequency FFT algorithm is used to compute $H(k)$. This yields $H(k)$ in bit-reversed order, which is the way it is stored in memory.

In the overlap-save method, the first $M-1$ data points of each data block are the last $M-1$ data points of the previous data block. Each data block contains L new data points, such that $N = L + M - 1$. The N-point DFT of each data block is performed by the FFT algorithm. If the decimation-in-frequency algorithm is employed, the input data block requires no shuffling and the values of the DFT occur in bit-reversed order. Since this is exactly the order of $H(k)$, we can multiply the DFT of the data, say $X_m(k)$, with $H(k)$ and thus the result

$$Y_m(k) = H(k)X_m(k)$$

is also in bit-reversed order.

The inverse DFT (IDFT) can be computed by use of an FFT algorithm that takes the input in bit-reversed order and produces an output in normal order. Thus there is no need to shuffle any block of data either in computing the DFT or the IDFT.

If the overlap-add method is used to perform the linear filtering, the computational method using the FFT algorithm is basically the same. The only difference is that the N-point data blocks consist of L new data points and $M-1$ additional zeros. After the IDFT is computed for each data block, the N-point filtered blocks are overlapped as indicated in Section 5.3.2, and the $M-1$ overlapping data points between successive output records are added together.

UNIT - V
IIR AND FIR DIGITAL FILTERS

The transfer function is obtained by taking Z transform of finite sample impulse response. The filters designed by using finite samples of impulse response are called FIR filters. Some of the advantages of FIR filter are linear phase, both recursive and non recursive, stable and round off noise can be made smaller. Some of the disadvantages of FIR filters are large amount of processing is required and non integral delay may lead to problems.

DESIGN OF FIR FILTERS

An FIR filter of length M with input $x(n)$ and output $y(n)$ is described by the difference equation

$$\begin{aligned} y(n) &= b_0x(n) + b_1x(n-1) + \cdots + b_{M-1}x(n-M+1) \\ &= \sum_{k=0}^{M-1} b_kx(n-k) \end{aligned} \tag{8.2.1}$$

where $\{b_k\}$ is the set of filter coefficients. Alternatively, we can express the output sequence as the convolution of the unit sample response $h(n)$ of the system with the input signal. Thus we have

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{8.2.2}$$

where the lower and upper limits on the convolution sum reflect the causality and finite-duration characteristics of the filter. Clearly, (8.2.1) and (8.2.2) are identical in form and hence it follows that $b_k = h(k)$, $k = 0, 1, \ldots, M-1$.

The filter can also be characterized by its system function

$$H(z) = \sum_{k=0}^{M-1} h(k)z^{-k} \tag{8.2.3}$$

which we view as a polynomial of degree $M-1$ in the variable z^{-1}. The roots of this polynomial constitute the zeros of the filter.

An FIR filter has linear phase if its unit sample response satisfies the condition

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, \ldots, M-1 \tag{8.2.4}$$

When the symmetry and antisymmetry conditions in (8.2.4) are incorporated into (8.2.3), we have

$$\begin{aligned} H(z) &= h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots + h(M-2)z^{-(M-2)} + h(M-1)z^{-(M-1)} \\ &= z^{-(M-1)/2}\left\{h\left(\frac{M-1}{2}\right) + \sum_{n=0}^{(M-3)/2} h(n)\left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right]\right\} \quad M \text{ odd} \\ &= z^{-(M-1)/2}\sum_{n=0}^{(M/2)-1} h(n)\left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right] \qquad M \text{ even} \end{aligned} \tag{8.2.5}$$

Now, if we substitute z^{-1} for z in (8.2.3) and multiply both sides of the resulting equation by $z^{-(M-1)}$, we obtain

$$z^{-(M-1)}H(z^{-1}) = \pm H(z) \tag{8.2.6}$$

When $h(n) = h(M-1-n)$, $H(\omega)$ can be expressed as

$$H(\omega) = H_r(\omega)e^{-j\omega(M-1)/2} \tag{8.2.7}$$

where $H_r(\omega)$ is a real function of ω and can be expressed as

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \qquad M \text{ odd} \tag{8.2.8}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \qquad M \text{ even} \tag{8.2.9}$$

The phase characteristic of the filter for both M odd and M even is

$$\Theta(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases} \tag{8.2.10}$$

When

$$h(n) = -h(M-1-n)$$

the unit sample response is *antisymmetric*. For M odd, the center point of the antisymmetric $h(n)$ is $n = (M-1)/2$. Consequently,

$$h\left(\frac{M-1}{2}\right) = 0$$

It is straightforward to show that the frequency response of an FIR filter with an antisymmetric unit sample response can be expressed as

$$H(\omega) = H_r(\omega)e^{j[-\omega(M-1)/2+\pi/2]} \tag{8.2.11}$$

where

$$H_r(\omega) = 2\sum_{n=0}^{(M-3)/2} h(n)\sin\omega\left(\frac{M-1}{2} - n\right) \qquad M \text{ odd} \tag{8.2.12}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\sin\omega\left(\frac{M-1}{2} - n\right) \qquad M \text{ even} \tag{8.2.13}$$

The phase characteristic of the filter for both M odd and M even is

$$\Theta(\omega) = \begin{cases} \dfrac{\pi}{2} - \omega\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ \dfrac{3\pi}{2} - \omega\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\omega) < 0 \end{cases} \tag{8.2.14}$$

The choice of a symmetric or antisymmetric unit sample response depends on the application. As we shall see later, a symmetric unit sample response is suitable for some applications, while an antisymmetric unit sample response is more suitable for other applications. For example, if $h(n) = -h(M-1-n)$ and M is odd, (8.2.12) implies that $H_r(0) = 0$ and $H_r(\pi) = 0$. Consequently, (8.2.12) is not suitable as either a lowpass filter or a highpass filter. Similarly, the antisymmetric unit sample response with M even also results in $H_r(0) = 0$, as can be easily verified from (8.2.13). Consequently, we would not use the antisymmetric condition in the design of a lowpass linear-phase FIR filter. On the other hand, the symmetry condition $h(n) = h(M-1-n)$ yields a linear-phase FIR filter with a nonzero response at $\omega = 0$, if desired, that is,

$$H_r(0) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n), \qquad M \text{ odd} \tag{8.2.15}$$

$$H_r(0) = 2\sum_{n=0}^{(M/2)-1} h(n), \qquad M \text{ even} \tag{8.2.16}$$

8.2.2 Design of Linear-Phase FIR Filters Using Windows

In this method we begin with the desired frequency response specification $H_d(\omega)$ and determine the corresponding unit sample response $h_d(n)$. Indeed, $h_d(n)$ is related to $H_d(\omega)$ by the Fourier transform relation

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)e^{-j\omega n} \tag{8.2.17}$$

where

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \tag{8.2.18}$$

Thus, given $H_d(\omega)$, we can determine the unit sample response $h_d(n)$ by evaluating the integral in (8.2.18).

In general, the unit sample response $h_d(n)$ obtained from (8.2.17) is infinite in duration and must be truncated at some point, say at $n = M - 1$, to yield an FIR filter of length M. Truncation of $h_d(n)$ to a length $M - 1$ is equivalent to multiplying $h_d(n)$ by a "rectangular window," defined as

$$w(n) = \begin{cases} 1, & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \tag{8.2.19}$$

Thus the unit sample response of the FIR filter becomes

$$\begin{aligned} h(n) &= h_d(n)w(n) \\ &= \begin{cases} h_d(n), & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{8.2.20}$$

It is instructive to consider the effect of the window function on the desired frequency response $H_d(\omega)$. Recall that multiplication of the window function $w(n)$ with $h_d(n)$ is equivalent to convolution of $H_d(\omega)$ with $W(\omega)$, where $W(\omega)$ is the frequency-domain representation (Fourier transform) of the window function, that is,

$$W(\omega) = \sum_{n=0}^{M-1} w(n)e^{-j\omega n} \tag{8.2.21}$$

Thus the convolution of $H_d(\omega)$ with $W(\omega)$ yields the frequency response of the (truncated) FIR filter. That is,

$$H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(v)W(\omega - v)dv \tag{8.2.22}$$

The Fourier transform of the rectangular window is

$$\begin{aligned} W(\omega) &= \sum_{n=0}^{M-1} e^{-j\omega n} \\ &= \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = e^{-j\omega(M-1)/2} \frac{\sin(\omega M/2)}{\sin(\omega/2)} \end{aligned} \tag{8.2.23}$$

This window function has a magnitude response

$$|W(\omega)| = \frac{|\sin(\omega M/2)|}{|\sin(\omega/2)|} \qquad -\pi \le \omega \le \pi \tag{8.2.24}$$

and a piecewise linear phase

$$\Theta(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & \text{when } \sin(\omega M/2) \ge 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi, & \text{when } \sin(\omega M/2) < 0 \end{cases} \tag{8.2.25}$$

The magnitude response of the window function is illustrated in Fig. 8.4 for $M = 31$ and 61. The width of the main lobe [width is measured to the first zero of $W(\omega)$]

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2}\right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{M-1}{2}\right)^2 - \left(n - \dfrac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\dfrac{M-1}{2}\right)\right]}$
Lanczos	$\left\{\dfrac{\sin\left[2\pi\left(n - \dfrac{M-1}{2}\right)\Big/(M-1)\right]}{2\pi\left(n - \dfrac{M-1}{2}\right)\Big/\left(\dfrac{M-1}{2}\right)}\right\}^L \quad L > 0$
Tukey	$1, \left\lvert n - \dfrac{M-1}{2}\right\rvert \le \alpha\dfrac{M-1}{2} \quad 0 < \alpha < 1$ $\dfrac{1}{2}\left[1 + \cos\left(\dfrac{n - (1+a)(M-1)/2}{(1-\alpha)(M-1)/2}\pi\right)\right]$ $\alpha(M-1)/2 \le \left\lvert n - \dfrac{M-1}{2}\right\rvert \le \dfrac{M-1}{2}$

8.2.3 Design of Linear-Phase FIR Filters by the Frequency-Sampling Method

In the frequency sampling method for FIR filter design, we specify the desired frequency response $H_d(\omega)$ at a set of equally spaced frequencies, namely

$$\omega_k = \tfrac{2\pi}{M}(k+\alpha) \qquad \begin{aligned} &k = 0, 1, \ldots, \frac{M-1}{2} \quad M \text{ odd} \\ &k = 0, 1, \ldots, \frac{M}{2} - 1 \quad M \text{ even} \\ &\alpha = 0 \quad \text{or} \quad \tfrac{1}{2} \end{aligned} \tag{8.2.30}$$

and solve for the unit sample response $h(n)$ of the FIR filter from these equally

spaced frequency specifications. To reduce sidelobes, it is desirable to optimize the frequency specification in the transition band of the filter. This optimization can be accomplished numerically on a digital computer by means of linear programming techniques as shown by Rabiner et al. (1970).

In this section we exploit a basic symmetry property of the sampled frequency response function to simplify the computations. Let us begin with the desired frequency response of the FIR filter, which is [for simplicity, we drop the subscript in $H_d(\omega)$],

$$H(\omega) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n} \tag{8.2.31}$$

Suppose that we specify the frequency response of the filter at the frequencies given by (8.2.30). Then from (8.2.31) we obtain

$$H(k+\alpha) \equiv H\left(\frac{2\pi}{M}(k+\alpha)\right)$$

$$H(k+\alpha) \equiv \sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M} \qquad k = 0, 1, \ldots, M-1 \tag{8.2.32}$$

It is a simple matter to invert (8.2.32) and express $h(n)$ in terms of $H(k+\alpha)$. If we multiply both sides of (8.2.32) by the exponential, $\exp(j2\pi km/M)$, $m = 0, 1, \ldots, M-1$, and sum over $k = 0, 1, \ldots, M-1$, the right-hand side of (8.2.32) reduces to $Mh(m)\exp(-j2\pi\alpha m/M)$. Thus we obtain

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M} \qquad n = 0, 1, \ldots, M-1 \tag{8.2.33}$$

The relationship in (8.2.33) allows us to compute the values of the unit sample response $h(n)$ from the specification of the frequency samples $H(k+\alpha)$, $k = 0, 1, \ldots, M-1$. Note that when $\alpha = 0$, (8.2.32) reduces to the discrete Fourier transform (DFT) of the sequence $\{h(n)\}$ and (8.2.33) reduces to the inverse DFT (IDFT).

Since $\{h(n)\}$ is real, we can easily show that the frequency samples $\{H(k+\alpha)\}$ satisfy the symmetry condition

$$H(k+\alpha) = H^*(M-k-\alpha) \tag{8.2.34}$$

This symmetry condition, along with the symmetry conditions for $\{h(n)\}$, can be used to reduce the frequency specifications from M points to $(M+1)/2$ points for M odd and $M/2$ points for M even. Thus the linear equations for determining $\{h(n)\}$ from $\{H(k+\alpha)\}$ are considerably simplified.

In particular, if (8.2.11) is sampled at the frequencies $\omega_k = 2\pi(k+\alpha)/M$, $k = 0, 1, \ldots, M-1$, we obtain

$$H(k+\alpha) = H_r\left(\frac{2\pi}{M}(k+\alpha)\right)e^{j[\beta\pi/2 - 2\pi(k+\alpha)(M-1)/2M]} \tag{8.2.35}$$

where $\beta = 0$ when $\{h(n)\}$ is symmetric and $\beta = 1$ when $\{h(n)\}$ is antisymmetric. A simplication occurs by defining a set of real frequency samples $\{G(k+m)\}$

$$G(k+\alpha) = (-1)^k H_r\left(\frac{2\pi}{M}(k+\alpha)\right) \qquad k = 0, 1, \ldots, M-1 \tag{8.2.36}$$

We use (8.2.36) in (8.2.35) to eliminate $H_r(\omega_k)$. Thus we obtain

$$H(k+\alpha) = G(k+\alpha)e^{j\pi k}e^{j[\beta\pi/2 - 2\pi(k+\alpha)(M-1)/2M]} \tag{8.2.37}$$

Now the symmetry condition for $H(k+\alpha)$ given in (8.2.34) translates into a corresponding symmetry condition for $G(k+\alpha)$, which can be exploited by substituting into (8.2.33), to simplify the expressions for the FIR filter impulse response $\{h(n)\}$ for the four cases $\alpha = 0$, $\alpha = \frac{1}{2}$, $\beta = 0$, and $\beta = 1$. The results are summarized in Table 8.3. The detailed derivations are left as exercises for the reader.

8.2.4 Design of Optimum Equiripple Linear-Phase FIR Filters

The window method and the frequency-sampling method are relatively simple techniques for designing linear-phase FIR filters. However, they also possess some minor disadvantages, described in Section 8.2.6, which may render them undesirable for some applications. A major problem is the lack of precise control of the critical frequencies such as ω_p and ω_s.

The filter design method described in this section is formulated as a Chebyshev approximation problem. It is viewed as an optimum design criterion in the sense that the weighted approximation error between the desired frequency response and the actual frequency response is spread evenly across the passband and evenly across the stopband of the filter minimizing the maximum error. The resulting filter designs have ripples in both the passband and the stopband.

To describe the design procedure, let us consider the design of a lowpass filter with passband edge frequency ω_p and stopband edge frequency ω_s. From the general specifications given in Fig. 8.2, in the passband, the filter frequency response satisfies the condition

$$1 - \delta_1 \le H_r(\omega) \le 1 + \delta_1 \qquad |\omega| \le \omega_p \tag{8.2.43}$$

Similarly, in the stopband, the filter frequency response is specified to fall between the limits $\pm\delta_2$, that is,

$$-\delta_2 \le H_r(\omega) \le \delta_2 \qquad |\omega| > \omega_s \tag{8.2.44}$$

Thus δ_1 represents the ripple in the passband and δ_2 represents the attenuation or ripple in the stopband. The remaining filter parameter is M, the filter length or the number of filter coefficients.

Case 1: Symmetric unit sample response $h(n) = h(M-1-n)$ and M Odd. In this case, the real-valued frequency response characteristic $H_r(\omega)$ is

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \tag{8.2.45}$$

If we let $k = (M-1)/2 - n$ and define a new set of filter parameters $\{a(k)\}$ as

$$a(k) = \begin{cases} h\left(\dfrac{M-1}{2}\right), & k = 0 \\ 2h\left(\dfrac{M-1}{2} - k\right), & k = 1, 2, \ldots, \dfrac{M-1}{2} \end{cases} \tag{8.2.46}$$

then (8.2.45) reduces to the compact form

$$H_r(\omega) = \sum_{k=0}^{(M-1)/2} a(k)\cos\omega k \tag{8.2.47}$$

Case 2: Symmetric unit sample response $h(n) = h(M-1-n)$ and M Even. In this case, $H_r(\omega)$ is expressed as

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\omega\left(\frac{M-1}{2} - n\right) \tag{8.2.48}$$

Again, we change the summation index from n to $k = M/2 - n$ and define a new set of filter parameters $\{b(k)\}$ as

$$b(k) = 2h\left(\frac{M}{2} - k\right), k = 1, 2, \ldots, M/2 \tag{8.2.49}$$

With these substitutions (8.2.48) becomes

$$H_r(\omega) = \sum_{k=1}^{M/2} b(k)\cos\omega\left(k - \frac{1}{2}\right) \tag{8.2.50}$$

In carrying out the optimization, it is convenient to rearrange (8.2.50) further into the form

$$H_r(\omega) = \cos\frac{\omega}{2}\sum_{k=0}^{(M/2)-1} \tilde{b}(k)\cos\omega k \tag{8.2.51}$$

where the coefficients $\{\tilde{b}(k)\}$ are linearly related to the coefficients $\{b(k)\}$. In fact, it can be shown that the relationship is

$$\begin{aligned} \tilde{b}(0) &= \tfrac{1}{2}b(1) \\ \tilde{b}(k) &= 2b(k) - \tilde{b}(k-1) \qquad k = 1, 2, 3, \ldots, \frac{M}{2} - 2 \\ \tilde{b}\left(\frac{M}{2} - 1\right) &= 2b\left(\frac{M}{2}\right) \end{aligned} \tag{8.2.52}$$

Case 3: Antisymmetric unit sample response $h(n) = -h(M-1-n)$ and M Odd. The real-valued frequency response characteristic $H_r(\omega)$ for this case is

$$H_r(\omega) = 2\sum_{n=0}^{(M-3)/2} h(n)\sin\omega\left(\frac{M-1}{2}-n\right) \tag{8.2.53}$$

If we change the summation in (8.2.53) from n to $k = (M-1)/2 - n$ and define a new set of filter parameters $\{c(k)\}$ as

$$c(k) = 2h\left(\frac{M-1}{2}-k\right) \qquad k = 1, 2, \ldots, (M-1)/2 \tag{8.2.54}$$

then (8.2.53) becomes

$$H_r(\omega) = \sum_{k=1}^{(M-1)/2} c(k)\sin\omega k \tag{8.2.55}$$

As in the previous case, it is convenient to rearrange (8.2.55) into the form

$$H_r(\omega) = \sin\omega \sum_{k=0}^{(M-3)/2} \tilde{c}(k)\cos\omega k \tag{8.2.56}$$

Case 4: Antisymmetric unit sample response $h(n) = -h(M-1-n)$ and M Even. In this case, the real-valued frequency response characteristic $H_r(\omega)$ is

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\sin\omega\left(\frac{M-1}{2}-n\right) \tag{8.2.58}$$

A change in the summation index from n to $k = M/2 - n$ combined with a definition of a new set of filter coefficients $\{d(k)\}$, related to $\{h(n)\}$ according to

$$d(k) = 2h\left(\frac{M}{2}-k\right) \qquad k = 1, 2, \ldots, \frac{M}{2} \tag{8.2.59}$$

results in the expression

$$H_r(\omega) = \sum_{k=1}^{M/2} d(k)\sin\omega\left(k-\frac{1}{2}\right) \tag{8.2.60}$$

As in the previous two cases, we find it convenient to rearrange (8.2.60) into the form

$$H_r(\omega) = \sin\frac{\omega}{2}\sum_{k=0}^{(M/2)-1} \tilde{d}(k)\cos\omega k \tag{8.2.61}$$

Filter type	$Q(\omega)$	$P(\omega)$
$h(n) = h(M-1-n)$ M odd (case 1)	1	$\sum_{k=0}^{(M-1)/2} a(k)\cos\omega k$
$h(n) = h(M-1-n)$ M even (case 2)	$\cos\frac{\omega}{2}$	$\sum_{k=0}^{(M/2)-1} \tilde{b}(k)\cos\omega k$
$h(n) = -h(M-1-n)$ M odd (case 3)	$\sin\omega$	$\sum_{k=0}^{(M-3)/2} \tilde{c}(k)\cos\omega k$
$h(n) = -h(M-1-n)$ M even (case 4)	$\sin\frac{\omega}{2}$	$\sum_{k=0}^{(M/2)-1} \tilde{d}(k)\cos\omega k$

IIR FILTER DESIGN

DESIGN OF IIR FILTERS FROM ANALOG FILTERS

Just as in the design of FIR filters, there are several methods that can be used to design digital filters having an infinite-duration unit sample response. The techniques described in this section are all based on converting an analog filter into a digital filter. Analog filter design is a mature and well developed field, so it is not surprising that we begin the design of a digital filter in the analog domain and then convert the design into the digital domain.

An analog filter can be described by its system function.

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k} \tag{8.3.1}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are the filter coefficients, or by its impulse response, which is related to $H_a(s)$ by the Laplace transform

$$H_a(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt \tag{8.3.2}$$

Alternatively, the analog filter having the rational system function $H(s)$ given in (8.3.1), can be described by the linear constant-coefficient differential equation

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k} \tag{8.3.3}$$

where $x(t)$ denotes the input signal and $y(t)$ denotes the output of the filter.

Each of these three equivalent characterizations of an analog filter leads to alternative methods for converting the filter into the digital domain, as will be described in Sections 8.3.1 through 8.3.4. We recall that an analog linear time-invariant system with system function $H(s)$ is stable if all its poles lie in the left half of the s-plane. Consequently, if the conversion technique is to be effective, it should possess the following desirable properties:

1. The $j\Omega$ axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains.

2. The left-half plane (LHP) of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

We mentioned in the preceding section that physically realizable and stable IIR filters cannot have linear phase. Recall that a linear-phase filter must have a system function that satisfies the condition

$$H(z) = \pm z^{-N} H(z^{-1}) \tag{8.3.4}$$

where z^{-N} represents a delay of N units of time. But if this were the case, the filter would have a mirror-image pole outside the unit circle for every pole inside the unit circle. Hence the filter would be unstable. Consequently, a causal and stable IIR filter cannot have linear phase.

If the restriction on physical realizability is removed, it is possible to obtain a linear-phase IIR filter, at least in principle. This approach involves performing a time reversal of the input signal $x(n)$, passing $x(-n)$ through a digital filter $H(z)$, time-reversing the output of $H(z)$, and finally, passing the result through $H(z)$ again. This signal processing is computationally cumbersome and appears to offer no advantages over linear-phase FIR filters. Consequently, when an application requires a linear-phase filter, it should be an FIR filter.

In the design of IIR filters, we shall specify the desired filter characteristics for the magnitude response only. This does not mean that we consider the phase response unimportant. Since the magnitude and phase characteristics are related, as indicated in Section 8.1, we specify the desired magnitude characteristics and accept the phase response that is obtained from the design methodology.

8.3.1 IIR Filter Design by Approximation of Derivatives

One of the simplest methods for converting an analog filter into a digital filter is to approximate the differential equation in (8.3.3) by an equivalent difference equation. This approach is often used to solve a linear constant-coefficient differential equation numerically on a digital computer.

For the derivative $dy(t)/dt$ at time $t = nT$, we substitute the *backward difference* $[y(nT) - y(nT - 1)]/T$. Thus

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT - T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T} \tag{8.3.5}$$

where T represents the sampling interval and $y(n) \equiv y(nT)$. The analog differentiator with output $dy(t)/dt$ has the system function $H(s) = s$, while the digital system that produces the output $[y(n) - y(n-1)]/T$ has the system function $H(z) = (1 - z^{-1})/T$. Consequently, as shown in Fig. 8.29, the frequency-domain

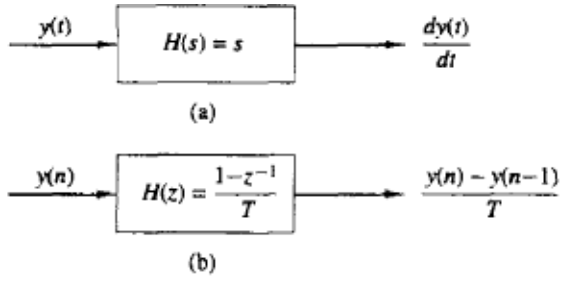


Figure 8.29 Substitution of the backward difference for the derivative implies the mapping $s = (1 - z^{-1})/T$.

equivalent for the relationship in (8.3.5) is

$$s = \frac{1 - z^{-1}}{T} \tag{8.3.6}$$

The second derivative $d^2y(t)/dt^2$ is replaced by the second difference, which is derived as follows:

$$\begin{aligned} \left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} &= \frac{d}{dt}\left[\frac{dy(t)}{dt}\right]_{t=nT} \\ &= \frac{[y(nT) - y(nT - T)]/T - [y(nT - T) - y(nT - 2T)]/T}{T} \\ &= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2} \end{aligned} \tag{8.3.7}$$

In the frequency domain, (8.3.7) is equivalent to

$$s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2} = \left(\frac{1 - z^{-1}}{T}\right)^2 \tag{8.3.8}$$

It easily follows from the discussion that the substitution for the kth derivative of $y(t)$ results in the equivalent frequency-domain relationship

$$s^k = \left(\frac{1 - z^{-1}}{T}\right)^k \tag{8.3.9}$$

Consequently, the system function for the digital IIR filter obtained as a result of the approximation of the derivatives by finite differences is

$$H(z) = H_a(s)|_{s=(1-z^{-1})/T} \tag{8.3.10}$$

where $H_a(s)$ is the system function of the analog filter characterized by the differential equation given in (8.3.3).

Let us investigate the implications of the mapping from the s-plane to the z-plane as given by (8.3.6) or, equivalently,

$$z = \frac{1}{1 - sT} \tag{8.3.11}$$

If we substitute $s = j\Omega$ in (8.2.11), we find that

$$z = \frac{1}{1 - j\Omega T}$$

$$= \frac{1}{1+\Omega^2T^2} + j\frac{\Omega T}{1+\Omega^2T^2} \tag{8.3.12}$$

As Ω varies from $-\infty$ to ∞, the corresponding locus of points in the z-plane is a circle of radius $\frac{1}{2}$ and with center at $z = \frac{1}{2}$, as illustrated in Fig. 8.30.

It is easily demonstrated that the mapping in (8.3.11) takes points in the LHP of the s-plane into corresponding points inside this circle in the z-plane and points in the RHP of the s-plane are mapped into points outside this circle. Consequently, this mapping has the desirable property that a stable analog filter is transformed into a stable digital filter. However, the possible location of the poles of the digital filter are confined to relatively small frequencies and, as a consequence, the mapping is restricted to the design of lowpass filters and bandpass filters having relatively small resonant frequencies. It is not possible, for example, to transform a highpass analog filter into a corresponding highpass digital filter.

In an attempt to overcome the limitations in the mapping given above, more complex substitutions for the derivatives have been proposed. In particular, an Lth-order difference of the form

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{1}{T}\sum_{k=1}^{L}\alpha_k\frac{y(nT+kT) - y(nT-kT)}{T} \tag{8.3.13}$$

has been proposed, where $\{\alpha_k\}$ are a set of parameters that can be selected to optimize the approximation. The resulting mapping between the s-plane and the z-plane is now

$$s = \frac{1}{T}\sum_{k=1}^{L}\alpha_k(z^k - z^{-k}) \tag{8.3.14}$$

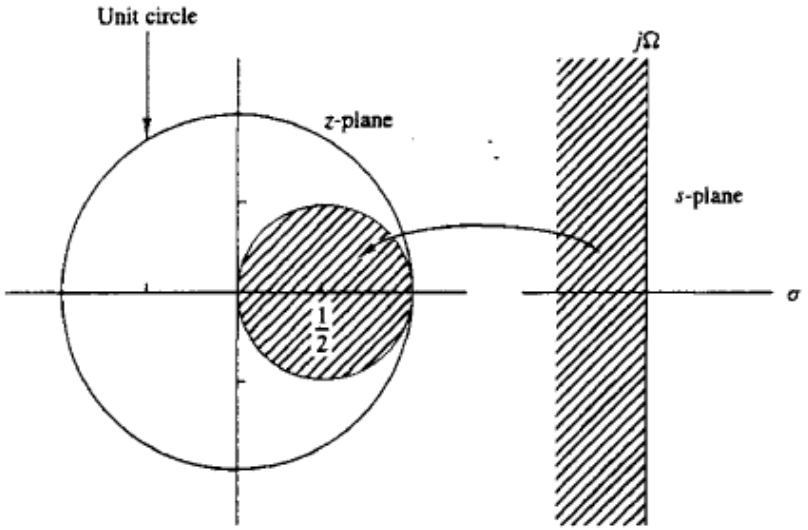


Figure 8.30 The mapping $s = (1 - z^{-1})/T$ takes LHP in the s-plane into points inside the circle of radius $\frac{1}{2}$ and center $z = \frac{1}{2}$ in the z-plane.

When $z = e^{j\omega}$, we have

$$s = j\frac{2}{T}\sum_{k=1}^{L}\alpha_k \sin \omega k \tag{8.3.15}$$

which is purely imaginary. Thus

$$\Omega = \frac{2}{T}\sum_{k=1}^{L}\alpha_k \sin \omega k \tag{8.3.16}$$

is the resulting mapping between the two frequency variables. By proper choice of the coefficients $\{\alpha_k\}$ it is possible to map the $j\Omega$-axis into the unit circle. Furthermore, points in the LHP in s can be mapped into points inside the unit circle in z.

Despite achieving the two desirable characteristics with the mapping of (8.3.16), the problem of selecting the set of coefficients $\{\alpha_k\}$ remains. In general, this is a difficult problem. Since simpler techniques exist for converting analog filters into IIR digital filters, we shall not emphasize the use of the Lth-order difference as a substitute for the derivative.

8.3.2 IIR Filter Design by Impulse Invariance

In the impulse invariance method, our objective is to design an IIR filter having a unit sample response $h(n)$ that is the sampled version of the impulse response of the analog filter. That is,

$$h(n) \equiv h(nT) \qquad n = 0, 1, 2, \ldots \tag{8.3.17}$$

where T is the sampling interval.

To examine the implications of (8.3.17), we refer back to Section 4.2.9. Recall that when a continuous time signal $x_a(t)$ with spectrum $X_a(F)$ is sampled at a rate $F_s = 1/T$ samples per second, the spectrum of the sampled signal is the periodic repetition of the scaled spectrum $F_s X_a(F)$ with period F_s. Specifically, the relationship is

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a[(f - k)F_s] \tag{8.3.18}$$

where $f = F/F_s$ is the normalized frequency. Aliasing occurs if the sampling rate F_s is less than twice the highest frequency contained in $X_a(F)$.

Expressed in the context of sampling the impulse response of an analog filter with frequency response $H_a(F)$, the digital filter with unit sample response $h(n) \equiv h_a(nT)$ has the frequency response

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f - k)F_s] \tag{8.3.19}$$

or, equivalently,

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s] \tag{8.3.20}$$

or

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right) \tag{8.3.21}$$

Figure 8.31 depicts the frequency response of a lowpass analog filter and the frequency response of the corresponding digital filter.

It is clear that the digital filter with frequency response $H(\omega)$ has the frequency response characteristics of the corresponding analog filter if the sampling interval T is selected sufficiently small to completely avoid or at least minimize the effects of aliasing. It is also clear that the impulse invariance method is inappropriate for designing highpass filters due the to spectrum aliasing that results from the sampling process.

To investigate the mapping of points between the z-plane and the s-plane implied by the sampling process, we rely on a generalization of (8.3.21) which relates the z-transform of $h(n)$ to the Laplace transform of $h_a(t)$. This relationship is

$$H(z)|_{z=e^{sT}} = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi k}{T}\right) \tag{8.3.22}$$

where

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$$

$$H(z)|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n)e^{-sTn} \tag{8.3.23}$$

Note that when $s = j\Omega$, (8.3.22) reduces to (8.3.21), where the factor of j in $H_a(\Omega)$ is suppressed in our notation.

Let us consider the mapping of points from the s-plane to the z-plane implied by the relation

$$z = e^{sT} \tag{8.3.24}$$

If we substitute $s = \sigma + j\Omega$ and express the complex variable z in polar form as $z = re^{j\omega}$, (8.3.24) becomes

$$re^{j\omega} = e^{\sigma T}e^{j\Omega T}$$

Clearly, we must have

$$\begin{aligned} r &= e^{\sigma T} \\ \omega &= \Omega T \end{aligned} \tag{8.3.25}$$

Consequently, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$. When $\sigma = 0$, we have $r = 1$. Therefore, the LHP in s is mapped inside the unit circle in z and the RHP in s is mapped outside the unit circle in z.

Also, the $j\Omega$-axis is mapped into the unit circle in z as indicated above. However, the mapping of the $j\Omega$-axis into the unit circle is not one-to-one. Since ω is unique over the range $(-\pi, \pi)$, the mapping $\omega = \Omega T$ implies that the interval $-\pi/T \leq \Omega \leq \pi/T$ maps into the corresponding values of $-\pi \leq \omega \leq \pi$. Furthermore, the frequency interval $\pi/T \leq \Omega \leq 3\pi/T$ also maps into the interval $-\pi \leq \omega \leq \pi$ and, in general, so does the interval $(2k-1)\pi/T \leq \Omega \leq (2k+1)\pi/T$, when k is an integer. Thus the mapping from the analog frequency Ω to the frequency variable ω in the digital domain is many-to-one, which simply reflects the effects of aliasing due to sampling. Figure 8.32 illustrates the mapping from the s-plane to the z-plane for the relation in (8.3.24).

To explore further the effect of the impulse invariance design method on the characteristics of the resulting filter, let us express the system function of the analog filter in partial-fraction form. On the assumption that the poles of the analog filter are distinct, we can write

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \tag{8.3.26}$$

where $\{p_k\}$ are the poles of the analog filter and $\{c_k\}$ are the coefficients in the partial-fraction expansion. Consequently,

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t} \qquad t \geq 0 \tag{8.3.27}$$

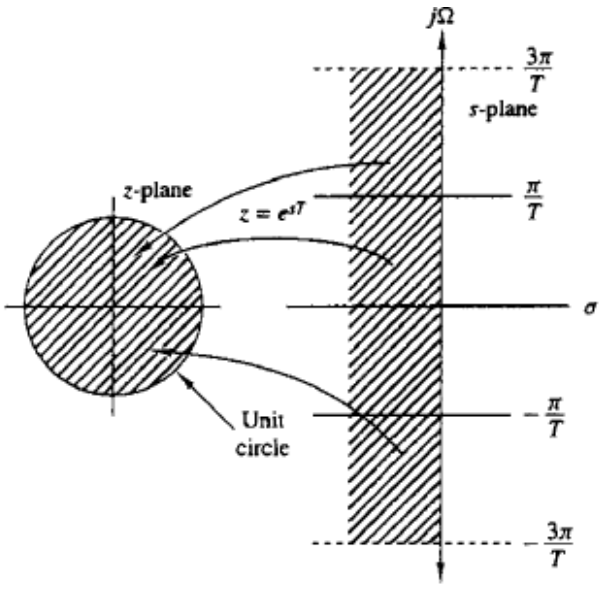


Figure 8.32 The mapping of $z = e^{sT}$ maps strips of width $2\pi/T$ (for $\sigma < 0$) in the s-plane into points in the unit circle in the z-plane.

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$\begin{aligned} h(n) &= h_a(nT) \\ &= \sum_{k=1}^{N} c_k e^{p_k T n} \end{aligned} \tag{8.3.28}$$

Now, with the substitution of (8.3.28), the system function of the resulting digital IIR filter becomes

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h(n) z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k T n} \right) z^{-n} \\ &= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n \end{aligned} \tag{8.3.29}$$

The inner sum in (8.3.29) converges because $p_k < 0$ and yields

$$\sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n = \frac{1}{1 - e^{p_k T} z^{-1}} \tag{8.3.30}$$

Therefore, the system function of the digital filter is

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}} \tag{8.3.31}$$

We observe that the digital filter has poles at

$$z_k = e^{p_k T} \qquad k = 1, 2, \ldots, N \tag{8.3.32}$$

8.3.3 IIR Filter Design by the Bilinear Transformation

The IIR filter design techniques described in the preceding two sections have a severe limitation in that they are appropriate only for lowpass filters and a limited class of bandpass filters.

In this section we describe a mapping from the s-plane to the z-plane, called the bilinear transformation, that overcomes the limitation of the other two design

methods described previously. The bilinear transformation is a conformal mapping that transforms the $j\Omega$-axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Furthermore, all points in the LHP of s are mapped inside the unit circle in the z-plane and all points in the RHP of s are mapped into corresponding points outside the unit circle in the z-plane.

The bilinear transformation can be linked to the trapezoidal formula for numerical integration. For example, let us consider an analog linear filter with system function

$$H(s) = \frac{b}{s + a} \tag{8.3.33}$$

This system is also characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \tag{8.3.34}$$

Instead of substituting a finite difference for the derivative, suppose that we integrate the derivative and approximate the integral by the trapezoidal formula. Thus

$$y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0) \tag{8.3.35}$$

where $y'(t)$ denotes the derivative of $y(t)$. The approximation of the integral in (8.3.35) by the trapezoidal formula at $t = nT$ and $t_0 = nT - T$ yields

$$y(nT) = \frac{T}{2}[y'(nT) + y'(nT - T)] + y(nT - T) \tag{8.3.36}$$

Now the differential equation in (8.3.34) evaluated at $t = nT$ yields

$$y'(nT) = -ay(nT) + bx(nT) \tag{8.3.37}$$

We use (8.3.37) to substitute for the derivative in (8.3.36) and thus obtain a difference equation for the equivalent discrete-time system. With $y(n) \equiv y(nT)$ and $x(n) \equiv x(nT)$, we obtain the result

$$\left(1 + \frac{aT}{2}\right) y(n) - \left(1 - \frac{aT}{2}\right) y(n - 1) = \frac{bT}{2}[x(n) + x(n - 1)] \tag{8.3.38}$$

The z-transform of this difference equation is

$$\left(1 + \frac{aT}{2}\right) Y(z) - \left(1 - \frac{aT}{2}\right) z^{-1}Y(z) = \frac{bT}{2}(1 + z^{-1})X(z)$$

Consequently, the system function of the equivalent digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(bT/2)(1 + z^{-1})}{1 + aT/2 - (1 - aT/2)z^{-1}}$$

or, equivalently,

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + a} \tag{8.3.39}$$

Clearly, the mapping from the s-plane to the z-plane is

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{8.3.40}$$

This is called the *bilinear transformation*.

Although our derivation of the bilinear transformation was performed for a first-order differential equation, it holds, in general, for an Nth-order differential equation.

To investigate the characteristics of the bilinear transformation, let

$$z = re^{j\omega}$$

$$s = \sigma + j\Omega$$

Then (8.3.40) can be expressed as

$$\begin{aligned} s &= \frac{2}{T}\frac{z-1}{z+1} \\ &= \frac{2}{T}\frac{re^{j\omega} - 1}{re^{j\omega} + 1} \\ &= \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right) \end{aligned}$$

Consequently,

$$\sigma = \frac{2}{T}\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} \tag{8.3.41}$$

$$\Omega = \frac{2}{T}\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega} \tag{8.3.42}$$

First, we note that if $r < 1$, then $\sigma < 0$, and if $r > 1$, then $\sigma > 0$. Consequently, the LHP in s maps into the inside of the unit circle in the z-plane and the RHP in s maps into the outside of the unit circle. When $r = 1$, then $\sigma = 0$ and

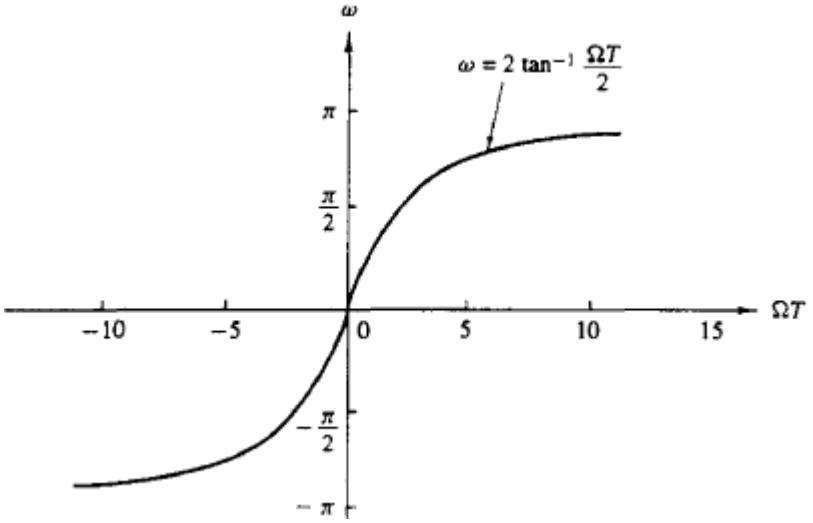
$$\begin{aligned} \Omega &= \frac{2}{T}\frac{\sin\omega}{1 + \cos\omega} \\ &= \frac{2}{T}\tan\frac{\omega}{2} \end{aligned} \tag{8.3.43}$$

or, equivalently,

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \tag{8.3.44}$$

The relationship in (8.3.44) between the frequency variables in the two domains is illustrated in Fig. 8.36. We observe that the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$. However, the mapping is highly nonlinear. We observe a frequency compression or *frequency warping*, as it is usually called, due to the nonlinearity of the arctangent function.

It is also interesting to note that the bilinear transformation maps the point $s = \infty$ into the point $z = -1$. Consequently, the single-pole lowpass filter in



Chebyshev filters. There are two types of Chebyshev filters. Type I Chebyshev filters are all-pole filters that exhibit equiripple behavior in the passband and a monotonic characteristic in the stopband. On the other hand, the family of type II Chebyshev filters contains both poles and zeros and exhibits a monotonic behavior in the passband and an equiripple behavior in the stopband. The zeros of this class of filters lie on the imaginary axis in the s-plane.

The magnitude squared of the frequency response characteristic of a type I Chebyshev filter is given as

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_p)} \tag{8.3.51}$$

where ϵ is a parameter of the filter related to the ripple in the passband and $T_N(x)$ is the Nth-order Chebyshev polynomial defined as

$$T_N(x) = \begin{cases} \cos(N \cos^{-1} x), & |x| \leq 1 \\ \cosh(N \cosh^{-1} x), & |x| > 1 \end{cases} \tag{8.3.52}$$

The Chebyshev polynomials can be generated by the recursive equation

$$T_{N+1}(x) = 2x\,T_N(x) - T_{N-1}(x) \qquad N = 1, 2, \ldots \tag{8.3.53}$$

where $T_0(x) = 1$ and $T_1(x) = x$. From (8.3.53) we obtain $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, and so on.

Some of the properties of these polynomials are as follows:

1. $|T_N(x)| \leq 1$ for all $|x| \leq 1$.
2. $T_N(1) = 1$ for all N.
3. All the roots of the polynomial $T_N(x)$ occur in the interval $-1 \leq x \leq 1$.

The filter parameter ϵ is related to the ripple in the passband, as illustrated in Fig. 8.39, for N odd and N even. For N odd, $T_N(0) = 0$ and hence $|H(0)|^2 = 1$. On the other hand, for N even, $T_N(0) = 1$ and hence $|H(0)|^2 = 1/(1 + \epsilon^2)$. At the band edge frequency $\Omega = \Omega_p$, we have $T_N(1) = 1$, so that

$$\frac{1}{\sqrt{1+\epsilon^2}} = 1 - \delta_1$$

or, equivalently,

$$\epsilon^2 = \frac{1}{(1-\delta_1)^2} - 1 \tag{8.3.54}$$

where δ_1 is the value of the passband ripple.

The poles of a type I Chebyshev filter lie on an ellipse in the s-plane with major axis

$$r_1 = \Omega_p \frac{\beta^2 + 1}{2\beta} \tag{8.3.55}$$

and minor axis

$$r_2 = \Omega_p \frac{\beta^2 - 1}{2\beta} \tag{8.3.56}$$

where β is related to ϵ according to the equation

$$\beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{1/N} \tag{8.3.57}$$

The pole locations are most easily determined for a filter of order N by first locating the poles for an equivalent Nth-order Butterworth filter that lie on circles of radius r_1 or radius r_2, as illustrated in Fig. 8.40. If we denote the angular positions of the poles of the Butterworth filter as

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N} \qquad k = 0, 1, 2, \ldots, N-1 \tag{8.3.58}$$

then the positions of the poles for the Chebyshev filter lie on the ellipse at the coordinates (x_k, y_k), $k = 0, 1, \ldots, N-1$, where

$$\begin{aligned} x_k &= r_2 \cos\phi_k, \qquad k = 0, 1, \ldots, N-1 \\ y_k &= r_1 \sin\phi_k, \qquad k = 0, 1, \ldots, N-1 \end{aligned} \tag{8.3.59}$$

A type II Chebyshev filter contains zeros as well as poles. The magnitude squared of its frequency response is given as

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2[T_N^2(\Omega_s/\Omega_p)/T_N^2(\Omega_s/\Omega)]} \tag{8.3.60}$$

where $T_N(x)$ is, again, the Nth-order Chebyshev polynomial and Ω_s is the stopband frequency as illustrated in Fig. 8.41. The zeros are located on the imaginary axis at the points

$$s_k = j\frac{\Omega_s}{\sin\phi_k} \qquad k = 0, 1, \ldots, N-1 \tag{8.3.61}$$

The poles are located at the points (v_k, w_k), where

$$v_k = \frac{\Omega_s x_k}{\sqrt{x_k^2 + y_k^2}} \qquad k = 0, 1, \ldots, N-1 \tag{8.3.62}$$

$$w_k = \frac{\Omega_s y_k}{\sqrt{x_k^2 + y_k^2}} \qquad k = 0, 1, \ldots, N-1 \tag{8.3.63}$$

where $\{x_k\}$ and $\{y_k\}$ are defined in (8.3.59) with β now related to the ripple in the stopband through the equation

$$\beta = \left[\frac{1 + \sqrt{1 - \delta_2^2}}{\delta_2}\right]^{1/N} \tag{8.3.64}$$

$$N = \frac{\log\left[\left(\sqrt{1-\delta_2^2} + \sqrt{1 - \delta_2^2(1+\epsilon^2)}\right)/\epsilon\delta_2\right]}{\log\left[(\Omega_s/\Omega_p) + \sqrt{(\Omega_s/\Omega_p)^2 - 1}\right]}$$

$$= \frac{\cosh^{-1}(\delta/\epsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

where, by definition, $\delta_2 = 1/\sqrt{1+\delta^2}$.

Frequency Transformations in the Analog Domain

Type of transformation	Transformation	Band edge frequencies of new filter
Lowpass	$s \longrightarrow \frac{\Omega_p}{\Omega_p'} s$	Ω_p'
Highpass	$s \longrightarrow \frac{\Omega_p \Omega_p'}{s}$	Ω_p'
Bandpass	$s \longrightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$	Ω_l, Ω_u
Bandstop	$s \longrightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$	Ω_l, Ω_u

Frequency Transformations in the Digital Domain

Type of transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \dfrac{z^{-1} - a}{1 - az^{-1}}$	ω_p' = band edge frequency of new filter $a = \dfrac{\sin[(\omega_p - \omega_p')/2]}{\sin[(\omega_p + \omega_p')/2]}$
Highpass	$z^{-1} \longrightarrow -\dfrac{z^{-1} + a}{1 + az^{-1}}$	ω_p' = band edge frequency new filter $a = -\dfrac{\cos[(\omega_p + \omega_p')/2]}{\cos[(\omega_p - \omega_p')/2]}$
Bandpass	$z^{-1} \longrightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = -2\alpha K/(K + 1)$ $a_2 = (K - 1)/(K + 1)$ $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \cot\dfrac{\omega_u - \omega_l}{2} \tan\dfrac{\omega_p}{2}$
Bandstop	$z^{-1} \longrightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = -2\alpha/(K + 1)$ $a_2 = (1 - K)/(1 + K)$ $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \tan\dfrac{\omega_u - \omega_l}{2} \tan\dfrac{\omega_p}{2}$

****

UNIT -V

STRUCTURES OF FIR AND IIR SYSTEMS

STRUCTURES FOR THE REALIZATION OF DISCRETE-TIME SYSTEMS

The major factors that influence our choice o f a specific realization are computational complexity, memory requirements, and finite-word-length effects in the computations.

STRUCTURES FOR FIR SYSTEMS

In general, an FIR system is described by the difference equation

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \tag{7.2.1}$$

or, equivalently, by the system function

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k} \tag{7.2.2}$$

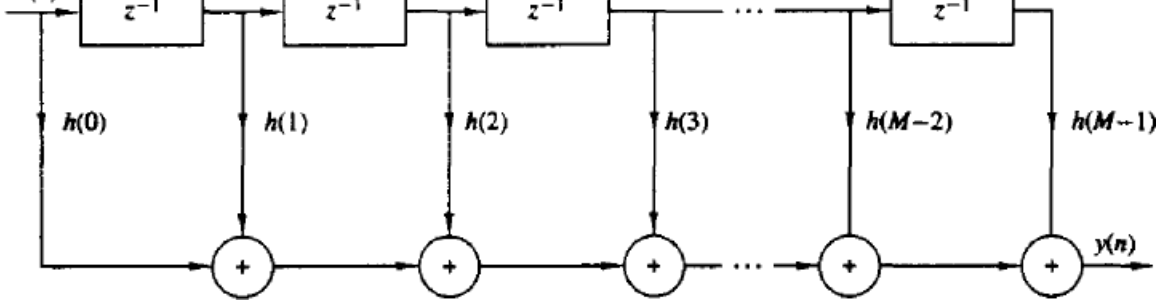
Furthermore, the unit sample response of the FIR system is identical to the coefficients $\{b_k\}$, that is,

$$h(n) = \begin{cases} b_n, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

Direct-Form Structure

The direct form realization follows immediately from the non recursive difference equation givenbelow.

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$



Cascade-Form Structures

The cascaded realization follows naturally system function given by equation. It is simple matter tofactor H(z) into second order FIR system

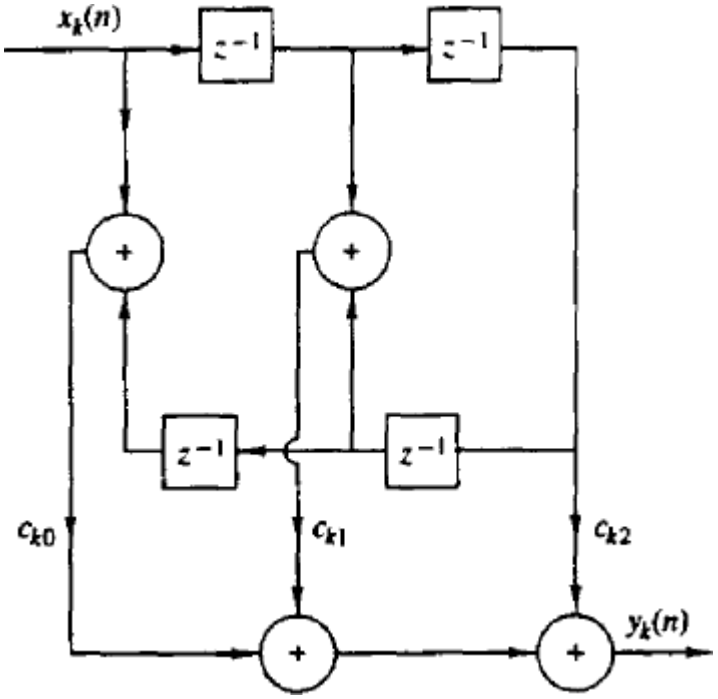
So that

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

where

$$H_k(z) = b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2} \qquad k = 1, 2, \ldots, K$$

$$H_k(z) = c_{k0}(1 - z_k z^{-1})(1 - z_k^* z^{-1})(1 - z^{-1}/z_k)(1 - z^{-1}/z_k^*)$$
$$= c_{k0} + c_{k1}z^{-1} + c_{k2}z^{-2} + c_{k1}z^{-3} + z^{-4}$$



Frequency-Sampling Structures

The frequency-sampling realization is an alternative structure for an FIR filter in which the parameters that characterize the filter are the values o f the desired frequency response instead of the impulse response *h(n).* To derive the frequency sampling structure, we specify the desired frequency response at a set o f equally spaced frequencies, namely

$$\omega_k = \frac{2\pi}{M}(k + \alpha) \qquad k = 0, 1, \ldots, \frac{M-1}{2} \quad M \text{ odd}$$
$$k = 0, 1, \ldots, \frac{M}{2} - 1 \quad M \text{ even}$$
$$\alpha = 0 \text{ or } \tfrac{1}{2}$$

The frequency response of the system is given by

$$H(\omega) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n}$$

$$H(k+\alpha) = H\left(\frac{2\pi}{M}(k+\alpha)\right)$$

$$= \sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M} \qquad k = 0, 1, \ldots, M-1$$

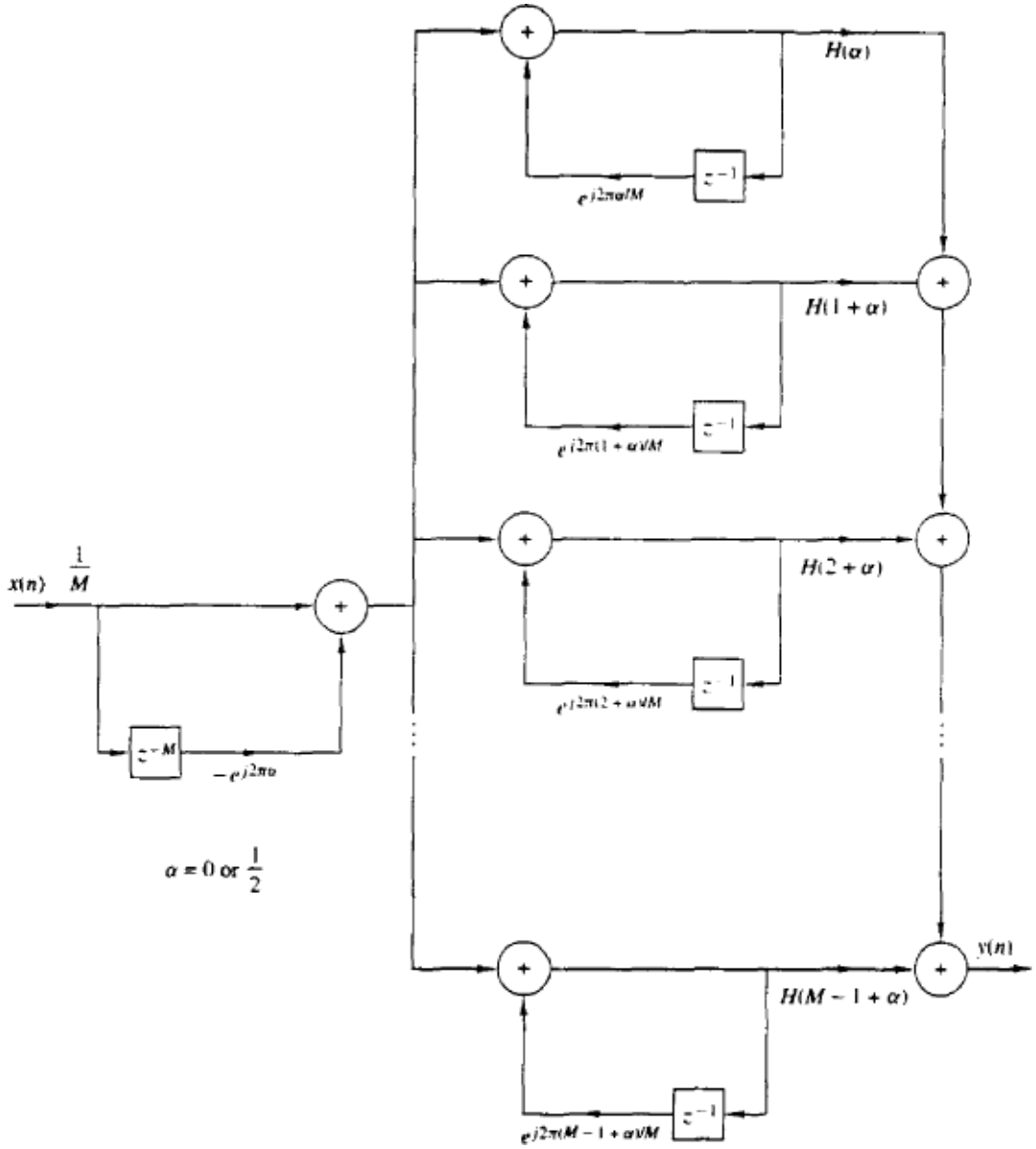
$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M} \qquad n = 0, 1, \ldots, M-1$$

$$H(z) = \sum_{n=0}^{M-1} h(n)z^{-n}$$

$$= \sum_{n=0}^{M-1}\left[\frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M}\right]z^{-n}$$

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha)\left[\frac{1}{M}\sum_{n=0}^{M-1}(e^{j2\pi(k+\alpha)/M}z^{-1})^n\right]$$

$$= \frac{1-z^{-M}e^{j2\pi\alpha}}{M}\sum_{k=0}^{M-1}\frac{H(k+\alpha)}{1-e^{j2\pi(k+\alpha)/M}z^{-1}}$$



Lattice Structure

In this section w e introduce another F IR filter structure, called the lattice filter or Lattice realization. Lattice filters are used extensively in digital speech processing And in the implementation of adaptive filters. Let us begin the development by considering a sequence of FIR filters with system functions

$$H_m(z) = A_m(z) \qquad m = 0, 1, 2, \ldots, M-1 \tag{7.2.17}$$

where, by definition, $A_m(z)$ is the polynomial

$$A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k} \qquad m \geq 1 \tag{7.2.18}$$

and $A_0(z) = 1$. The unit sample response of the mth filter is $h_m(0) = 1$ and $h_m(k) = \alpha_m(k)$, $k = 1, 2, \ldots, m$. The subscript m on the polynomial $A_m(z)$ denotes the degree of the polynomial. For mathematical convenience, we define $\alpha_m(0) = 1$.

If $\{x(n)\}$ is the input sequence to the filter $A_m(z)$ and $\{y(n)\}$ is the output sequence, we have

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k) x(n-k) \tag{7.2.19}$$

Next, let us consider an FIR filter for which $m = 2$. In this case the output from a direct-form structure is

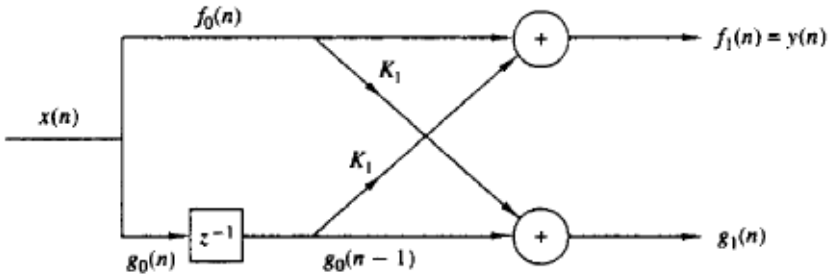
$$y(n) = x(n) + \alpha_2(1)x(n-1) + \alpha_2(2)x(n-2) \tag{7.2.22}$$

By cascading two lattice stages as shown in Fig. 7.10, it is possible to obtain the same output as (7.2.22). Indeed, the output from the first stage is

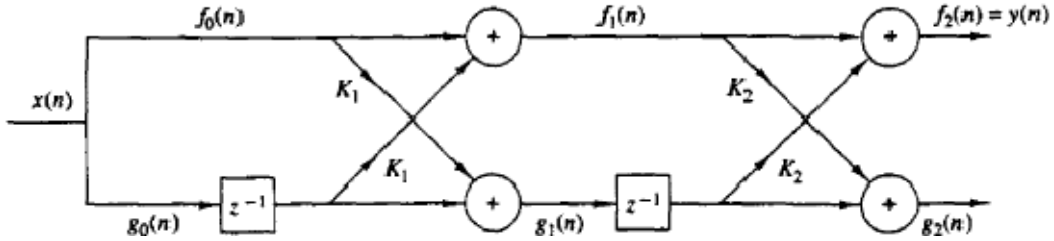
$$\begin{aligned} f_1(n) &= x(n) + K_1 x(n-1) \\ g_1(n) &= K_1 x(n) + x(n-1) \end{aligned} \tag{7.2.23}$$

The output from the second stage is

$$\begin{aligned} f_2(n) &= f_1(n) + K_2 g_1(n-1) \\ g_2(n) &= K_2 f_1(n) + g_1(n-1) \end{aligned} \tag{7.2.24}$$



$$f_0(n) = g_0(n) = x(n)$$
$$f_1(n) = f_0(n) + K_1 g_0(n-1) = x(n) + K_1 x(n-1)$$
$$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 x(n) + x(n-1)$$



$$\begin{aligned} f_2(n) &= x(n) + K_1 x(n-1) + K_2[K_1 x(n-1) + x(n-2)] \\ &= x(n) + K_1(1 + K_2)x(n-1) + K_2 x(n-2) \end{aligned}$$

The general form of lattice structure for m stage is given by'

$$f_0(n) = g_0(n) = x(n)$$
$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1) \qquad m = 1, 2, \ldots, M-1$$
$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \qquad m = 1, 2, \ldots, M-1$$

Conversion of lattice coefficients to direct-form filter coefficients. The direct-form FIR filter coefficients $\{\alpha_m(k)\}$ can be obtained from the lattice coefficients $\{K_i\}$ by using the following relations:

$$A_0(z) = B_0(z) = 1 \tag{7.2.47}$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z) \qquad m = 1, 2, \ldots, M-1 \tag{7.2.48}$$

$$B_m(z) = z^{-m} A_m(z^{-1}) \qquad m = 1, 2, \ldots, M-1 \tag{7.2.49}$$

Conversion of direct-form FIR filter coefficients to lattice coefficients. Suppose that we are given the FIR coefficients for the direct-form realization or, equivalently, the polynomial $A_m(z)$, and we wish to determine the corresponding lattice filter parameters $\{K_i\}$. For the m-stage lattice we immediately obtain the parameter $K_m = \alpha_m(m)$. To obtain K_{m-1} we need the polynomials $A_{m-1}(z)$ since, in general, K_m is obtained from the polynomial $A_m(z)$ for $m = M-1, M-2, \ldots, 1$. Consequently, we need to compute the polynomials $A_m(z)$ starting from $m = M-1$ and "stepping down" successively to $m = 1$.

$$K_m = \alpha_m(m) \quad \alpha_{m-1}(0) = 1$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - K_m\beta_m(k)}{1 - K_m^2}$$

$$= \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)} \qquad 1 \le k \le m-1$$

STRUCTURES FOR IIR SYSTEMS

In this section we consider different IIR system s structures described by the difference equation given by the system function. Just as in the case o f FIR system s, there are several types o f structures or realizations, including direct-form structures, cascade-form structures, lattice structures, and lattice-ladder structures. In addition, IIR systems lend them selves to a parallel form realization. We begin by describing two direct-form realizations.

DIRECT FORM STRUCTURES:

The rational system function as given by (7.1.2) that characterizes an IIR system can be viewed as two systems in cascade, that is,

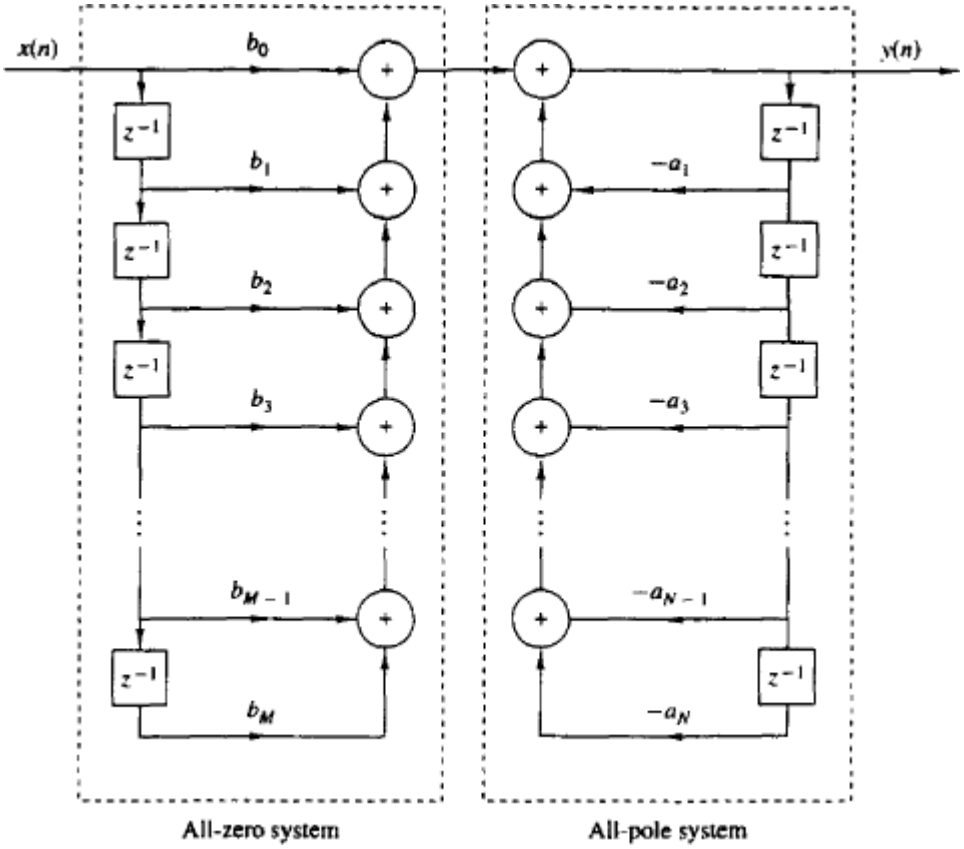
$$H(z) = H_1(z)H_2(z) \tag{7.3.1}$$

where $H_1(z)$ consists of the zeros of $H(z)$, and $H_2(z)$ consists of the poles of $H(z)$,

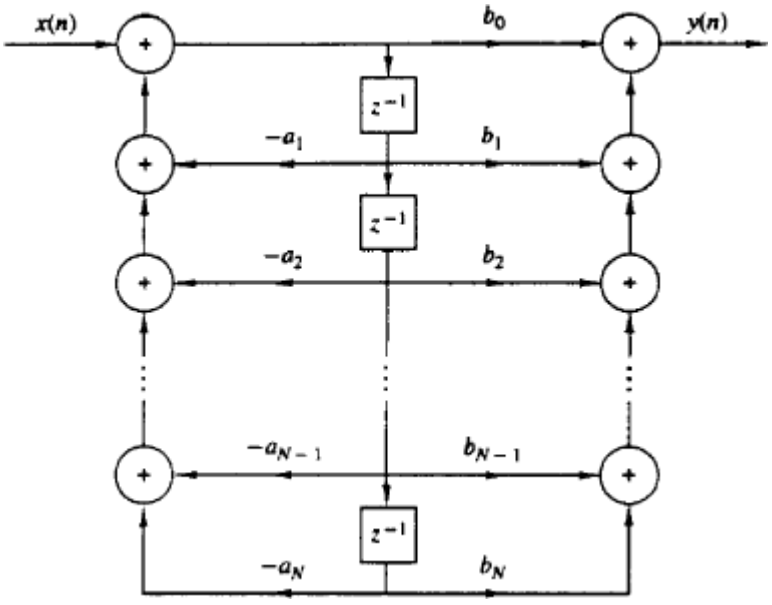
$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k} \tag{7.3.2}$$

and

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{7.3.3}$$

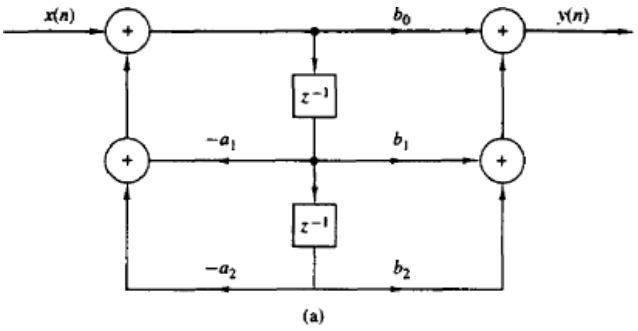


DIRECT FORM II

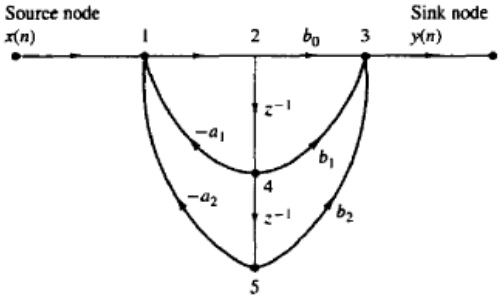


Signal Flow Graphs and Transposed Structures

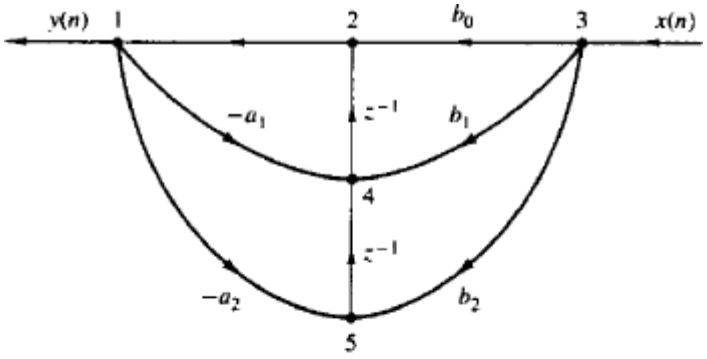
A signal flow graph provides an alternative, N but equivalent, graphical representation to a block diagram structure that we have been using to illustrate various system realizations. T he basic elements o f a flow graph are branches and nodes. A signal flow graph is basically a set o f directed branches that connect at nodes. By definition, the signal out of a branch is equal to the branch gain (system function) times the signal into the branch. Furthermore, the signal at anode o f a flow graph is equal to the sum o f the signals from all branches connecting to the node.



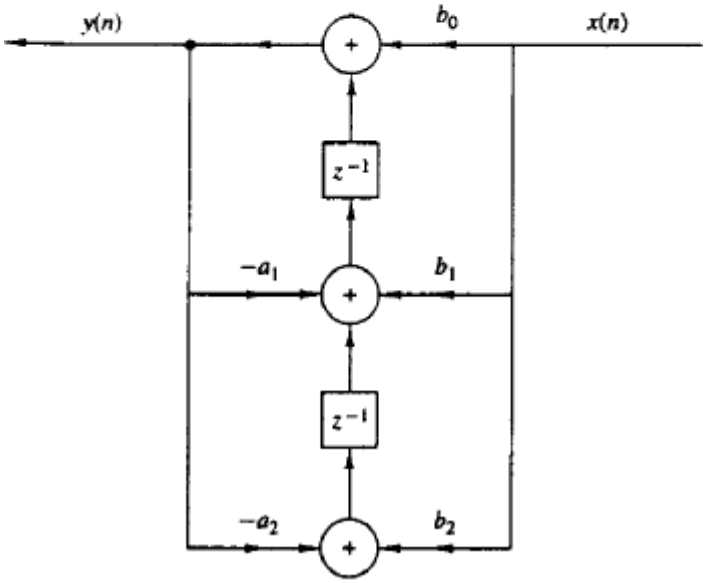
(a)



(b)



(a)



(b)

Cascade-Form Structures

Let us consider a high-order IIR system with system function given by equation. Without loss o f generality we assume that $N > M$ **. T h e system can be factored into a cascade o f second-order subsystem s, such that** $H(z)$ **can b e expressed as**

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

where K is the integer part of $(N+1)/2$. $H_k(z)$ has the general form

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$

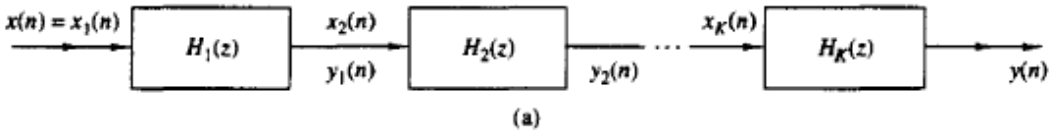
The general form of the cascade structure is illustrated in Fig. 7.19. If we use the direct form II structure for each of the subsystems, the computational algorithm for realizing the IIR system with system function $H(z)$ is described by the following set of equations.

$$y_0(n) = x(n) \tag{7.3.16}$$

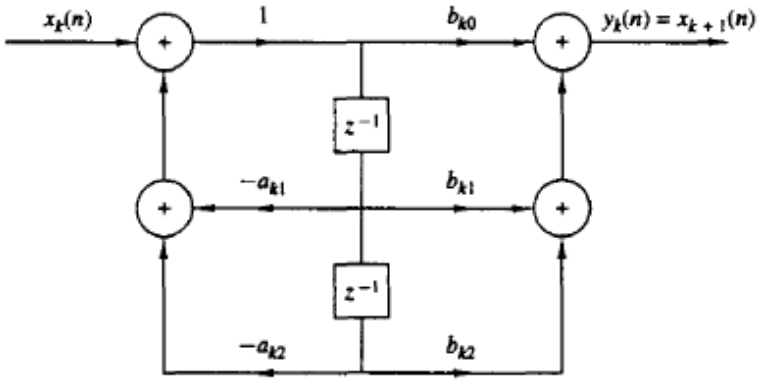
$$w_k(n) = -a_{k1}w_k(n-1) - a_{k2}w_k(n-2) + y_{k-1}(n) \qquad k = 1, 2, \ldots, K \tag{7.3.17}$$

$$y_k(n) = b_{k0}w_k(n) + b_{k1}w_k(n-1) + b_{k2}w_k(n-2) \qquad k = 1, 2, \ldots, K \tag{7.3.18}$$

$$y(n) = y_K(n) \tag{7.3.19}$$



(a)

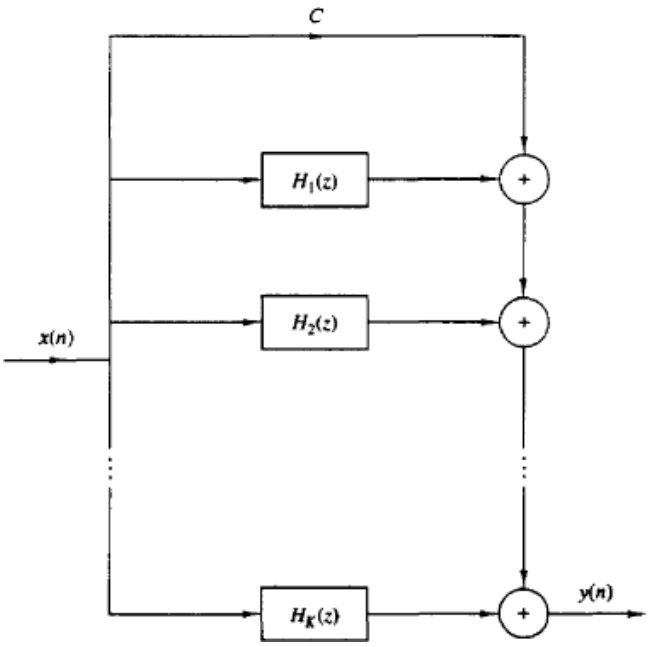


(b)

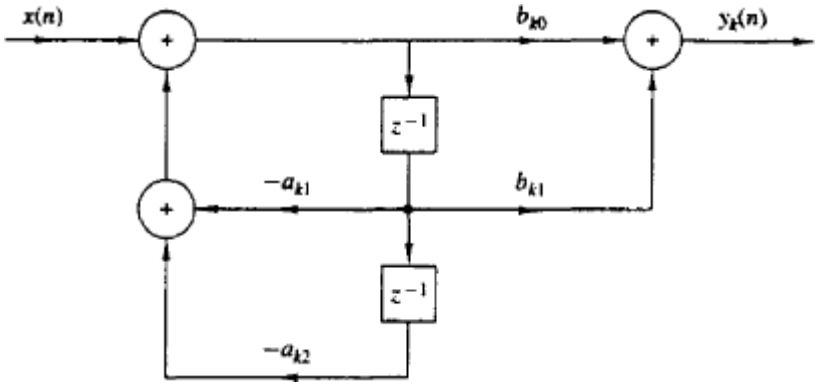
Parallel-Form Structures

A parallel-form realization o f an IIR system can be obtained by performing a partial-fraction expansion o f *H(z)* . Without loss o f generality, w e again assume that $N > M$ and that the poles are distinct. Then, by performing a partial-fraction expansion o f *H(z)*, we obtain the result

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$



The realization of second order form is given by



The general form of parallel form of structure is f\given by

$$w_k(n) = -a_{k1}w_k(n-1) - a_{k2}w_k(n-2) + x(n) \qquad k = 1, 2, \ldots, K$$

$$y_k(n) = b_{k0}w_k(n) + b_{k1}w_k(n-1) \qquad k = 1, 2, \ldots, K$$

$$y(n) = Cx(n) + \sum_{k=1}^{K} y_k(n)$$

Lattice and Lattice-Ladder Structures for IIR Systems

In Section 7.2.4 we developed a lattice filter structure that is equivalent to an FIR system. In this section we extend the development to IIR systems.

Let us begin with an all-pole system with system function

$$H(z) = \frac{1}{1 + \sum_{k=1}^{N} a_N(k)z^{-k}} = \frac{1}{A_N(z)} \tag{7.3.26}$$

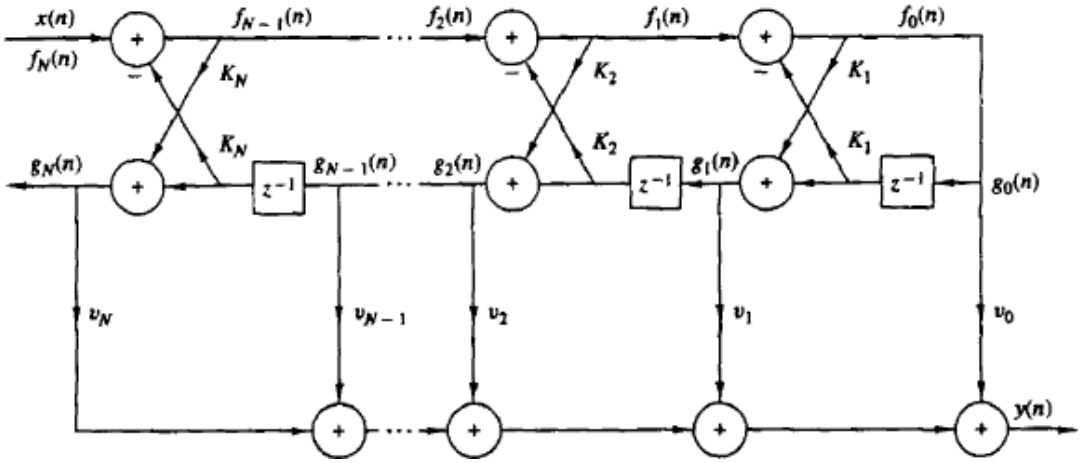
The direct form realization of this system is illustrated in Fig. 7.23. The difference equation for this IIR system is

$$y(n) = -\sum_{k=1}^{N} a_N(k)y(n-k) + x(n) \tag{7.3.27}$$

It is interesting to note that if we interchange the roles of input and output [i.e., interchange $x(n)$ with $y(n)$ in (7.3.27)], we obtain

$$x(n) = -\sum_{k=1}^{N} a_N(k)x(n-k) + y(n)$$

or, equivalently,

$$y(n) = x(n) + \sum_{k=1}^{N} a_N(k)x(n-k) \tag{7.3.28}$$

We note that the equation in (7.3.28) describes an FIR system having the system function $H(z) = A_N(z)$, while the system described by the difference equation in (7.3.27) represents an IIR system with system function $H(z) = 1/A_N(z)$.