(20A01402) Structural Analysis-I

- **PREPARED BY**

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COURSE OUTCOMES

CO1	Apply energy theorems for analysis of indeterminate structures
CO2	Analyze indeterminate structures with yielding of supports
CO3	Analyze beams and portal frames using slope deflection method
CO4	Analyze beams and portal frames using moment distribution methods
CO5	Analyze bending moment, normal thrust and radial shear in the arches

Structural Analysis-I

UNIT – I: Basic Analysis of Indeterminate Structures

Introduction-Strain energy in linear elastic system, expression of strain energy due to axial load, bending moment and shear force – Castigliano's first theorem - Deflections of simple beams and pin jointed trusses - Indeterminate Structural Analysis – Determination of static and kinematic indeterminacies – Solution of trusses up to two degrees of internal and external indeterminacy – Castigliano's second theorem.

UNIT - II Fixed Beams & Continuous Beams

Introduction to statically indeterminate beams- theorem of three moments-uniformly distributed load, central point load, eccentric point load, number of point loads, uniformly varying load, couple and combination of loads – Shear force and Bending moment diagrams –effect of sinking of support, effect of rotation of a support.

UNIT - III Slope-Deflection Method

Introduction- derivation of slope deflection equation- application to continuous beams with and without settlement of supports- Analysis of single bay, single storey, portal frame including side sway.

UNIT - IV Moment Distribution Method

Introduction to moment distribution method- application to continuous beams with and without settlement of supports. Analysis of single storey ,portal frames – including Sway

UNIT - V Arches

Introduction- hinges-transfer of load to arches-linear arch-hinges in the arch-arch action-Horizontal force – three hinged arches – circular arches – springs at different level-Two hinged arches- two hinged circular arches – fixed arches (only theory) - Temperature stresses in arches.

Textbooks:

1. C. S. Reddy, “Basic Structural Analysis”, Tata McGraw Hill
2. S. Ramamurtham, “Theory of Structures”, Dhanpat Rai Publishing Company (p) Ltd, 2009

Reference Books:

1. Timoshenko & Young, “Theory of Structures”, Tata McGraw Hill
2. S.S. Bhavikatti, “Structural analysis”, Volume 1 and 2, Vikas publishing house pvt. Ltd.
3. Dr.Vaidyanathan, Dr.P.Perumal, “Comprehensive structural analysis”, Vol-II, Laxmi Publications (P) Ltd.
4. Junarkar S. B., “Structural Mechanics”, Vol I & II, Charotar Publishers

Energy Methods

Introduction

- In mechanics, Energy is defined as the capacity to do work, and work is the product of the force and the distance it moves along its direction.
- In solid deformable bodies, the stresses multiplied by the respective areas are the forces and the deformation are the distances.
- The product of the force and deformations is the internal work done in a body by externally applied forces.
- The internal work done is stored in the body as the internal elastic energy of deformation or the elastic strain energy.

Conservation of energy, work and strain

- Conservation of energy is one of the basic law of physics and in a closed system consisting of a structure and the applied force must obeys this law.

$W = E_s + E_l$

W = Work Performed

E_s = Energy stored in the body

E_l = Energy loss

- Now in a structure, work is performed by the external load moving through a distance and the energy is stored due to elastic deformation of the members.
- If the structure is static there is no kinetic energy in the system with no energy loss due to heat, permanent set etc. The equation reduces to

$W = E_s$

E_s = Elastic strain energy also denoted by “U”

Hence for a conservational structural system

$W = U$

Strain energy/unit volume = $u = 1/2 \times \sigma \times \varepsilon$

Total Strain energy = $U = ½ \int \sigma \times \varepsilon \times dv$

where, σ = stress, ε = strain

Real work and Complimentary work

- Work = Force × Displacement
- The work done as the force F moves through a distance dΔ

  ΔW = F × dΔ

  Total work done = W = ∫ F × dΔ
- If force "F" is three dimensional with components F_x, F_y and F_z

  Total work done W = $\int F_x \times d\Delta_x + \int F_y \times d\Delta_y + \int F_z \times d\Delta_z$

  This work is known as Real work as shown in Fig. 1.

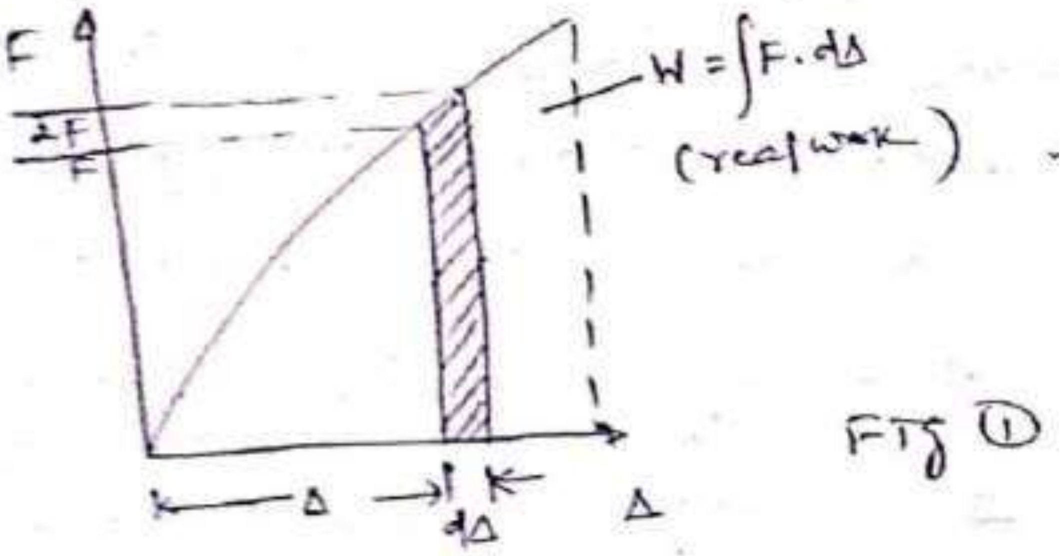


FIG ①

Complimentary work:

$\Delta W_c = \Delta \times dF$

Total Complimentary work, $W_c = \int \Delta \times dF$ as in Fig. 2.

- It is the area above the load deflection curve.
- In linear elastic analysis, load – deflection curve is linear as shown in Fig. 3.

Real work = Complimentary work

$W = W_c = ½ F \times \Delta$

Area below the graph = Area above the graph

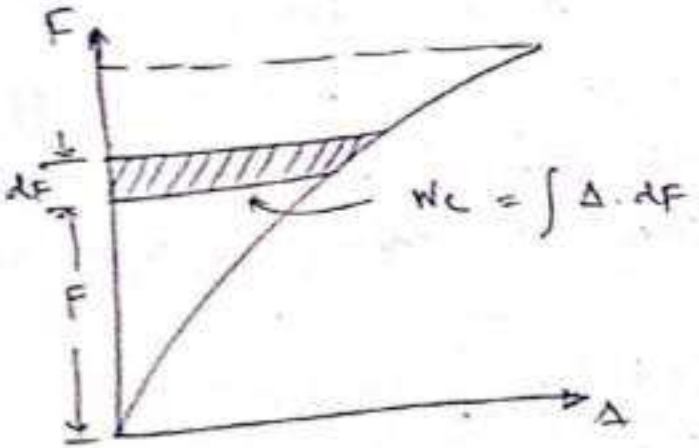


fig ②

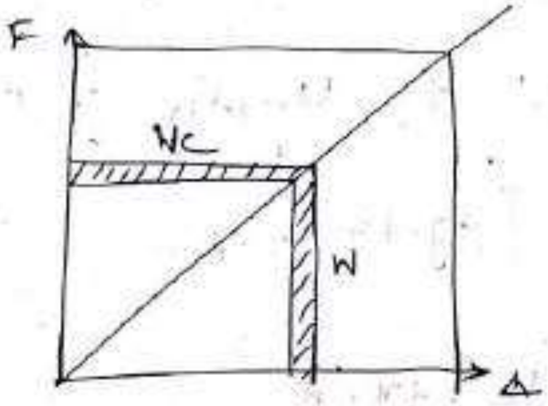
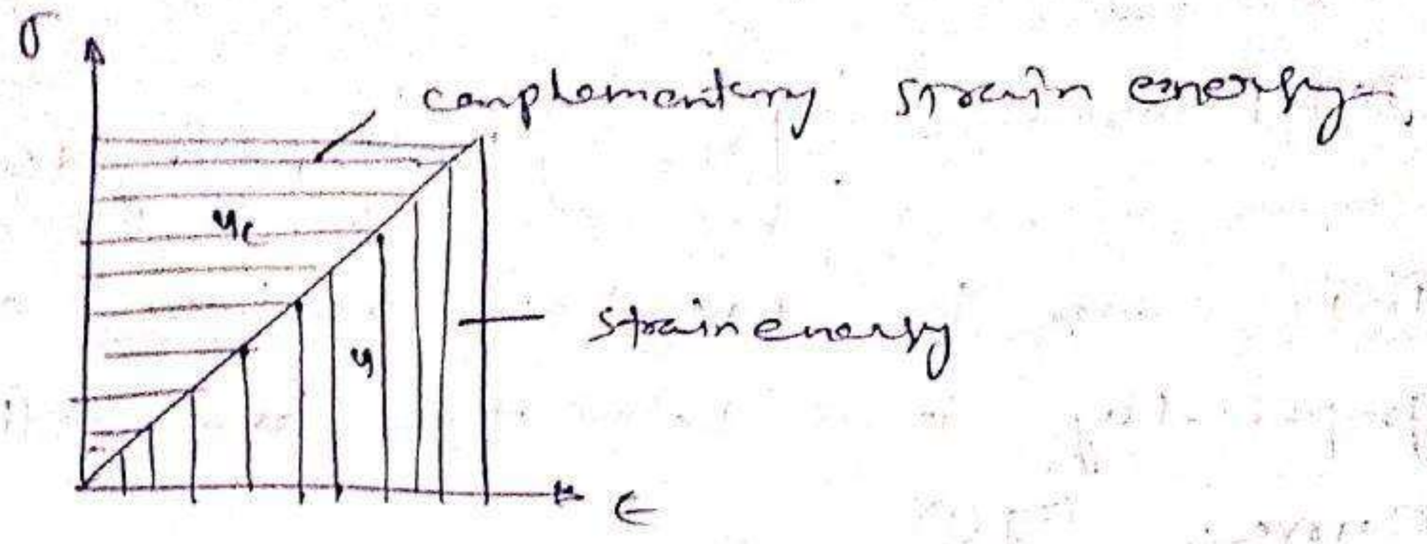


Fig ③

Expression of strain energy for linear elastic members

- Axial loaded members
- Members under Bending moment
- Members under torsional moment on a circular cross section
- Members under shear force on a rectangular section

- Axial loaded members:



$$u = \frac{1}{2}\,\sigma.\epsilon$$

u = Strain energy
Per unit volume

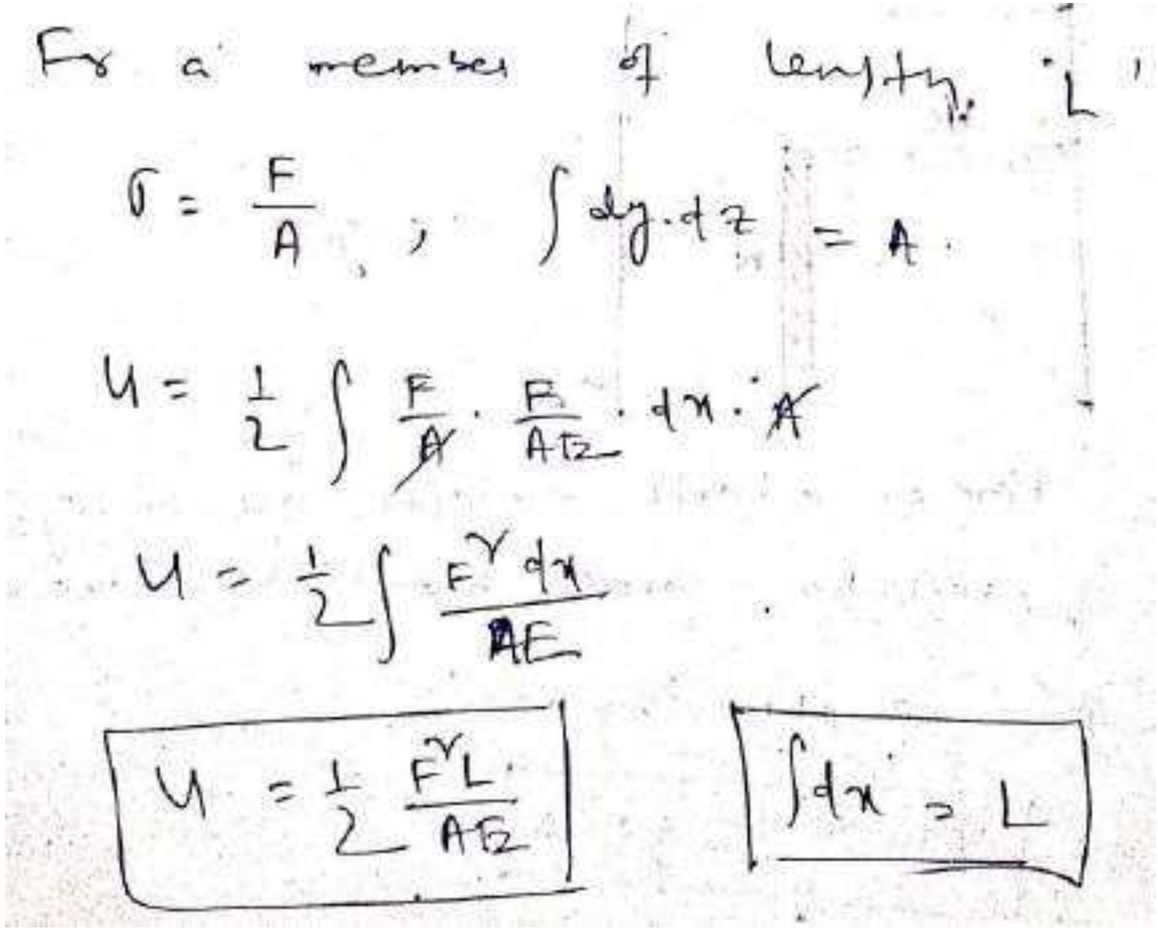
$$\mathcal{U} = \frac{1}{2}\int \sigma.\epsilon.dV.$$

$\mathcal{U}$ = Total strain
energy.

$$U = \frac{1}{2}\int \sigma.\frac{\sigma}{E}.dx.dy.dz.$$

F = External force or load, A = Area of a bar, L = Length of a bar

E = Modulus of elasticity



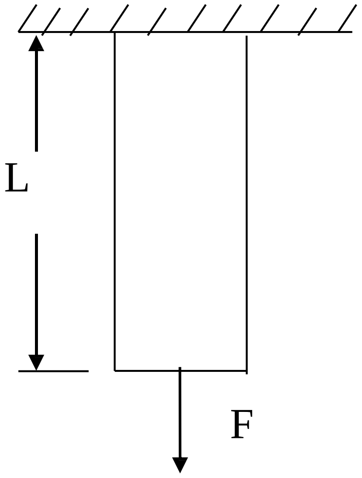
For a member of length 'L'

$$\sigma = \frac{F}{A} \quad ; \quad \int dy \cdot dz = A .$$

$$U = \frac{1}{2} \int \frac{F}{\cancel{A}} \cdot \frac{F}{AE} \cdot dx \cdot \cancel{A}$$

$$U = \frac{1}{2} \int \frac{F^2 \, dx}{AE}$$

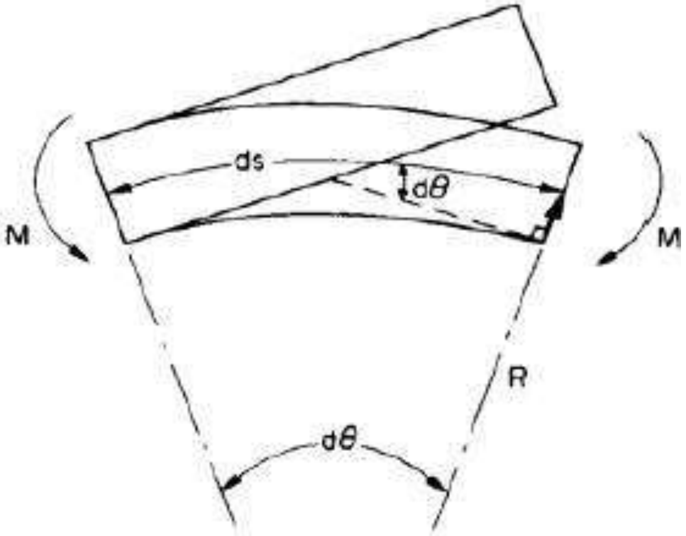
$$\boxed{U = \frac{1}{2} \frac{F^2 L}{AE}} \qquad \boxed{\int dx = L}$$



• Members under Bending moment:

From the pure bending, we know **M/I = σ/y = E/R**
where. M = Bending moment, I = moment of inertia, σ = Bending stress, y = most distant point from the neutral axis, E = modulus of elasticity, R = Radius of curvature

Strain energy = work done = $\frac{1}{2}$ moment x angle turned through (in radians)



$$\sigma = \frac{M}{I} \cdot y$$

$$U = \frac{1}{2} \int \frac{M}{I} \cdot y \cdot \frac{M}{I} \cdot \frac{y}{E} \cdot dx \cdot dy \cdot dz$$

$$= \frac{1}{2E} \int \frac{M^2 \cdot dx}{I^2} \int y^2 \cdot dy \cdot dz$$

$$= \frac{1}{2E} \int \frac{M^2 \cdot dx}{I^2} \cdot I$$

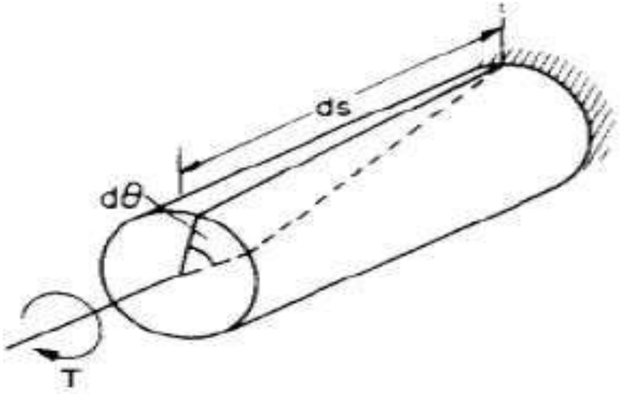
$$= \frac{1}{2E} \int \frac{M^2 \cdot dx}{I}$$

$$\boxed{U = \frac{1}{2EI} \int M^2 \cdot dx}$$

- Members under torsional moment on a circular cross section:

$$\text{Strain energy} = \text{work done} = \frac{1}{2} T\, d\Theta$$

$$U = ½ \int \text{shear stress} \times \text{shear strain} \times \text{volume}$$



$$\tau_{xy} = \frac{T.r}{J}$$

$$\gamma_{xy} . G = \frac{T.r}{J}$$

$$\gamma_{xy} = \frac{T.r}{GJ}$$

G = Rigidity modulus
τ_{xy} = Shear stress
γ_{xy} = Shear strain

$$\boxed{G = \frac{\tau_{xy}}{\gamma_{xy}}}$$

$$U = \frac{1}{2} \int \tau_{xy} . \gamma_{xy} . dx . dy . dz$$

$$= \frac{1}{2} \int \frac{T.r}{J} . \frac{Tr}{GJ}\, dx . dy . dz .$$

T = Torsional moment
r = Radius of shaft
J = Polar moment of Inertia.

$$= \frac{1}{2G} \int \frac{T^2}{J^2} dx \int r^2 \, dy \, dz$$

$$= \frac{1}{2G} \int \frac{T^2}{J^2} dx \, . \, J$$

$$\boxed{U = \frac{1}{2GJ} \int T^2 dx}$$

- Members under shear force on a rectangular section:

V = shear force, I = moment of inertia, b = width of the section, G = shear modulus

$$\tau_{xy} = \frac{VQ}{Ib} \quad , \quad \gamma_{xy} = \frac{VQ}{Ib.G}$$

$$U = \frac{1}{2}\int \frac{VQ}{I.b} . \frac{VQ}{Ib.G} . dx.dy.dz$$

$$= \frac{1}{2G}\int \frac{V^2 Q^2}{I^2 b^2} .dx.dy.dz.$$

$$= \frac{1}{2G}\int V^2.dx \int \frac{Q^2.dy.dz}{I^2 b^2}$$

Now $\int \frac{Q^2 \cdot dy \cdot dz}{I^2 \cdot b^2} = \frac{1}{I^2} \int \left(\frac{Q}{b}\right)^2 \cdot dA = \frac{1}{A_s}$

A_s is called effective shear area.

$$\boxed{U = \frac{1}{2G} \cdot \int \frac{V^2 \cdot dx}{A_s}}$$

Deflection by Strain Energy Method

- This is also known as real work methods since work done by actual loads are considered.
- From the law of conservation of energy

Strain energy = Real work done by loads

$$U = \sum_{0}^{n} \frac{1}{2} P \Delta$$

This method is used for finding deflection in structure only under the following situations:

- The structure is subjected to a single concentrated load.
- Deflection required is at the loaded point and is in the direction of load.

Deflection by Strain Energy Method

Deflection by Strain Energy Method

- This Method is also called ‘Real Work Method’.
- Since, work done by the actual loads are considered.
- From the law of conservation of energy,

Strain Energy (U) = Real work done by loads

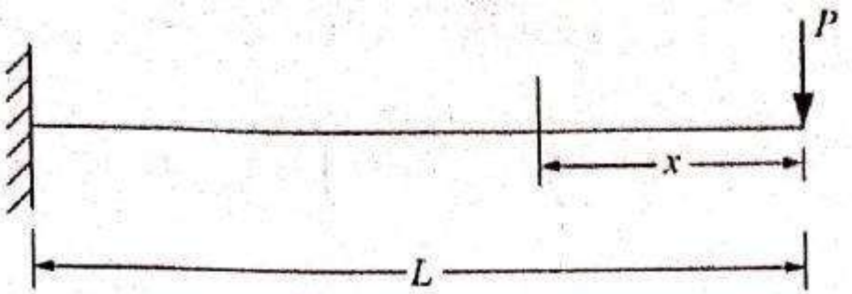
$$U = \sum_{0}^{n} \frac{1}{2} P\Delta$$

- This equation can be used to find out the deflection in beams and frames subjected to bending stresses.

Strain energy method can be used for finding deflection under the following situations:

- The structure is subjected to a concentrated load.
- Deflection required is at the loaded point and is in the direction of load.

Q1. Using strain energy method determine the deflection of the free end of a cantilever of length '*L*' subjected to a concentrated load '*P*' at the free end.



Solution The bending moment at a distance x from the free end is,

$$M = Px$$

$$\therefore \quad \text{Strain Energy (S.E.)} = \int_0^L \frac{M^2}{2EI} dx$$

$$= \int_0^L \frac{P^2 x^2}{2EI} dx$$

$$= \frac{P^2}{2EI} \left[\frac{x^3}{3} \right]_0^L$$

$$= \frac{P^2 L^3}{6EI}$$

Work done by the load = $\frac{1}{2}P\Delta$, where Δ is the deflection at the free end.

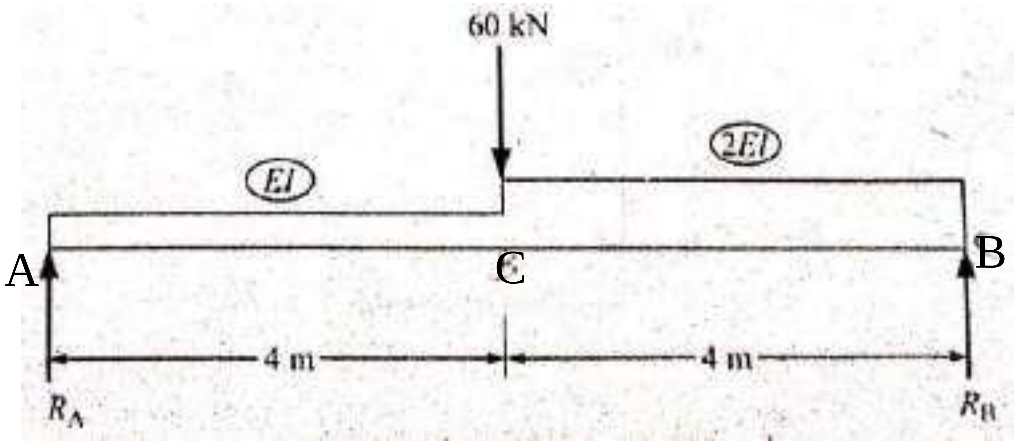
Therefore, from conservation of energy,

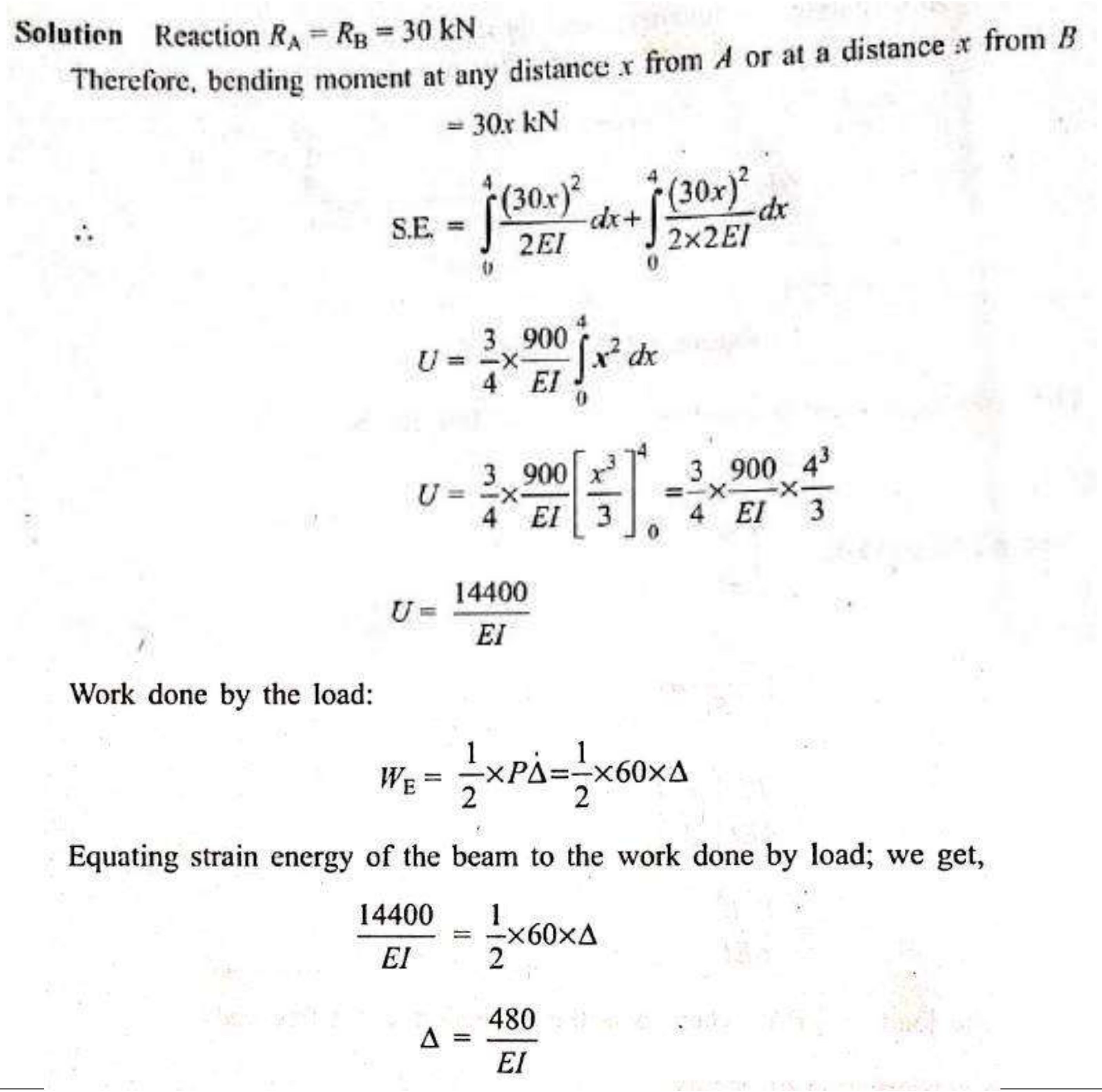
S.E. = Work done by external loads

$$\frac{P^2L^3}{6EI} = \frac{1}{2}P\Delta$$

$$\Delta = \frac{PL^3}{3EI}$$

Q2. Using strain energy method determine the deflection under 60 kN load in the beam shown in Figure.





Solution Reaction $R_A = R_B = 30$ kN

Therefore, bending moment at any distance x from A or at a distance x from B

$$= 30x \text{ kN}$$

$$\therefore \quad \text{S.E.} = \int_0^4 \frac{(30x)^2}{2EI} dx + \int_0^4 \frac{(30x)^2}{2\times 2EI} dx$$

$$U = \frac{3}{4}\times\frac{900}{EI}\int_0^4 x^2\, dx$$

$$U = \frac{3}{4}\times\frac{900}{EI}\left[\frac{x^3}{3}\right]_0^4 = \frac{3}{4}\times\frac{900}{EI}\times\frac{4^3}{3}$$

$$U = \frac{14400}{EI}$$

Work done by the load:

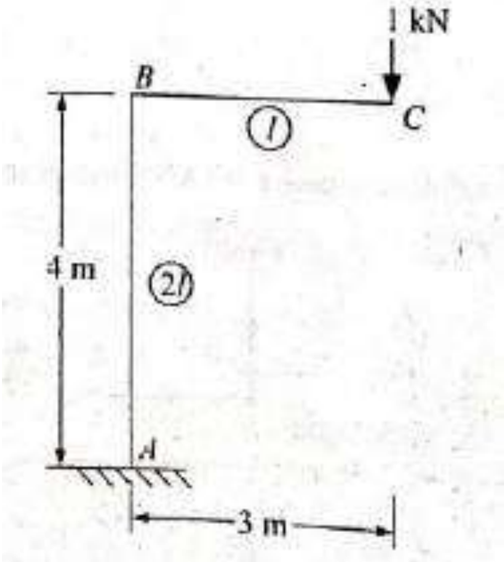
$$W_E = \frac{1}{2}\times P\Delta = \frac{1}{2}\times 60\times\Delta$$

Equating strain energy of the beam to the work done by load; we get,

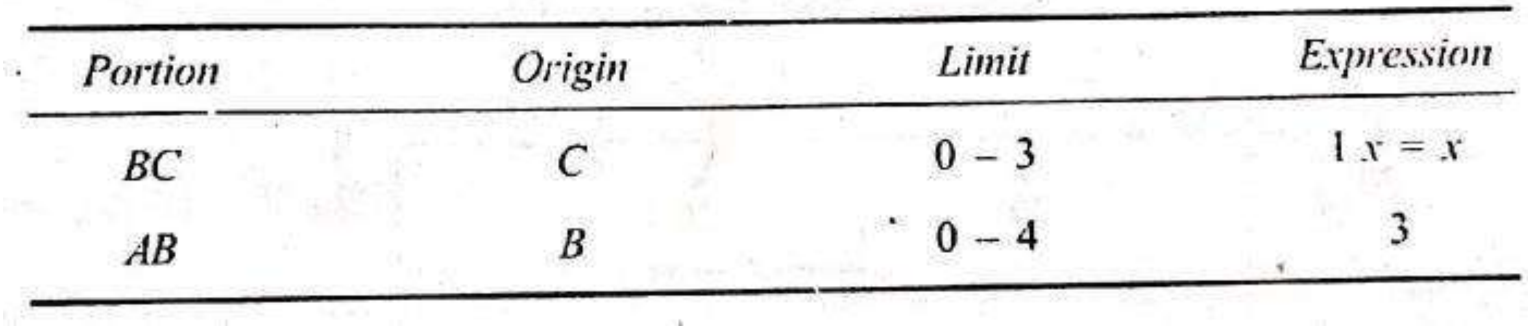
$$\frac{14400}{EI} = \frac{1}{2}\times 60\times\Delta$$

$$\Delta = \frac{480}{EI}$$

Q3. Using strain energy method determine the vertical deflection of point '*C*' in the frame shown in Figure. *E* = 200 kN/mm^2 and *I* = 30 × 10^6 mm^4.



- The details of bending moment expressions for various portion of the structure is calculated individually for member BC than for member AB, and given data in Tabular form:



Portion	*Origin*	*Limit*	*Expression*
BC	*C*	0 – 3	1 x = x
AB	*B*	0 – 4	3

$$\text{S.E} = \int_0^3 \frac{(x)^2}{2EI}dx + \int_0^4 \frac{(3)^2}{2E\times 2I}dx$$

$$= \frac{1}{2EI}\left[\frac{x^3}{3}\right]_0^3 + \frac{1}{4EI}[9x]_0^4$$

$$= \frac{1}{6EI}\times 3^3 + \frac{1}{4EI}\times 9\times 4$$

$$= \frac{13.5}{EI}$$

$$\text{Work done} = \frac{1}{2}\times 1\times \Delta = \frac{\Delta}{2}$$

Note: As the bending moment is given in kN and metres, *EI* should be used as kNm^2.
i.e. 1 kNmm^2 = 1× 10^{-6} kNm^2

Equating work done to strain energy, we get

$$\frac{\Delta}{2} = \frac{13.5}{EI}$$

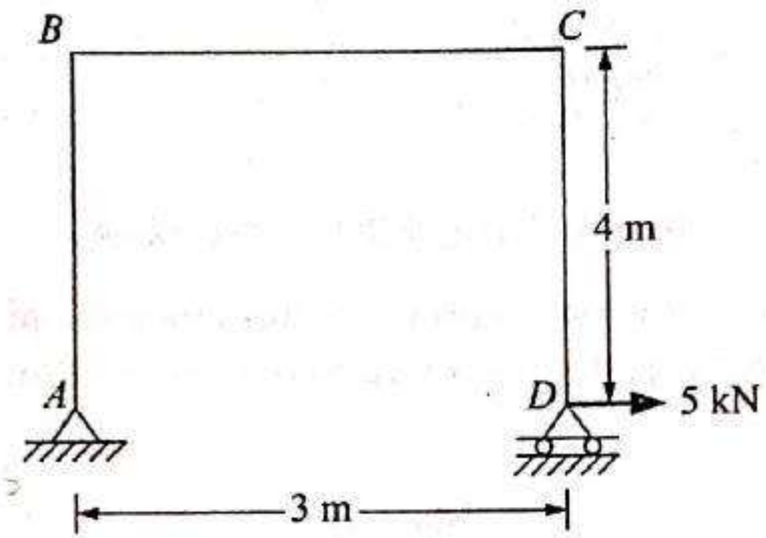
$$\Delta = \frac{27}{EI}$$

$$\text{EI} = 200 \times 30 \times 10^6 \times 10^{-6} = 6000 \text{ kNm}^2$$

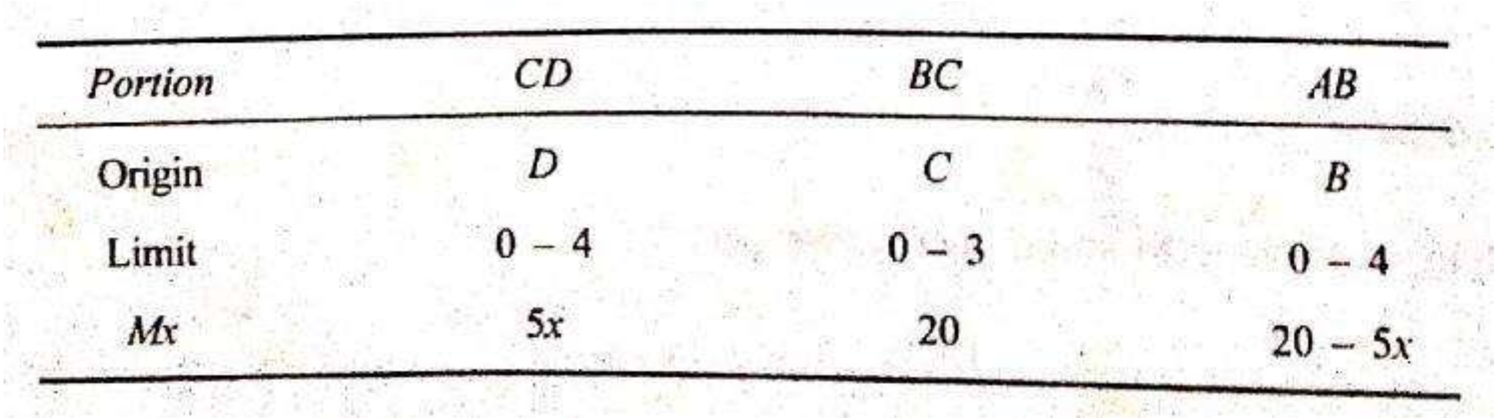
$$\Delta = \frac{27}{6000} \text{ m}$$

$$= 0.045 \text{ m} = 4.5 \text{ mm}$$

Q4. Using strain energy method determine the horizontal deflection of the roller end ‘*D*’ of the portal frame shown in Figure. *EI* = 8000 kNm^2 throughout.



- The details of bending moment expressions for various portion of the structure is calculated individually for member CD, BC than for member AB, and given data in Tabular form:



Portion	*CD*	*BC*	*AB*
Origin	*D*	*C*	*B*
Limit	0 – 4	0 – 3	0 – 4
Mx	5*x*	20	20 – 5*x*

$$S.E = \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(20-5x)^2}{2EI}dx$$

$$= \frac{1}{2EI}\left[\frac{25x^3}{3}\right]_0^4 + \frac{1}{2EI}[400x]_0^3 + \frac{1}{2EI}\left[400x - 200\frac{x^2}{2} + \frac{25x^3}{3}\right]_0^4$$

$$= \frac{266.67}{EI} + \frac{600}{EI} + \frac{1}{2EI}\left[1600 - 1600 + \frac{25\times 64}{3}\right]$$

$$= \frac{1133.33}{EI}$$

Work done $= \frac{1}{2}\times P\times\Delta = \frac{1}{2}\times 5\Delta = 2.5\Delta$

Equating S.E. to work done, we get, $2.5\Delta = \frac{1133.33}{EI}$

$$\Delta = \frac{453.33}{EI} = \frac{453.33}{8000} = 0.0567\,\text{m}$$

$$= 56.7\,\text{mm}$$

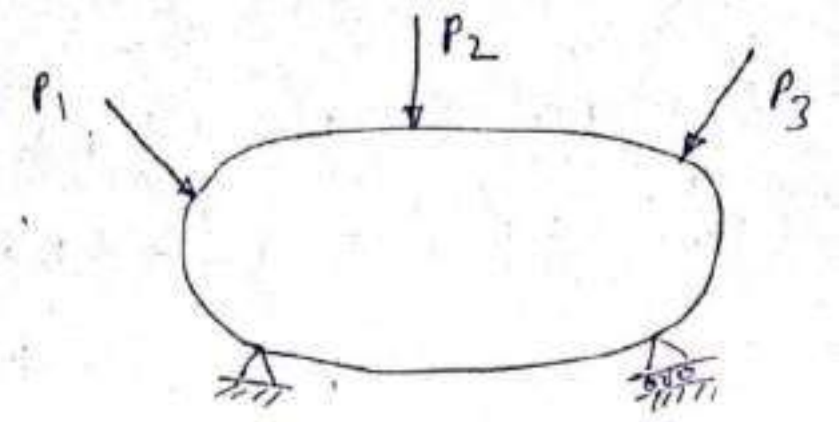
Castigliano's Theorem

Castigliano’s First theorem

- The first theorem of Castigliano states that the partial derivative of the total strain energy in any structure with respect to applied force or moment gives the displacement or rotation respectively at the point of application of the force or moment in the direction of the applied force or moment.

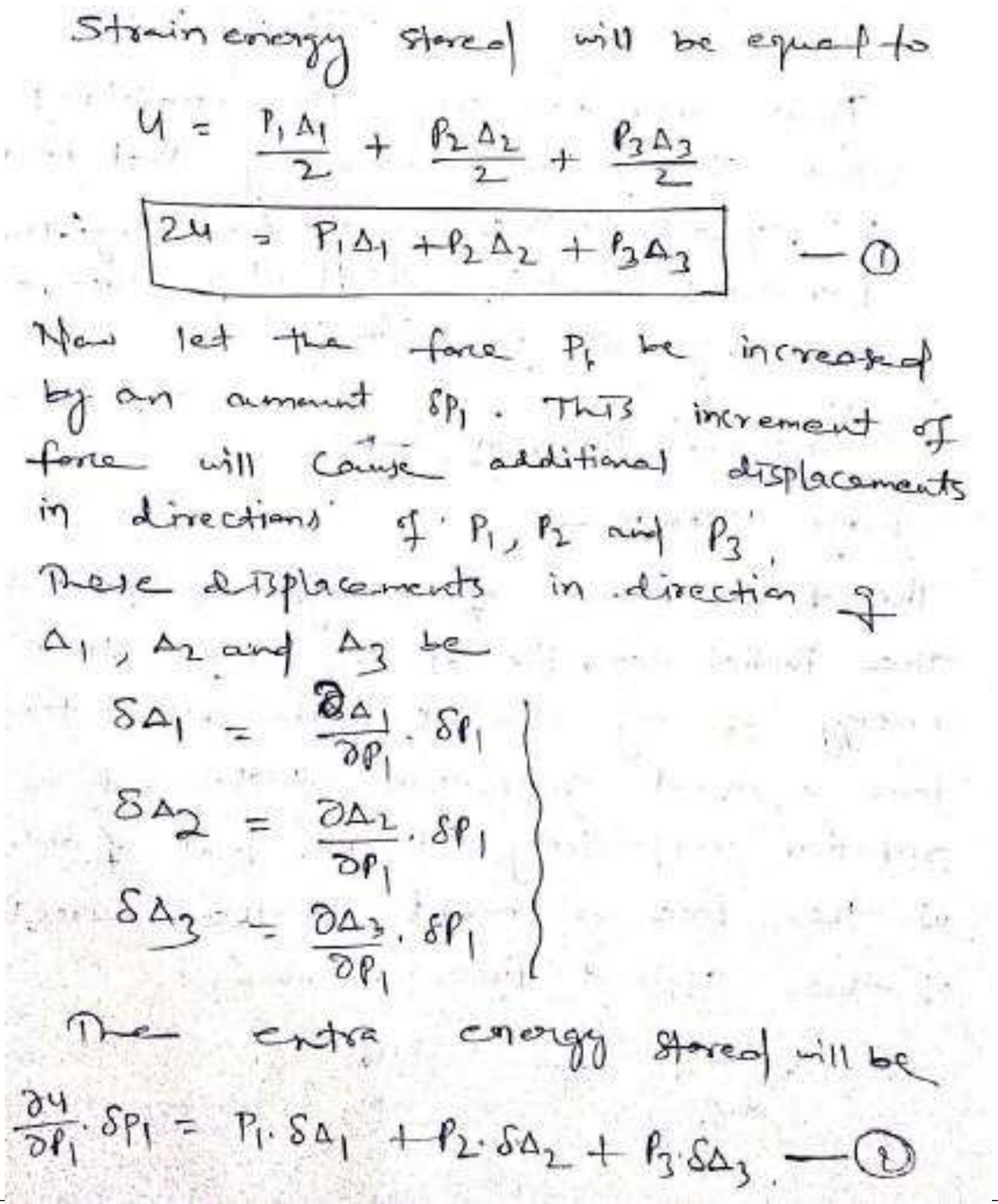
$$\frac{dy}{dP_1} = \Delta_1 \qquad \frac{dy}{dM_2} = \theta_2$$

Proof of the First Theorem



Consider a body subjected to forces P_1, P_2 and P_3 as shown in figure.

Let the displacements be Δ_1, Δ_2 and Δ_3 in the directions of P_1, P_2 and P_3 respectively.



Strain energy stored will be equal to

$$U = \frac{P_1\Delta_1}{2} + \frac{P_2\Delta_2}{2} + \frac{P_3\Delta_3}{2}$$

$$\therefore \boxed{2U = P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3} \quad — ①$$

Now let the force P_1 be increased by an amount δP_1. This increment of force will cause additional displacements in directions of P_1, P_2 and P_3. These displacements in direction of Δ_1, Δ_2 and Δ_3 be

$$\left.\begin{aligned} \delta\Delta_1 &= \frac{\partial\Delta_1}{\partial P_1}\cdot\delta P_1 \\ \delta\Delta_2 &= \frac{\partial\Delta_2}{\partial P_1}\cdot\delta P_1 \\ \delta\Delta_3 &= \frac{\partial\Delta_3}{\partial P_1}\cdot\delta P_1 \end{aligned}\right\}$$

The extra energy stored will be

$$\frac{\partial U}{\partial P_1}\cdot\delta P_1 = P_1\cdot\delta\Delta_1 + P_2\cdot\delta\Delta_2 + P_3\cdot\delta\Delta_3 \quad — ②$$

Put the value of $\delta\Delta_1$, $\delta\Delta_2$ and $\delta\Delta_3$ in equation ② we get

$$\frac{\partial y}{\partial P_1}\cdot\delta P_1 = P_1\cdot\frac{\partial\Delta_1}{\partial P_1}\cdot\delta P_1 + P_2\cdot\frac{\partial\Delta_2}{\partial P_1}\delta P_1 + P_3\cdot\frac{\partial\Delta_3}{\partial P_1}\cdot\delta P_1$$

$$\boxed{\therefore \frac{\partial y}{\partial P_1} = P_1\cdot\frac{\partial\Delta_1}{\partial P_1} + P_2\cdot\frac{\partial\Delta_2}{\partial P_1} + P_3\cdot\frac{\partial\Delta_3}{\partial P_1}} \quad \text{—③}$$

Differentiating equation ① w.r.T. P_1

$$2\cdot\frac{\partial y}{\partial P_1} = \Delta_1 + P_1\cdot\frac{\partial\Delta_1}{\partial P_1} + P_2\cdot\frac{\partial\Delta_2}{\partial P_1} + P_3\cdot\frac{\partial\Delta_3}{\partial P_1} \quad \text{—④}$$

Subtracting ③ from ④ we get,

$$\boxed{\frac{\partial y}{\partial P_1} = \Delta_1}$$ Proof of 1st theorem.

Thus the Partial derivative of Strain energy with respect to P_1 gives displacement in the direction of P_1.

Castigliano's Second Theorem

- The second theorem of castigliano states that the work done by external forces in a structure will be minimum.

- The Theorem is very much useful in analysis of statically indeterminate structures.

Let W = Work done by external forces on a structure

U = Strain energy stored in the structure

W_1 = Work done by reactive forces

Strain Energy = $U = W + W_1$

$$W = U - W_1$$

By Castigliano's 2nd theorem 'W' should be minimum.

Thus the partial derivative of the work done with respect to external forces will be zero.

- In case the supports are unyielding, the work done by reactive forces will be zero.

- Strain energy stored is equal to the work done by external forces will be minimum.

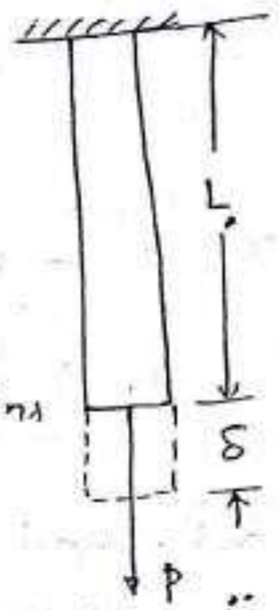
- Thus the partial derivative of strain energy with respect to redundant reaction will be zero.

- Castigliano’s First theorem helps in determining deflection of a structure and the Second theorem helps in determining redundant reaction components.

Law of Conservation of Energy

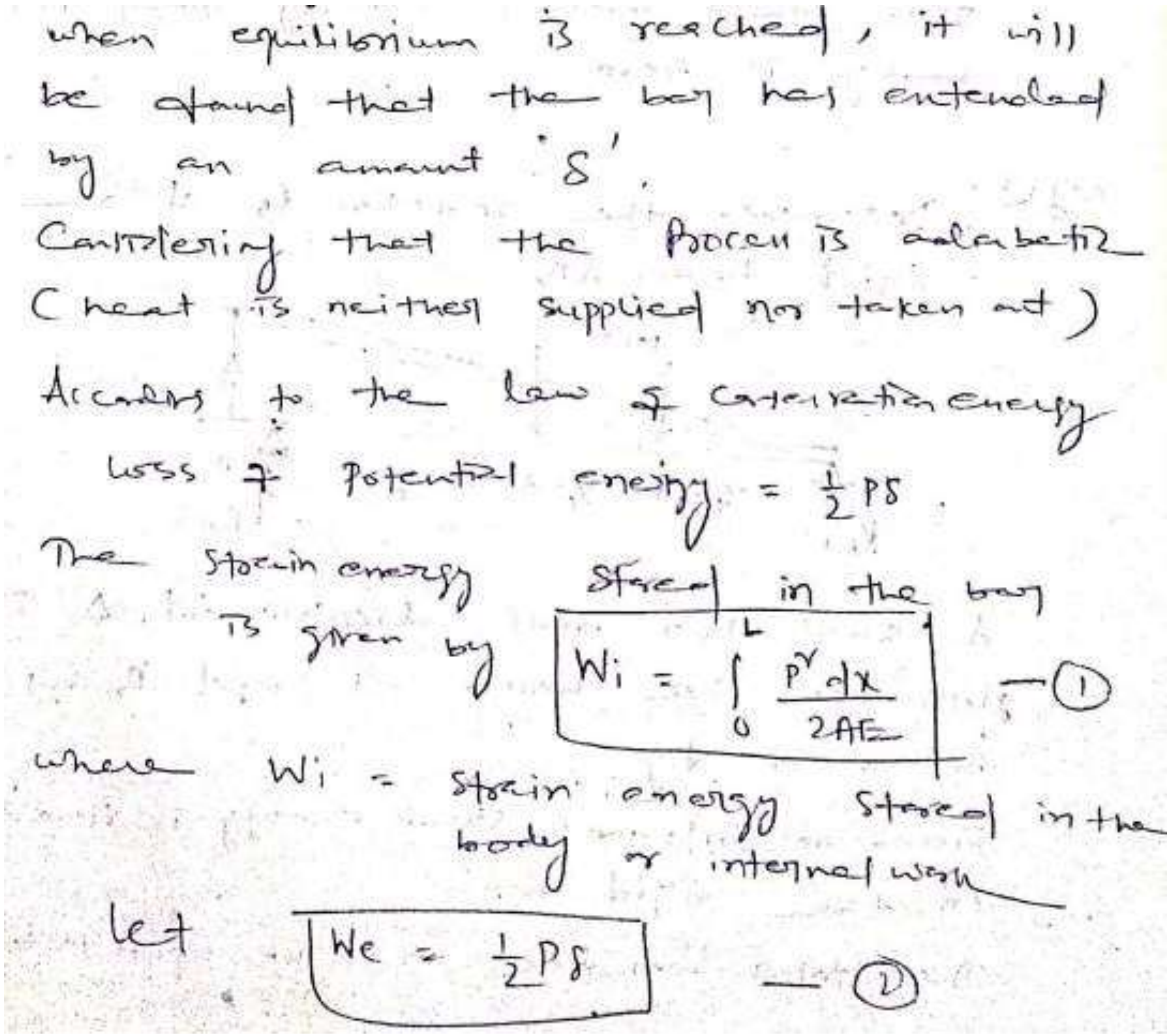
This is the basic law of physics. energy is neither created nor can be destroyed.

If a structure and external loads acting on it are isolated, such that these neither receive nor give out energy, then the total energy of the system remains constant.



①

A typical application of the law of conservation of energy can be applied to a bar subjected to an axial load 'P' gradually applied as shown in fig ①.



when equilibrium is reached, it will be found that the bar has extended by an amount 'δ'.

Considering that the process is adiabatic (heat is neither supplied nor taken out)

According to the law of conservation energy

loss of potential energy $= \frac{1}{2} P\delta$.

The strain energy stored in the bar is given by

$$\boxed{W_i = \int_0^L \frac{P^2\,dx}{2AE}} \quad — ①$$

where W_i = strain energy stored in the body or internal work

let $\boxed{W_e = \frac{1}{2} P\delta}$ — ②

where We = external work done.

$$\therefore \quad W_e + W_i = -\frac{1}{2} P\delta + \int_0^L \frac{P^2\,dx}{2AE} = 0. \quad \text{—③}$$

< The minus sign for external work is to take into account the loss of Potential energy >.

$$\text{or} \quad -W_e + W_i = 0. \quad \text{—④}$$

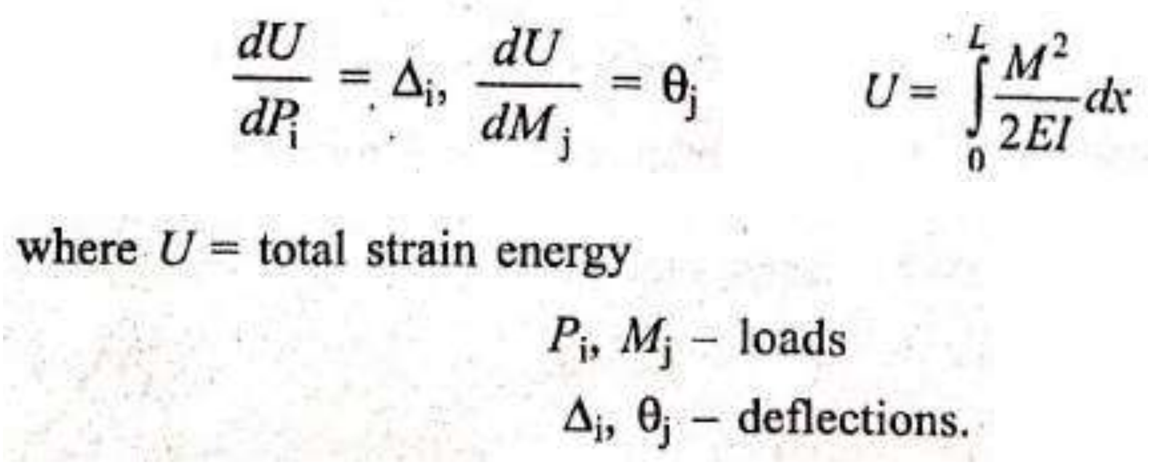
$$\text{or,} \quad \boxed{W_e = W_i} \quad \text{—⑤}$$

$$\boxed{\text{External workdone} = \text{Internal strain energy}}$$

Deflection by Castigliano's Method

Deflection by Castigliano’s Method:

Castigliano’s theorem may be represented by



$$\frac{dU}{dP_i} = \Delta_i, \quad \frac{dU}{dM_j} = \theta_j \qquad U = \int_0^L \frac{M^2}{2EI} dx$$

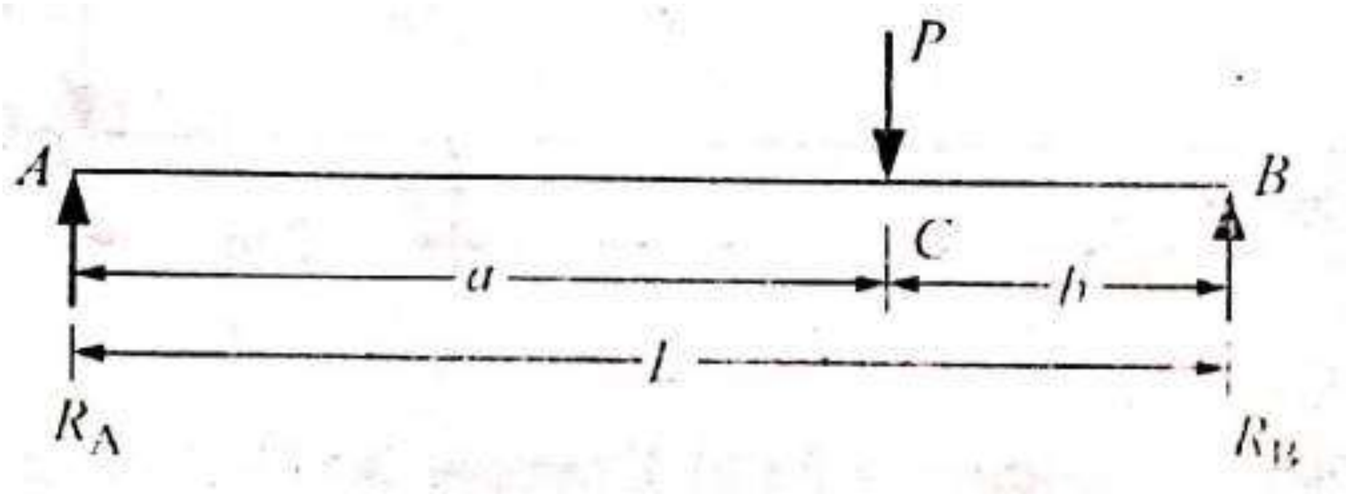
where U = total strain energy

P_i, M_j – loads

Δ_i, θ_j – deflections.

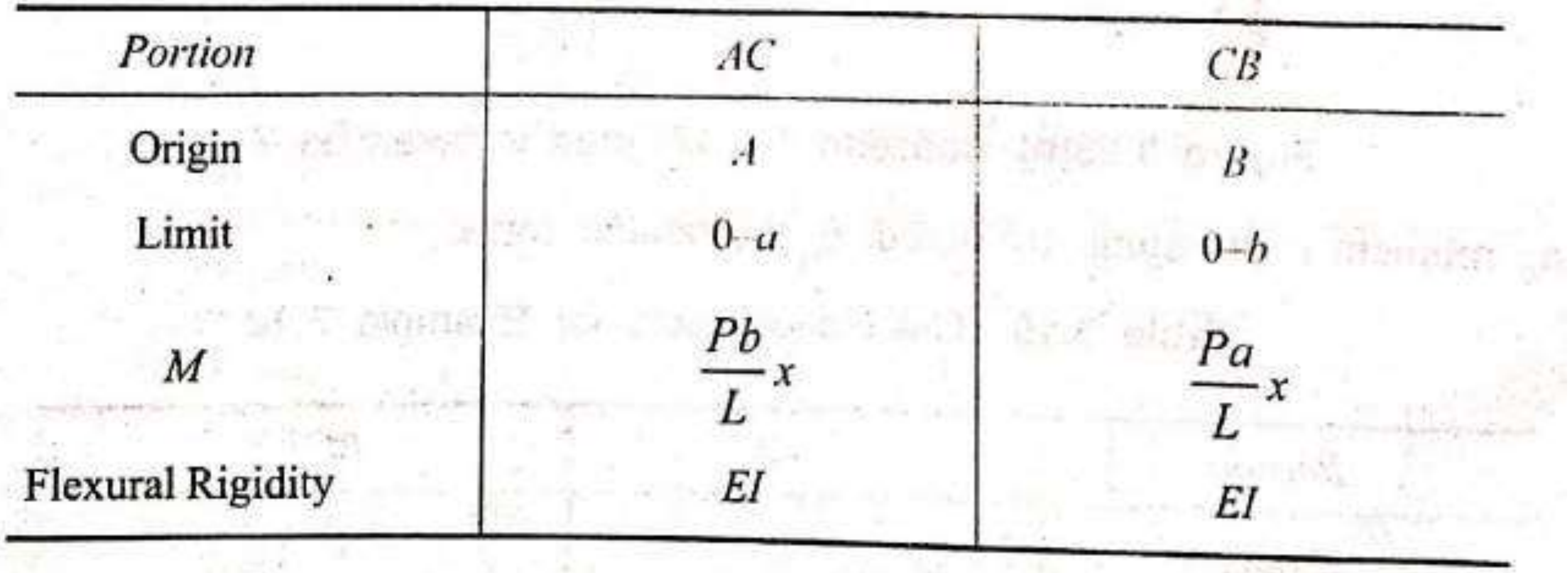
- If a load is acting at a point and is in the desired direction, the general expression for bending moment to cover the entire structure is to be find out.
- The strain energy for the entire structure is differentiated with respect to load (*P* = Load or *M* = Moment) to get the desired deflection.
- If the load is not acting, a dummy load (*P* or *M*) is applied and then the bending moment expressions is to be find out.
- If dummy load is used, First differentiate w.r.t the dummy load, then substitute dummy load as zero and then integrate w.r.t ‘x’ .

Q1. A simply supported beam of span ‘*L*’, carries a concentrated load ‘*P*’ at a distance ‘*a*’ from the left hand side as shown in Figure. Using Castigliano’s theorem determine the deflection under the load. Assume uniform flexural rigidity.



First determine the reaction by taking moment from any one support,

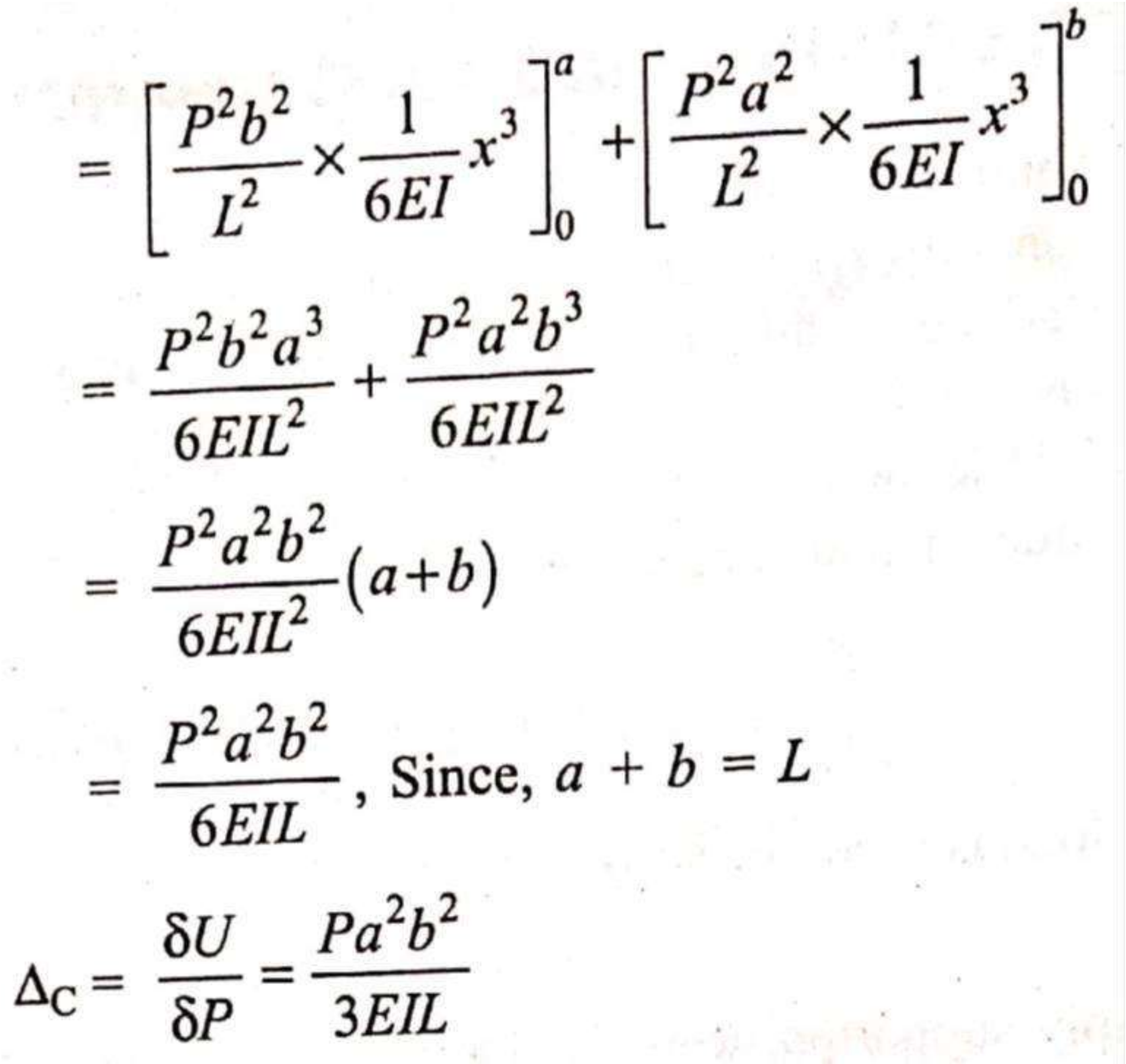
- Reaction at A, $R_A = \dfrac{Pb}{L}$
- Reaction at B, $R_B = \dfrac{Pa}{L}$
- Find out the expression for moment in a Tabular form for portion BC and then AC.



Portion	*AC*	*CB*
Origin	A	B
Limit	$0-a$	$0-b$
M	$\frac{Pb}{L}x$	$\frac{Pa}{L}x$
Flexural Rigidity	EI	EI

The strain energy of the Beam = $U = \int_0^L \frac{M^2}{2EI} dx$

$$U = \int_0^a \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI} dx + \int_0^b \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI} dx$$



$$= \left[\frac{P^2b^2}{L^2} \times \frac{1}{6EI} x^3\right]_0^a + \left[\frac{P^2a^2}{L^2} \times \frac{1}{6EI} x^3\right]_0^b$$

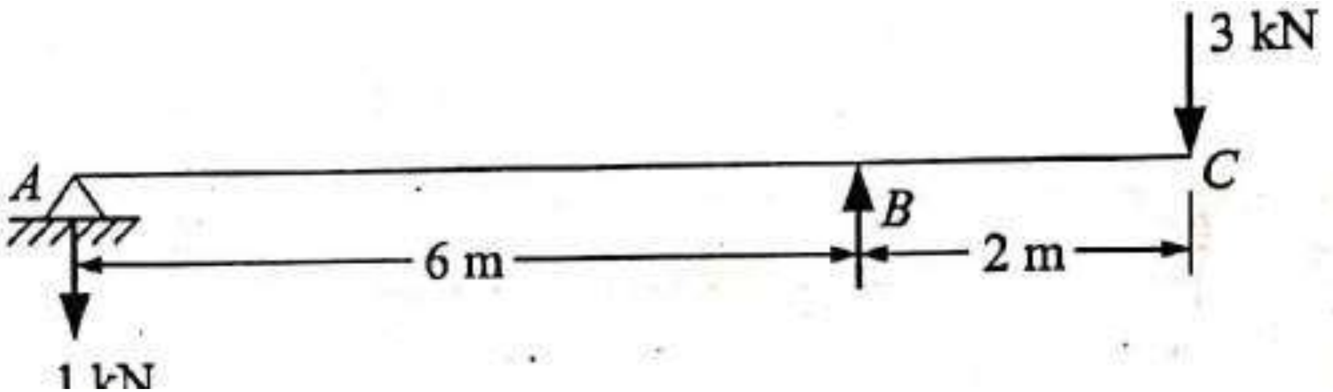
$$= \frac{P^2b^2a^3}{6EIL^2} + \frac{P^2a^2b^3}{6EIL^2}$$

$$= \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$= \frac{P^2a^2b^2}{6EIL}, \text{ Since, } a + b = L$$

$$\Delta_C = \frac{\delta U}{\delta P} = \frac{Pa^2b^2}{3EIL}$$

Q2. Determine the vertical deflection at the free end and rotation at ‘*A*’ in the over hanging beam shown in Figure. Use Castigliano’s theorem. Assume uniform flexural rigidity.



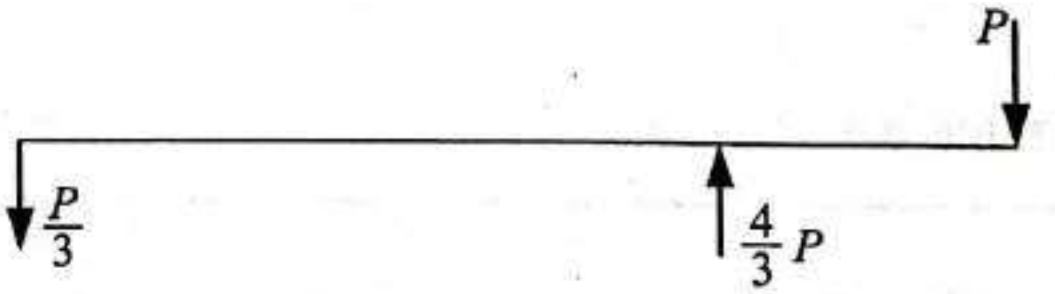
Deflection at ‘C’ = Δ_c

- Taking force P = 3 kN and moment about *A*,

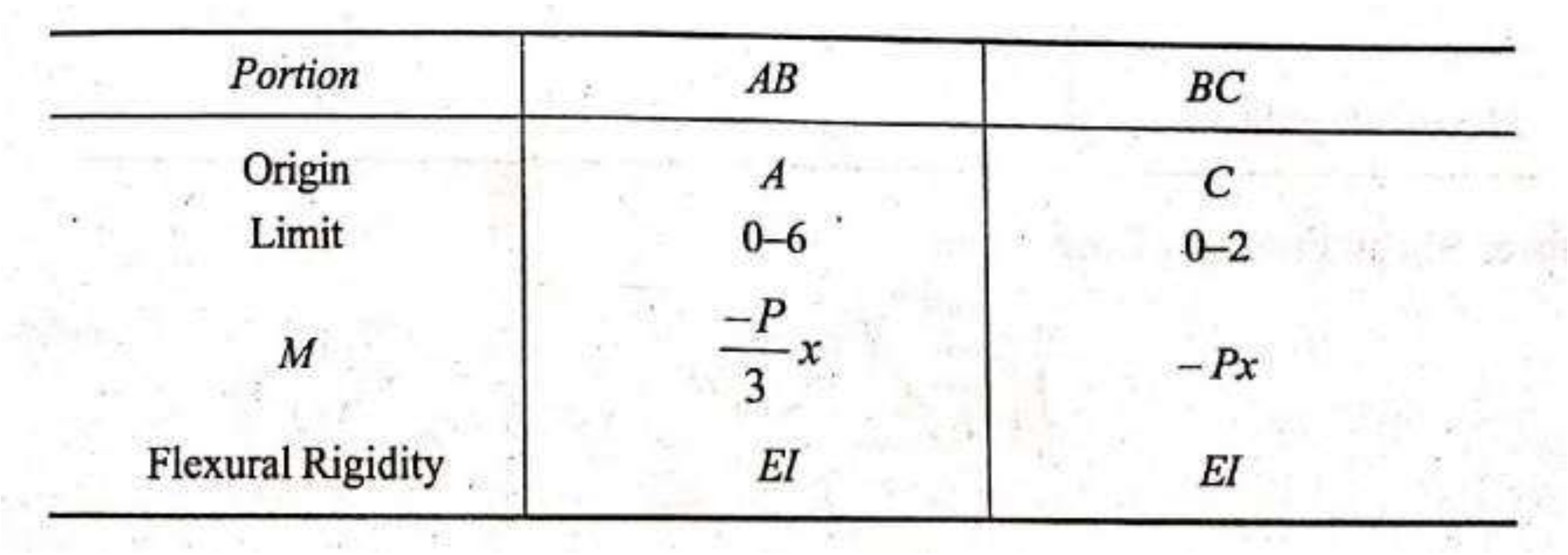
$$R_B \times 6 = P \times 8$$

$$R_B = \frac{4}{3} P \uparrow$$

$$R_A = \frac{P}{3} \downarrow$$



Bending moment expression for over hanging beam for portion AB and BC is noted in the Tabular form.



Portion	*AB*	*BC*
Origin	*A*	*C*
Limit	0–6	0–2
M	$\frac{-P}{3}x$	$-Px$
Flexural Rigidity	*EI*	*EI*

$$U = \int \frac{M^2}{2EI}\,dx$$

$$= \int_0^6 \frac{P^2x^2}{9} \times \frac{1}{2EI}\,dx + \int_0^2 \frac{P^2x^2}{2EI}\,dx$$

$$= \frac{P^2}{18EI}\left[\frac{x^3}{3}\right]_0^6 + \left[\frac{P^2x^3}{6EI}\right]_0^2$$

$$= \frac{4P^2}{EI} + \frac{4}{3} \times \frac{P^2}{EI}$$

$$= \frac{5.333P^2}{EI}$$

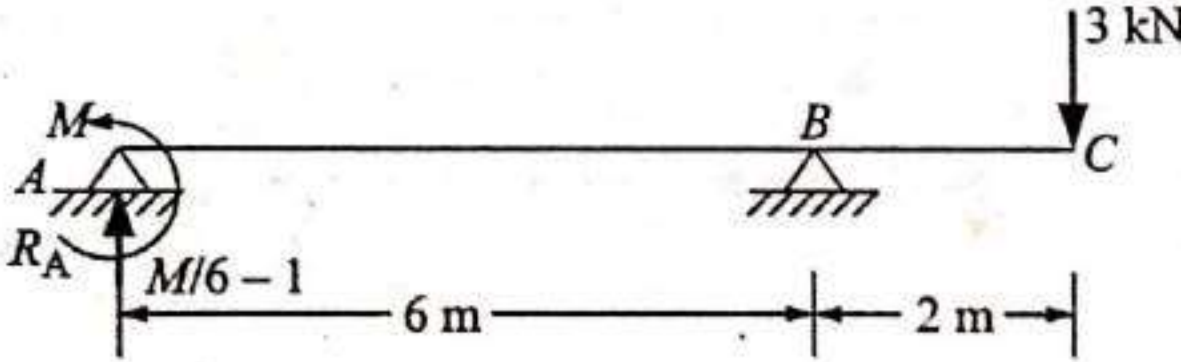
$$\Delta_C = \frac{dU}{dP} = \frac{10.667P}{EI}$$

Substituting $P = 3$ kN, we get

$$\Delta_C = \frac{32}{EI}$$

Rotation at A = θ_A

- Apply dummy moment, ‘*M*’ at *A* as shown in Figure



$$\Sigma M_B = 0, \text{ gives}$$

$$R_A = \frac{M-6}{6} = \frac{M}{6} - 1$$

Portion	*AB*	*BC*
Origin	*A*	*C*
Limit	0 – 6	0 – 2
M	$\left(\frac{M}{6} - 1\right)x - M$	$-3x$

$$U = \int_0^6 \left[\left(\frac{M}{6}-1\right)x - M\right]_0^2 \frac{1}{2EI}\,dx + \int_0^2 \frac{(-3x)^2}{2EI}\,dx$$

$$\frac{dU}{dM} = \int_0^6 2\left[\left(\frac{M}{6}-1\right)x - M\right]\left(\frac{x}{6}-1\right)\frac{dx}{2EI} + 0$$

Since, ‘*M*’ is a dummy moment, its value is substituted as zero, and then integrated

$$\frac{dU}{dM} = \theta_A = \frac{1}{EI}\int_0^6 (-x)\left(\frac{x}{6}-1\right)dx$$

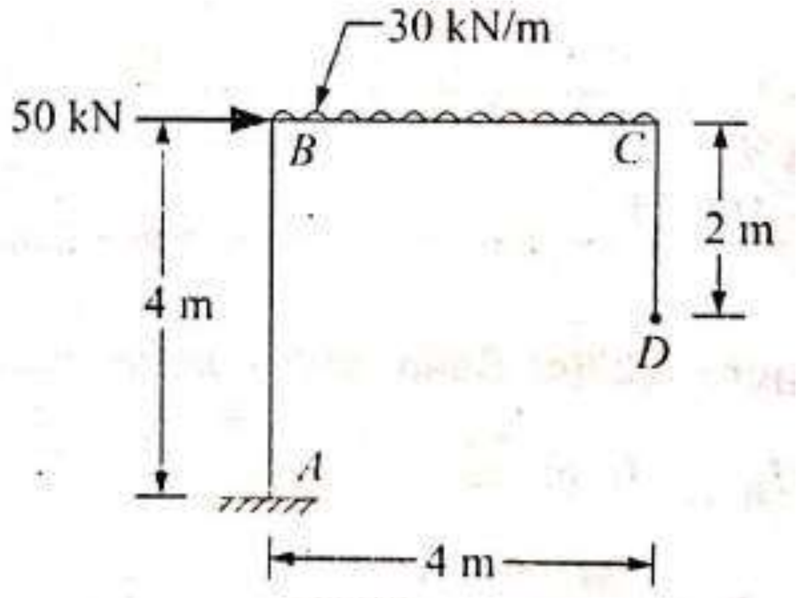
$$= \frac{1}{EI}\int_0^6 \left(-\frac{x^2}{6}+x\right)dx$$

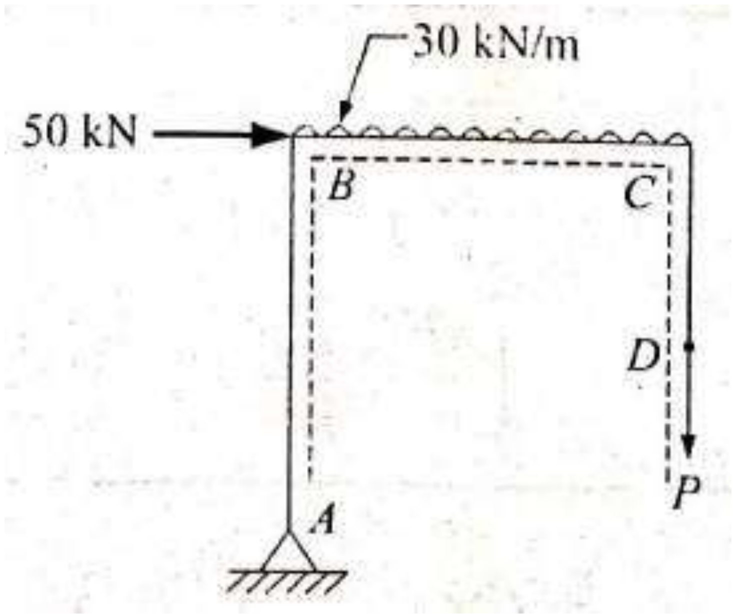
$$= \frac{1}{EI}\left(-\frac{x^3}{18}+\frac{x^2}{2}\right)_0^6 dx$$

$$= \frac{6}{EI}$$

Note: First differentiate w.r.t the dummy load, then substitute dummy load as zero and then integrate w.r.t ‘*x*’.

Q2. Determine the vertical and horizontal deflection at the free end ‘*D*’ in the frame shown in Figure. Use Castigliano’s theorem. Take *EI* = 12×10^{13} Nmm^2.





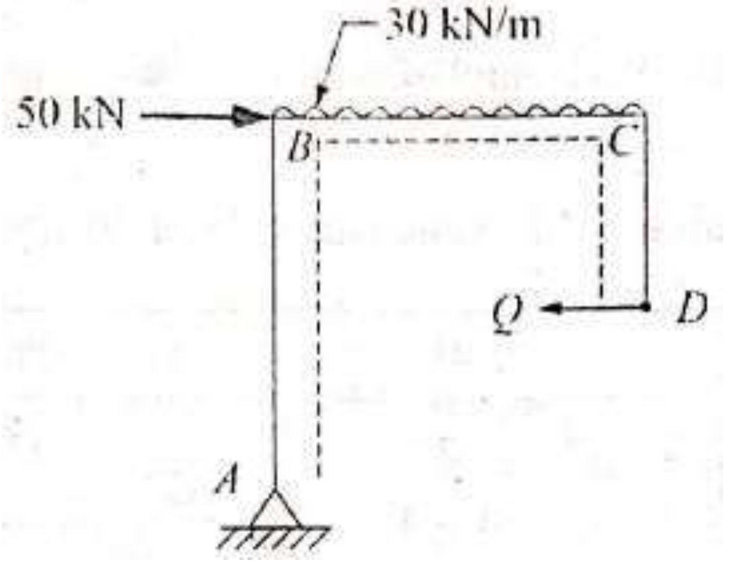
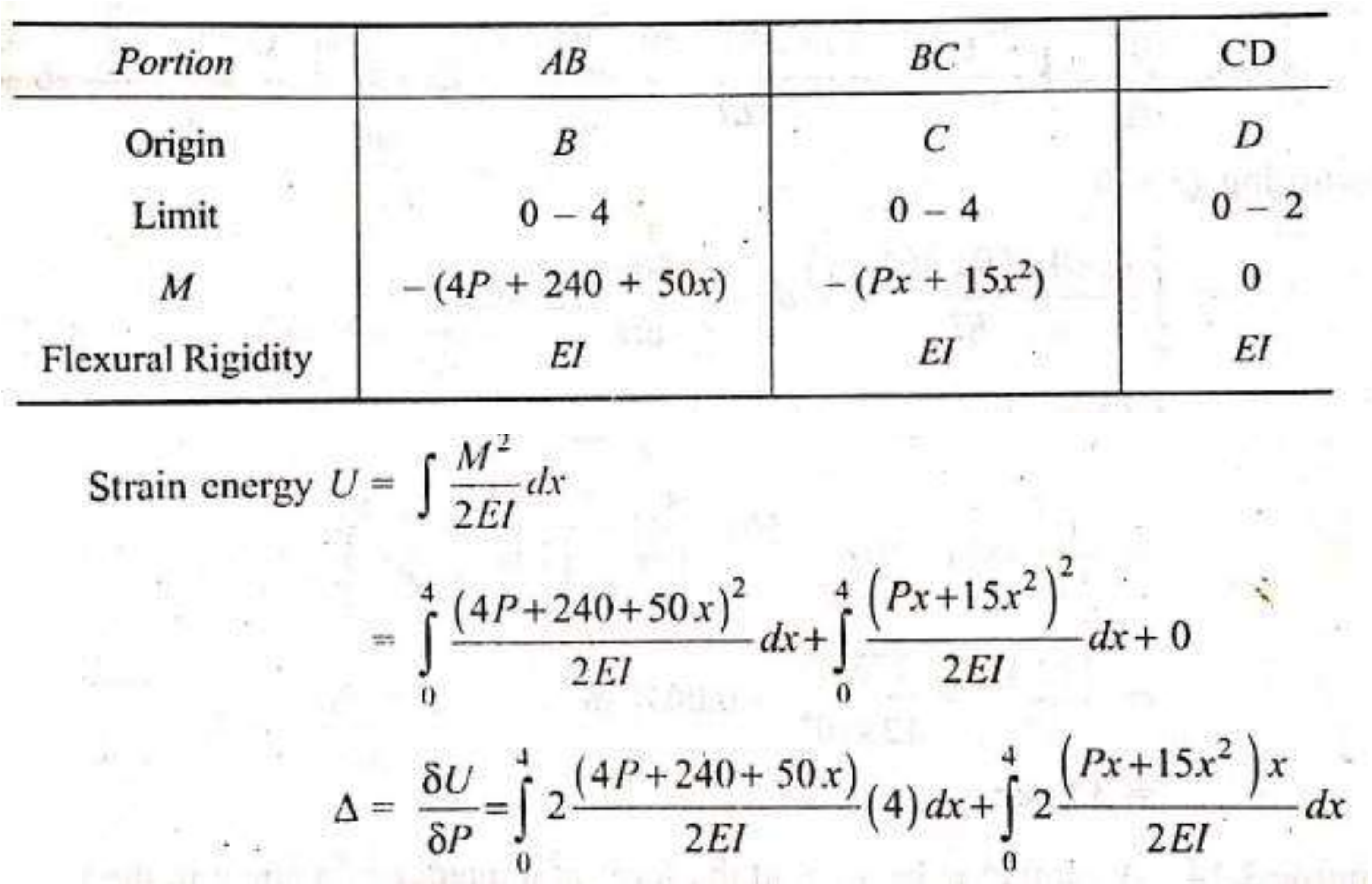


Figure 2: Frame with dummy horizontal load ‘*Q*’ at ‘*D*’

Vertical Deflection:

- Since, there is no load at ‘*D*’ in vertical direction, a dummy load ‘*P*’ is applied at ‘*D*’ in vertical direction in addition to given loads as shown in Figure 1. The moment expressions are presented in a tabular form.

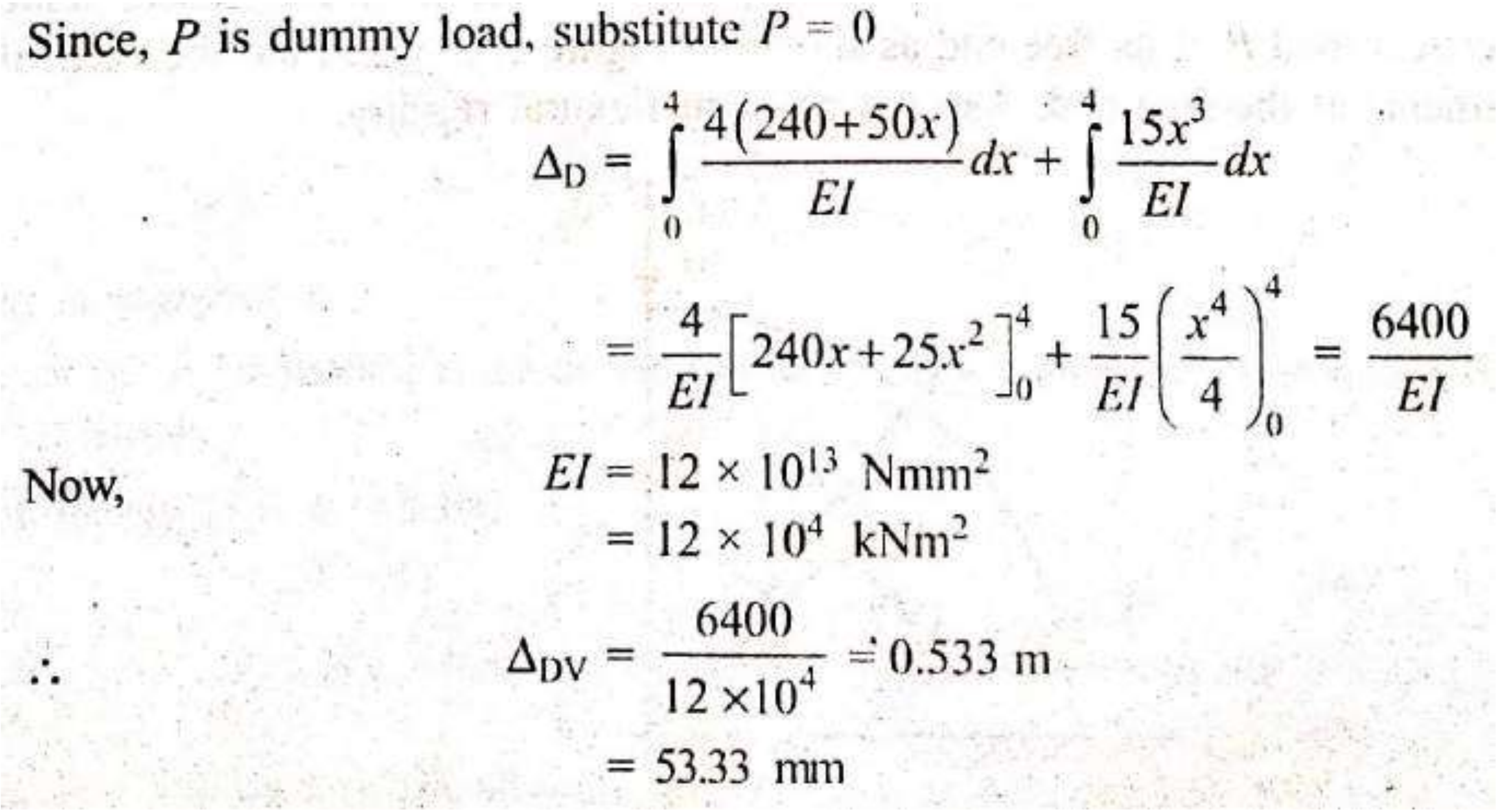


Portion	*AB*	*BC*	CD
Origin	*B*	*C*	*D*
Limit	0 – 4	0 – 4	0 – 2
M	$-(4P + 240 + 50x)$	$-(Px + 15x^2)$	0
Flexural Rigidity	*EI*	*EI*	*EI*

$$\text{Strain energy } U = \int \frac{M^2}{2EI}dx$$

$$= \int_0^4 \frac{(4P+240+50x)^2}{2EI}dx + \int_0^4 \frac{\left(Px+15x^2\right)^2}{2EI}dx + 0$$

$$\Delta = \frac{\delta U}{\delta P} = \int_0^4 2\frac{(4P+240+50x)}{2EI}(4)\,dx + \int_0^4 2\frac{\left(Px+15x^2\right)x}{2EI}dx$$



Since, P is dummy load, substitute $P = 0$

$$\Delta_D = \int_0^4 \frac{4(240+50x)}{EI} dx + \int_0^4 \frac{15x^3}{EI} dx$$

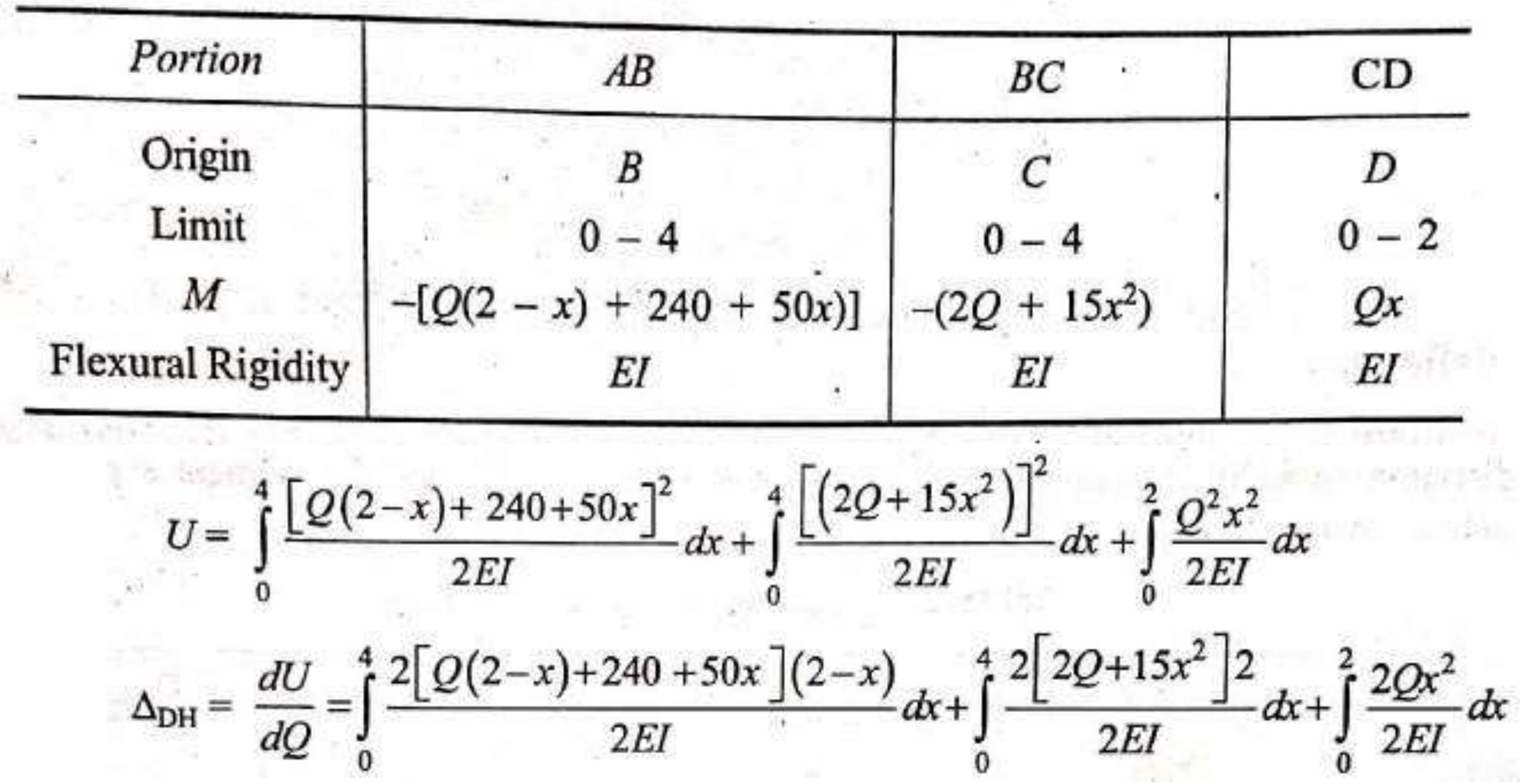
$$= \frac{4}{EI}\left[240x + 25x^2\right]_0^4 + \frac{15}{EI}\left(\frac{x^4}{4}\right)_0^4 = \frac{6400}{EI}$$

Now, $$EI = 12 \times 10^{13} \text{ Nmm}^2$$
$$= 12 \times 10^4 \text{ kNm}^2$$

$\therefore$ $$\Delta_{DV} = \frac{6400}{12 \times 10^4} \doteq 0.533 \text{ m}$$
$$= 53.33 \text{ mm}$$

Horizontal Deflection:

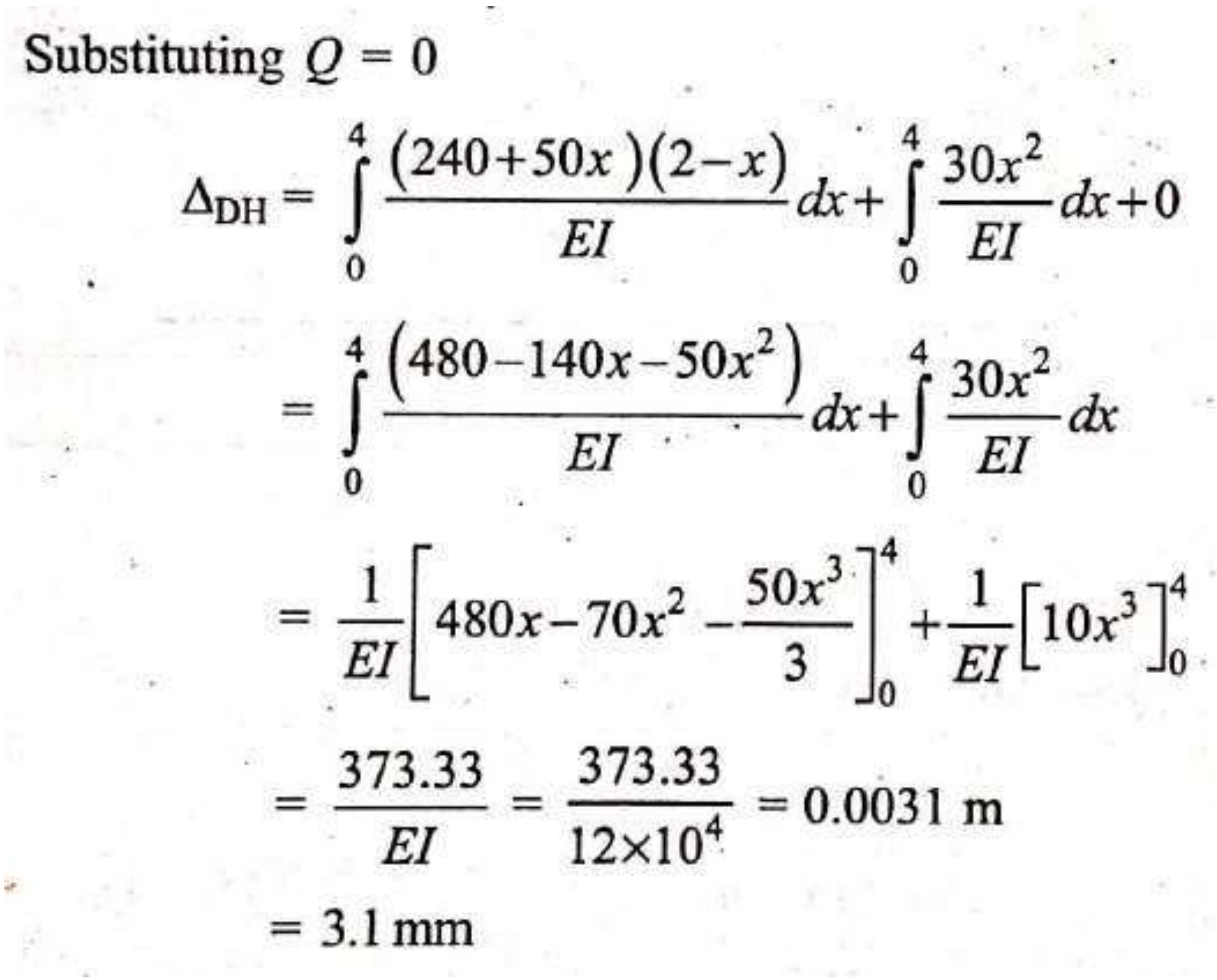
- Since, there is no load at ‘*D*’ in horizontal direction, a dummy load ‘*Q*’ is applied at ‘*D*’ in horizontal direction in addition to given loads as shown in Figure 2. The moment expressions are presented in a tabular form.



Portion	*AB*	*BC*	CD
Origin	*B*	*C*	*D*
Limit	0 – 4	0 – 4	0 – 2
M	$-[Q(2-x)+240+50x)]$	$-(2Q+15x^2)$	Qx
Flexural Rigidity	*EI*	*EI*	*EI*

$$U=\int_0^4 \frac{\left[Q(2-x)+240+50x\right]^2}{2EI}dx+\int_0^4 \frac{\left[\left(2Q+15x^2\right)\right]^2}{2EI}dx+\int_0^2 \frac{Q^2x^2}{2EI}dx$$

$$\Delta_{\mathrm{DH}}=\frac{dU}{dQ}=\int_0^4 \frac{2\left[Q(2-x)+240+50x\right](2-x)}{2EI}dx+\int_0^4 \frac{2\left[2Q+15x^2\right]2}{2EI}dx+\int_0^2 \frac{2Qx^2}{2EI}dx$$



Substituting $Q = 0$

$$\Delta_{\mathrm{DH}} = \int_0^4 \frac{(240+50x)(2-x)}{EI}dx + \int_0^4 \frac{30x^2}{EI}dx + 0$$

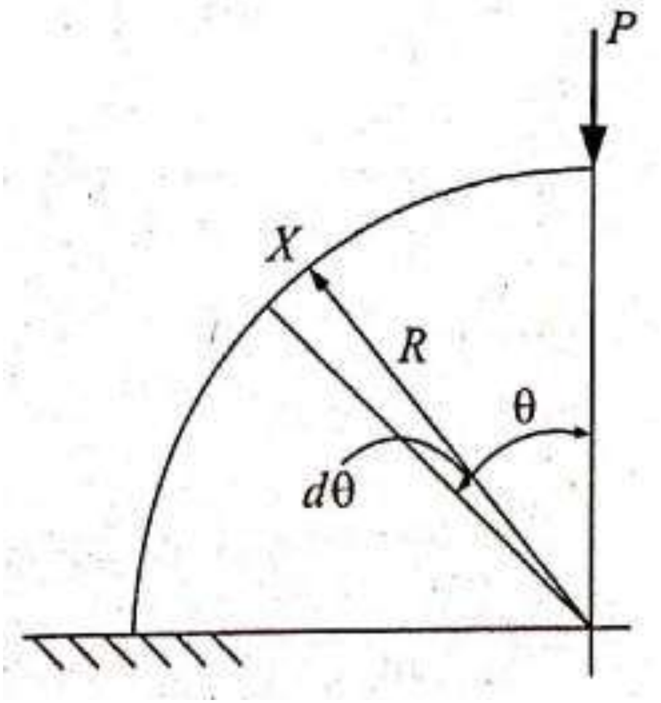
$$= \int_0^4 \frac{\left(480-140x-50x^2\right)}{EI}dx + \int_0^4 \frac{30x^2}{EI}dx$$

$$= \frac{1}{EI}\left[480x - 70x^2 - \frac{50x^3}{3}\right]_0^4 + \frac{1}{EI}\left[10x^3\right]_0^4$$

$$= \frac{373.33}{EI} = \frac{373.33}{12\times10^4} = 0.0031 \text{ m}$$

$$= 3.1 \text{ mm}$$

Q2. A cantilever beam is in the form of a quarter of a circle in the vertical plane and is subjected to a vertical load ‘P’ at its free end as shown in Figure. Find the vertical and horizontal deflections at the free end. Use Castigliano’s theorem. Assume uniform flexural rigidity.



Vertical Deflection of free end:

- Consider the section at ‘x’ as shown in Figure 1. The Bending moment at the section ‘x’ is

$$M = PR \sin \theta$$

Strain energy in the elemental length '$R\,d\theta$' is

$$= \left(\frac{M^2}{2EI}\right) R\,d\theta$$

$$= \frac{P^2 R^2 \sin^2\theta}{2EI} R\,d\theta$$

$$= \frac{P^2 R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$$

$$U = \int_0^{\pi/2} \frac{P^2 R^3}{2EI} \times \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{P^2 R^3}{4EI}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi P^2 R^3}{8EI}$$

$$\Delta_V = \frac{\delta U}{dP} = \frac{\pi P R^3}{4EI}$$

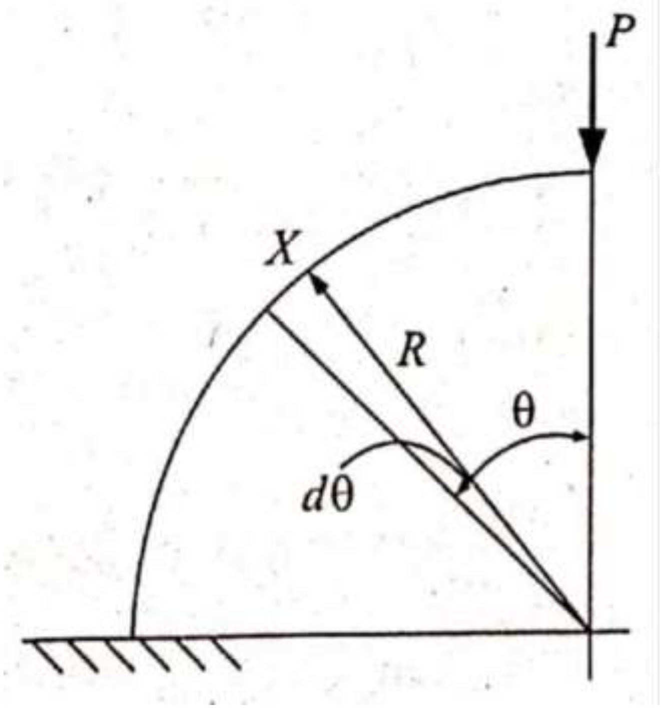


Figure 1: Cantilever curved beam

Horizontal Deflection:

Since, there is no horizontal force at the free end, apply a dummy horizontal force ‘Q’, as shown in Figure 2.

The bending moment at section ‘x’ is

$$M = PR \sin \theta + QR\,(1 - \cos \theta)$$

Strain Energy (U) =

$$U = \int_{0}^{\pi/2} \frac{\left[PR\sin\theta + QR(1-\cos\theta)\right]^2}{2EI} R\,d\theta$$

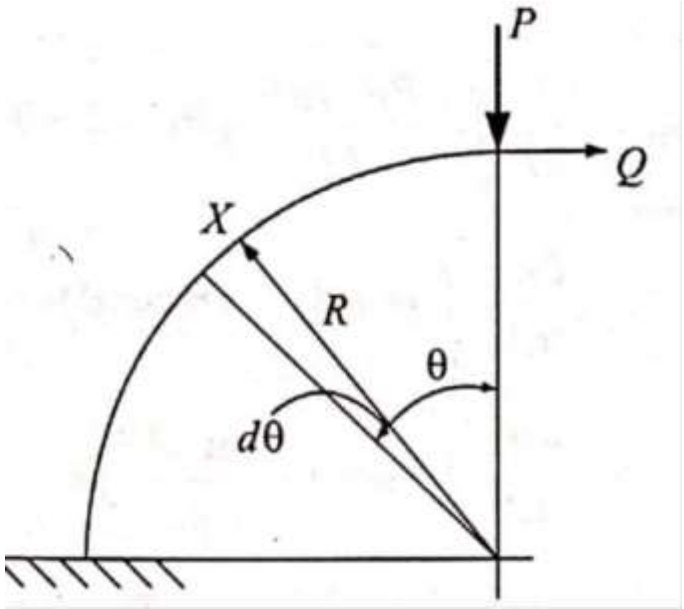


Figure 2: Cantilever curved beam with dummy load ‘Q’ at the free end

Horizontal Displacement = Δ_H

$$\Delta_H = \frac{\delta U}{\delta Q} = \int_{0}^{\pi/2} \frac{\left[PR\sin\theta + QR(1-\cos\theta)\right]}{EI}\left[R(1-\cos\theta)\right] R\,d\theta$$

Substituting, Q = 0, in above equation

$$\Delta_H = \frac{1}{EI}\int_0^{\pi/2}\left(\frac{PR\sin\theta}{EI}\right)[R(1-\cos\theta)]R\,d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}(\sin\theta - \sin\theta\cos\theta)d\theta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)d\theta$$

$$= \frac{PR^3}{EI}\left(\cos\theta - \frac{\cos 2\theta}{4}\right)_0^{\pi/2}$$

$$= \frac{PR^3}{EI}\left(0 + \frac{1}{4} - 1 + \frac{1}{4}\right)$$

$$= -\frac{PR^3}{2EI}$$

i.e., $\Delta_H = \frac{PR^3}{2EI}$, towards support

Deflection by Unit Load Method

Deflection by Unit Load Method

- This method is applicable to beam and rigid frame where only flexural effect is considered.
- In the analysis, the effect of axial force and shear forces are neglected.
- The deflection at any point can be find out by:

$$\Delta = \int_0^L \frac{Mm}{EI} dx$$

Where, *M* = Bending moment at the section due to the external forces

m = Bending moment at the section due to unit loading

E = Modulus of Elasticity

I = Moment of Inertia of the section

Q1. Determine the deflection at the free end of the over hanging beam shown in Figure by unit load method.

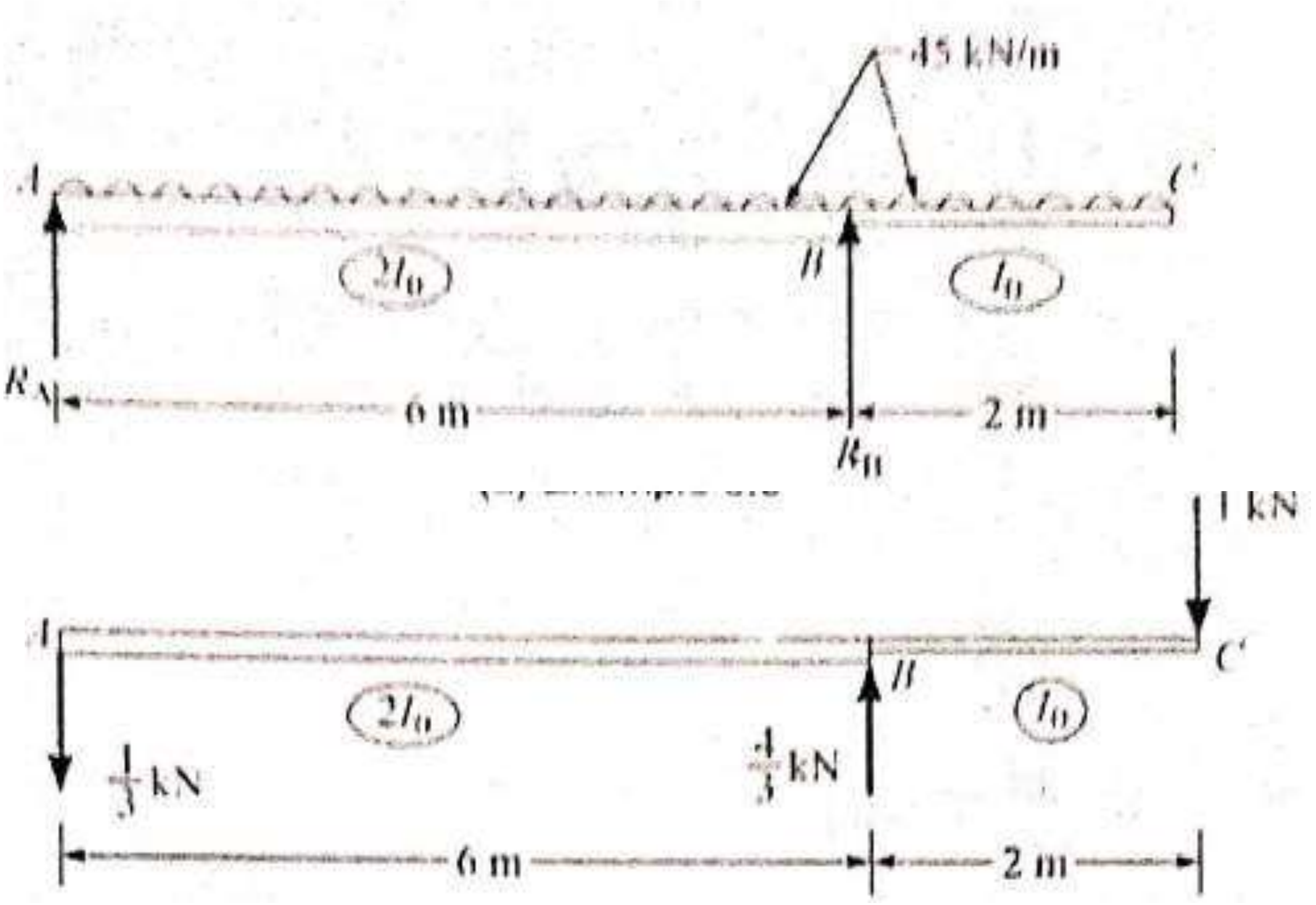


Figure 1: Beam with unit load at ‘*C*’

- Find out the reactions due to external forces, taking moment about A

$$\sum M_A = 0, \text{ gives}$$

$$R_B \times 6 = 45 \times 8 \times 4$$

$$R_B = 240 \text{ kN}$$

$$\sum V = 0, \text{ gives}$$

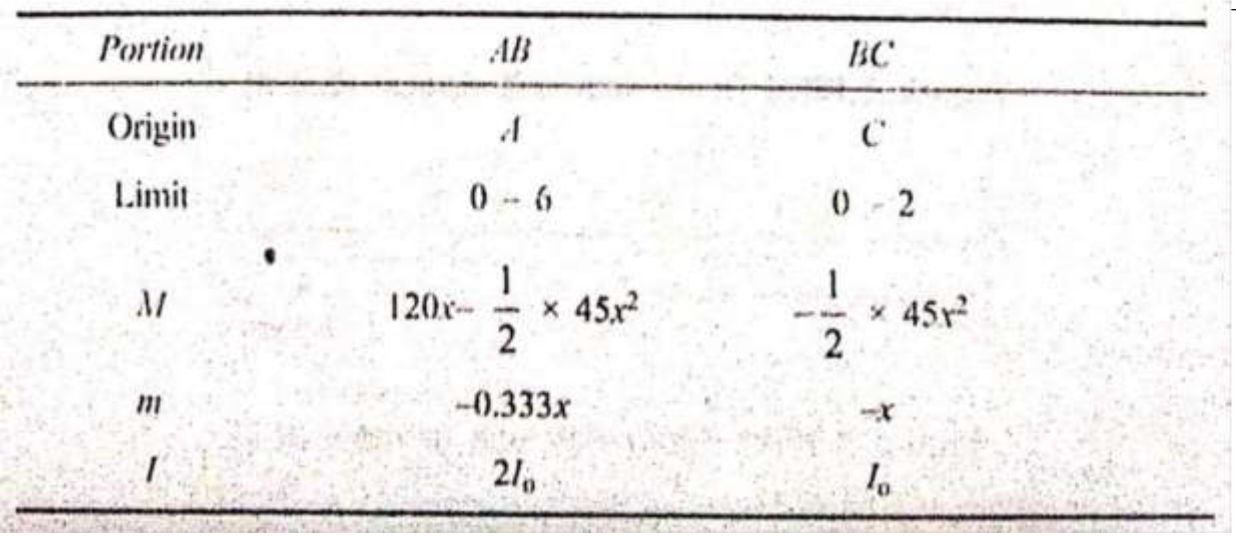
$$R_A = 45 \times 8 - 240 = 120 \text{ kN}$$

- Find out the reactions, when unit load acting at ‘*C*’

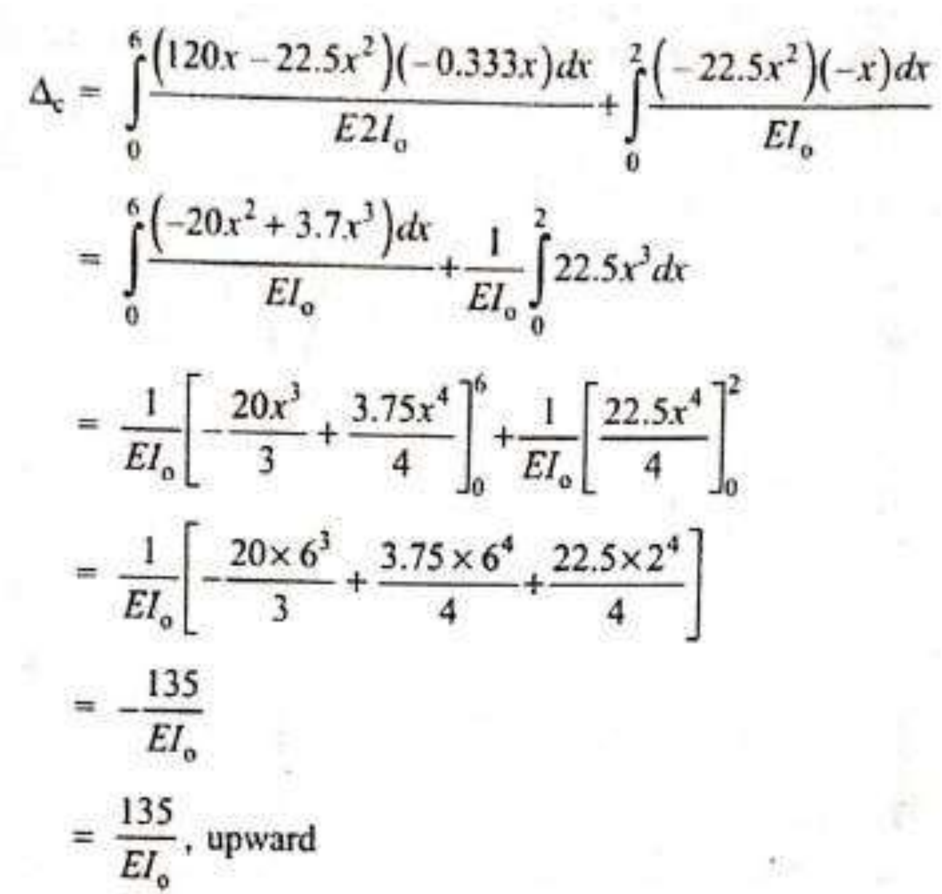
$$R_B = \frac{1\times 8}{6} = 1.333 \text{ kN}$$

$$R_A = 0.333 \text{ kN} \downarrow$$

- Taking sagging moment as positive and hogging moment as negative, find out the expressions for moments in various portions of the beam due to external loading and unit force where the deflection is to be determined in a Tabular form.



Portion	*AB*	*BC*
Origin	*A*	*C*
Limit	0 – 6	0 – 2
M	$120x - \frac{1}{2} \times 45x^2$	$-\frac{1}{2} \times 45x^2$
m	$-0.333x$	$-x$
I	$2I_0$	I_0



$$\Delta_c = \int_0^6 \frac{\left(120x - 22.5x^2\right)\left(-0.333x\right)dx}{E2I_o} + \int_0^2 \frac{\left(-22.5x^2\right)\left(-x\right)dx}{EI_o}$$

$$= \int_0^6 \frac{\left(-20x^2 + 3.7x^3\right)dx}{EI_o} + \frac{1}{EI_o}\int_0^2 22.5x^3 dx$$

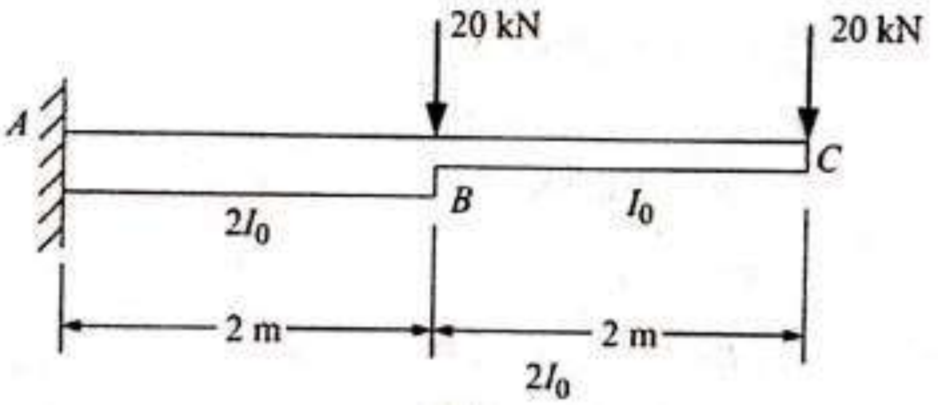
$$= \frac{1}{EI_o}\left[-\frac{20x^3}{3} + \frac{3.75x^4}{4}\right]_0^6 + \frac{1}{EI_o}\left[\frac{22.5x^4}{4}\right]_0^2$$

$$= \frac{1}{EI_o}\left[-\frac{20 \times 6^3}{3} + \frac{3.75 \times 6^4}{4} + \frac{22.5 \times 2^4}{4}\right]$$

$$= -\frac{135}{EI_o}$$

$$= \frac{135}{EI_o}, \text{ upward}$$

Q2. Determine the deflection and rotation at the free end of the cantilever beam shown in Figure by unit load method. Given $E = 200000\ \text{N/mm}^2$ and $I = 12 \times 10^6\ \text{mm}^4$



- Find out the deflection and rotation at the free end of the cantilever beam, apply unit load for deflection and unit moment for rotation at the free end of the beam as shown in Figure.

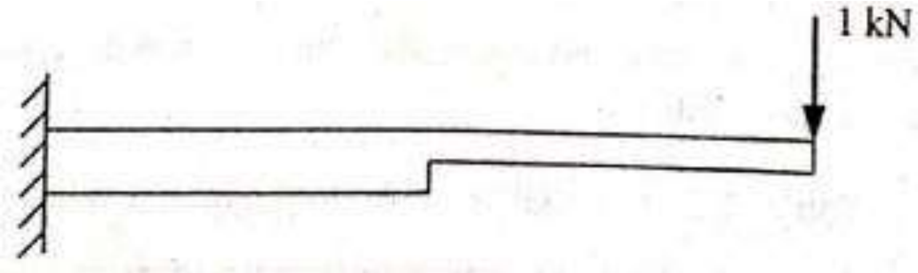


Figure 1: Beam with unit vertical load at '*C*'

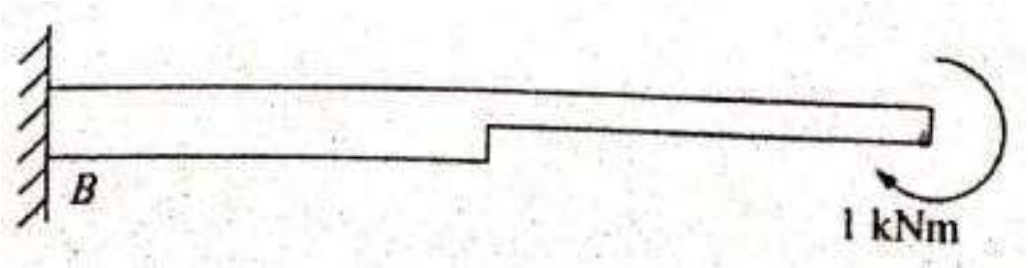
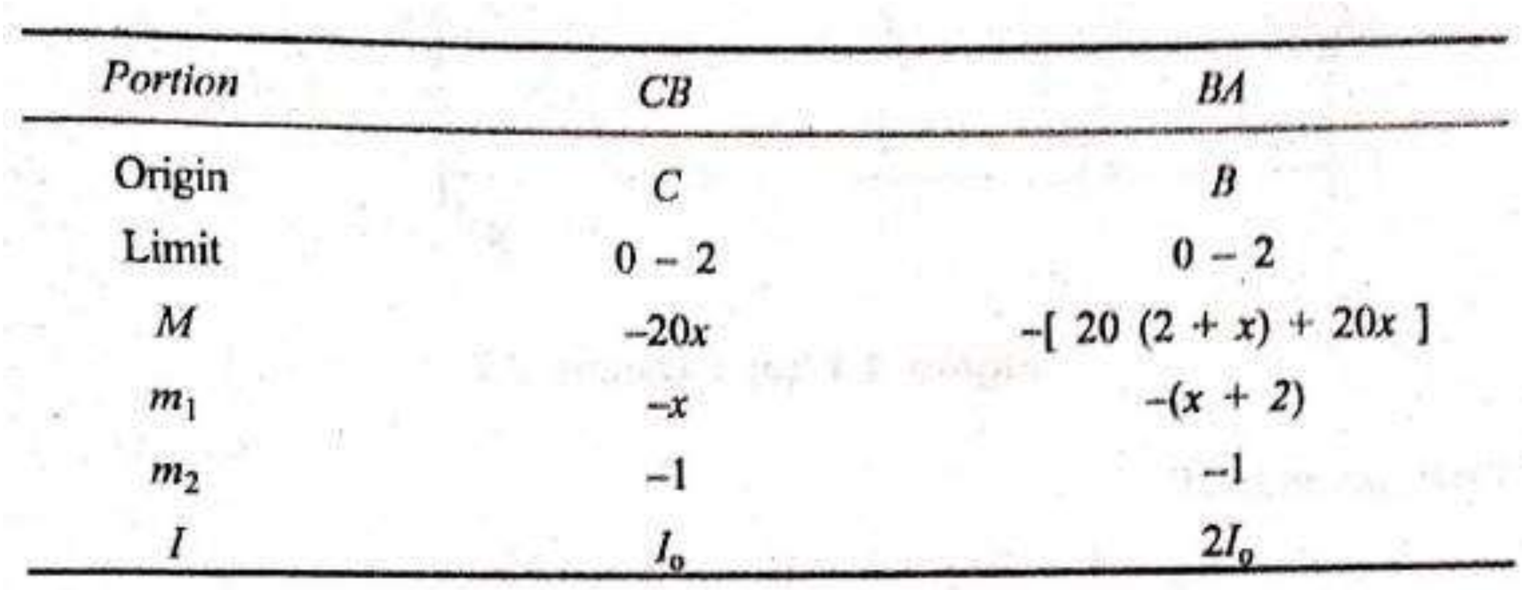


Figure 2: Beam with unit moment at ‘*C*’

- The bending moment expressions can be calculated by
- *M* for external given load, m_1 for unit vertical load at ‘C’ and m_2 for unit moment at ‘C’ for various portion of cantilever beam and tabulated below.



Portion	CB	BA
Origin	C	B
Limit	0 – 2	0 – 2
M	$-20x$	$-[\ 20\ (2 + x) + 20x\ ]$
m_1	$-x$	$-(x + 2)$
m_2	-1	-1
I	I_0	$2I_0$

Vertical deflection at 'C' = $\Delta = \int_0^L \frac{Mm_1}{EI}\,dx$

$$= \int_0^2 \frac{(-20x)(-x)}{EI_o}\,dx + \int_0^2 \frac{\left[20(2+x)+20x\right](x+2)}{E2I_o}\,dx$$

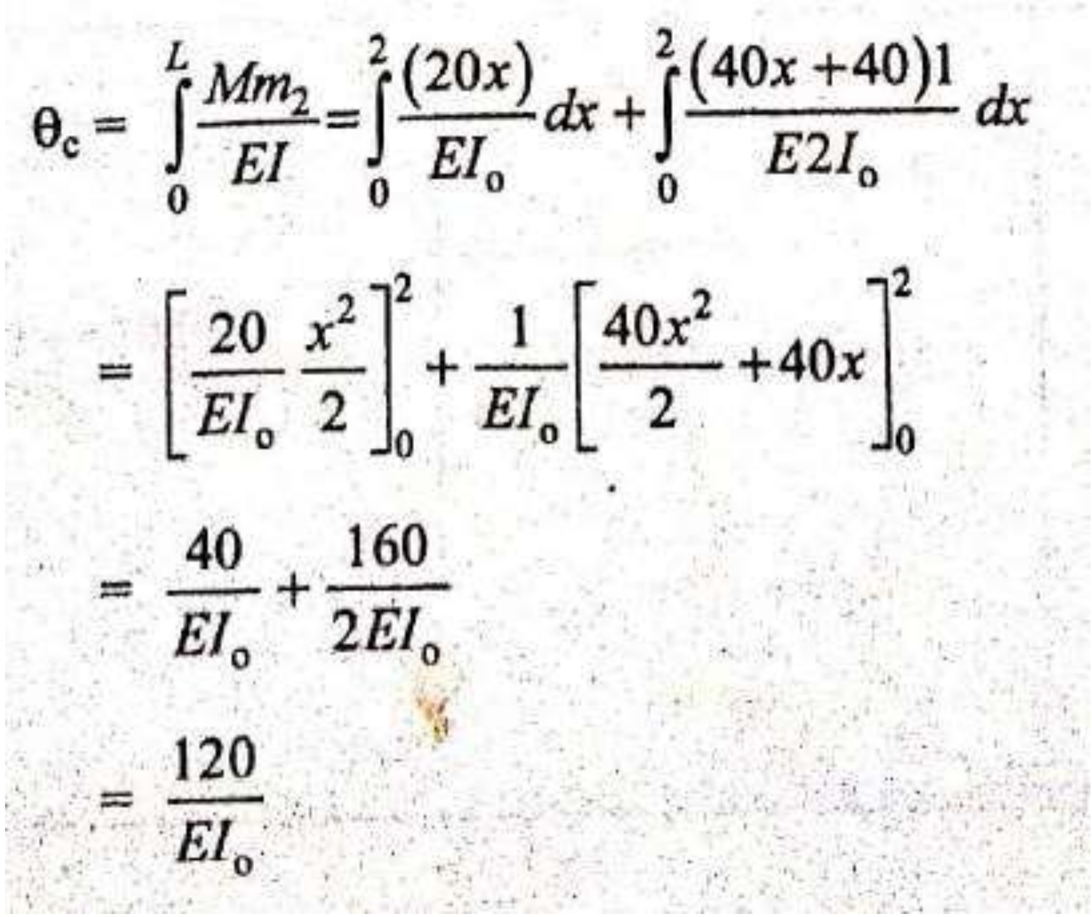
$$= \int_0^2 \frac{20x^2}{EI_o}\,dx + \int_0^2 \frac{(40x+40)(x+2)}{2EI_o}\,dx$$

$$= \left[\frac{20}{3}\frac{x^3}{EI_o}\right]_0^2 + \frac{1}{2EI_o}\left[\frac{40x^3}{3} + \frac{120x^2}{2} + 80x\right]_0^2$$

$$= \frac{53.333}{EI_o} + \frac{1}{EI_o}[253.333]$$

$$= \frac{306.67}{EI_o}$$

Rotation at ‘*C*’ =



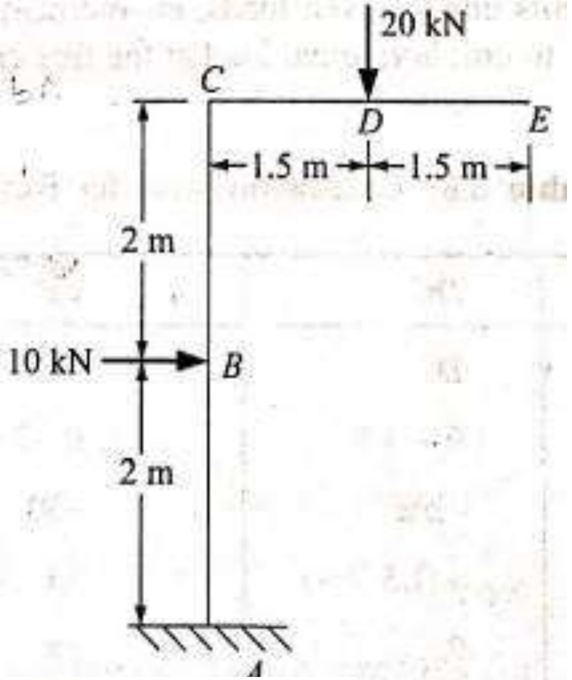
$$\theta_c = \int_0^L \frac{Mm_2}{EI} = \int_0^2 \frac{(20x)}{EI_o}dx + \int_0^2 \frac{(40x+40)1}{E2I_o}dx$$

$$= \left[\frac{20}{EI_o}\frac{x^2}{2}\right]_0^2 + \frac{1}{EI_o}\left[\frac{40x^2}{2}+40x\right]_0^2$$

$$= \frac{40}{EI_o} + \frac{160}{2EI_o}$$

$$= \frac{120}{EI_o}$$

Q3. Determine the vertical and horizontal deflection at the free end of the bent shown in Figure by unit load method. Assume uniform flexural rigidity *EI* throughout.



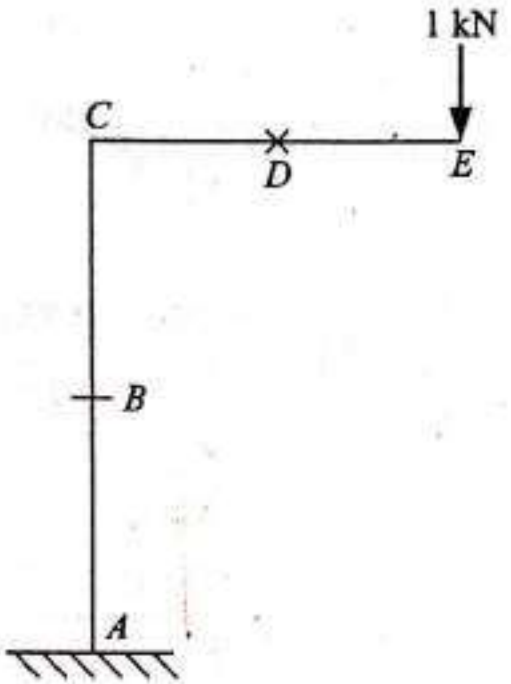


Figure 1: Frame with unit vertical load at ‘*E*’

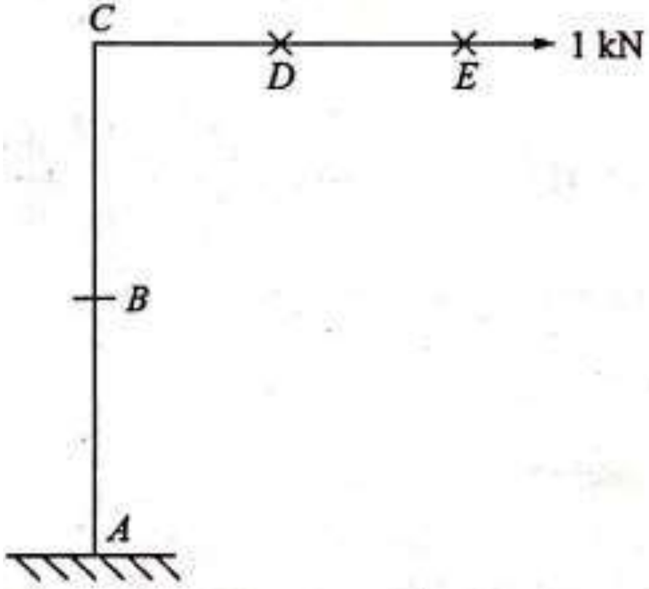
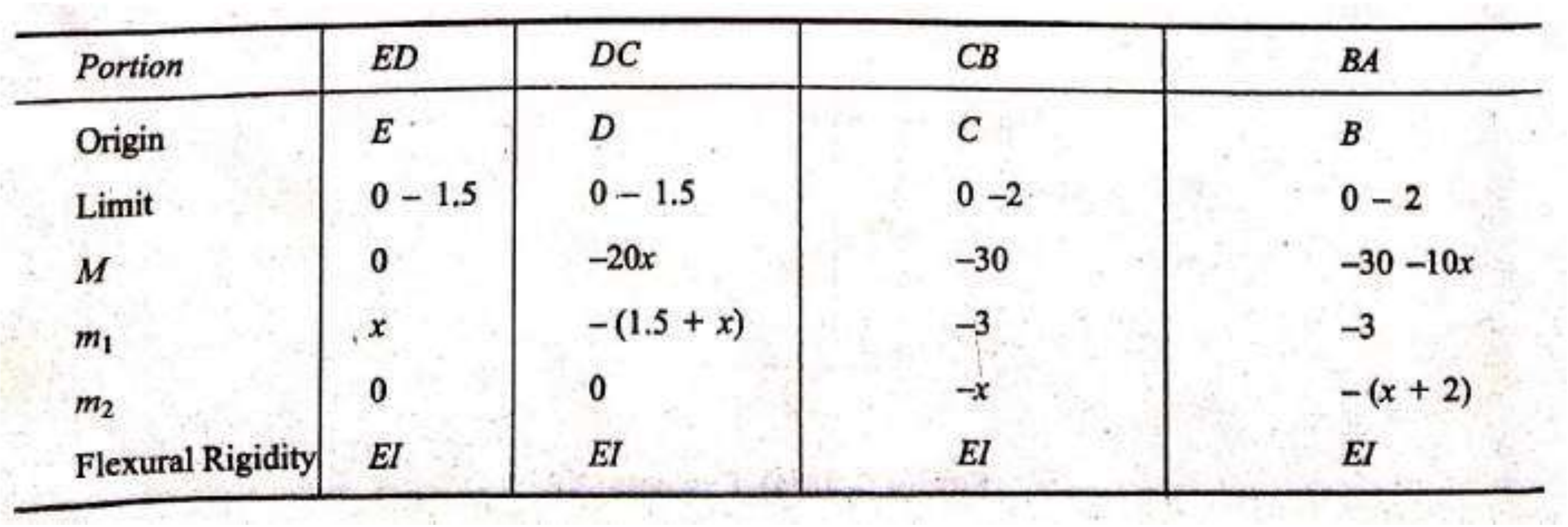


Figure 2: Frame with unit horizontal load at ‘*E*’

- Find out the expressions in Tabular form for moment '*M*' due to external loads, m_1 due to the unit vertical load present at the free end (Figure 1) and m_2 due to the unit horizontal load present at the free end (Figure 2) of the bent.



Portion	*ED*	*DC*	*CB*	*BA*
Origin	*E*	*D*	*C*	*B*
Limit	0 – 1.5	0 – 1.5	0 –2	0 – 2
M	0	–20x	–30	–30 –10x
m_1	x	$-(1.5 + x)$	–3	–3
m_2	0	0	$-x$	$-(x + 2)$
Flexural Rigidity	*EI*	*EI*	*EI*	*EI*

Note: Moment carrying tension on dotted side is taken as positive

Vertical deflection at ‘*E*’ = Δ_{EV}

$$EI\Delta_{EV} = \int Mm_1\, dx$$

$$= 0 + \int_0^{1.5} 20x(1.5+x)\,dx + \int_0^{2} 90dx + \int_0^{2} (90+30x)\,dx$$

$$= \int_0^{1.5} \left(30x+20x^2\right)dx + \int_0^{2} 90dx + \int_0^{2} (90+30x)\,dx$$

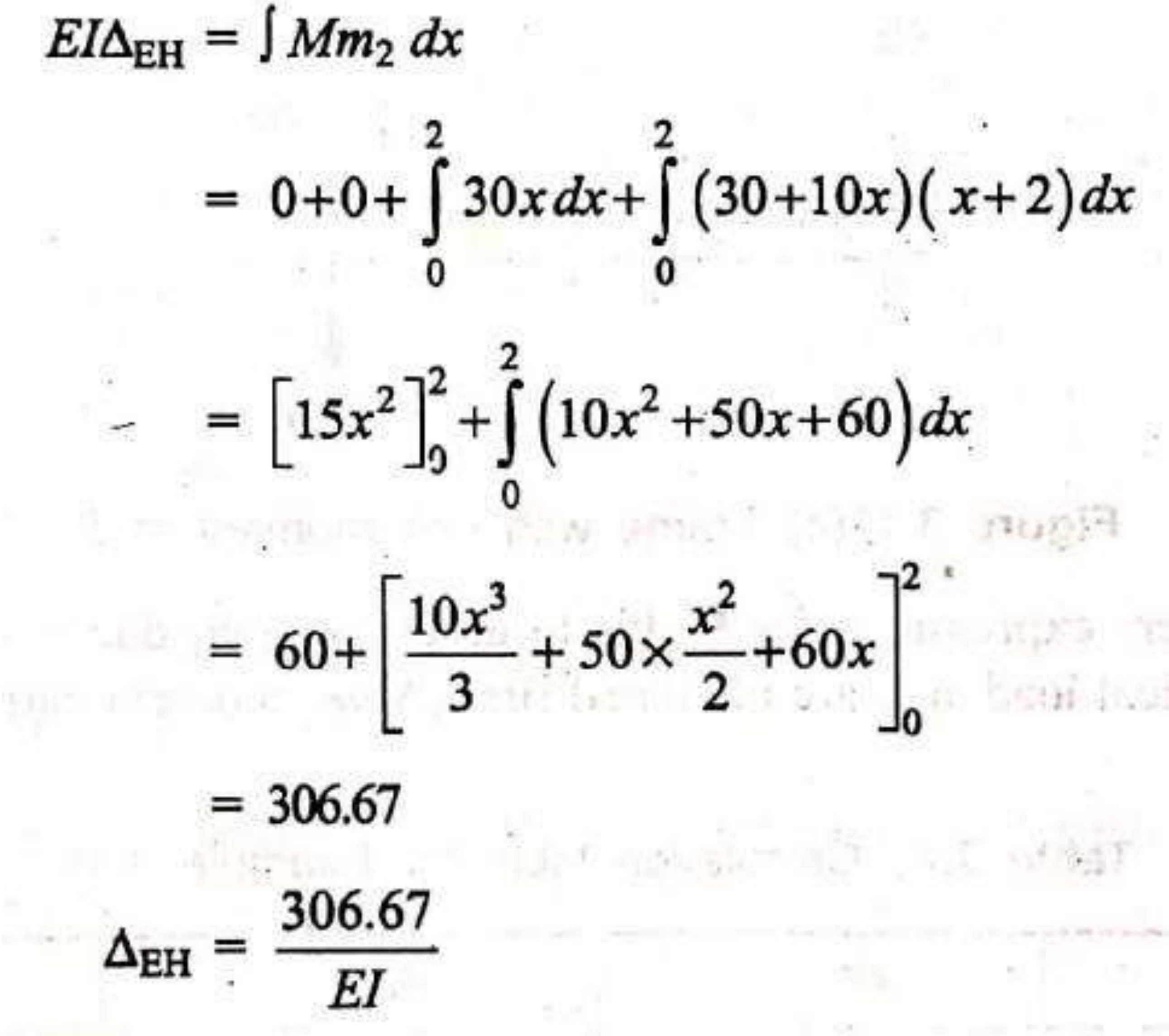
$$= \left[\frac{30x^2}{2} + \frac{20x^3}{3}\right]_0^{1.5} + [90x]_0^2 + \left[90x + \frac{30x^2}{2}\right]_0^2$$

$$= 56.25 + 180 + 240$$

$$= 476.25$$

$$\Delta_{EV} = \frac{476.25}{EI}$$

Horizontal Deflection at ‘*E*’ = Δ_{EH}



$$EI\Delta_{EH} = \int Mm_2\,dx$$

$$= 0+0+\int_0^2 30x\,dx+\int_0^2 (30+10x)(x+2)\,dx$$

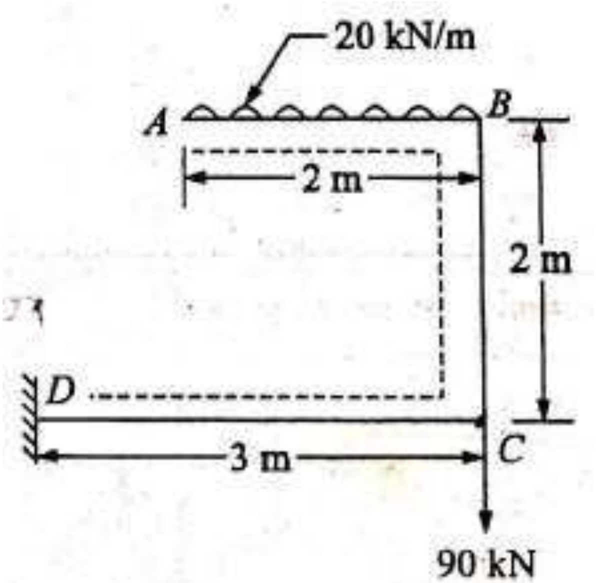
$$= \left[15x^2\right]_0^2+\int_0^2 \left(10x^2+50x+60\right)dx$$

$$= 60+\left[\frac{10x^3}{3}+50\times\frac{x^2}{2}+60x\right]_0^2$$

$$= 306.67$$

$$\Delta_{EH} = \frac{306.67}{EI}$$

Q4. Determine the vertical deflections at A and C in the frame shown in Figure by unit load method. Take E = 200 GPa, I = 150×10^4 mm^4.



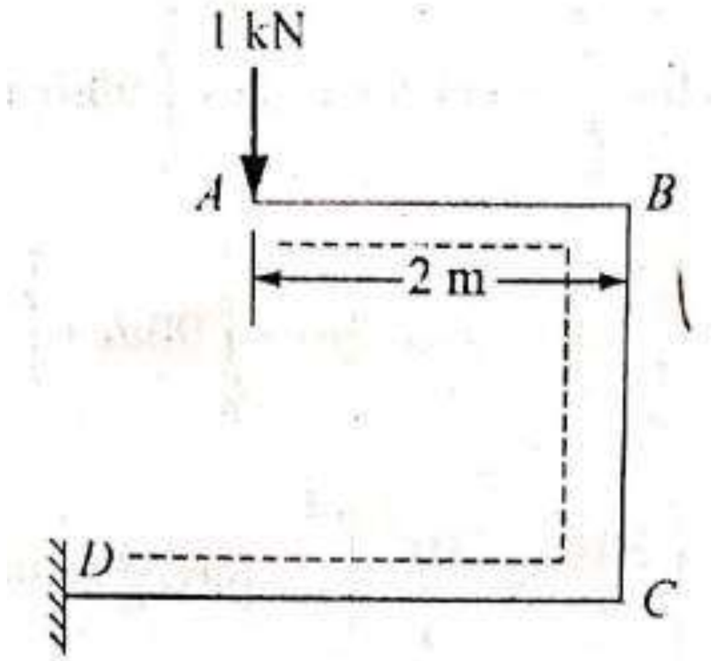


Figure 1: Frame with unit vertical lod at ‘*A*’

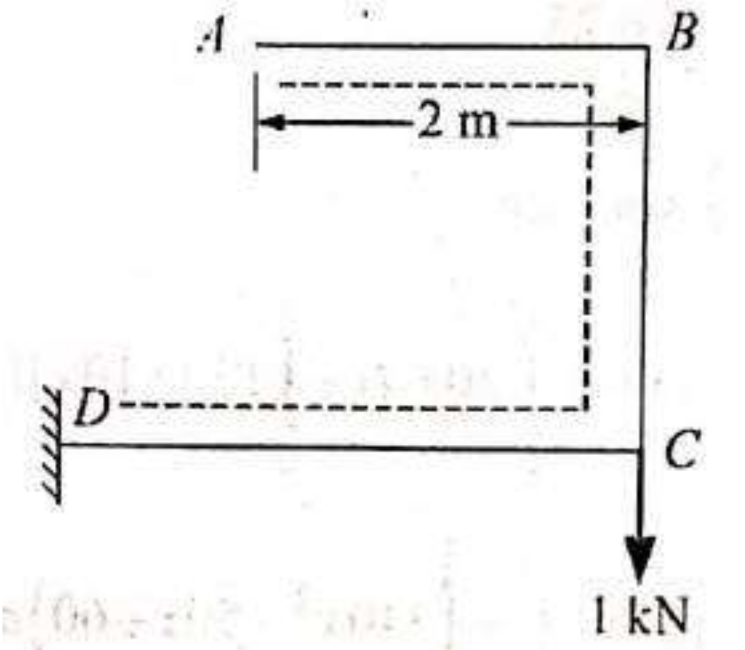
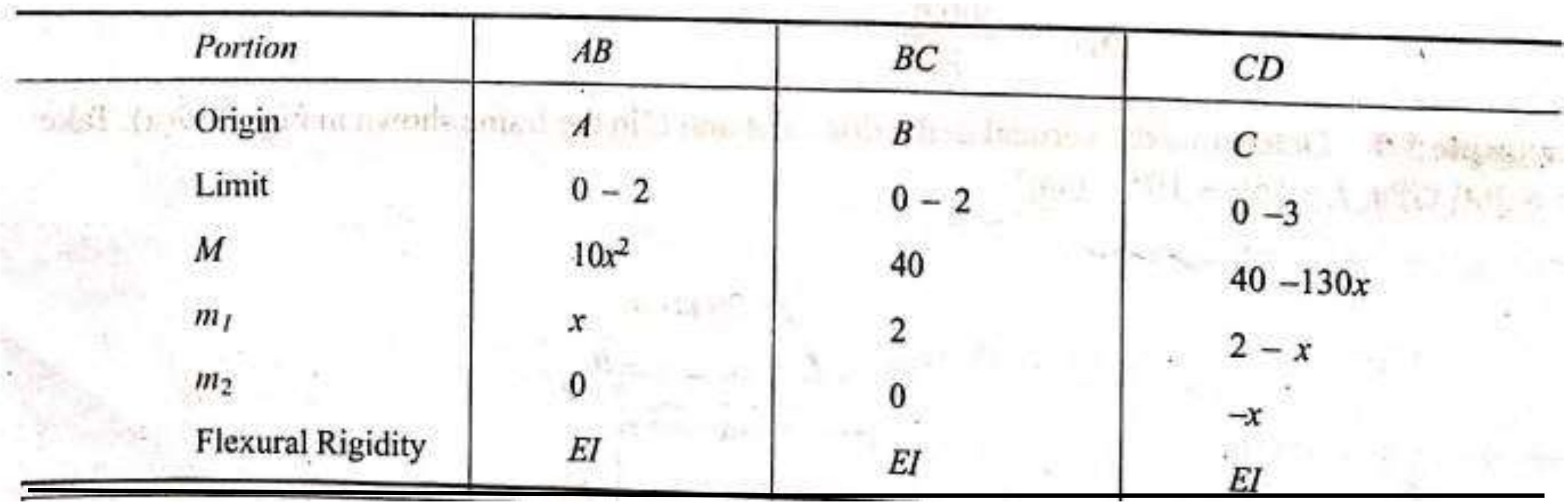


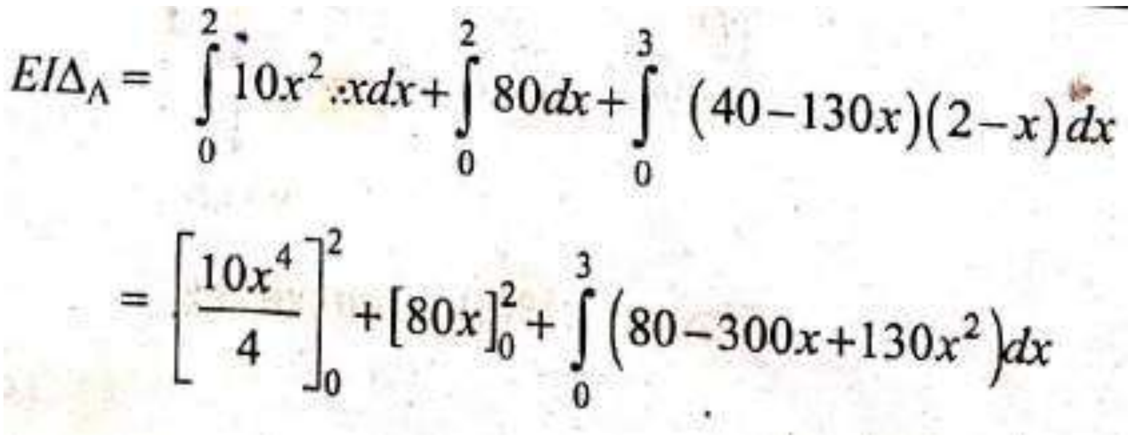
Figure 2: Frame with unit vertical load at ‘*C*’

- The bending moment expressions for 'M' due to given load, m_1 due to unit vertical load at A and m_2 due to unit vertical load at C are Tabulated below.



Portion	AB	BC	CD
Origin	A	B	C
Limit	0 – 2	0 – 2	0 –3
M	$10x^2$	40	$40 - 130x$
m_1	x	2	$2 - x$
m_2	0	0	$-x$
Flexural Rigidity	EI	EI	EI

Vertical deflection at A = Δ_A



$$EI\Delta_A = \int_0^2 10x^2 .x dx + \int_0^2 80 dx + \int_0^3 (40-130x)(2-x) dx$$

$$= \left[\frac{10x^4}{4}\right]_0^2 + [80x]_0^2 + \int_0^3 \left(80-300x+130x^2\right) dx$$

$$= \frac{10(2^4)}{4} + 80(2) + \left[80x - 300\frac{x^2}{2} + \frac{130x^3}{3}\right]_0^3$$

$$= 260$$

$$E = 240 \text{ GPa} = 240 \times 10^9 \text{ N/m}^2$$

$$I = 150 \times 10^4 \text{ mm}^4 = 150 \times 10^4 \times 10^{-12} \text{ m}^4$$

$$= 150 \times 10^{-8} \text{ m}^4$$

$$\therefore \Delta = \frac{260}{240 \times 10^9 \times 150 \times 10^{-8}} = 7.222 \times 10^{-4} \text{ m}$$

$$= 0.722 \text{ mm}$$

Vertical Deflection at C = Δ_c

$$EI\Delta_c = \int Mm_2\,dx$$

$$= 0+0+\int_0^3 (40-130x)(-x)\,dx$$

$$= \int_0^3 \left(-40x+130x^2\right)dx$$

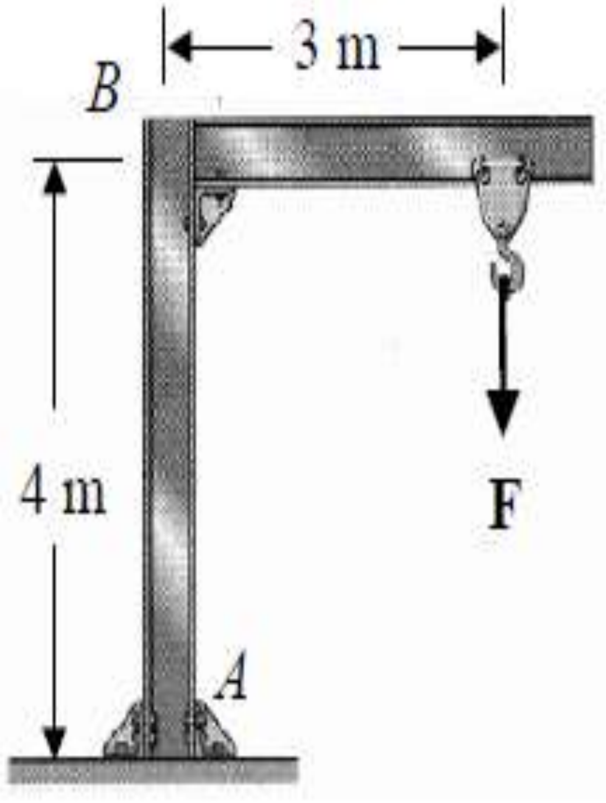
$$= \left[-20x^2+130\frac{x^3}{3}\right]_0^3$$

$$= 990$$

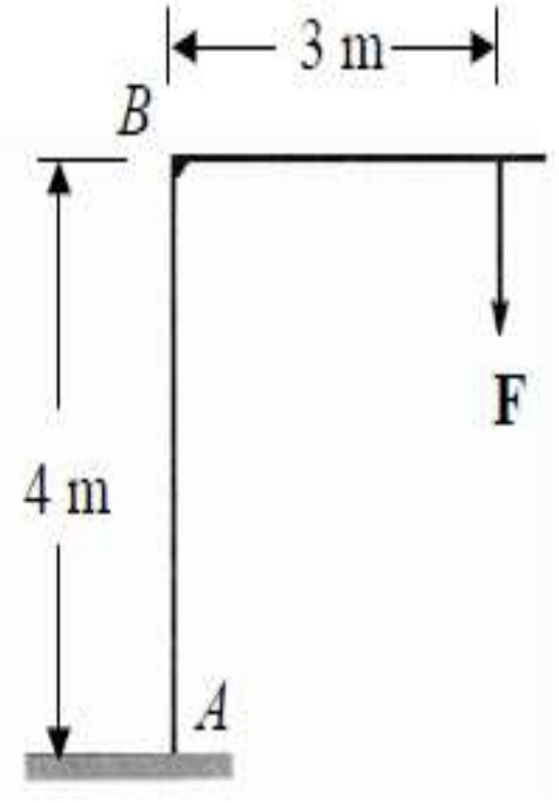
$$\Delta_C = \frac{990}{240\times10^9\times150\times10^{-8}} = 2.75\times10^{-3}\ \text{m}$$

$$= 2.75\ \text{mm}$$

DETREMINATE & INDETREMINATE STRUCTURES



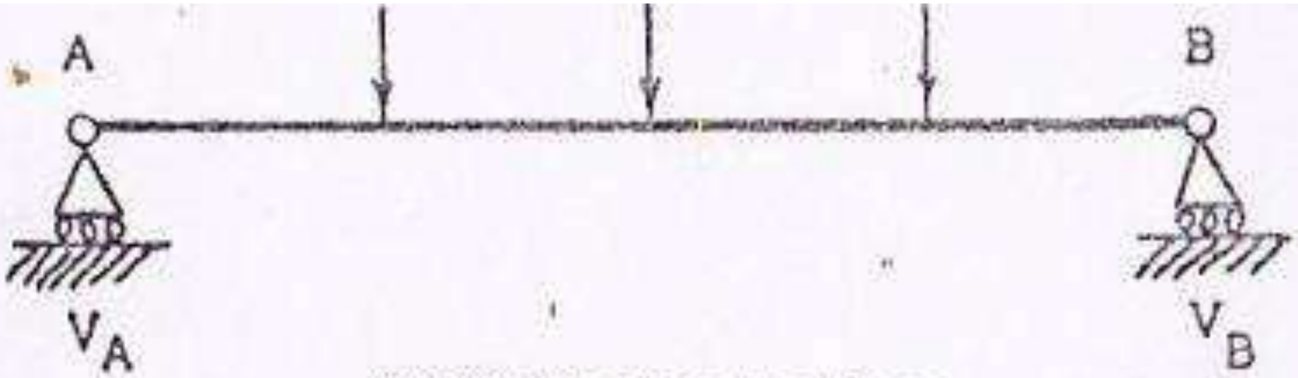
actual structure



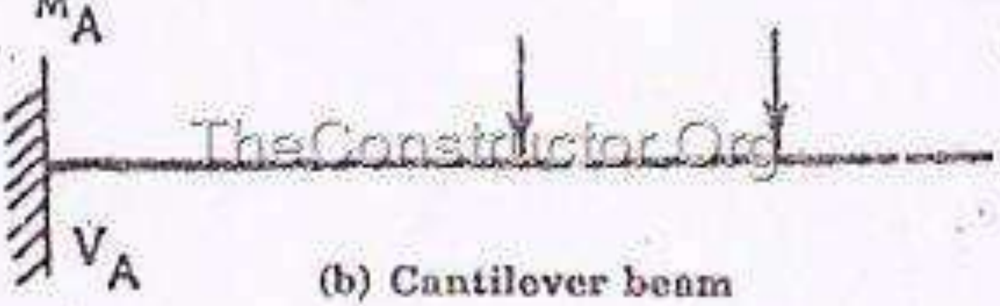
idealized structure

Determinate structures

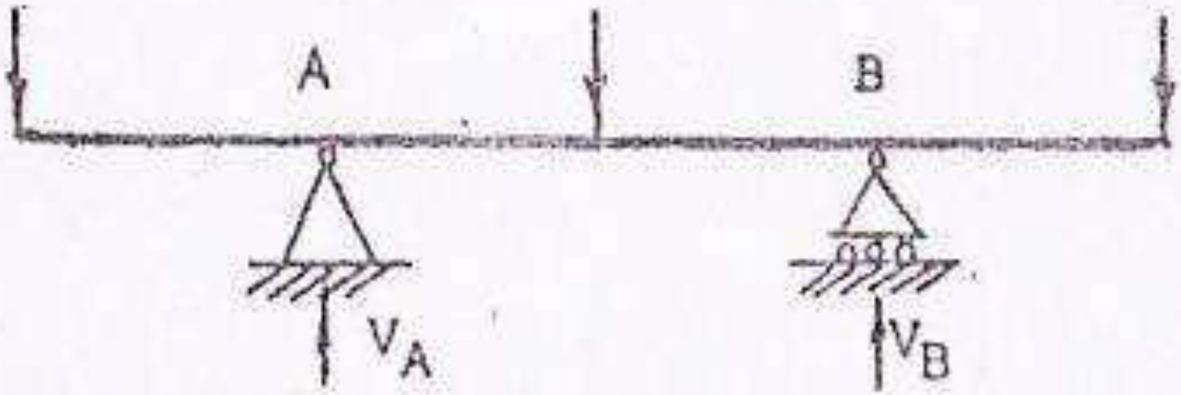
- **Determinate structures** are analysed just by the use of basic equilibrium equations. By this analysis, the unknown reactions are found for the further determination
- Example of determinate structures are: **simply supported beams, cantilever beams, single and double overhanging beams, three hinged arches** etc



(a) Simply supported beam



(b) Cantilever beam



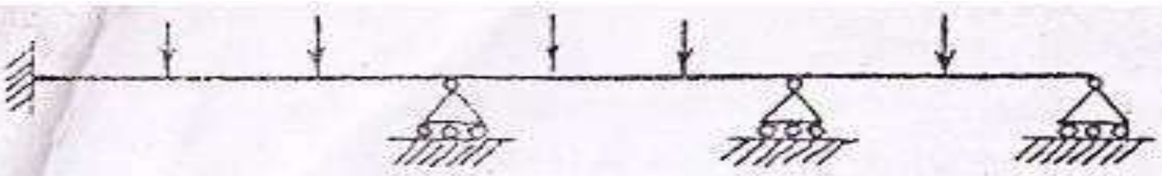
Indeterminate Structures:

- **Redundant or indeterminate structures** are not capable of being analysed by mere use of basic equilibrium equations.
- Along with the basic equilibrium equations, some extra conditions are required to be used like **compatibility conditions** of deformations etc to get the unknown reactions for drawing **bending moment and shear force diagrams**.
- Examples of indeterminate structures are: **fixed beams, continuous beams, fixed arches, two hinged arches, portals, multistoried frames**, etc.

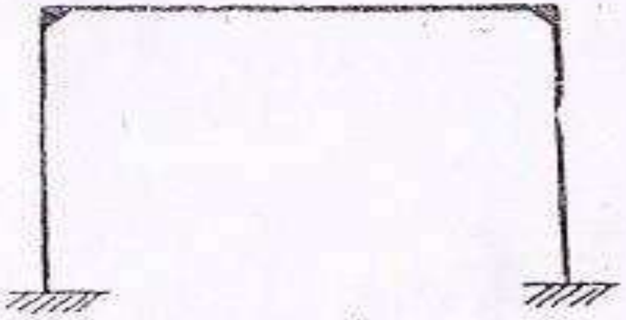
- a structure is termed as statically indeterminate, if it can not be analysed from principles of statics alone, i.e. $\sum H = 0, \sum V = 0, \sum M = 0$
- A statically indeterminate structure may be classified as:

1. Externally indeterminate, (example: continuous beams and frames shown in figure-1(a) and (b)).

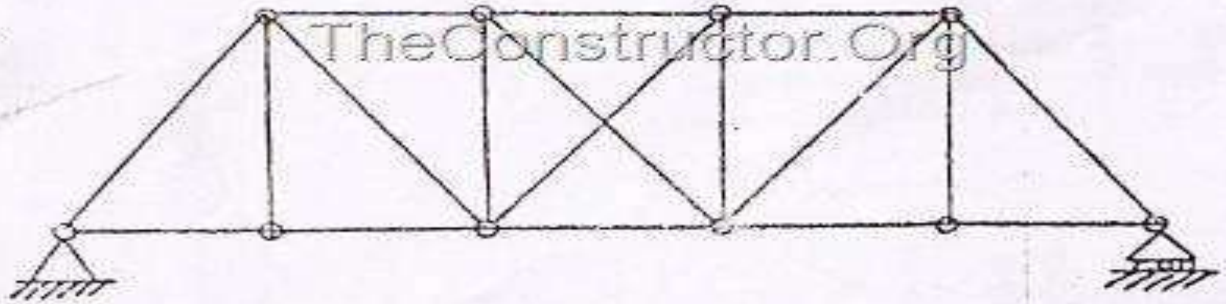
2. Internally indeterminate, (example: trusses shown in figure-1(c) and (d)).



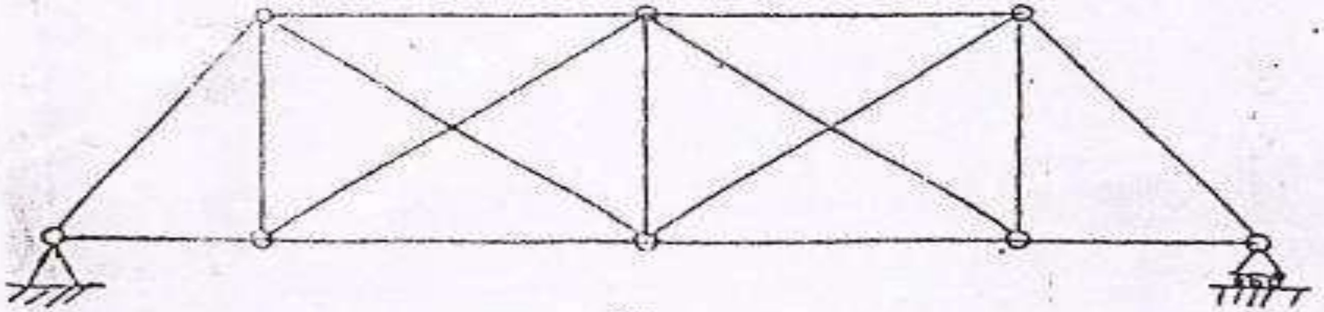
(a)



(b)



(c)



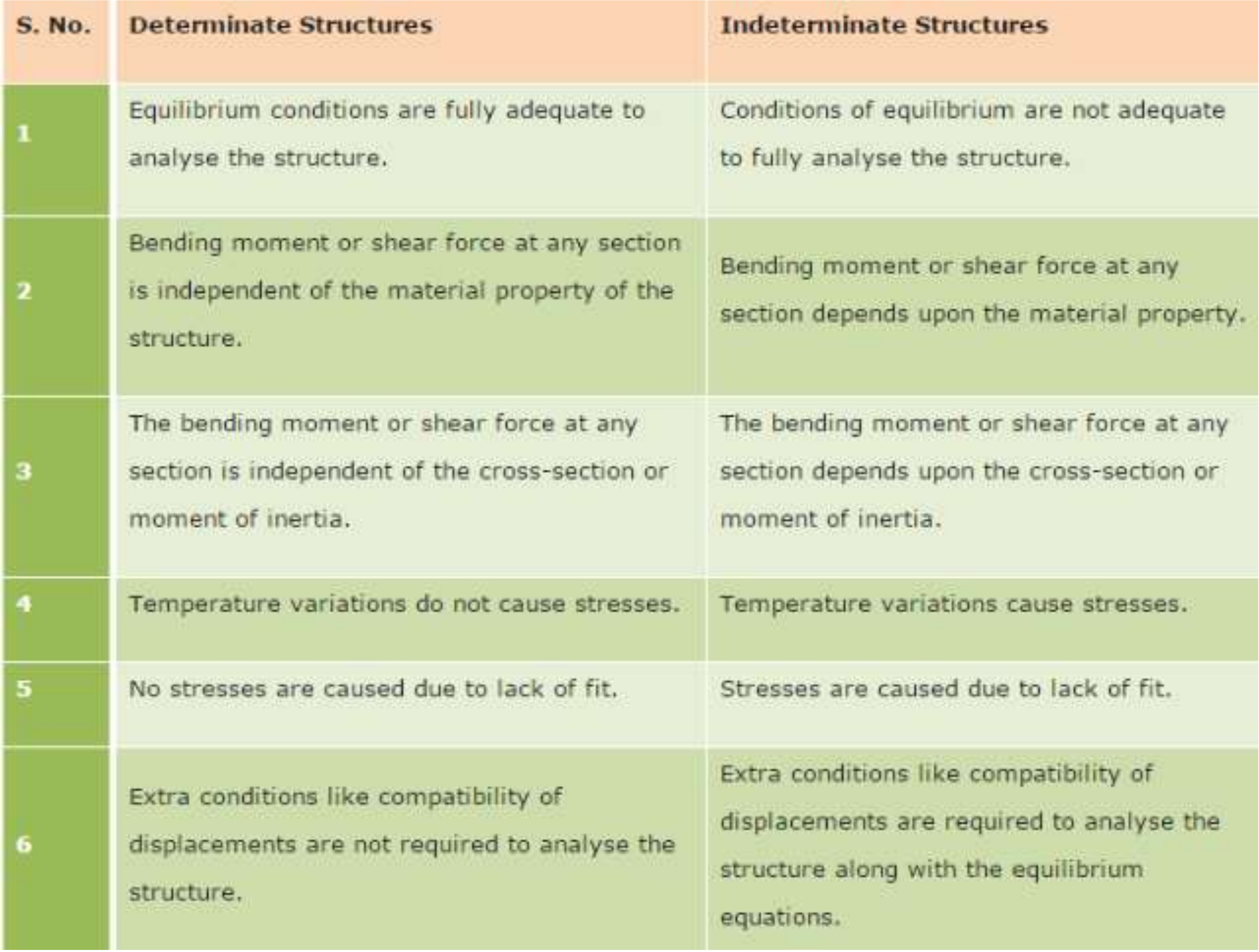
(d)

EXTERNALLY INDETERMINATE STRUCTURES:

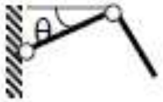
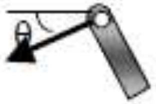
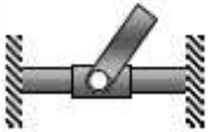
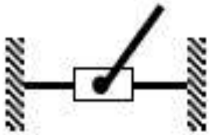
- A structure is usually externally indeterminate or redundant if the reactions at the supports can not be determined by using three equations of equilibrium, i.e.$\sum H = 0, \sum V = 0, \sum M = 0$
- If however a beam rests on more than two supports or in addition any of the end support is fixed, there are more than two reactions to be determined.
- These reactions can not be determined by conditions of equilibrium alone.
- The degree of indeterminacy or redundancy is given by the number of extra or redundant reactions to be determined.

Internally indeterminate

- If structure is externally determinate but it is not possible to determine all internal forces then structure is said to be internally indeterminate.

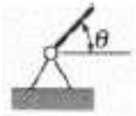


S. No.	Determinate Structures	Indeterminate Structures
1	Equilibrium conditions are fully adequate to analyse the structure.	Conditions of equilibrium are not adequate to fully analyse the structure.
2	Bending moment or shear force at any section is independent of the material property of the structure.	Bending moment or shear force at any section depends upon the material property.
3	The bending moment or shear force at any section is independent of the cross-section or moment of inertia.	The bending moment or shear force at any section depends upon the cross-section or moment of inertia.
4	Temperature variations do not cause stresses.	Temperature variations cause stresses.
5	No stresses are caused due to lack of fit.	Stresses are caused due to lack of fit.
6	Extra conditions like compatibility of displacements are not required to analyse the structure.	Extra conditions like compatibility of displacements are required to analyse the structure along with the equilibrium equations.

Type of Connection	Idealized Symbol	Reaction	Number of Unknowns
(1)  Light cable			One unknown. The reaction is a force that acts in the direction of the cable or link.
(2)  rollers rockers			One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact
(3) 			One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact
(4) 			One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact

(5)

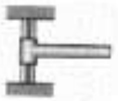
Smooth pin or hinge



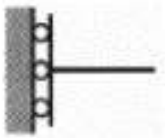
Two unknowns. The reactions are two force components.

(6)

slider



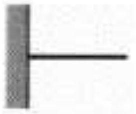
fixed-connected collar

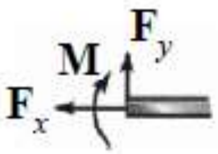


Two unknowns. The reactions are a force and moment.

(7)

fixed support





Three unknowns. The reactions are the moment and the two force components.

DOI = 6 − 3

= 3

Ex

DOI = 4 − 3

= 1

DOI = 7 − 3 = 4

Ex

DOI = 6 − 3 = 3

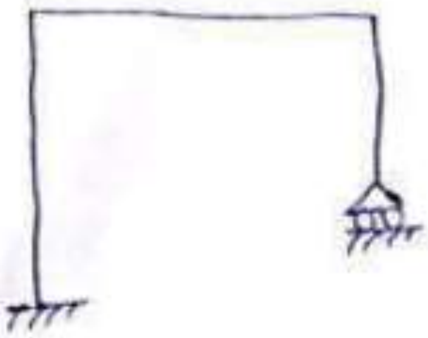
Note :— If there is external hinges are provided, there will be an additional equilibrium equation (moment at hinge = 0) for eac hinge.

Ex:-

DOI = 7 − 3 − 2 = 2

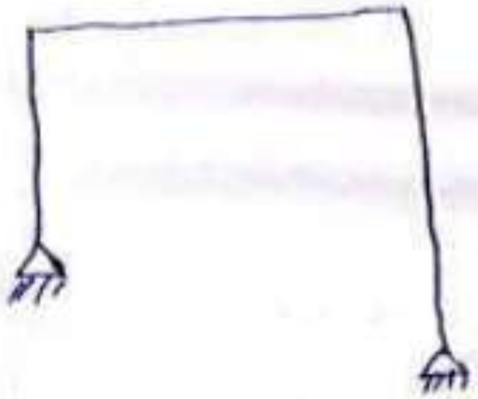
DOI = 6 − 3 − 1 = 2

Ex



DOI = 4-3 = 1

Ex



DOI = 4-3 = 1

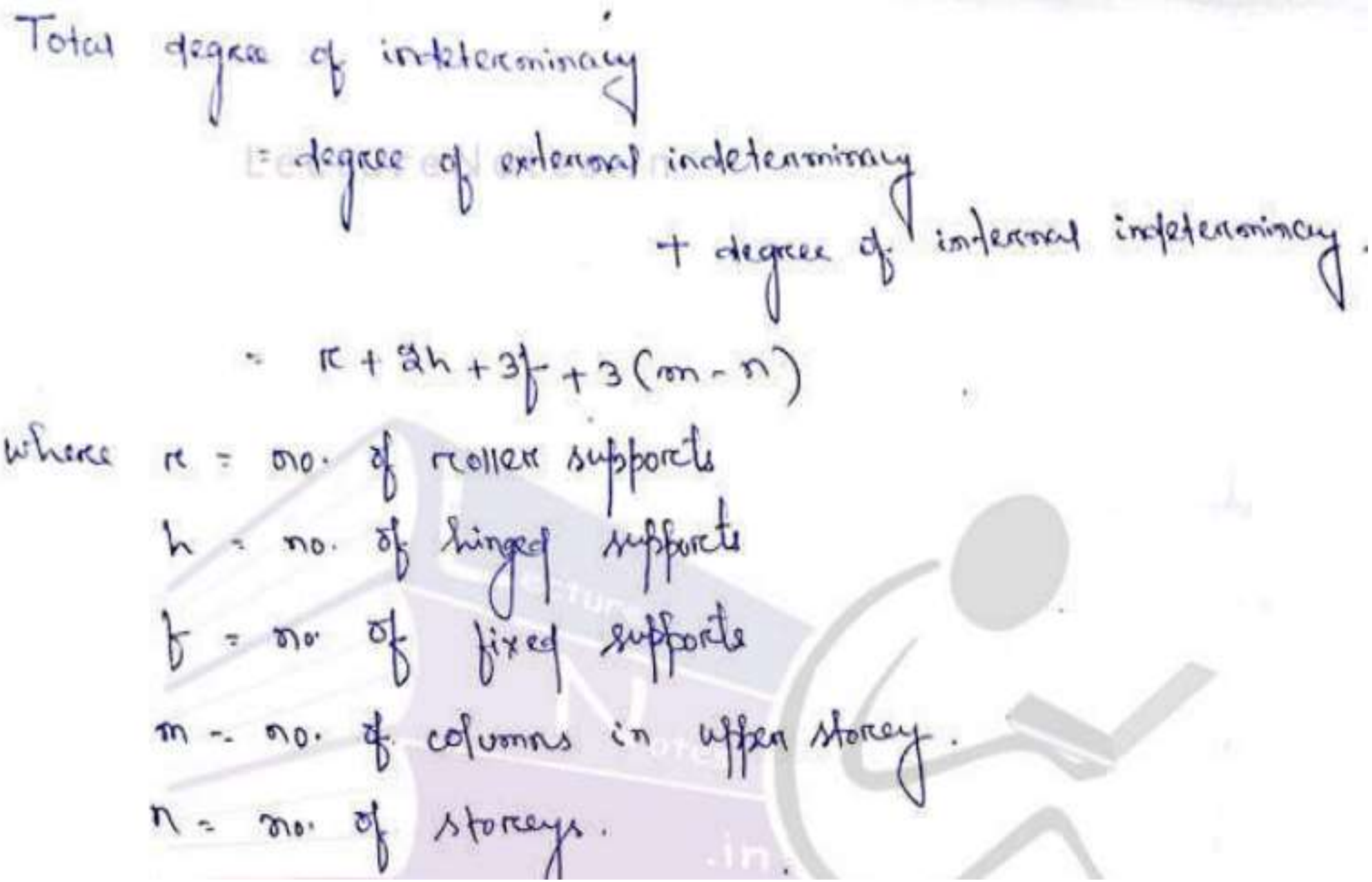
x

DOI = 5-3 = 2

Ex

DOI = 6-3
= 3

Method 1



Total degree of indeterminacy

= degree of external indeterminacy + degree of internal indeterminacy.

$= r + 2h + 3f + 3(m - n)$

where r = no. of roller supports

h = no. of hinged supports

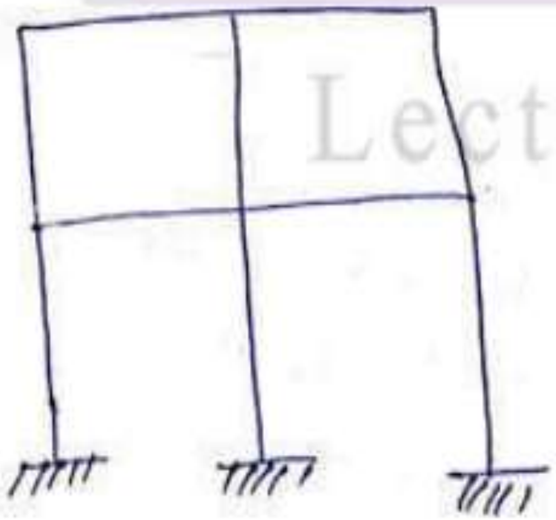
f = no. of fixed supports

m = no. of columns in upper storey.

n = no. of storeys.

n = no. of storeys.

Ex :-



r = 0

h = 0

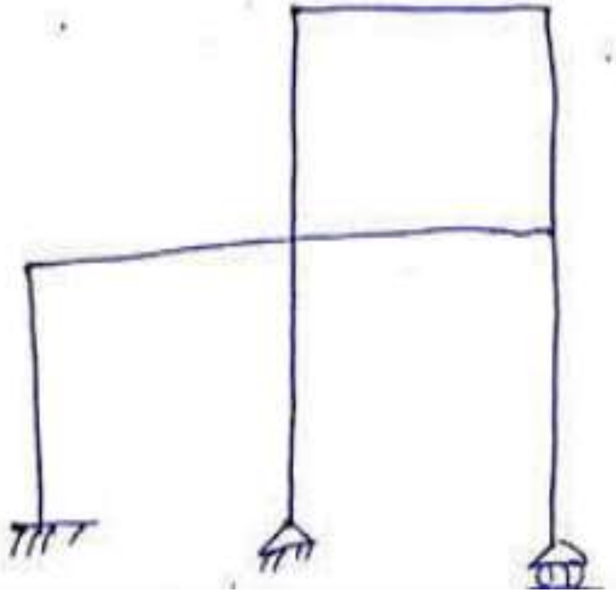
f = 3

m = 3

n = 2

$DOI = 0 + 0\times2 + 3\times3$
$+ 3(3-2)$
$= 9+3$
$= 12$

Ex :-



r = 1

h = 1

f = 1

m = 2

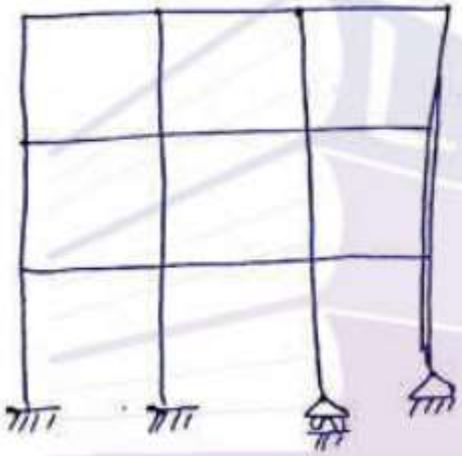
n = 2

$DOI = 1 + 2\times1 + 3\times1$
$+ 3(2-2)$
$= 6$

Second Method

Total degree of indeterminacy $= rc + 2h + 3f - 3 + 3$ (no. of bays in upper storey)

Ex:-



$r = 1$

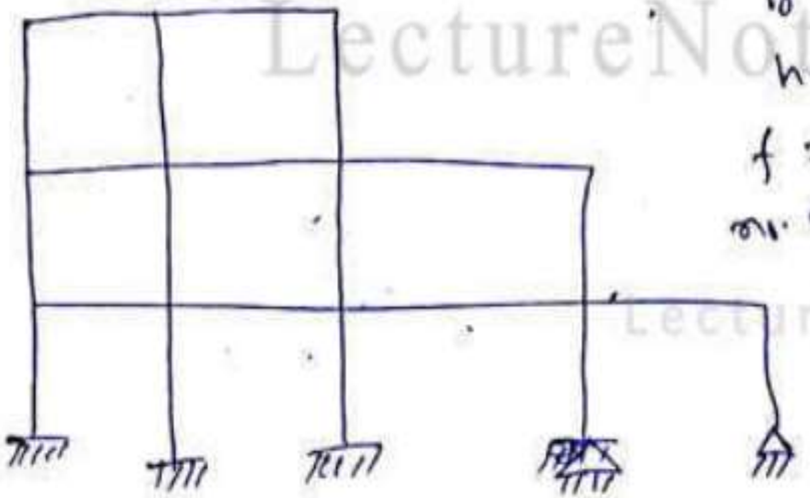
$h = 1$

$f = 2$

no. of bays $= 6$

$$DOI = 1 + 2\times1 + 3\times2 - 3 + 3(6)$$
$$= 1 + 2 + 6 - 3 + 18$$
$$= 24$$

Ex:-



$r = 0$

$h = 2$

$f = 3$

no. of bays $= 5$

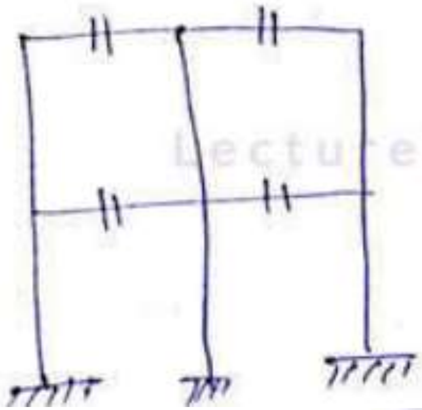
$$DOI = 0 + 2\times2 + 3\times3 - 3 + 3\times5$$
$$= 4 + 9 - 3 + 15$$
$$= 25$$

Third Method :-

Total degree of indeterminacy

$= 3 \times$ no. of cut section − no. of releases at the supports.

Ex :-

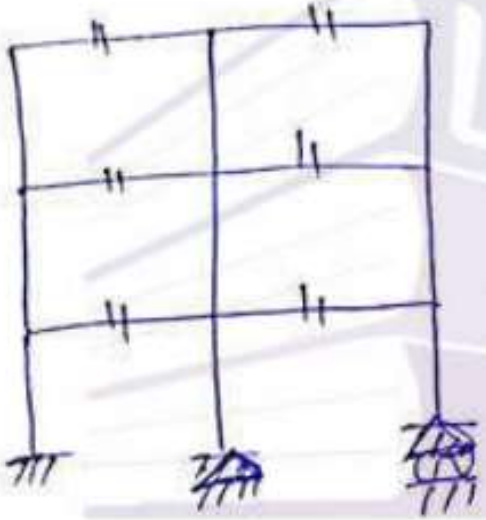


$DOI = 3 \times 4 - 0$

$= 12$

no. of cuts = 4

Ex



$DOI = 3 \times 6 - 2 - 1$

$= 15$

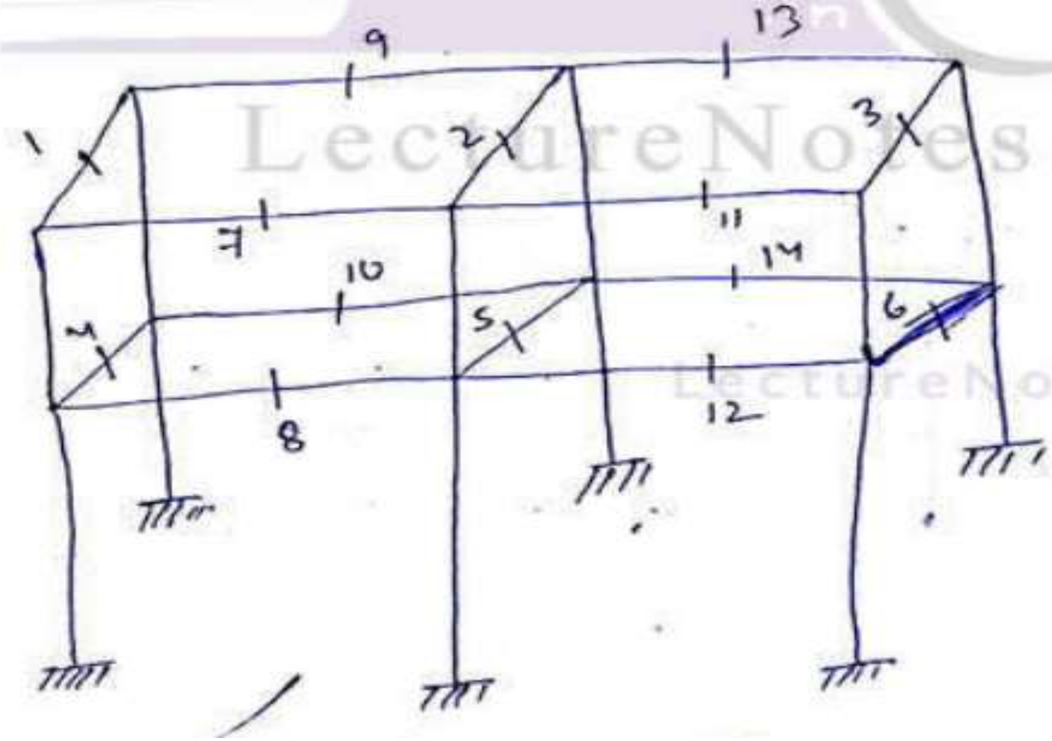
(b) Space frame :—

Total ~~no.~~ degree of indeterminacy = 6 x no. of cut sections — no. of releases at support.

~~no. of reaction components + no. of unknown at cuts~~

~~— no. of independent eqn.~~

Ex:-



no. of cut section = 14

DOI = 6×14 − 0
= 84

* Determination of degree of indeterminacy for pin jointed frame.

$$DOI = (m + r) - 2j \quad (\text{for plane frame})$$

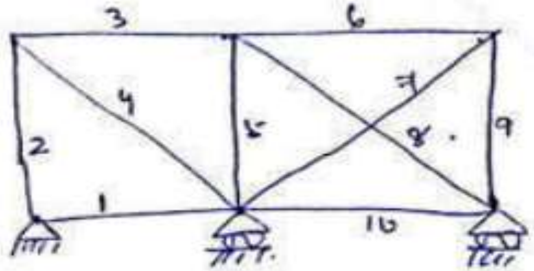
$$= (m + r) - 3j \quad (\text{for space frame})$$

$m \rightarrow$ no. of member forces.

$r \rightarrow$ reaction components

$j \rightarrow$ no. of joints of pin jointed frame.

Ex.-



$m = 10$

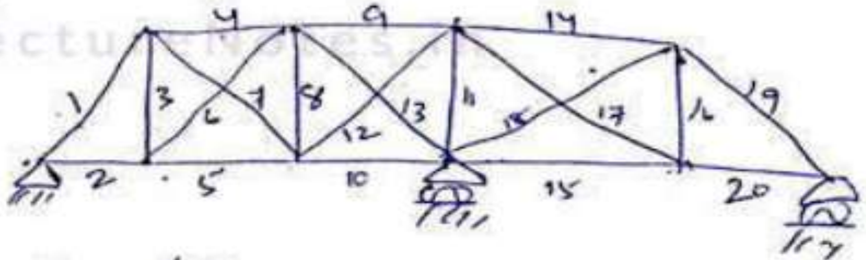
$r = 4$

$j = 6$

$DOI = (10 + 4) - 2 \times 6$

$= 2$

ex



$m = 20$

$r = 4$

$j = 10$

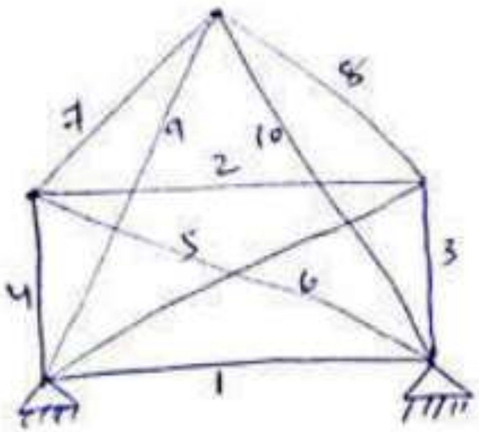
$DOI = (m + r) - 2j$

$= 20 + 4 - 2 \times 10$

$= 24 - 20$

$= 4$

e.g



$m = 10$

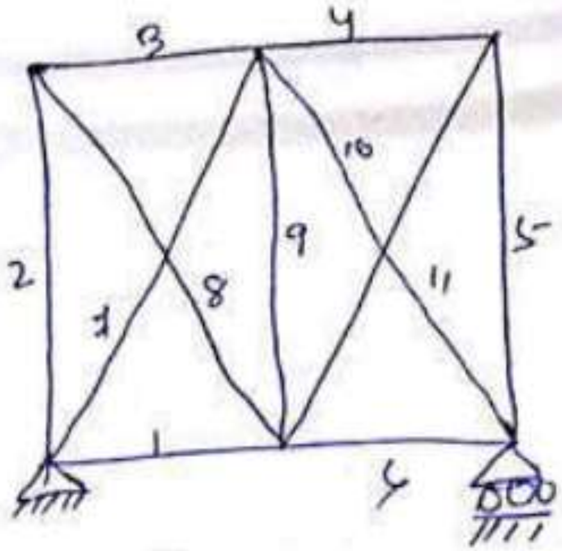
$r = 4$

$j = 5$

$DOI = 10 + 4 - 10$

$= 4$

e.g



$m = 11$

$r = 3$

$j = 6$

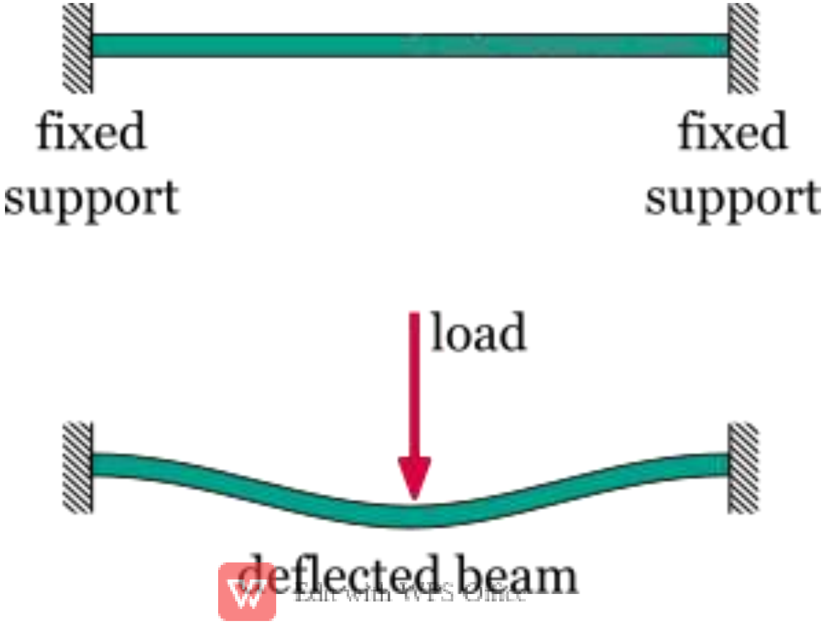
$DOI = 11 + 3 - 2 \times 6$

$= 2$

FIXED BEAMS

FIXED BEAMS

- A beam whose both ends are fixed is known as a fixed beam. Fixed beam is also called as built-in or encaster beam.
- Incase of fixed beam both its ends are rigidly fixed and the slope and deflection at the fixed ends are zero



Advantages of fixed beams

(i)For the same loading, the maximum deflection of a fixed beam is less than that of a simply supported beam.

(ii)For the same loading, the fixed beam is subjected to lesser maximum bending moment.

(iii)The slope at both ends of a fixed beam is zero.

(iv)The beam is more stable and stronger.

Disadvantages of a fixed beam

(i) Large stresses are set up by temperature changes.

(ii) Special care has to be taken in aligning supports accurately at the same level.

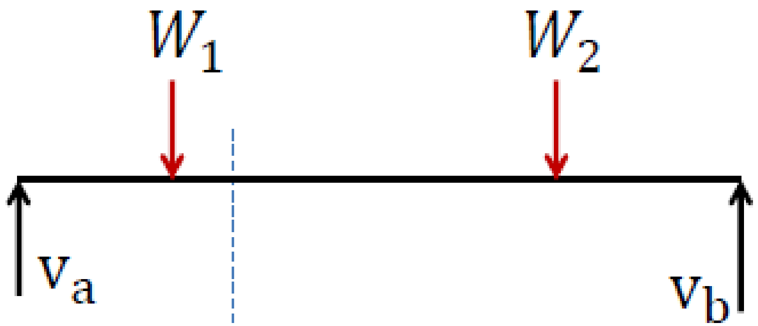
(iii) Large stresses are set if a little sinking of one support takes place.

(iv) Frequent fluctuations in loading render the degree of fixity at the ends very uncertain

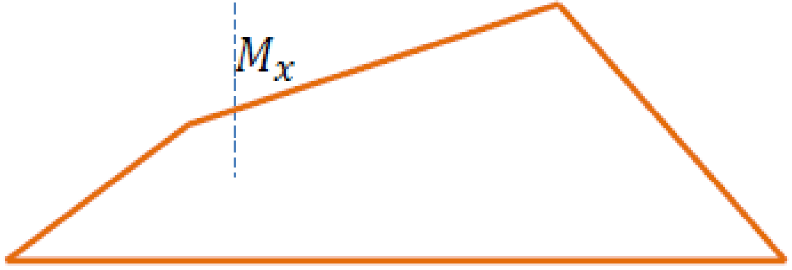
The beam may be analyzed in the following stages.

(i) Let us first consider the beam as Simply supported.

Let v_a and v_b be the vertical reactions at the supports A and B. Figure (ib) shows the bending moment diagram for this condition. At any section the bending moment M_x is a sagging moment.



(ia) Freely supported condition



(ib) Free B.M.D.

- (ii) Now let us consider the effect of end couples M_A and M_B alone.

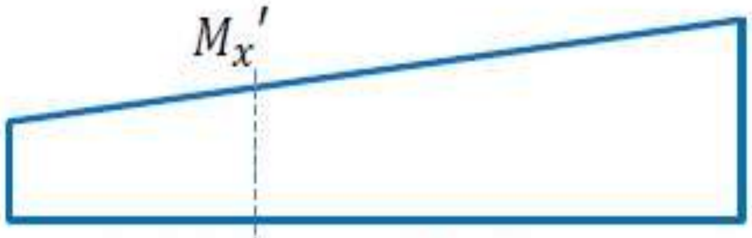
Let v be the reaction at each end due to this condition.

Suppose $M_B > M_A$.

Then $V = \frac{M_B - M_A}{L}$.

If $M_B > M_A$ the reaction V is upwards at B and downwards at A.

(iia) Effect of end couples

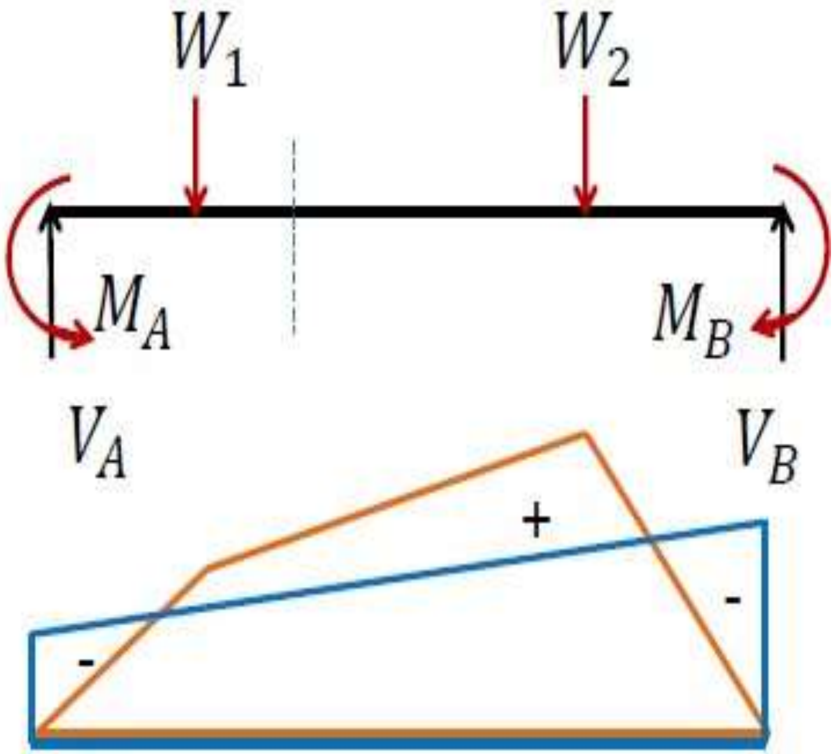


(iib) Fixed B.M.D.

Fig (iib). Shows the bending moment diagram for this condition.

At any section the bending moment $M_{x'}$ is hogging moment.

- Now the final bending moment diagram can be drawn by combining the above two B.M. diagrams as shown in Fig. (iiib)



(iiib) Resultant B.M.D.

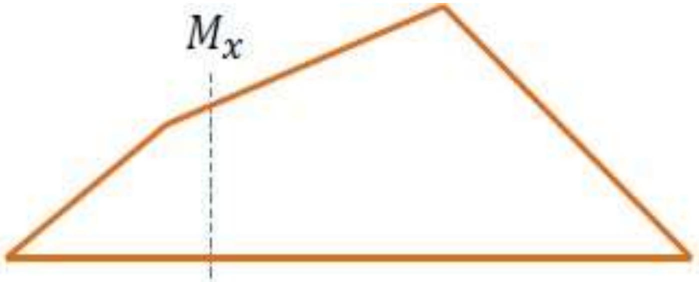
Now the final reaction $V_A = v_a - v$

and $V_B = v_b + v$

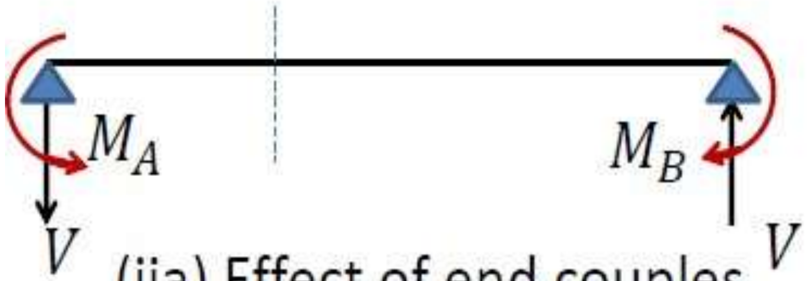
The actual bending moment at any section X, distance x from the end A is given by,

$$EI\frac{d^2y}{dx^2} = M_x - M_x'$$

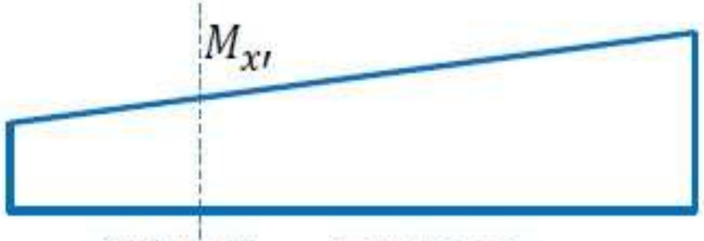
(ia) Freely supported condition



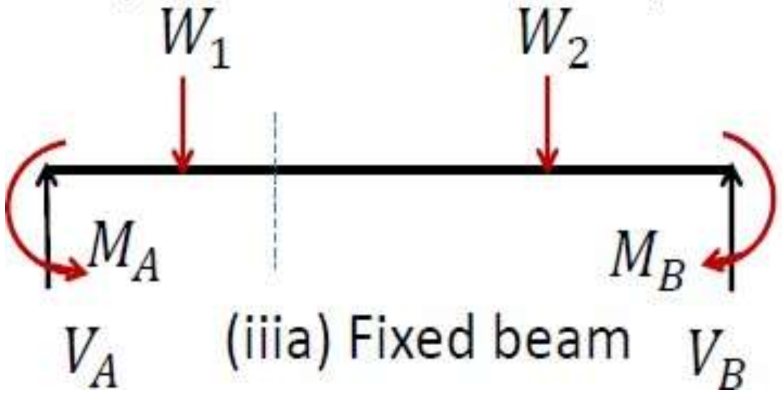
(ib) Free B.M.D.



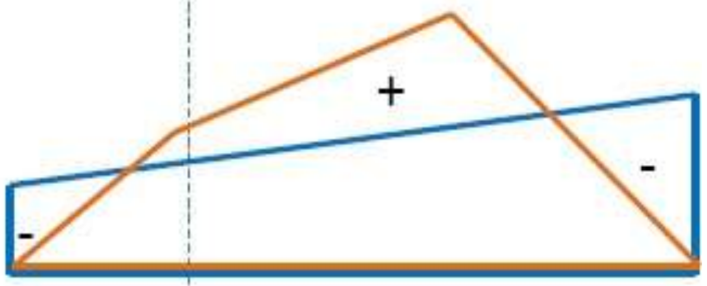
(iia) Effect of end couples



(iib) Fixed B.M.D.



(iiia) Fixed beam



(iiib) Resultant B.M.D.

$$EI\frac{d^2y}{dx^2} = M_x - M_x{}'$$

- Integrating, we get,
- $EI\left[\frac{dy}{dx}\right]_0^l = \int_0^l M_x dx - \int_0^l M_x' dx$
- But at x=0, $\frac{dy}{dx} = 0$

  and at $x = l, \frac{dy}{dx} = 0$

Further $\int_0^l M_x dx$ = area of the Free BMD = a

$$\int_0^l M_x' dx = \text{area of the fixed B.M.D} = a'$$

Substituting in the above equation, we get,

$0 = a - a'$

$$\therefore a = a'$$

$$a = a'$$

∴ Area of the free B.M.D. = Area of the fixed B.M.D.

Again consider the relation,

$$EI\,\frac{d^2y}{dx^2} = M_x - M_x{}'$$

$Multying\ by\ x\ we\ get,$

$$EIx\,\frac{d^2y}{dx^2} = M_x x - M'_x x$$

- Integrating we get,
- $\int_0^l EIx\,\frac{d^2y}{dx^2} = \int_0^l M_x x\,dx\ - \int_0^l M'_x x\,dx$
- $\therefore\ EI\left[x\frac{dy}{dx} - y\right]_0^l$ = a$\bar{x}$-a'$\bar{x}^{+}$
- Where $\bar{x}$= distance of the centroid of the free B.M.D. from A.

  and $\bar{x}'$= distance of the centroid of the fixed B.M.D. from A.

- Further at x=0, y=0 and $\frac{dy}{dx} = 0$
- and at x=l, y=0 and $\frac{dy}{dx} = 0.$
- Substituting in the above relation, we have

$$0 = a\bar{x}\text{-}a'\bar{x}^{+}$$

$$a\bar{x}\text{=}a'\bar{x}^{+}$$

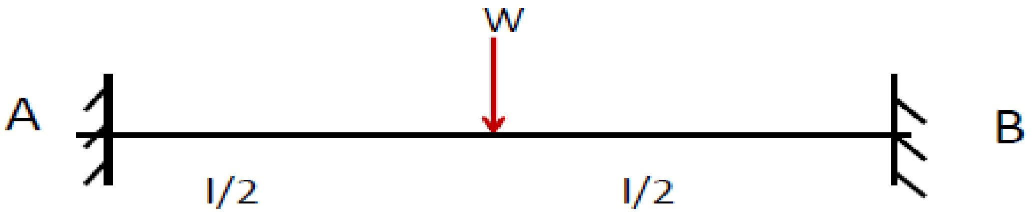
or $\bar{x} = \bar{x}$

∴ The distance of the centroid of the free B.M.D . From A= The distance of the centroid of the fixed B.M.D. from A.

$$\therefore a = a'$$

$$\bar{x} = \bar{x}$$

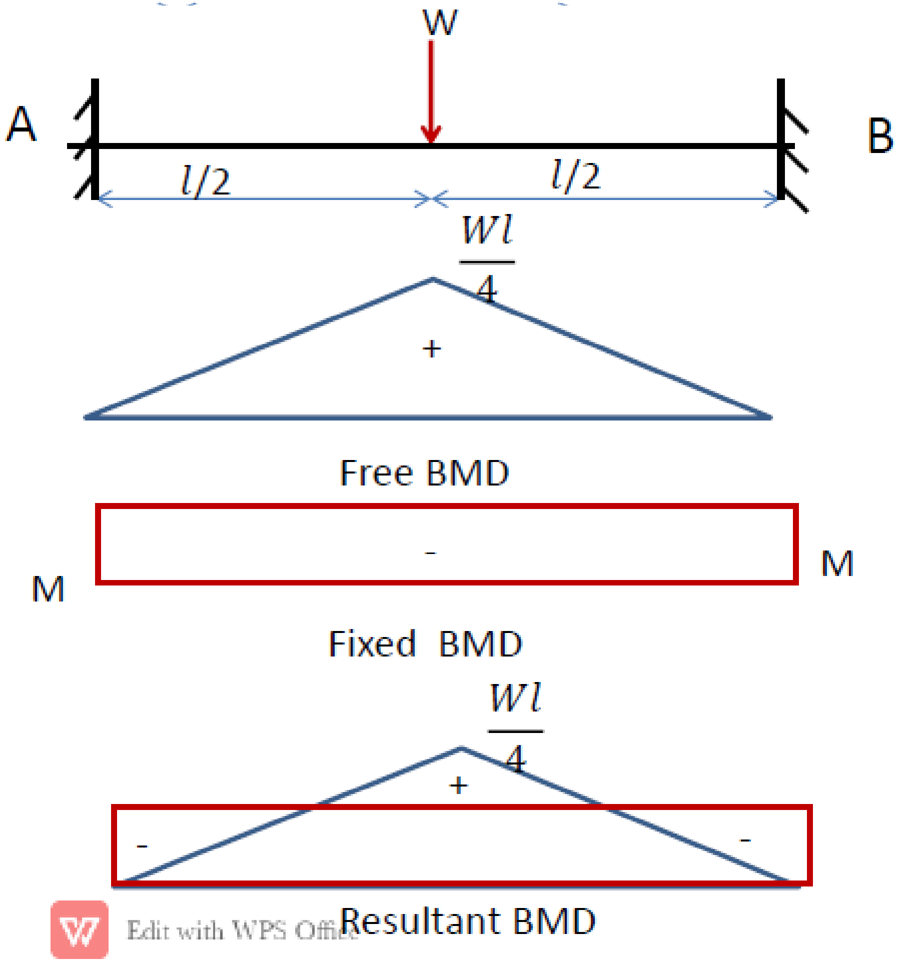
- Find the fixed end moments of a fixed beam subjected to a point load at the center.



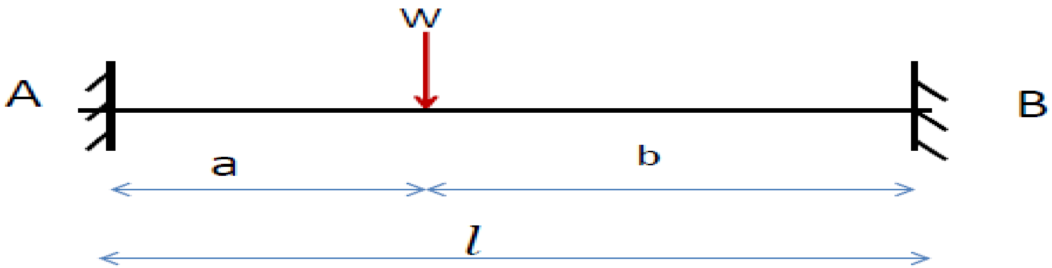
- $A' = A$

$$M \times l = \frac{1}{2} \times l \times \frac{Wl}{4}$$

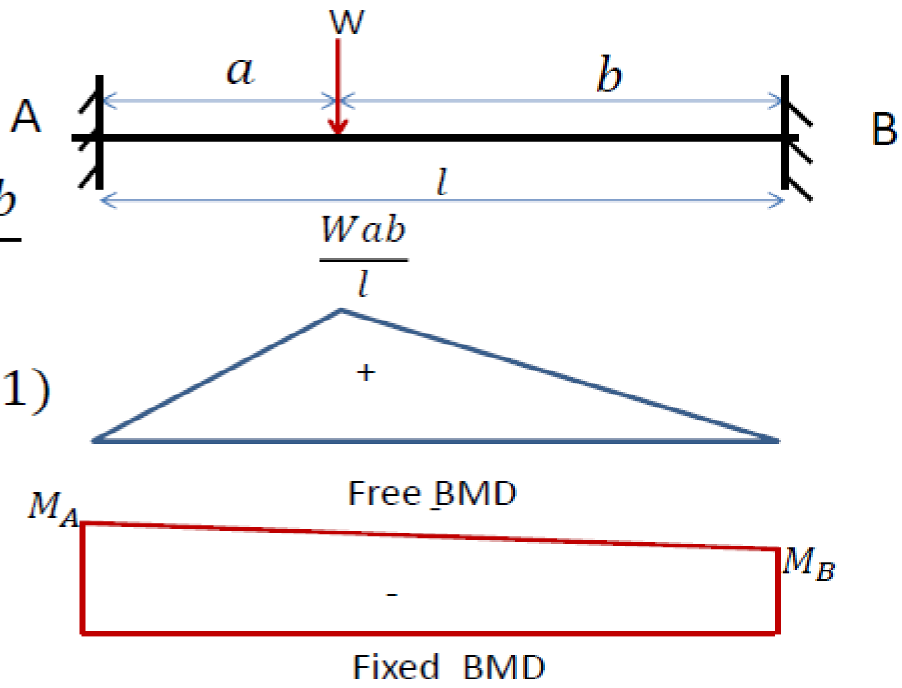
$$M = \frac{Wl}{8} = M_A = M_B$$



- Find the fixed end moments of a fixed beam subjected to a eccentric point load.



- $A' = A$



$$\frac{M_A + M_B}{2} \times l = \frac{1}{2} \times l \times \frac{Wab}{l}$$

$$M_A + M_B = \frac{Wab}{l} ----(1)$$

- $x' = x$

$$\frac{M_A + 2M_B}{M_A + M_B} \times \frac{l}{3} = \frac{l + a}{3}$$

$$M_B = M_A \times \frac{a}{l - a}$$

$$M_B = \frac{a}{l} ---(2)$$

$$M_A + M_B = \frac{Wab}{l} ----(1)$$

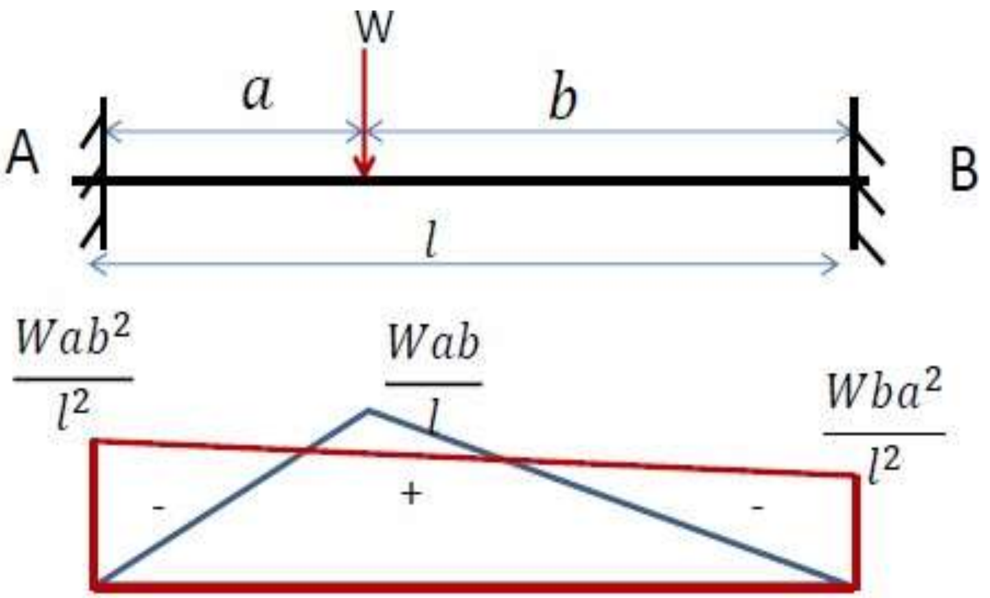
$$M_B = M_A \times \frac{a}{b} ---(2)$$

By substituting (2) in (1),

$$M_A = \frac{Wab^2}{l^2}$$

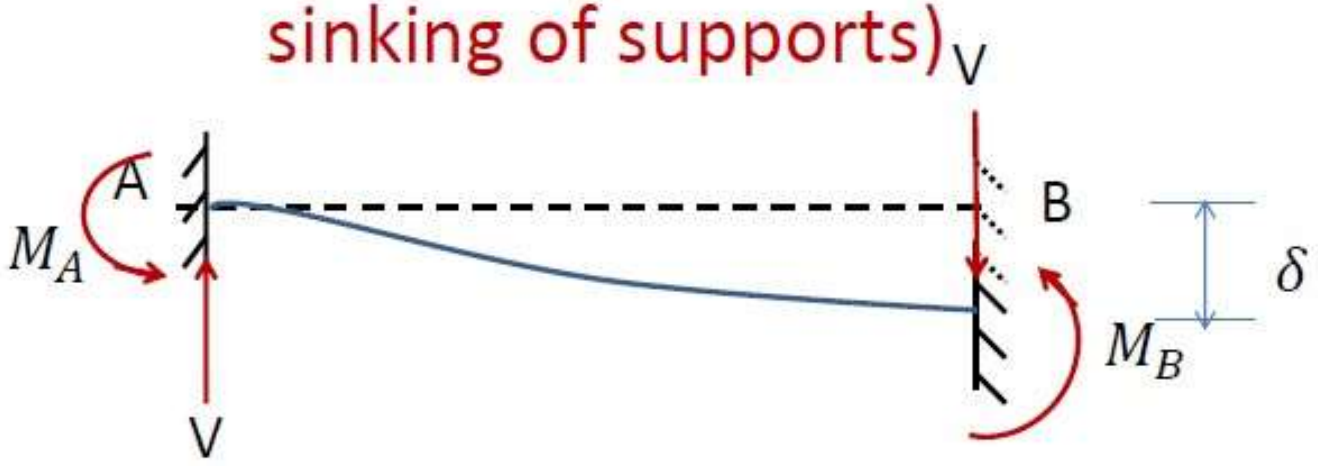
From (2),

$$M_B = \frac{Wba^2}{l^2}$$



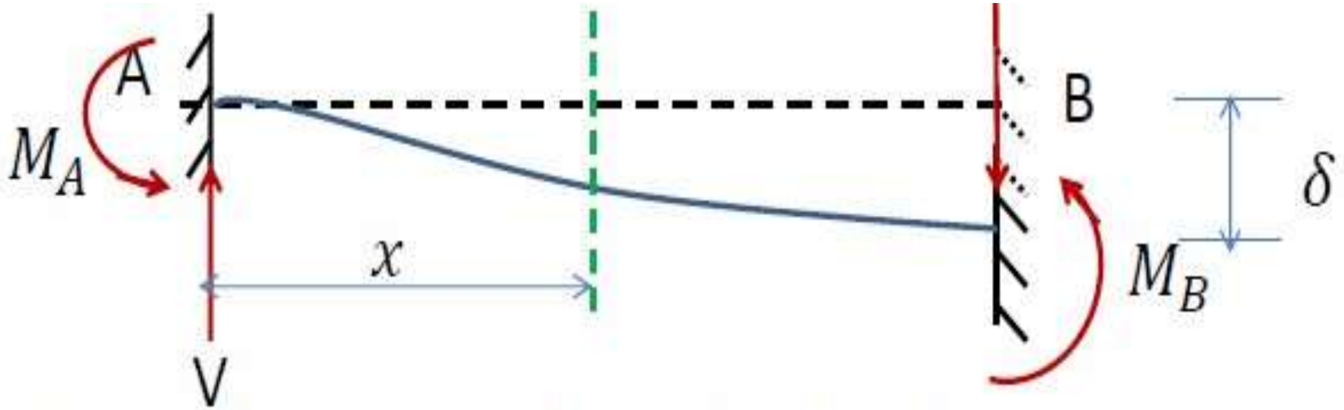
Resultant BMD

Fixed beam with ends at different levels (Effect of sinking of supports)



M_A is negative (hogging) and M_B is positive (sagging). Numerically M_A and M_B are equal.

Let V be the reaction at each support.



Consider any section distance x from the end A.

Since the rate of loading is zero, we have, with the usual notations

$$EI\frac{d^4y}{dx^4} = 0$$

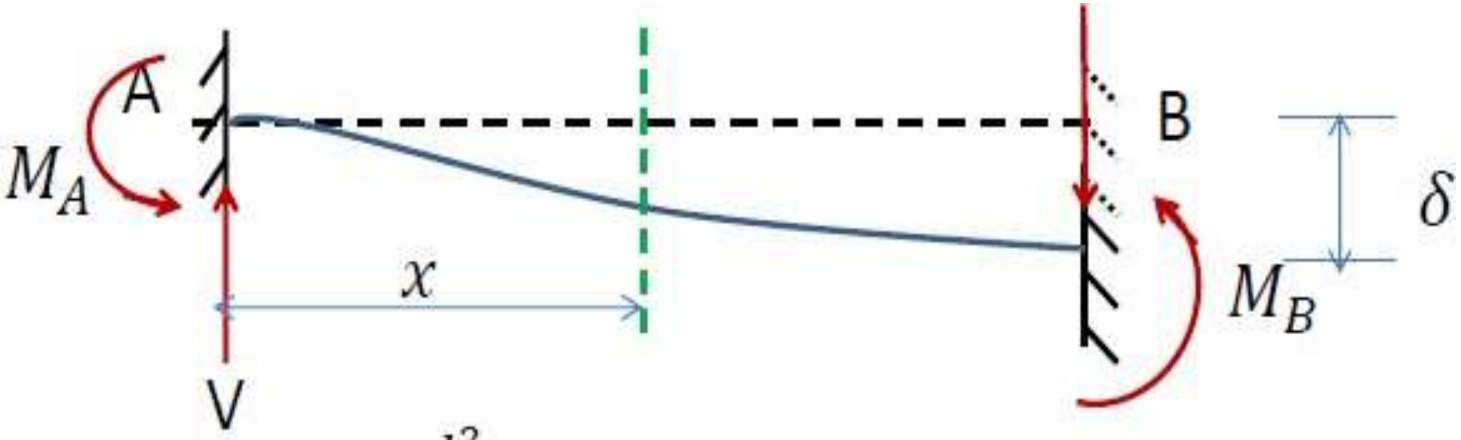
Integrating, we get,

Shear force = $EI\frac{d^3y}{dx^3} = C_1$

Where C_1 is a constant

At $x = 0$, $S.F. = +\text{V}$

$$\therefore C_1 = \text{V}$$



B.M. at any section = $EI\frac{d^2y}{dx^2} = Vx + C_1$

At $x = 0$, $B.M. = -M_A$

$$\therefore\ C_2 = -M_A$$

$$\therefore EI\frac{d^2y}{dx^2} = Vx - M_A$$

Integrating again,

$EI\frac{dy}{dx} = \frac{V}{2}x^2 - M_Ax + C_3$ (Slope equation)

But at $x = 0$, $\frac{dy}{dx} = 0$ $\quad\therefore C_3 = 0$

Integrating again,

$EI\ y = \frac{Vx^3}{6} - \frac{M_A x^2}{2} + C_4$ ------ (Deflection equation)

But at $x = 0, y = 0$

$$\therefore C_4 = 0$$

At $x = l, y = -\delta$

$$-EI\ \delta = \frac{Vl^3}{6} - \frac{M_A l^2}{2} ---------(\text{i})$$

But we also know that at B, $x = l\ and\ \frac{dy}{dx} = 0$

And substitute in slope Eq. $EI\frac{dy}{dx} = \frac{V}{2}x^2 - M_A x$

$$\therefore 0 = \frac{Vl^2}{2} - M_A l$$

$$\therefore V = \frac{2M_A}{l} ---------(\text{ii})$$

Substituting in deflection Eq.(i) i.e., $-EI\ \delta = \frac{Vl^3}{6} - \frac{M_A l^2}{2}$;we have,

$$-EI\ \delta = \frac{2M_A}{l} \times \frac{l^3}{6} - \frac{M_A l^2}{2}$$

$$EI\ \delta = \frac{M_A l^2}{6}$$

$$\therefore M_A = \frac{6EI\delta}{l^2}$$

Hence the law for the bending moment at any section distant x from A is given by,

$$M = EI\frac{d^2y}{dx^2} = \mathrm{V}x - M_A$$

$$\therefore M = \frac{2M_A}{l}x - \frac{6EI\delta}{l^2}$$

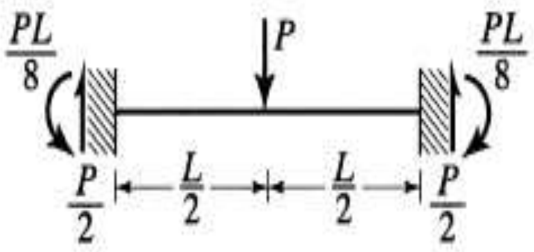
$$\text{But for B. M. at B, put x} = \text{l},$$

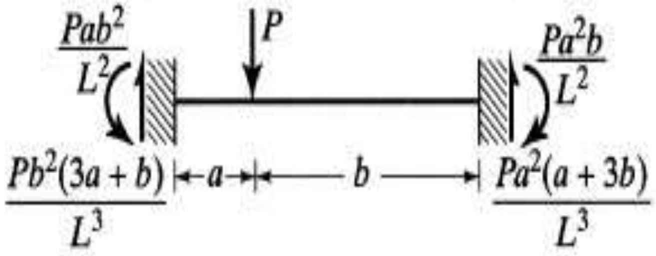
$$\therefore M_B = \frac{2M_A}{l} \times l - \frac{6EI\delta}{l^2} = \frac{12EI\delta}{l^2} - \frac{6EI\delta}{l^2} = \frac{6EI\delta}{l^2}$$

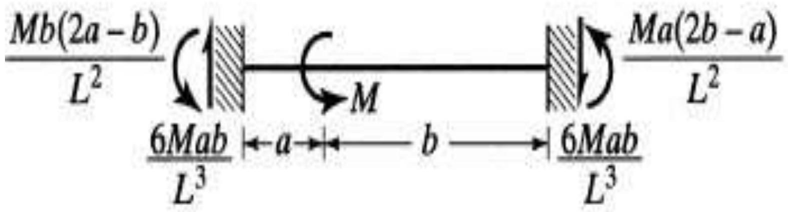
Hence when the ends of a fixed beam are at different levels,

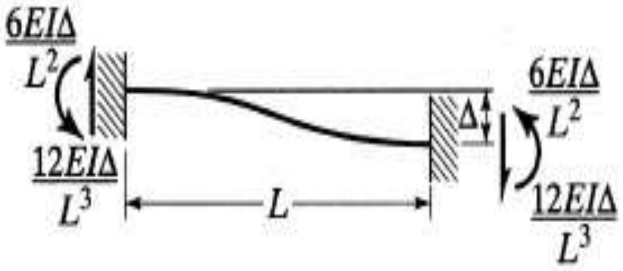
The fixing moment at each end = $\frac{6EI\delta}{l^2}$ numerically.

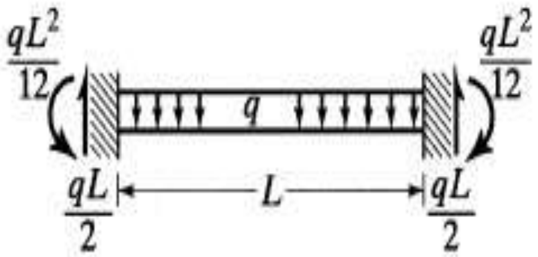
At the higher end this moment is a hogging moment and at the lower end this moment is a sagging moment.

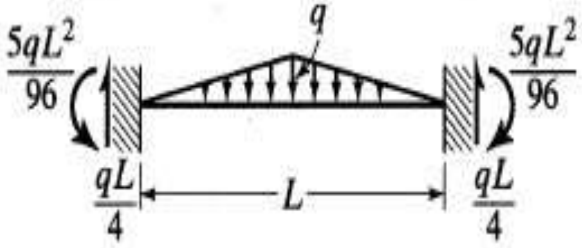


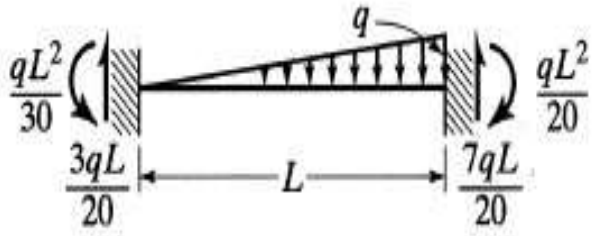


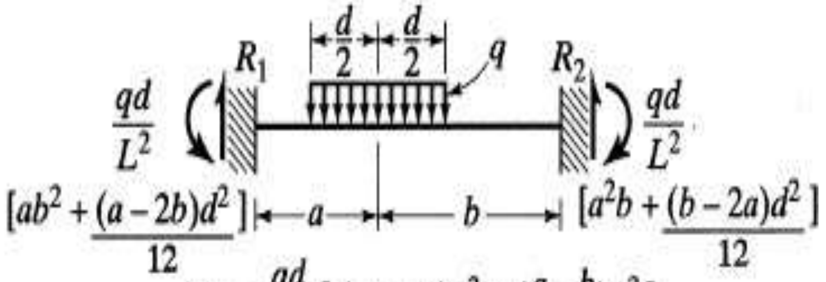








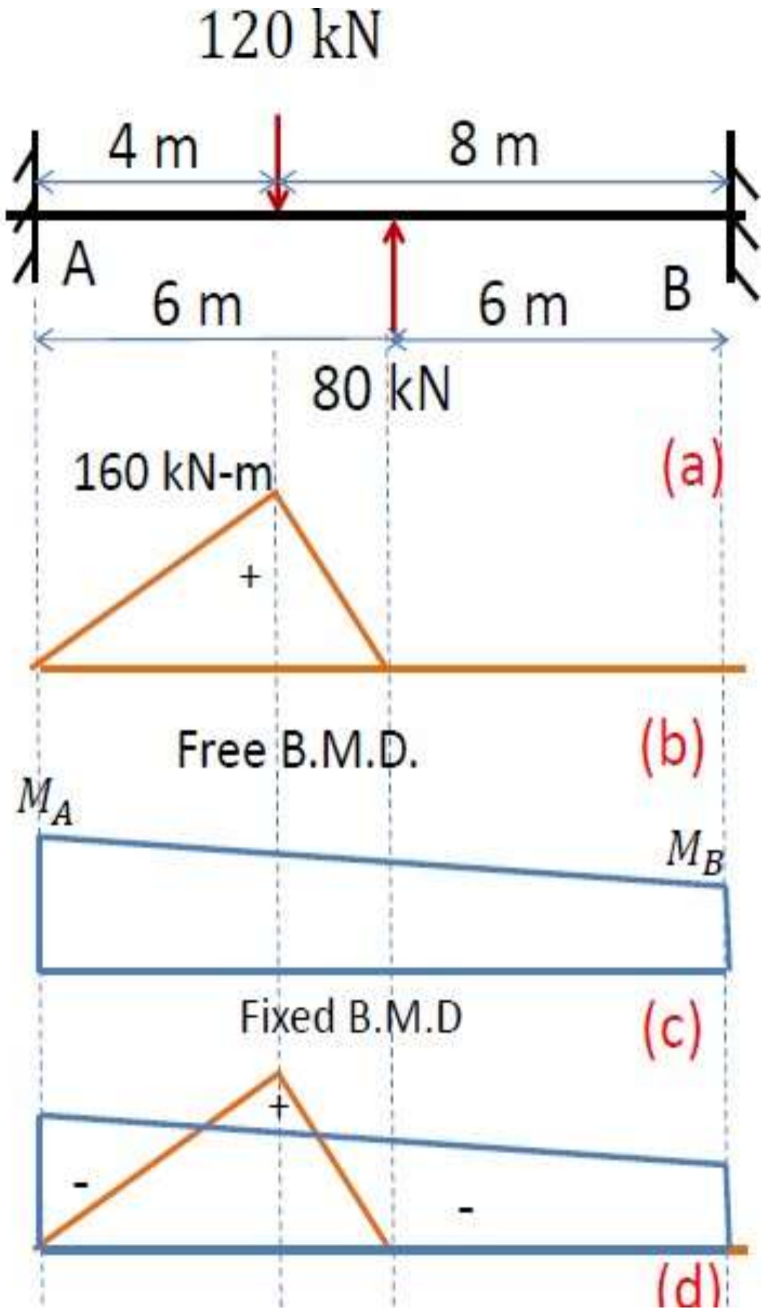




$$R_1 = \frac{qd}{L^3}\left[\,(2a+L)b^2 + \left(\frac{a-b}{4}\right)d^2\right]$$

$$R_2 = \frac{qd}{L^3}\left[\,(2b+L)a^2 - \left(\frac{a-b}{4}\right)d^2\right]$$

- Solution:
- The M (Free B.M.) and M’ (Fixed B.M.) diagrams have been shown in Fig.(b) and (c) respectively.



For the M-Diagram:

$$A = \frac{1}{2} \times 6 \times 160 = 480\ kNm$$

For the M’ diagram:

$$A' = \frac{M_A + M_B}{2} \times 12 = 6(M_A + M_B)$$

- Area of the fixed B.M. D. = Area of the free B.M.D.

A’ = A

$$6(M_A + M_B) = 480$$

$$M_A + M_B = 80\text{-----------}(1)$$

The distance of the centroid of the free B.M. D. from A = The distance of the centroid of the fixed B.M.D. from A.

i.e., $x = x'$

$$\frac{6+4}{3} = \left(\frac{M_A + 2M_B}{M_A + M_B}\right) \times \frac{12}{3}$$

$$(M_A + 2M_B)12 = (M_A + M_B)10$$

$$12M_A + 24\,M_B - 10\,M_A - 10M_B = 0$$

$$2M_A + 14\,M_B = 0$$

$$M_A = -7M_B \;\; -----(2)$$

- Substitute $M_A = -7M_B$ in equation (1)

$$-7M_B + M_B = 80$$

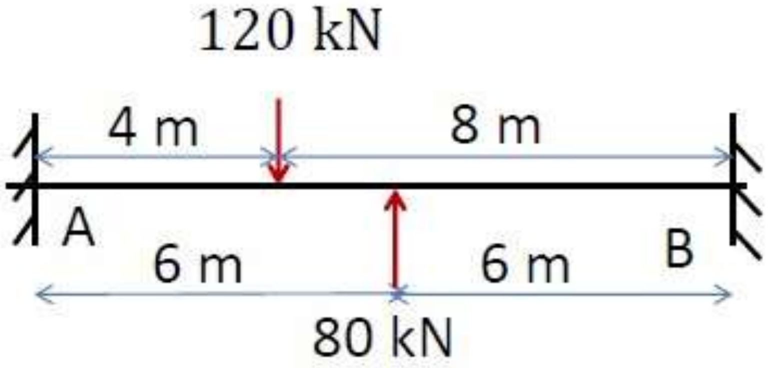
$$\therefore M_B = \frac{-80}{6} = -13.33$$

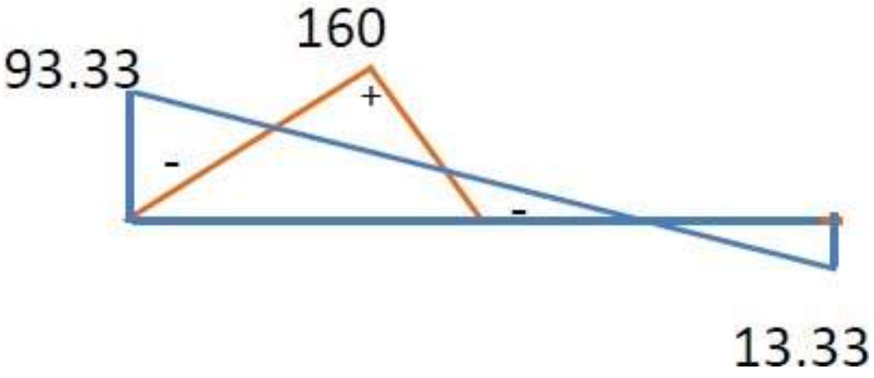
$$M_B = -13.33 \text{ kNm}$$

$$M_A = -7M_B$$

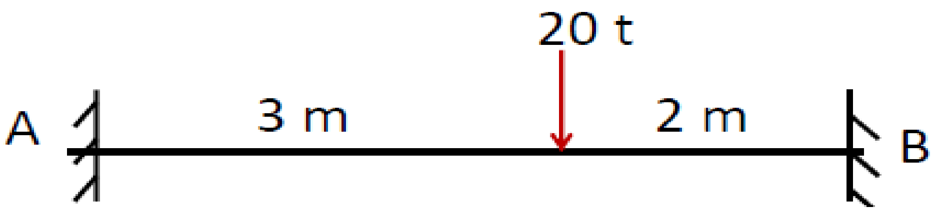
$$= -7(-13.33) = 93.33$$

$$\therefore M_A = 93.33\ kNm$$

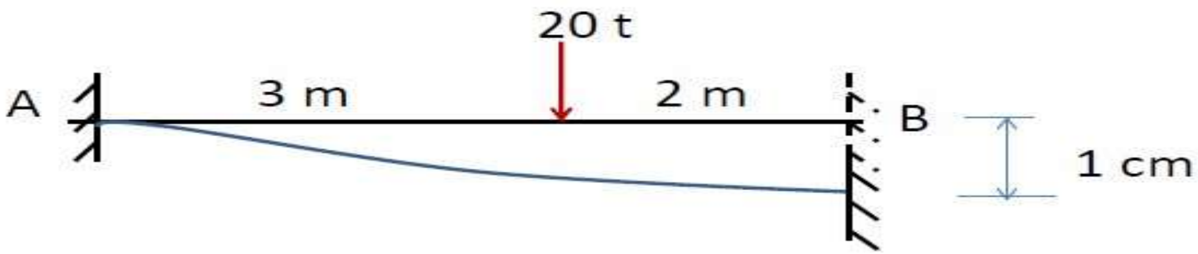




- A fixed beam of span 5 metres carries a concentrated load of 20 t at 3 meters from the left end. If the right end sinks by 1 cm, find the fixing moments at the supports. For the beam section take I=30,000 cm^4 and E=2x10^3 t/cm^2. Find also the reaction at the supports.

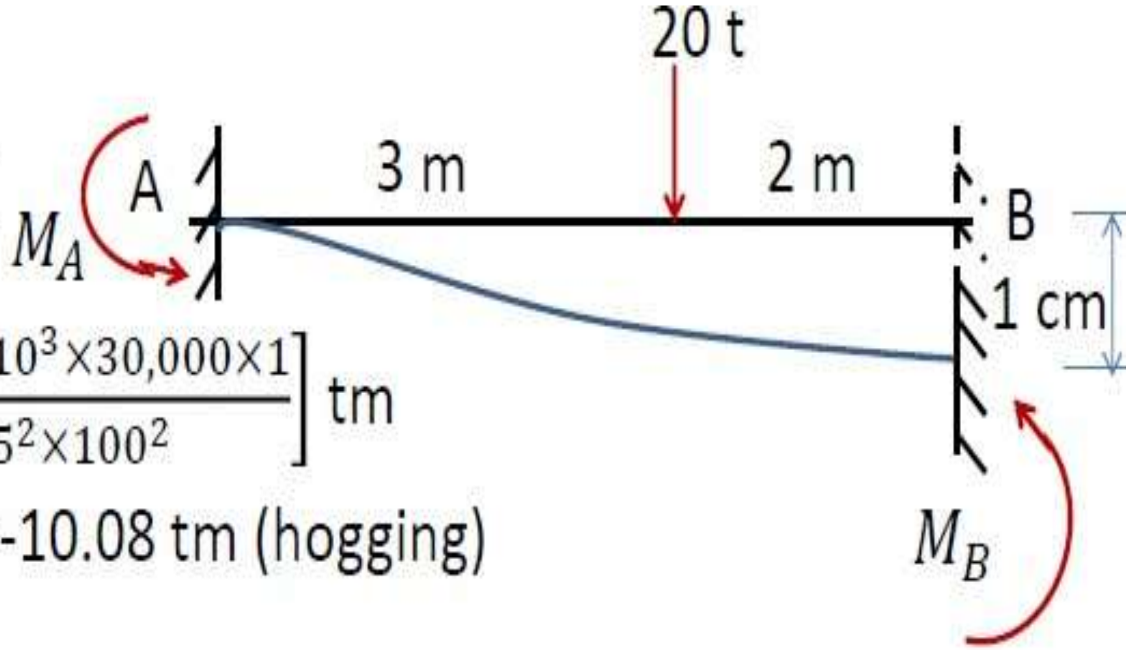


- A fixed beam of span 5 metres carries a concentrated load of 20 t at 3 meters from the left end.

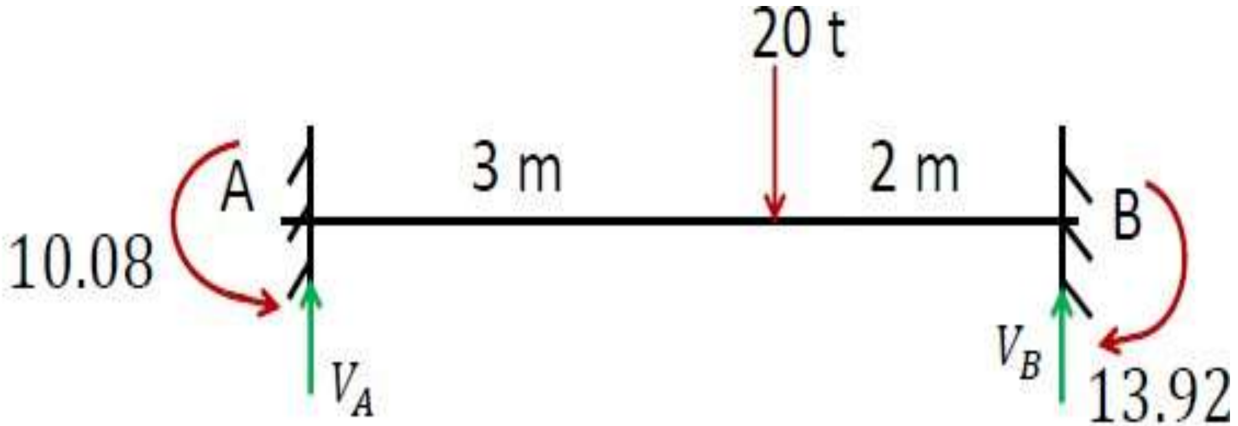


- The right end sinks by 1 cm, find the fixing moments at the supports.

- $M_A = -\frac{Wab^2}{l^2} - \frac{6EI\delta}{l^2}$
- $= -\left[\frac{20\times3\times2^2}{5^2} + \frac{6\times2\times10^3\times30{,}000\times1}{5^2\times100^2}\right]$ tm
- $= -[9.6 + 0.48]$ tm=-10.08 tm (hogging)



- $M_B = -\frac{Wba^2}{l^2} + \frac{6EI\delta}{l^2}$
- $= \left[-\frac{20\times2\times3^2}{5^2} + \frac{6\times2\times10^3\times30{,}000\times1}{5^2\times100^2}\right]$ tm
- $= [-14.4 + 0.48]$ tm= -13.92 tm (hogging)



- **Reaction at A:**
- $\sum M_B=0,$
- $V_A \times 5 + 13.92 - 10.08 - (20 \times 2) = 0$
- $\therefore V_A = 7.232$ t
- **Reaction at B:**
- $\therefore V_B = 20 - 7.232 = 12.768$ t.

Continuous Beams

Introduction:

- ☐ Beams are made continuous over the supports to increase structural integrity.
- ☐ A continuous beam provides an alternate load path in the case of failure at a section.
- ☐ In regions with high seismic risk, continuous beams and frames are preferred in buildings and bridges.
- ☐ A continuous beam is a statically indeterminate structure.
-

The advantages of a continuous beam as compared to a simply supported beam are as follows

1) For the same span and section, vertical load capacity is more.

2) Mid span deflection is less.

3) The depth at a section can be less than a simply supported beam for the same span. Else, for the same depth the span can be more than a simply supported beam.

⇒ The continuous beam is economical in material.

4) There is redundancy in load path.

⇒ Possibility of formation of hinges in case of an extreme event.

5) Requires less number of anchorages of tendons.

6) For bridges, the number of deck joints and bearings are reduced.

⇒ Reduced maintenance

There are of course several disadvantages of a continuous beam as compared to a simply supported beam.

1) Difficult analysis and design procedures.

2) Difficulties in construction, especially for precast members.

3) Increased frictional loss due to changes of curvature in the tendon profile.

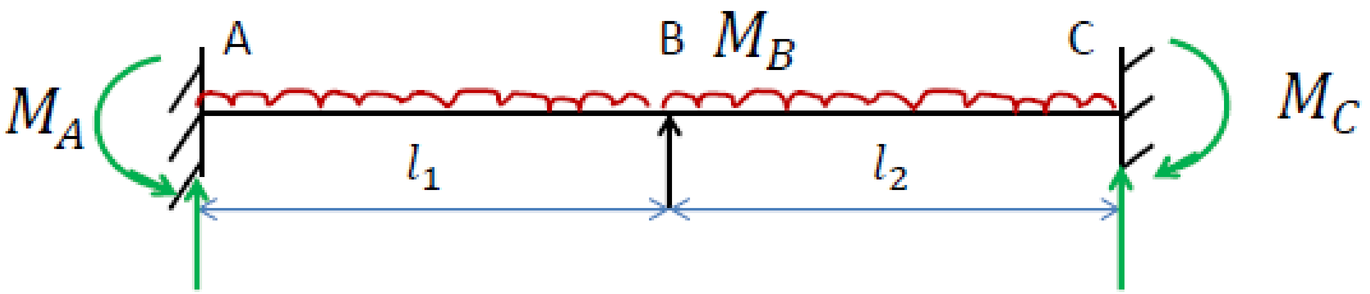
4) Increased shortening of beam, leading to lateral force on the supporting columns.

5) Secondary stresses develop due to time dependent effects like creep and shrinkage, settlement of support and variation of temperature.

6) The concurrence of maximum moment and shear near the supports needs proper detailing of reinforcement.

7) Reversal of moments due to seismic force requires proper analysis and design.

Clapeyron’s theorem of three moments



- As shown in above Figure, AB and BC are any two successive spans of a continuous beam subjected to an external loading.

- If the extreme ends A and C fixed supports, the support moments M_A, M_B and M_C at the supports A, B and C are given by the relation,

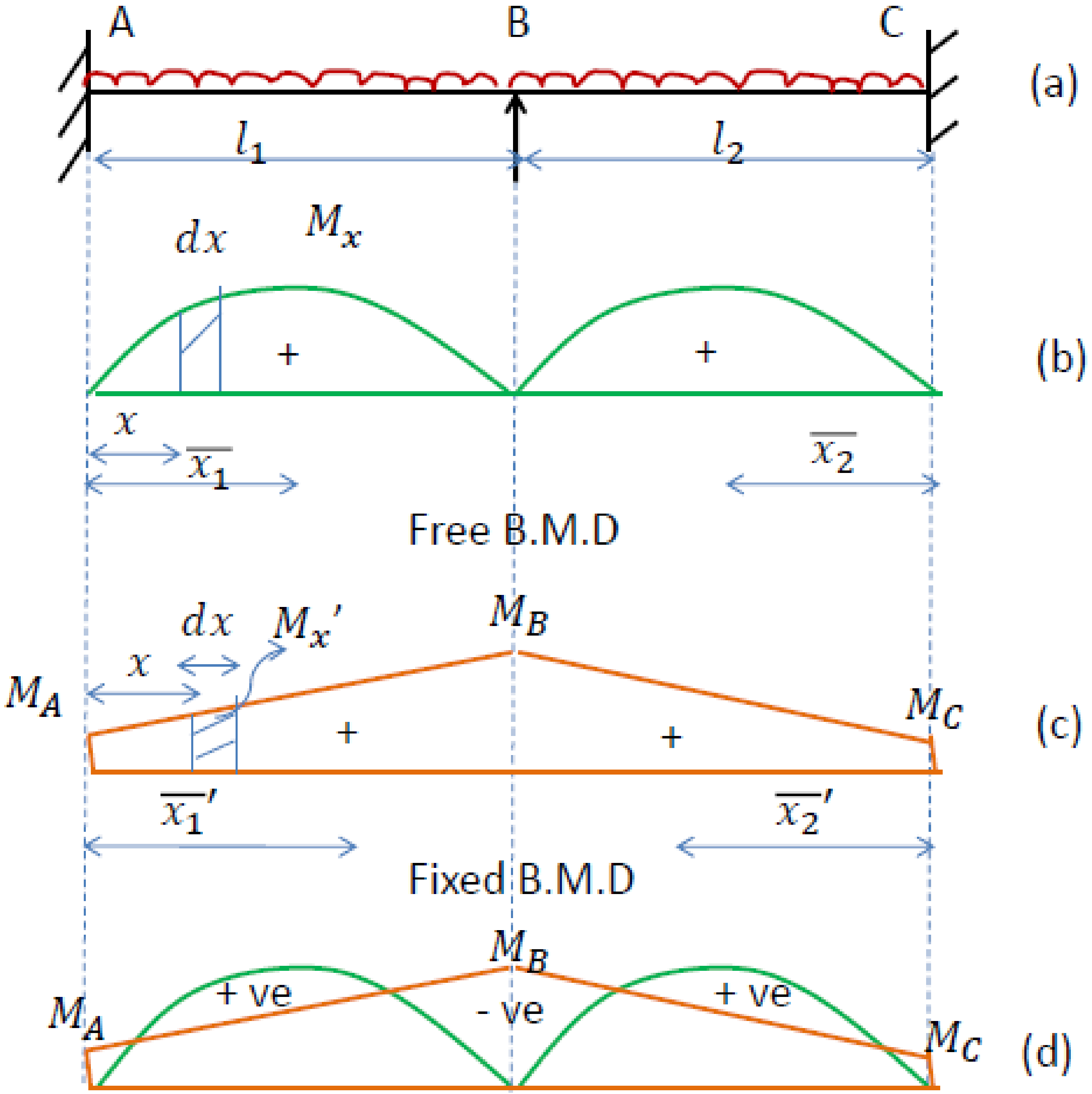
$$M_A l_1 + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

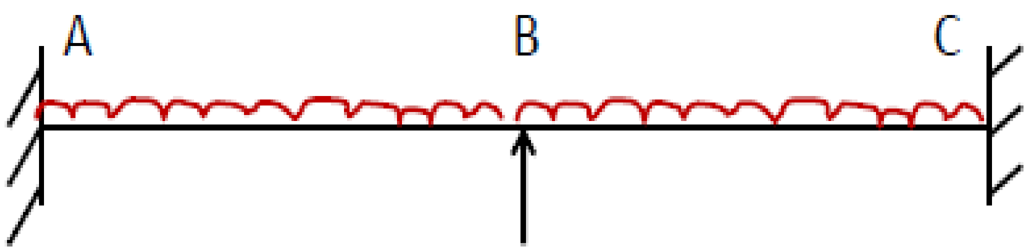
$$M_A l_1 + 2M_B(l_1 + l_2) + M_C(l_2) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

- Where,
- a_1 = area of the free B.M. diagram for the span AB.
- a_2 = area of the free B.M. diagram for the span BC.

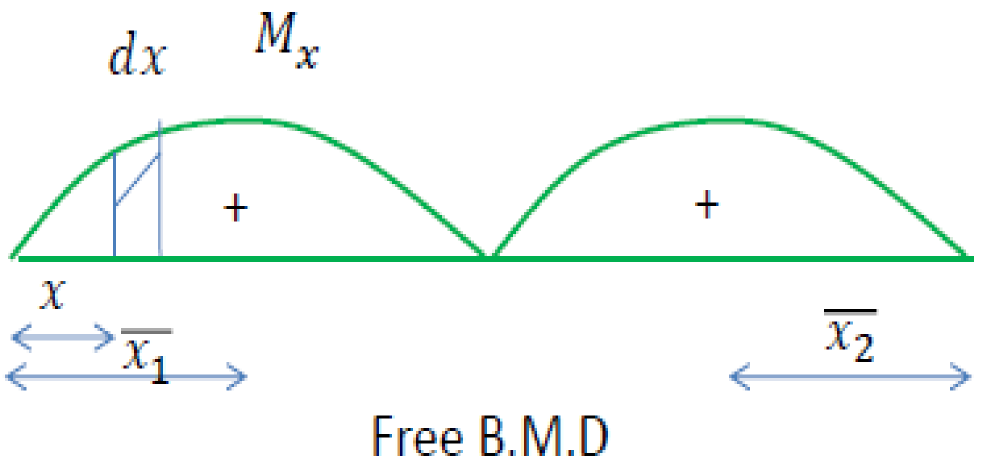
- $\overline{x_1}$= Centroidal distance of free B.M.D on AB from A.

- $\overline{x_2}$= Centroidal distance of free B.M.D on BC from C.

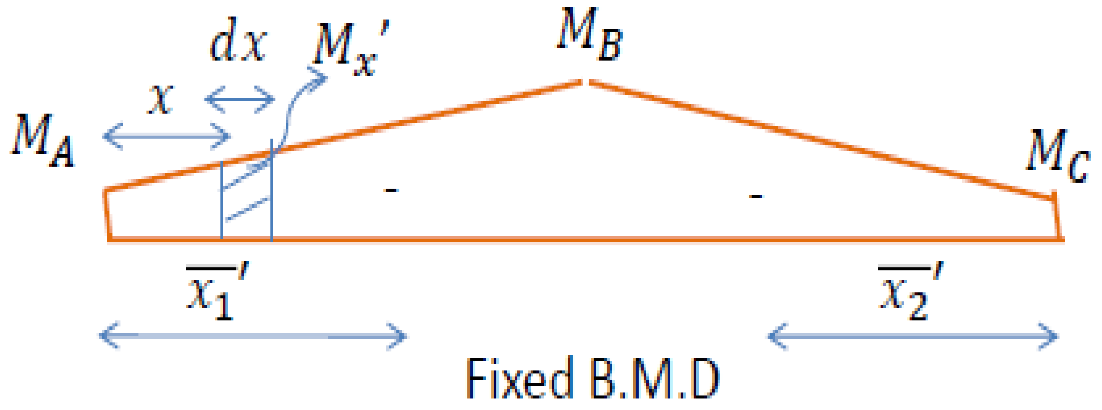

A
B
C
(a)
l_1
l_2
dx
M_x
+
+
(b)
x
$\overline{x_1}$
$\overline{x_2}$
Free B.M.D
M_B
dx
M_x'
x
M_A
M_C
+
+
(c)
$\overline{x_1}'$
$\overline{x_2}'$
Fixed B.M.D
M_B
M_A
+ ve
- ve
+ ve
M_C
(d)



(a)The given beam



(b) Free B.M.D.



(c) Fixed B.M.D.

- Consider the span AB:
- Let at any section in AB distant x from A the free and fixed bending moments be M_x and M_x' respectively.

- Hence the net bending moment at the section is given by

$$EI\frac{d^2y}{dx^2} = M_x - M_x'$$

- Multiplying by x, we get

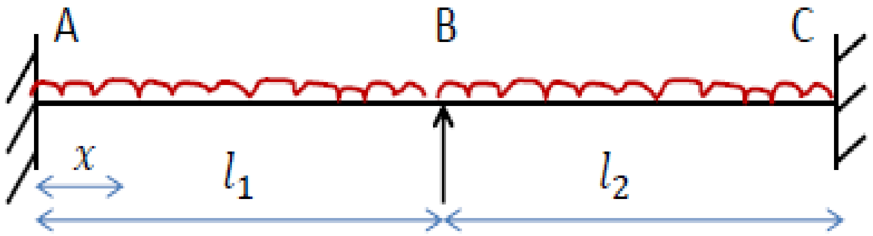
$$EIx\frac{d^2y}{dx^2} = M_x x - M_x' x$$

- $EIx\frac{d^2y}{dx^2} = M_x x - M_x{}'x$

- Integrating from $x = 0 \; to \; x = l_1$, we get,

$$EI\int_0^{l_1} x\frac{d^2y}{dx^2} = \int_0^{l_1} M_x x\,dx - \int_0^{l_1} M_x{}'x\,dx$$

$$EI\left[x.\frac{dy}{dx} - y\right]_0^{l_1} = \int_0^{l_1} M_x x\,dx - \int_0^{l_1} M_x{}'x\,dx \qquad ---(1)$$



- But it may be such that

  At $x = 0$, deflection $y = 0$

- At $x = l_1, y = 0;$ *and slope at B forAB,* $\frac{dy}{dx} = \theta_{BA}$

- $\int_0^{l_1} M_x x\, dx = a_1\overline{x_1}$ = Moment of the free B. M. D. on AB about A .

- $\int_0^{l_1} M_x' x\, dx = a_1' \overline{x_1}' =$ Moment of the fixed B. M. D. on AB about A.

$$EI\left[x.\frac{dy}{dx}-y\right]_0^{l_1}=\int_0^{l_1}M_x x\,dx-\int_0^{l_1}M_x{}'x\,dx\quad --(1)$$

- Therefore the equation (1) is simplified as,

$EI\;[l_1\theta_{BA}-0]\;=a_1\overline{x_1}\;-a_1'\overline{x_1}'.$

But a_1' = area of the fixed B.M.D. on AB = $\frac{(M_A+M_B)}{2}l_1$

$\overline{x_1}'$ = Centroid of the fixed B. M. D. from A = $\frac{(M_A+2M_B)}{M_A+M_B}\frac{l_1}{3}$

- Therefore,

$$a_1^{'}\overline{x_1}^{'} = \frac{(M_A + M_B)}{2} l_1 \times \left(\frac{M_A + 2M_B}{M_A + M_B}\right) \frac{l_1}{3} = (M_A + 2M_B)\frac{l_1^2}{6}$$

$$EI\, l_1 \theta_{BA} = a_1\overline{x_1} - (M_A + 2M_B)\frac{l_1^2}{6}$$

$$6EI\, \theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - (M_A + 2M_B)l_1 \quad ----(2)$$

Similarly by considering the span BC and taking C as origin it can be shown that,

$$6EI\, \theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - (M_C + 2M_B)l_2 \quad ----(3)$$

θ_{BC} = slope for span CB at B

- But $\theta_{BA} = -\theta_{BC}$ as the direction of x from A for the span AB, and from C for the span CB are in opposite direction.
- And hence, $\theta_{BA} + \theta_{BC} = 0$

$$6EI\,\theta_{BA} = \frac{6a_1\overline{x_1}}{l_1} - (M_A + 2M_B)l_1 \qquad ----(2)$$

$$6EI\,\theta_{BC} = \frac{6a_2\overline{x_2}}{l_2} - (M_C + 2M_B)l_2 \qquad ----(3)$$

- Adding equations (2) and (3), we get

$$EI\,\theta_{BA} + 6EI\,\theta_{BC} = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - (M_A + 2M_B)l_1 - (M_C + 2M_B)l_2$$

$$6EI(\theta_{BA} + \theta_{BC}) = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - [M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2]$$

$$0 = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2} - [M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2]$$

$$[M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2] = \frac{6a_1\overline{x_1}}{l_1} + \frac{6a_2\overline{x_2}}{l_2}$$

Moment-Distribution Method

Structural Analysis
By
Aslam Kassimali

Theory of Structures-II

M Shahid Mehmood

Department of Civil Engineering

Moment-Distribution Method

- Classical method.
- Used for Beams and Frames.
- Developed by Hardy Cross in 1924.
- Used by Engineers for analysis of small structures.
- It does not involve the solution of many simultaneous equations.

Moment-Distribution Method

- For beams and frames without sidesway, it does not involve the solution of simultaneous equations.

- For frames with sidesway, number of simultaneous equations usually equals the number of independent joint translations.

- In this method, Moment Equilibrium Equations of joints are solved iteratively by considering the moment equilibrium at one joint at a time, while the remaining joints are considered to be restrained.

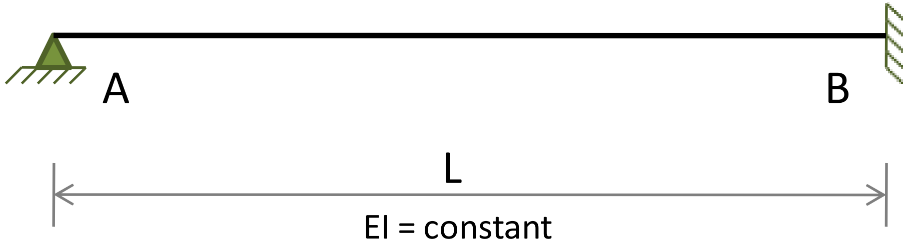
Definitions and Terminology

Sign Convention

- Counterclockwise member end moments are considered positive.
- Clockwise moments on joints are considered positive.

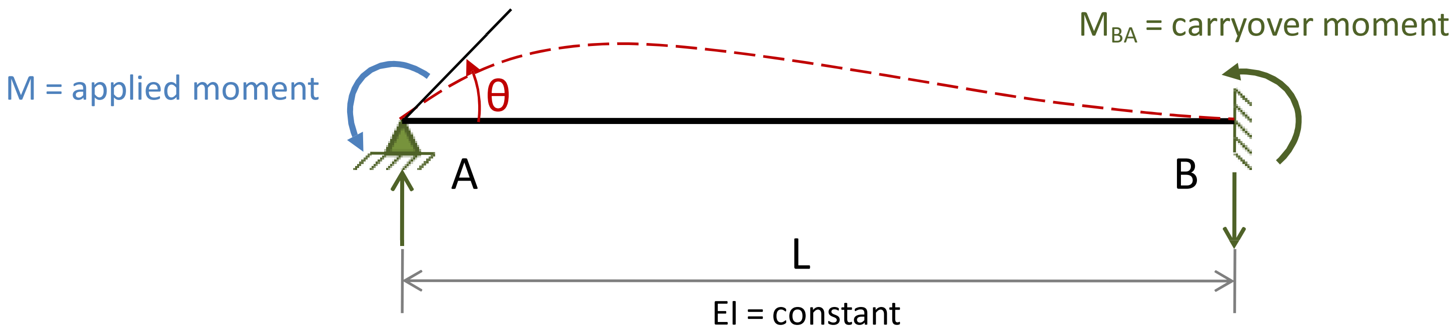
Member Stiffness

- Consider a prismatic beam AB, which is hinged at end A and fixed at end B.



Member Stiffness

If we apply a moment M at the end A, the beam rotates by an angle θ at the hinged end A and develops a moment M_{BA} at the fixed end B, as shown.



The relationship between the applied moment M and the rotation θ can be established using the slope-deflection equation.

Member Stiffness

By substituting $M_{nf} = M$, $\theta_n = \theta$, and $\theta_f = \Psi = FEM_{nf} = 0$ into the slope-deflection equation, we obtain

$$M = \left(\frac{4EI}{L}\right)\theta \qquad (1)$$

"The bending stiffness, $\bar{K}$, of a member is defined as the moment that must be applied at an end of the member to cause a unit rotation of that end."

By setting $\theta = 1$ rad in Eq. 1, we obtain the expression for the bending stiffness of the beam of figure to be

$$\bar{K} = \frac{4EI}{L} \qquad (2)$$

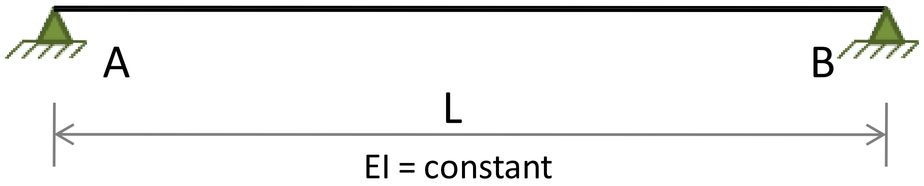
Member Stiffness

when the modulus of elasticity for all the members of a structure is the same (constant), it is usually convenient to work with the relative bending stiffness of members in the analysis.

"The relative bending stiffness, K, of a member is obtained by dividing its bending stiffness, $\overline{K}$ *, by 4E."*

$$K = \frac{\overline{K}}{4E} = \frac{I}{L} \tag{3}$$

- Now suppose that the far end B of the beam is hinged as shown.

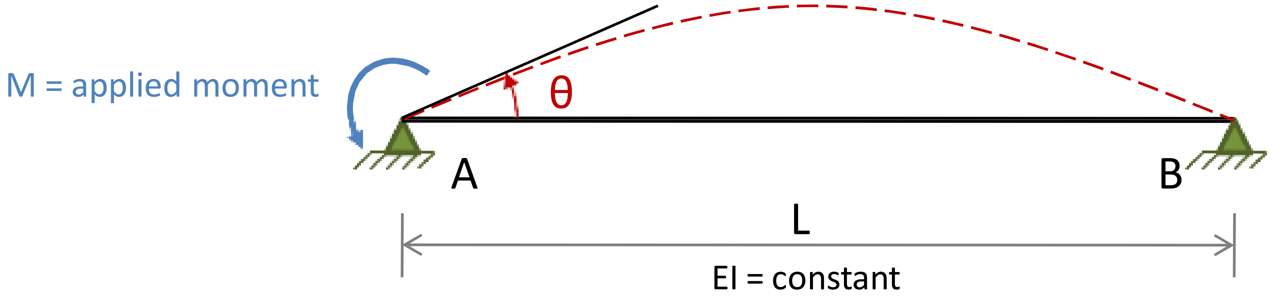


Member Stiffness

The relationship between the applied moment M and the rotation θ of the end A of the beam can now be determined by using the modified slope-deflection equation.

By substituting $M_{rh} = M$, $\theta_r = \theta$, and $\Psi = FEM_{rh} = FEM_{hr} = 0$ into MSDE, we obtain

$$M = \left(\frac{3EI}{L}\right)\theta \qquad (4)$$



Member Stiffness

By setting $\theta = 1$ rad, we obtain the expression for the bending stiffness of the beam of figure to be

$$\bar{K} = \frac{3EI}{L} \tag{5}$$

A comparison of Eq. 2 & Eq. 5 indicates that the stiffness of the beam is reduced by 25% when the fixed support at B is replaced by a hinged support.

The relative bending stiffness of the beam can now be obtained by dividing its bending stiffness by 4E.

$$K = \frac{\bar{K}}{4E} = \frac{3}{4}\left(\frac{I}{L}\right) \tag{6}$$

Member Stiffness

Relationship b/w applied end moment M and the rotation θ

$$M = \begin{cases} \left(\dfrac{4EI}{L}\right)\theta & \text{if far end of member is fixed} \\ \left(\dfrac{3EI}{L}\right)\theta & \text{if far end of member is hinged} \end{cases} \qquad (7)$$

Bending stiffness of a member

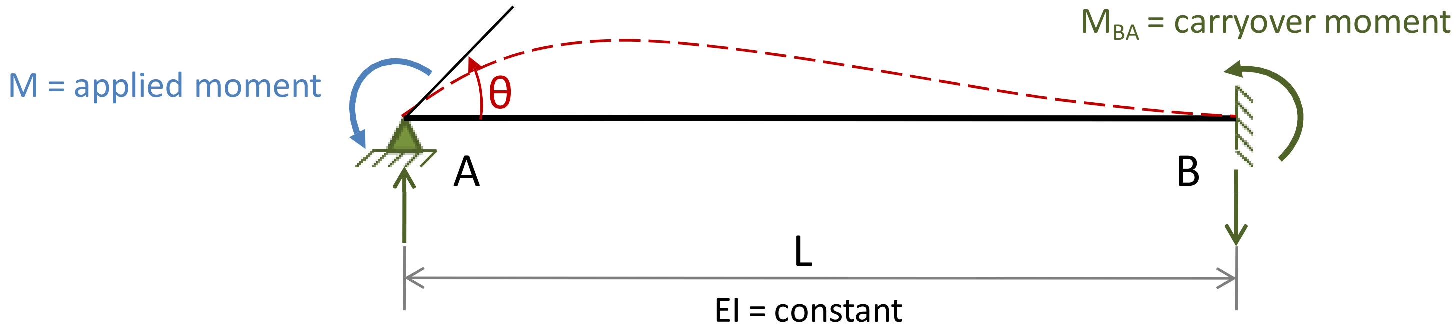
$$\overline{K} = \begin{cases} \dfrac{4EI}{L} & \text{if far end of member is fixed} \\ \dfrac{3EI}{L} & \text{if far end of member is hinged} \end{cases} \qquad (8)$$

Relative bending stiffness of a member

$$K = \begin{cases} \dfrac{I}{L} & \text{if far end of member is fixed} \\ \dfrac{3}{4}\dfrac{I}{L} & \text{if far end of member is hinged} \end{cases} \qquad (9)$$

Carryover Moment

Let us consider again the hinged-fixed beam of Figure.



When a moment M is applied at the hinged end A of the beam, a moment M_{BA} develops at the fixed end B.

The moment M_{BA} is termed the *carryover moment*.

Carryover Moment

To establish the relationship b/w the applied moment M and the carryover moment M_{BA}, we write the slope deflection equation for M_{BA} by substituting $M_{nf} = M_{BA}$, $\theta_f = \theta$, and $\theta_n = \Psi = FEM_{nf} = 0$ into SDE

$$M_{BA} = \left(\frac{2EI}{L}\right)\theta \tag{10}$$

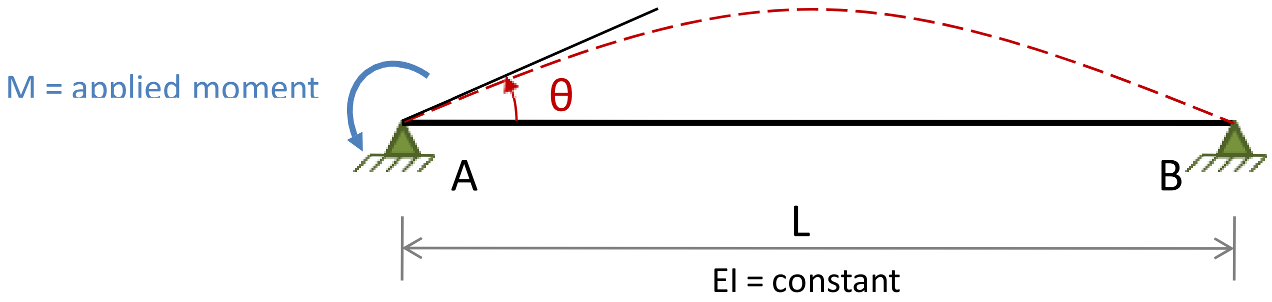
By substituting $\theta = ML/(4EI)$ from Eq. 1 into Eq. 10, we obtain

$$M_{BA} = \frac{M}{2} \tag{11}$$

Eq. 11 indicates, when a moment of magnitude M is applied at the hinged end of the beam, one-half of the applied moment is carried over to the far end, provided that the far end is fixed. The direction of M_{BA} and M is same.

Carryover Moment

When the far end of the beam is hinged as shown, the carryover moment M_{BA} is zero.



$$M_{BA} = \begin{cases} \dfrac{M}{2} & \text{if far end of member is fixed} \\ 0 & \text{if far end of member is hinged} \end{cases} \qquad (12)$$

Carryover Factor (COF)

"The ratio of the carryover moment to the applied moment (M_{BA}/M) is called the carryover factor of the member."

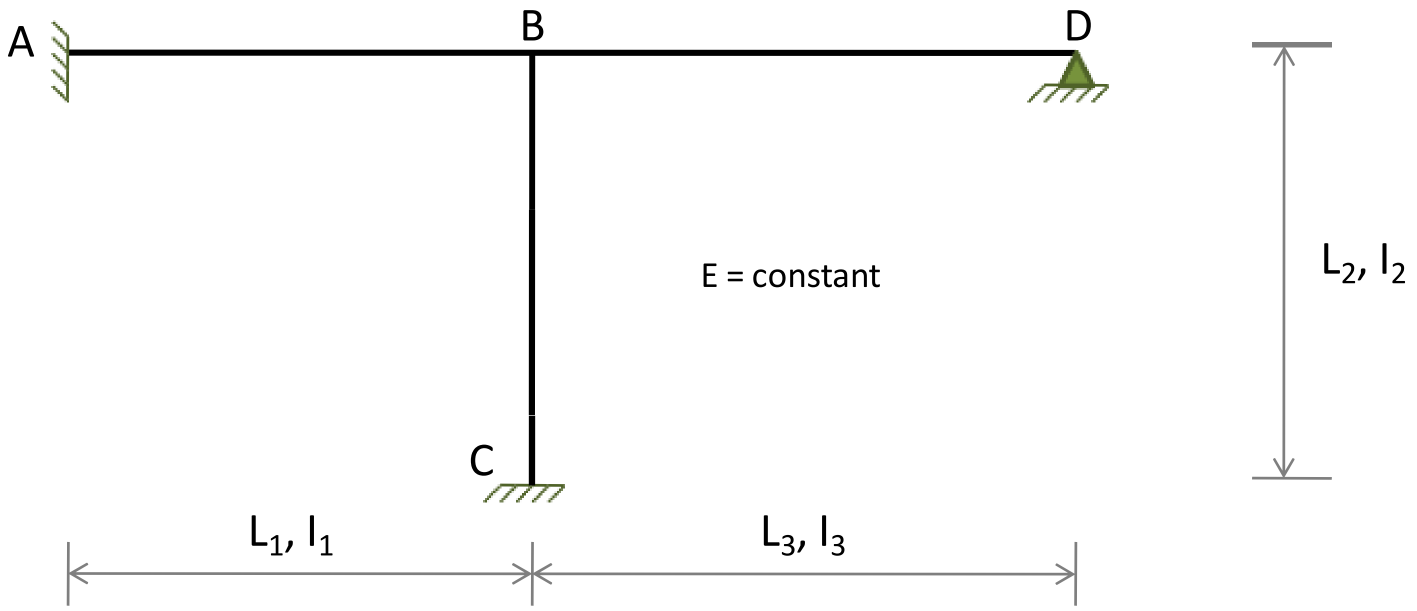
It represents the fraction of the applied moment M that is carried over to the far end of the member. By dividing Eq. 12 by M, we can express the carryover factor (COF) as

$$COF = \begin{cases} \frac{1}{2} & \text{if far end of member is fixed} \\ 0 & \text{if far end of member is hinged} \end{cases} \qquad (13)$$

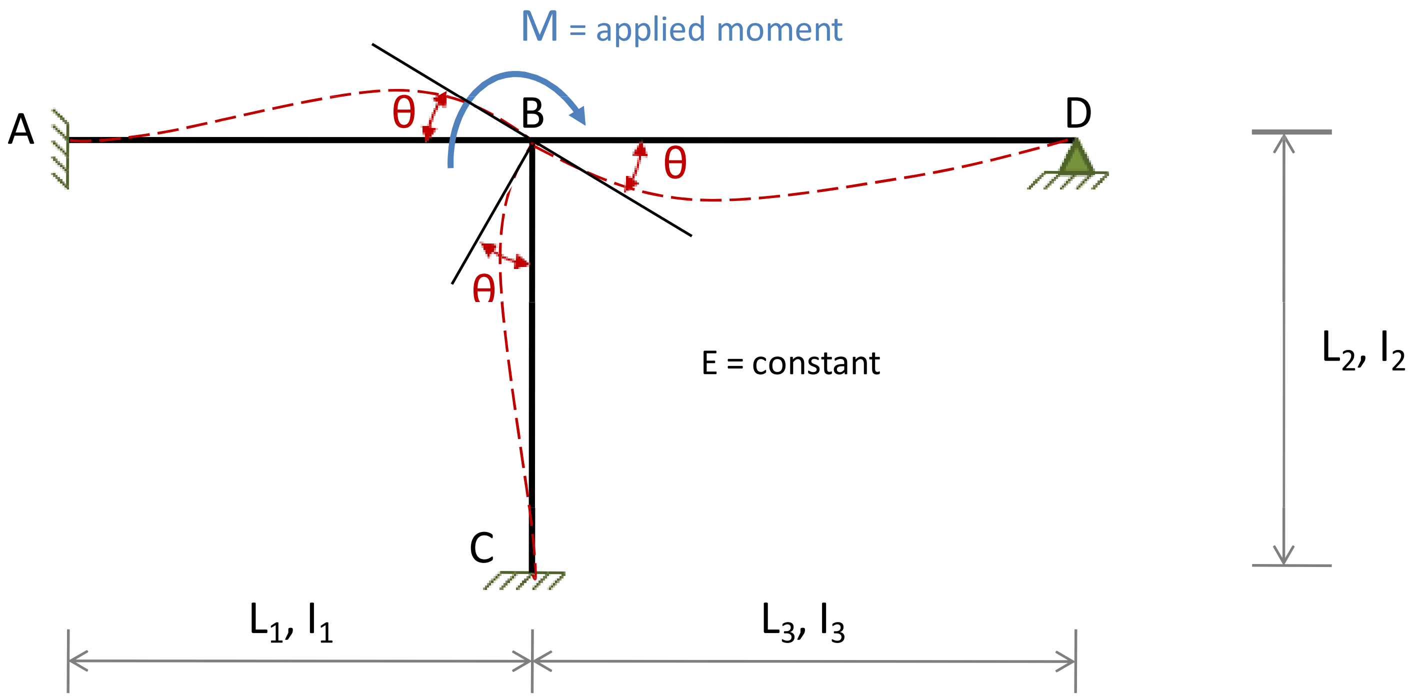
Distribution Factors

When analyzing a structure by the moment-distribution method, an important question that arises is how to distribute a moment applied at a joint among the various members connected to that joint.

Consider the three-member frame shown in figure below.



Suppose that a moment M is applied to the joint B, causing it to rotate by an angle θ as shown in figure below.

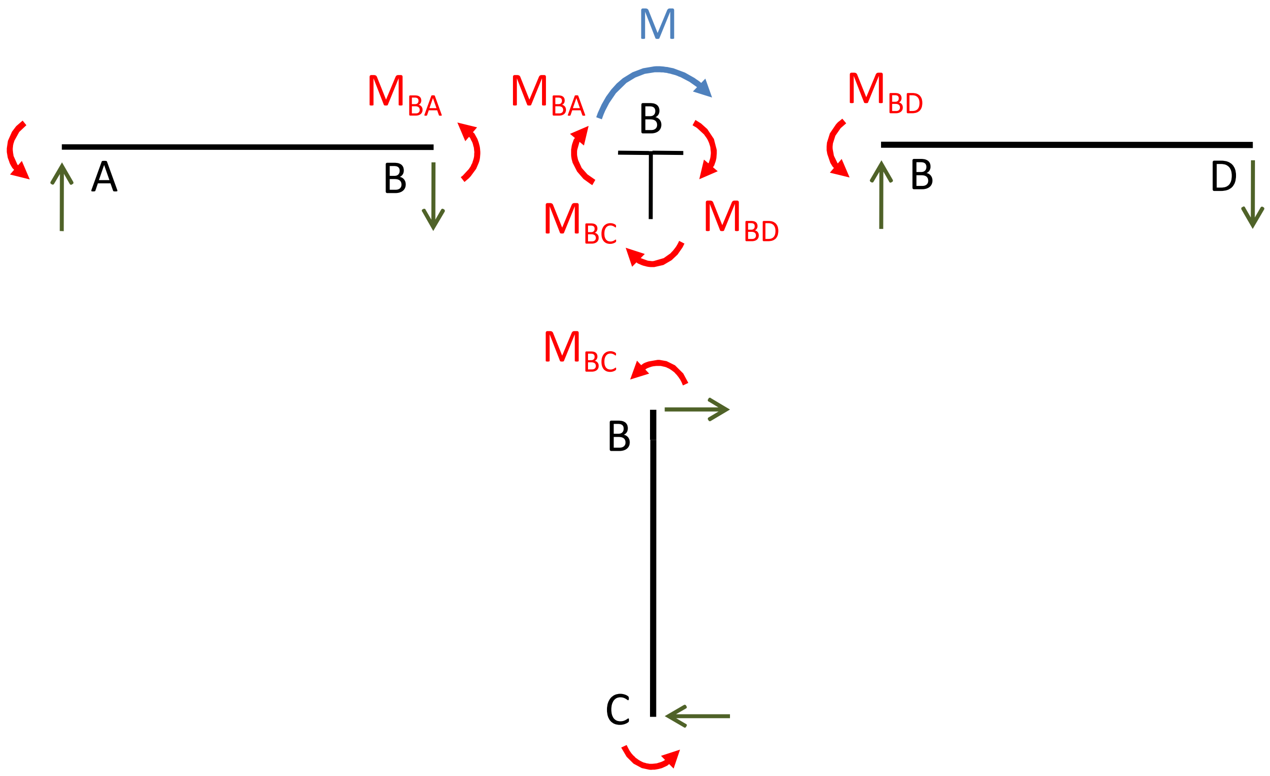


To determine what fraction of applied moment is resisted by each of the three members AB, BC, and BD, we draw free-body diagrams of joint B and of the three members AB, BC, and BD.

By considering the moment equilibrium of the free body of joint B ($\sum M_B = 0$), we write

$$M + M_{BA} + M_{BC} + M_{BD} = 0$$

$$M = -\left(M_{BA} + M_{BC} + M_{BD}\right) \tag{14}$$



Since members AB, BC, and BD are rigidly connected to joint B, the rotations of the ends B of these members are the same as that of the joint.

The moments at the ends B of the members can be expressed in terms of the joint rotation θ by applying Eq. 7.

Noting that the far ends A and C, respectively, of members AB and BC are fixed, whereas the far end D of member BD is hinged, we apply Eq. 7 through Eq. 9 to each member to obtain

$$M_{BA} = \left(\frac{4EI_1}{L_1} \right) \theta = \overline{K}_{BA} \theta = 4EK_{BA} \theta \qquad (15)$$

$$M_{BC} = \left(\frac{4EI_2}{L_2} \right) \theta = \overline{K}_{BC} \theta = 4EK_{BC} \theta \qquad (16)$$

$$M_{BD} = \left(\frac{3EI_3}{L_3} \right) \theta = \overline{K}_{BD} \theta = 4EK_{BD} \theta \qquad (17)$$

Substitution of Eq. 15 through Eq. 17 into the equilibrium equation Eq. 14 yields

$$
\begin{aligned}
M &= -\left(\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2} + \frac{3EI_3}{L_3}\right)\theta \\
&= -\left(\bar{K}_{BA} + \bar{K}_{BC} + \bar{K}_{BD}\right)\theta = -\left(\sum \bar{K}_B\right)\theta
\end{aligned}
\qquad (18)
$$

in which $\sum \bar{K}_B$ represents the sum of the bending stiffnesses of all the members connected to joint B.

"The rotational stiffness of a joint is defined as the moment required to cause a unit rotation of the joint."

From Eq. 18, we can see that the rotational stiffness of a joint is equal to the sum of the bending stiffnesses of all the members rigidly connected to the joint.

The negative sign in Eq. 18 appears because of the sign convention.

To express member end moments in terms of the applied moment M, we first rewrite Eq. 18 in terms of the relative bending stiffnesses of members as

$$M = -4E(K_{BA} + K_{BC} + K_{BD}) = -4E \sum K_B$$

$$\theta = -\frac{M}{4E\sum K_B} \tag{19}$$

By substituting Eq. 19 into Eqs. 15 through 17, we obtain

$$M_{BA} = -\left(\frac{K_{BA}}{\sum K_B}\right)M \tag{20}$$

$$M_{BC} = -\left(\frac{K_{BC}}{\sum K_B}\right)M \tag{21}$$

$$M_{BD} = -\left(\frac{K_{BD}}{\sum K_B}\right)M \qquad (22)$$

From Eqs. 20 through 22, we can see that the applied moment M is distributed to the three members in proportion to their relative bending stiffnesses.

“The ratio $K/\sum K_B$ for a member is termed the distribution factor of that member for end B, and it represents the fraction of the applied moment M that is distributed to end B of the member.”

Thus Eqs. 20 through 22 can be expressed as

$$M_{BA} = -DF_{BA}M \qquad (23)$$

$$M_{BC} = -DF_{BC}M \qquad (24)$$

$$M_{BD} = -DF_{BD}M \qquad (25)$$

in which $DF_{BA} = K_{BA}/\sum K_B$, $DF_{BC} = K_{BC}/\sum K_B$, and $DF_{BD} = K_{BD}/\sum K_B$, are the distribution factors for ends B of members AB, BC, and BD, respectively.

For example, if joint B of the frame is subjected to a clockwise moment of 150 k-ft ($M = 150$ k-ft) and if $L_1 = L_2 = 20$ ft, $L_3 = 30$ ft, and $I_1 = I_2 = I_3 = I$, so that

$$K_{BA} = K_{BC} = \frac{I}{20} = 0.05I$$

$$K_{BD} = \frac{3}{4}\left(\frac{I}{30}\right) = 0.025I$$

then the distribution factors for the ends B of members AB, BC, and BD are given by

$$DF_{BA} = \frac{K_{BA}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.05I}{(0.05+0.05+0.025)I} = 0.4$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.05I}{0.125I} = 0.4$$

$$DF_{BD} = \frac{K_{BD}}{K_{BA}+K_{BC}+K_{BD}} = \frac{0.025I}{0.125I} = 0.2$$

These distribution factors indicate that 40% of the 150 k-ft moment applied to joint B is exerted at end B of member AB, 40% at end B of member BC, and the remaining 20% at end B of member BD.

The moments at ends B of the three members are

$$M_{BA} = -DF_{BA}M = -0.4(150) = -60\ \text{k-ft} \qquad \text{or} \qquad 60\ \text{k-ft} \ ↻$$

$$M_{BC} = -DF_{BC}M = -0.4(150) = -60\ \text{k-ft} \qquad \text{or} \qquad 60\ \text{k-ft} \ ↻$$

$$M_{BD} = -DF_{BD}M = -0.2(150) = -30\ \text{k-ft} \qquad \text{or} \qquad 30\ \text{k-ft} \ ↻$$

Based on the foregoing discussion, we can state that, in general, *"the distribution factor (DF) for an end of a member that is rigidly connected to the adjacent joint equals the ratio of the relative bending stiffness of the member to the sum of the relative bending stiffnesses of all the members framing into the joint"*; that is

$$DF = \frac{K}{\sum K} \tag{26}$$

"The moment distributed to (or resisted by) a rigidly connected end of a member equals the distribution factor for that end times the negative of the moment applied to the adjacent joint."

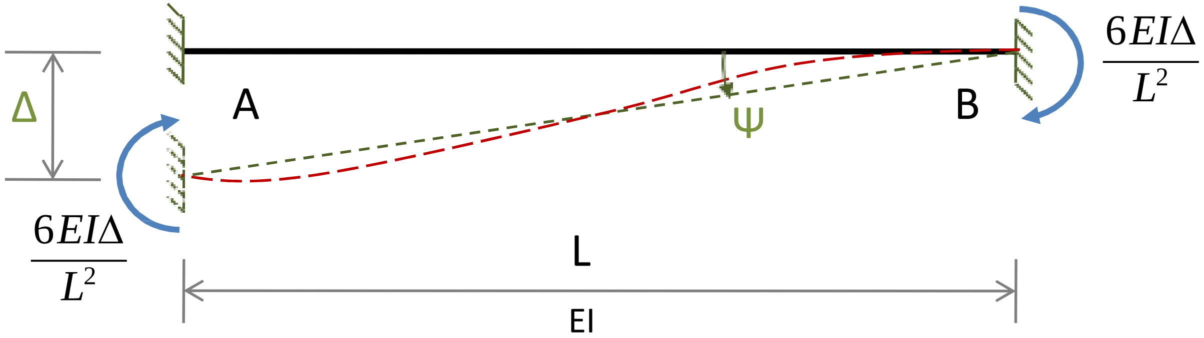
Fixed-End Moments

The fixed end moment expressions for some common types of loading conditions as well as for relative displacements of member ends are given inside the back cover of book.

In the MDM, the effects of joint translations due to support settlements and sidesway are also taken into account by means of fixed-end moments.

Consider the fixed beam of Figure.

A small settlement Δ of the left end A of the beam with respect to the right end B causes the beam's chord to rotate counterclockwise by an angle Ψ = Δ/L.

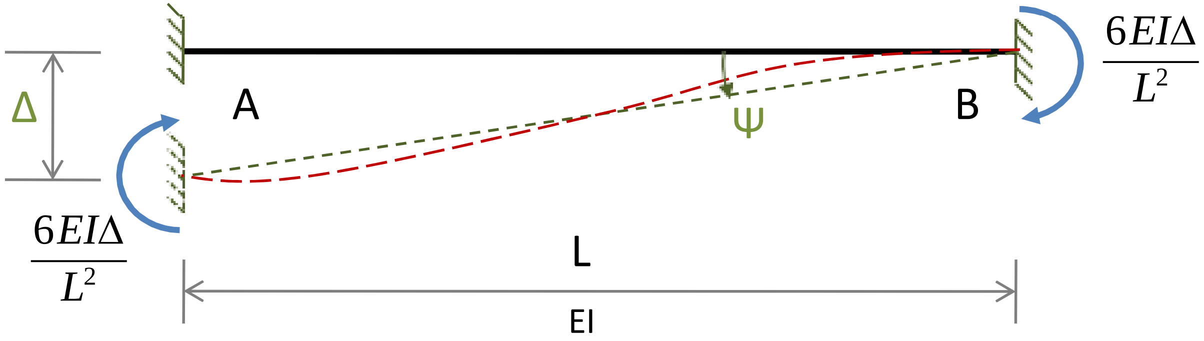


By writing the SDE for the two end moments with Ψ = Δ/L and by setting θ_A, θ_B, and FEM_{AB} and FEM_{BA} due to external loading, equal to zero, we obtain

$$FEM_{AB} = FEM_{BA} = -\frac{6EI\Delta}{L^2}$$

in which FEM_{AB} and FEM_{BA} denote the FEM due to the relative translation Δ between the two ends of the beam.

Note that the magnitudes as well as the directions of the two FEM are the same.



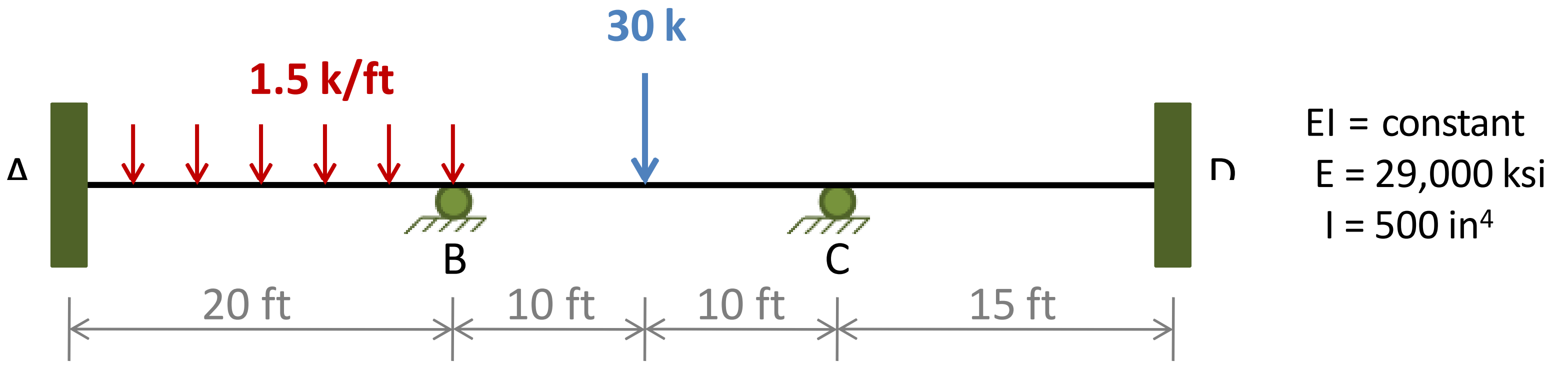
It can be seen from the figure that when a relative displacement causes a chord rotation in the CCW direction, then the two FEMs act in the CW (-ve) direction to maintain zero slopes at the two ends of the beam.

Conversely, if the chord rotation due to a relative displacement is CW, then both FEM act in CCW (+ve) direction.

Moment-Distribution Method

MDM	Moment Distribution Method
MD Table	Moment Distribution Table
COM	Carryover Moment
COF	Carryover Factor
DM	Distributed Moment
UM	Unbalanced Moment

Basic Concept of the Moment Distribution Method



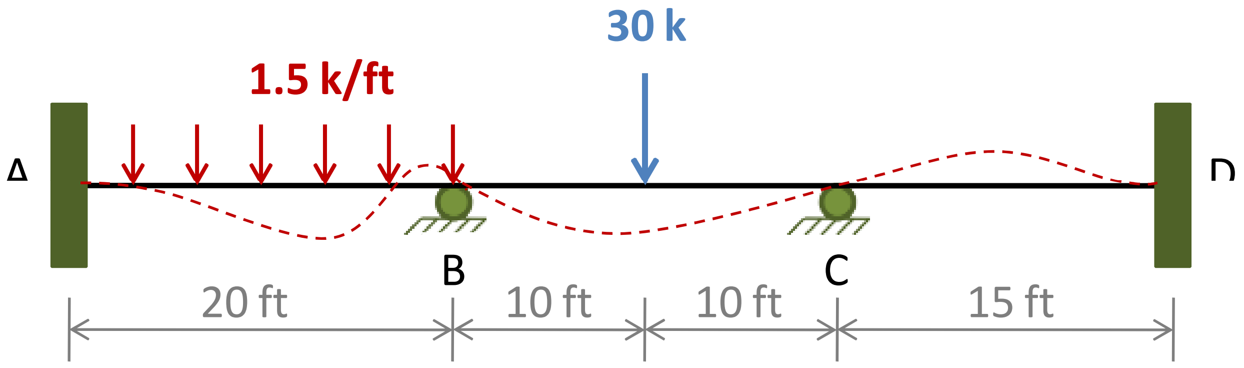
Distribution Factors

The first step in the analysis is to calculate the distribution factors at those joints of the structure that are free to rotate.

The distribution factor for an end of a member is equal to the relative bending stiffness of the member divided by the sum of relative bending stiffnesses of all the members connected to the joint.

$$DF = \frac{K}{\sum K} \tag{26}$$

Basic Concept of the Moment Distribution Method



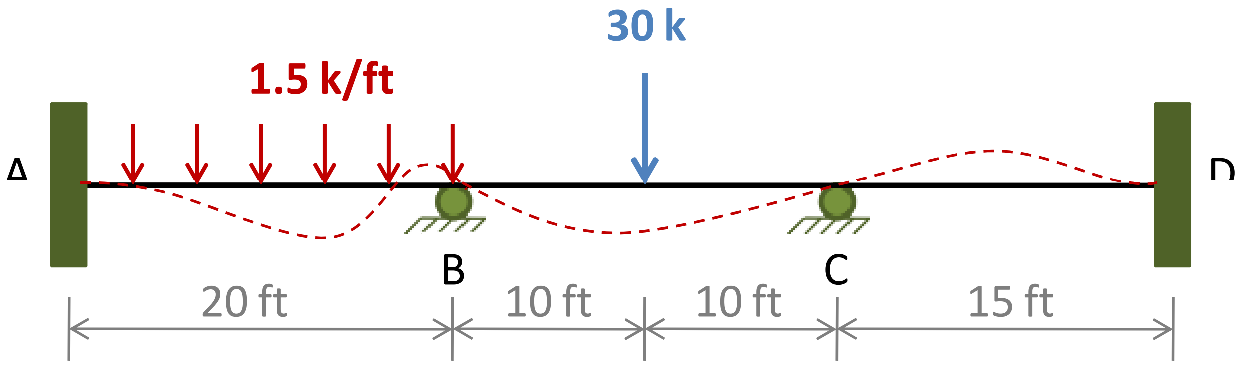
EI = constant
E = 29,000 ksi
I = 500 in^4

We can see that only joint B and C of the continuous beam are free to rotate. The distribution factors at joint B are

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{I/20}{2I/20} = 0.5$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{I/20}{2I/20} = 0.5$$

Basic Concept of the Moment Distribution Method



EI = constant
E = 29,000 ksi
I = 500 in^4

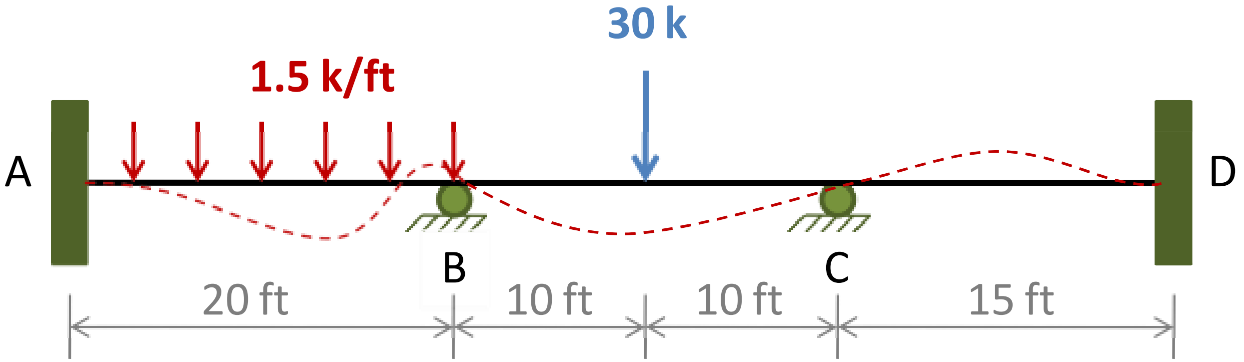
Similarly at joint C

$$DF_{CB} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{I/20}{(I/20)+(I/15)} = 0.429$$

$$DF_{CD} = \frac{K_{CD}}{K_{CB} + K_{CD}} = \frac{I/15}{(I/20)+(I/15)} = 0.571$$

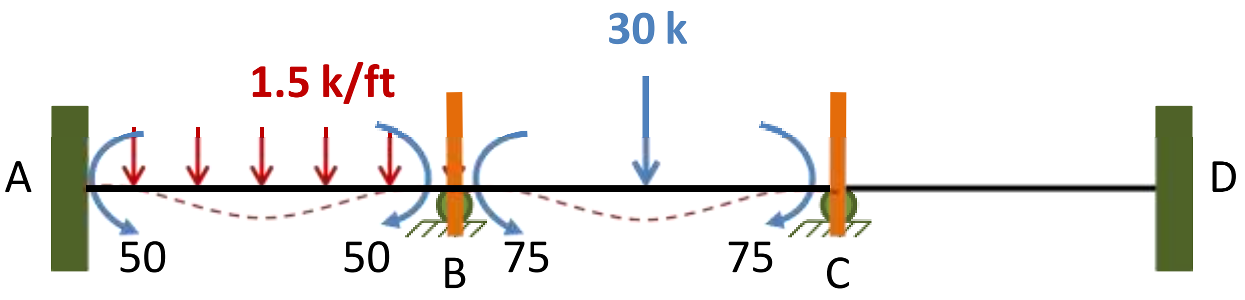
Note that the sum of distribution factors at each joint must always equal 1. The DF are recorded in boxes directly beneath the corresponding member ends on top of the MD Table.

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors

0.5	0.5	0.429	0.571



Fixed End Moments

Next, by assuming that joints B and C are restrained against rotation by imaginary clamps applied to them, we calculate the FEM that develop at the ends of each member. (1. line MD Table)

$$FEM_{AB} = \frac{1.5\ 20^{\ 2}}{12} = 50\ \mathrm{k - ft} \circlearrowleft \qquad \text{or} \qquad +50\ \mathrm{k\text{-}ft}$$

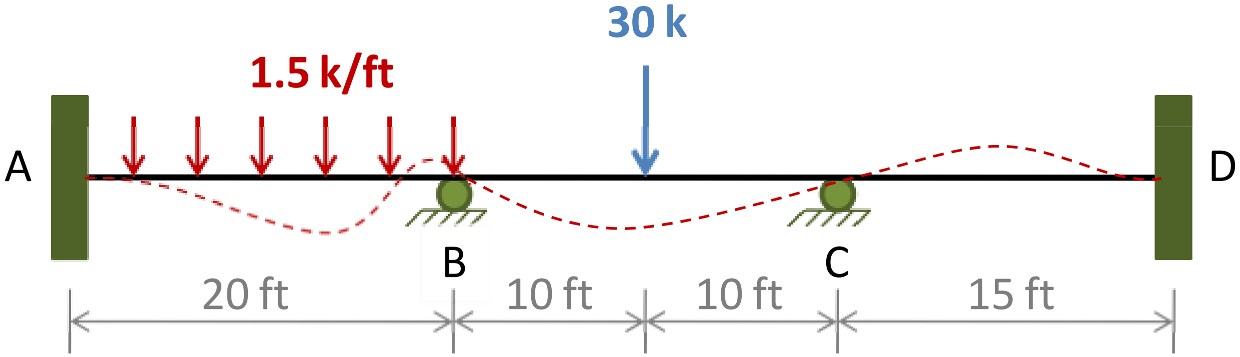
$$FEM_{BA} = \frac{1.5(20)^{2}}{12} = 50\ \mathrm{k - ft} \circlearrowright \qquad \text{or} \qquad -50\ \mathrm{k\text{-}ft}$$

$$FEM_{BC} = \frac{30(20)}{8} = 75\ \mathrm{k - ft} \circlearrowleft \qquad \text{or} \qquad +75\ \mathrm{k\text{-}ft}$$

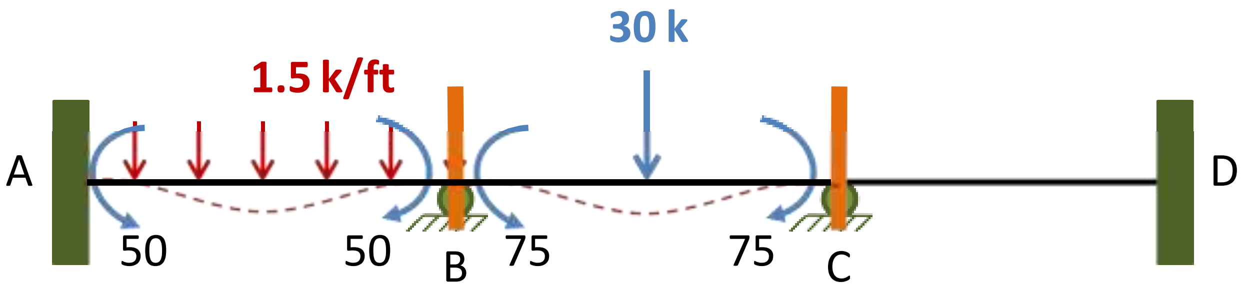
$$FEM_{CB} = 75\ \mathrm{k - ft} \circlearrowright \qquad \text{or} \qquad -75\ \mathrm{k\text{-}ft}$$

$$FEM_{CD} = FEM_{DC} = 0$$

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		

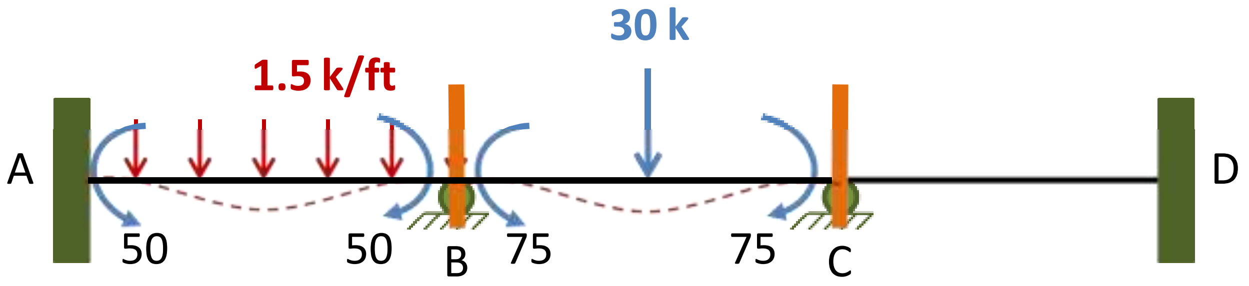


Balancing Joint *C*

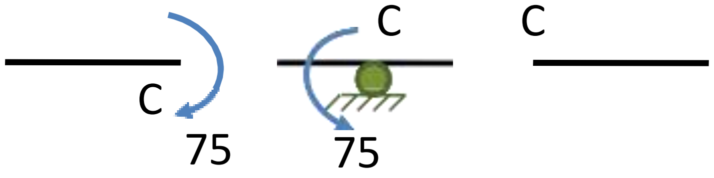
Since joints B and C are actually not clamped, we release them, one at a time. Let us begin at joint C.

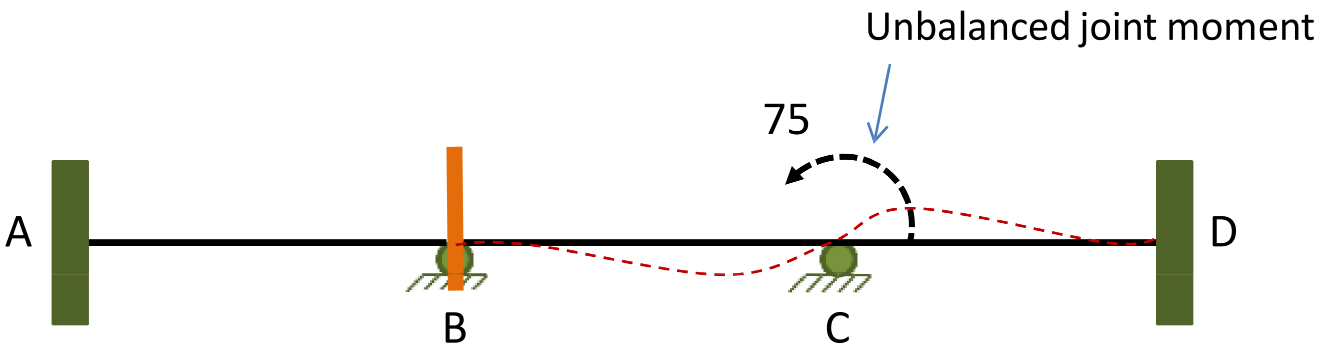
From fig. we can see that there is a -75 k-ft (clockwise) FEM at end C of member BC, whereas no moment exists at end C of member CD.

As long as joint C is restrained against rotation by the clamp, the -75 k-ft unbalanced moment is absorbed by the clamp.



When the imaginary clamp is removed to release the joint, the -75 k-ft unbalanced moment acts at the joint, causing it to rotate in the CCW direction until it is in equilibrium.

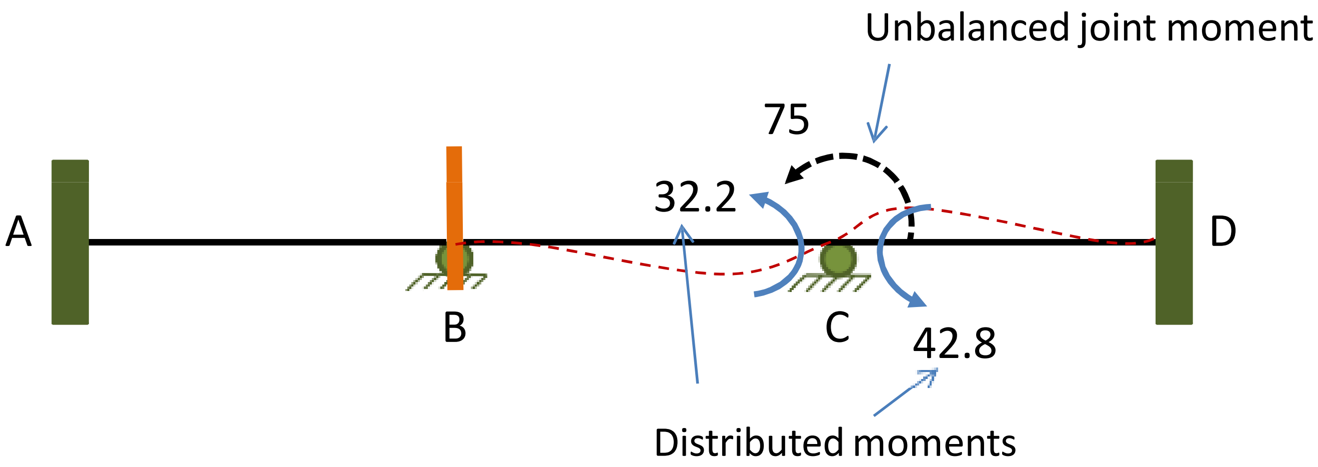




The rotation of joint C causes the distributed moments, DM_{CB} and DM_{CD}, to develop at ends C of members BC and CD, which can be evaluated by multiplying the negative of the unbalanced moment (+75 k-ft) by distribution factors DF_{CB} and DF_{CD}, respectively.

$$DM_{CB} = 0.429(+75) = +32.2\ \mathrm{k-ft}$$

$$DM_{CD} = 0.571(+75) = +42.8\ \mathrm{k-ft}$$

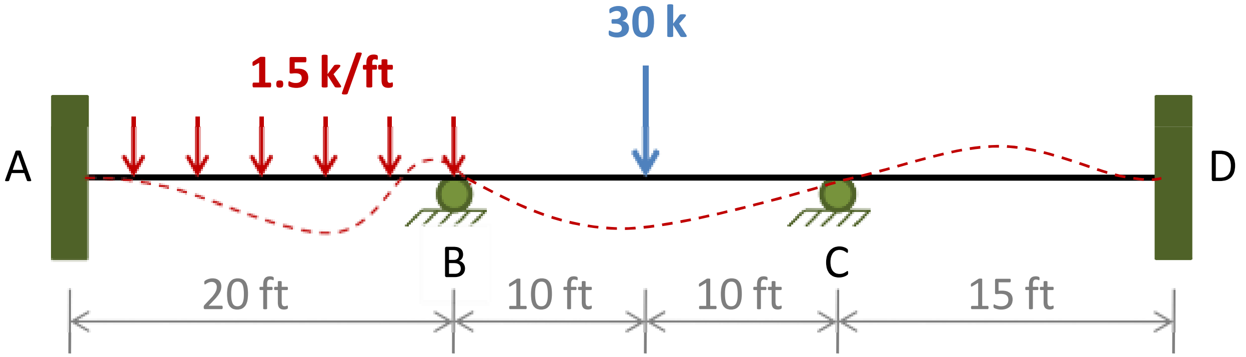


These distributed moments are recorded in line 2 of the MD Table, and a line is drawn beneath them to indicate that joint C is now balanced.

EI = constant

E = 29,000 ksi

I = 500 in^4



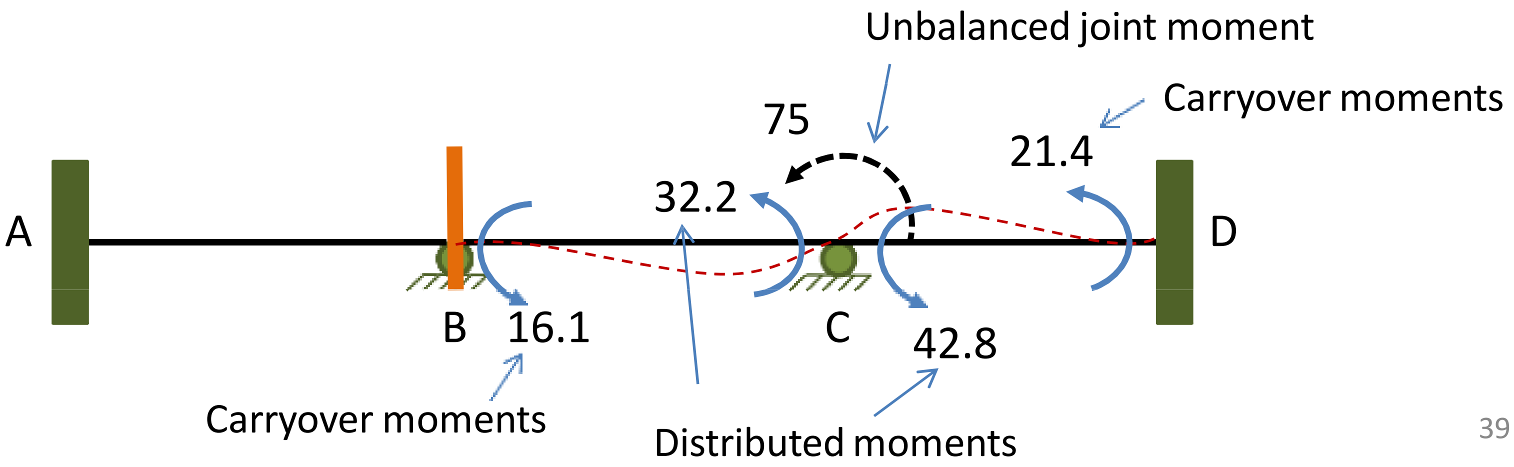
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover				+32.2	+42.8	

The DM at end C of member BC induces a COM at the far end B, which can be determined by multiplying the DM by the COF of the member.

Since joint B remains clamped, the COF of the member BC is ½ (Eq.13). Thus, COM at the end B of member BC is

$$COM_{BC} = COF_{CB}\left(DM_{CB}\right) = \frac{1}{2}\left(+32.2\right) = +16.1\ \text{k}-\text{ft}$$

$$COM_{DC} = COF_{CD}\left(DM_{CD}\right) = \frac{1}{2}\left(+42.8\right) = +21.4\ \text{k}-\text{ft}$$



Unbalanced joint moment

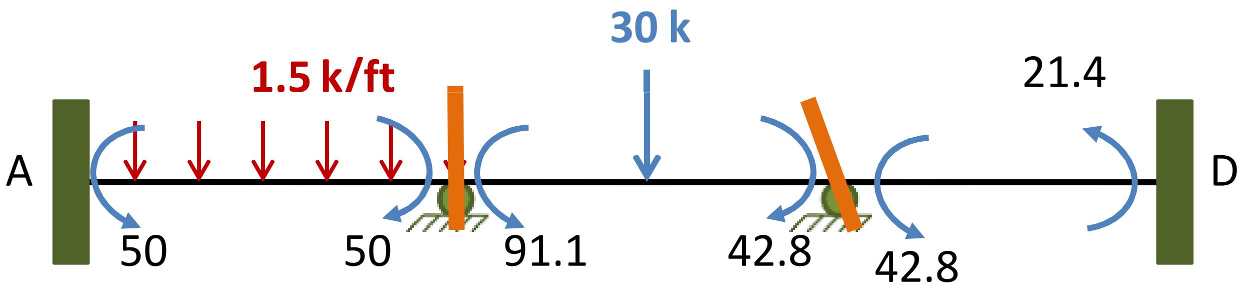
Carryover moments

Carryover moments

Distributed moments

These COM are recorded on the same line of the MD Table as the DM, with a horizontal arrow from each DM to its COM.

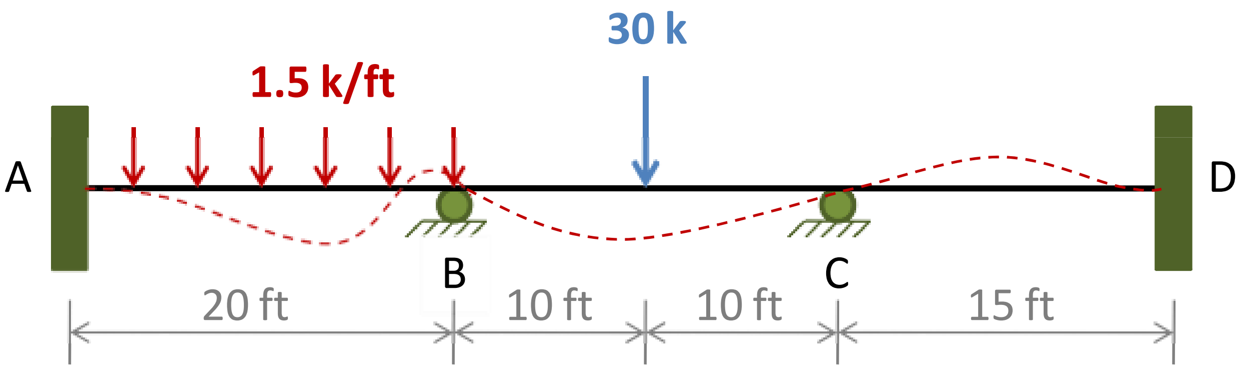
The total member end moments at this point in this analysis are depicted in Figure.



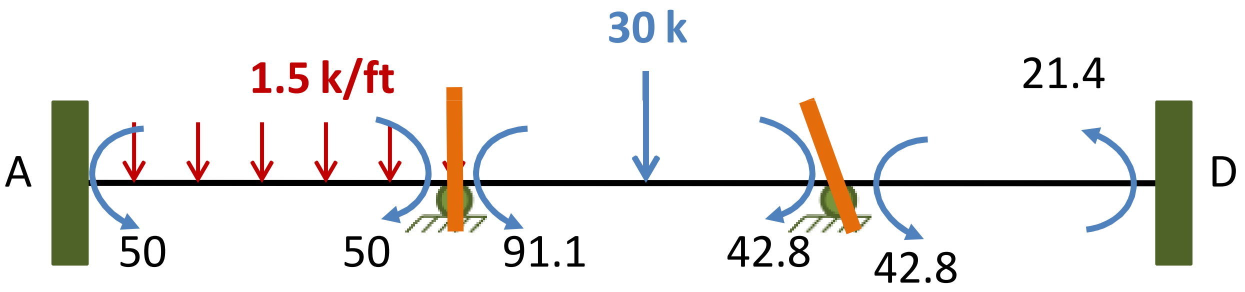
It can be seen that joint C is now in equilibrium, because it is subjected to two equal, but opposite moments.

Joint B, however, is not in equilibrium, and it needs to be balanced. Before we release joint B, an imaginary clamp is applied to joint C in its rotated position.

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1 ←	+32.2	+42.8 →	+24.1



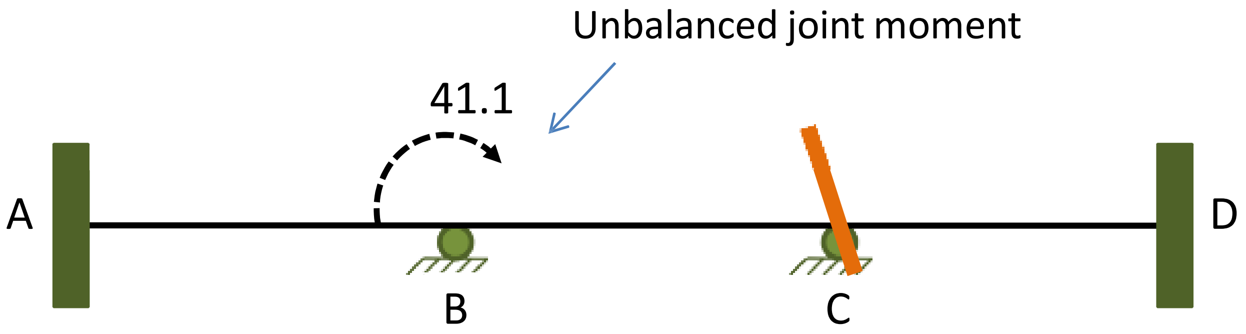
Balancing Joint *B*

Joint B is now released. The unbalanced moment at this joint is obtained by summing all the moments acting at the ends B of members AB and BC, which are rigidly connected to joint B.

From the MD Table (lines 1 & 2), we can see that there is a -50 k-ft FEM at end B of member AB, whereas the end B of member BC is subjected to a +75 k-ft FEM and a +16.1 k-ft COM. The unbalanced moment at joint B is

$$UM_B = -50 + 75 + 16.1 = +41.1 \text{ k} - \text{ft}$$

This UM causes joint B to rotate, as shown, and induces DM at ends B of member AB and BC.



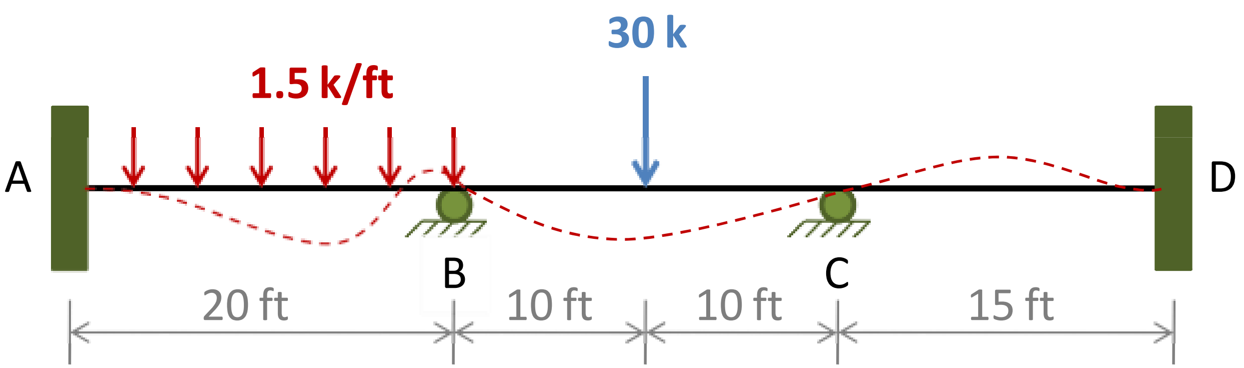
The DM are evaluated by multiplying the negative of the UM by the distribution factors:

$$DM_{BA} = 0.5(-41.1) = -20.6\,\mathrm{k-ft}$$

$$DM_{BC} = 0.5(-41.1) = -20.6\,\mathrm{k-ft}$$

These DM are recorded on line 3 of the MD Table and a line is drawn beneath them to indicate that joint B is now balanced.

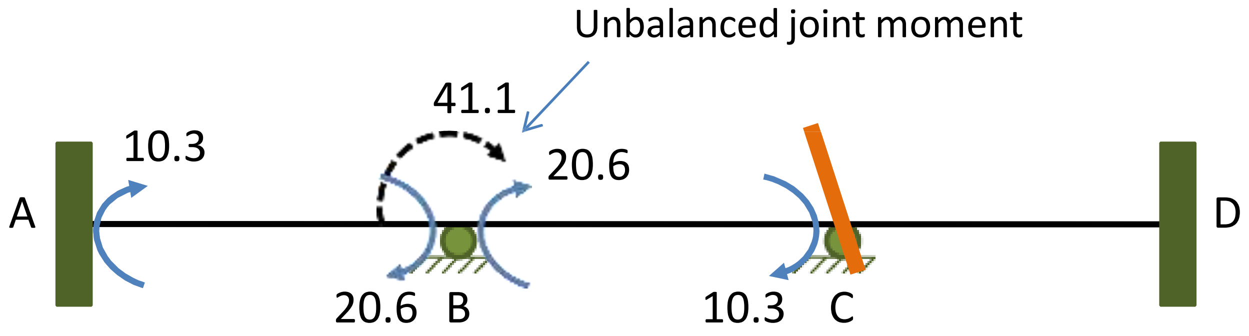
EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors

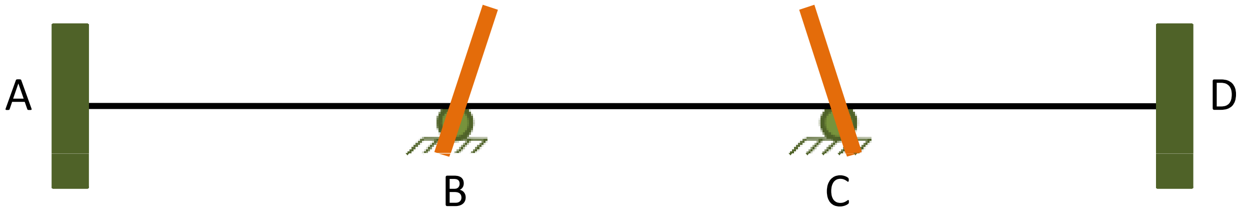
1. Fixed-end Moments
2. Balance joint C and carryover
3. Balance joint B and carryover

	0.5	0.5	0.429	0.571	
+50	-50	+75	-75		
		+16.1 ←	+32.2	+42.8 →	+24.1
-10.3 ←	-20.6	-20.6 →	-10.3		



One-half of the DM are then carried over to the far ends A and C of members AB and BC, respectively, as indicated by the horizontal arrows on line 3 of Table.

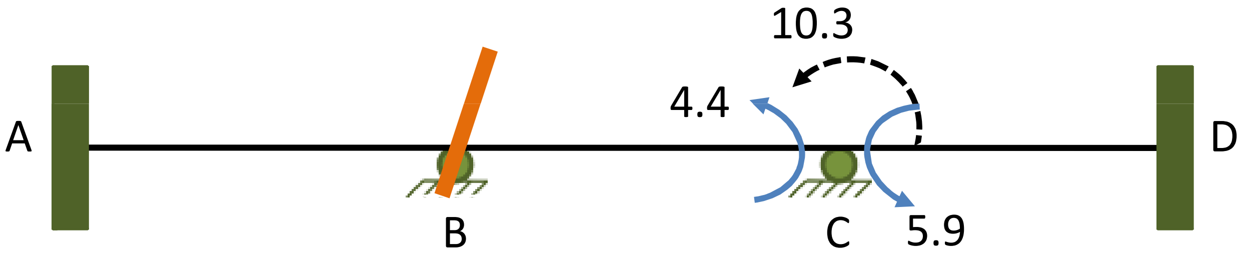
Joint B is then reclamped in its rotated position.



Balancing Joint *C*

With joint B now balanced, we can see from the MD Table (line 3) that, due to the carryover effect, there is a -10.3 k-ft UM at joint C.

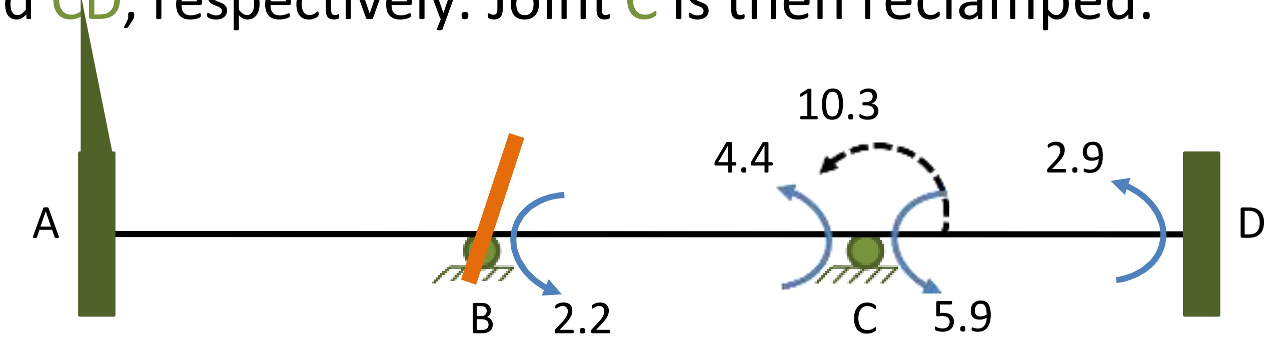
Recall that the moments above the horizontal line at joint C were balanced previously. Thus we release joint C again and distribute the UM to ends C of members BC and CD as



$$DM_{CB} = 0.429(+10.3) = +4.4\ \mathrm{k-ft}$$

$$DM_{CD} = 0.571(+10.3) = +5.9\ \mathrm{k-ft}$$

The DM are recorded on line 4 of the MD Table, and one-half of these moments are carried over to the ends B and D of members BC and CD, respectively. Joint C is then reclamped.



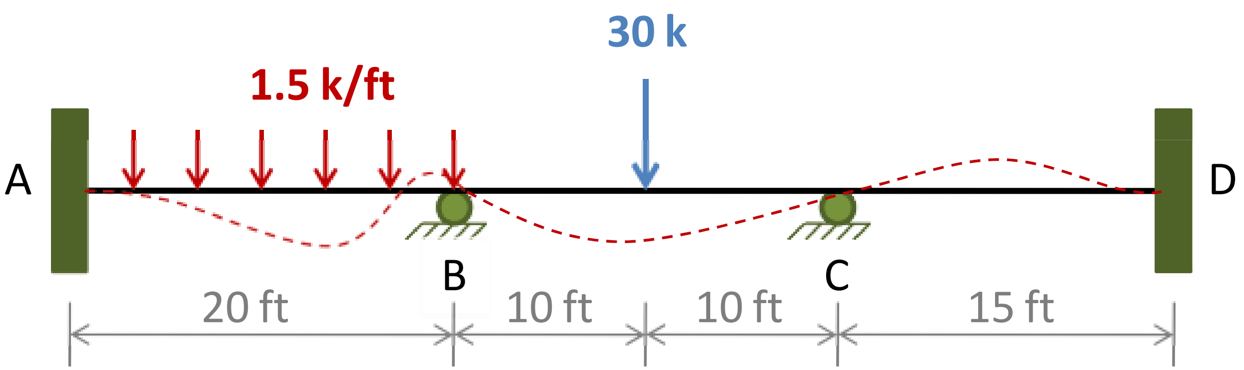
Balancing Joint *B*

The +2.2 k-ft UM at joint B (line 4) is balanced in a similar manner.

The DM and COM thus computed are shown on line 5 of the MD Table (slide 49).

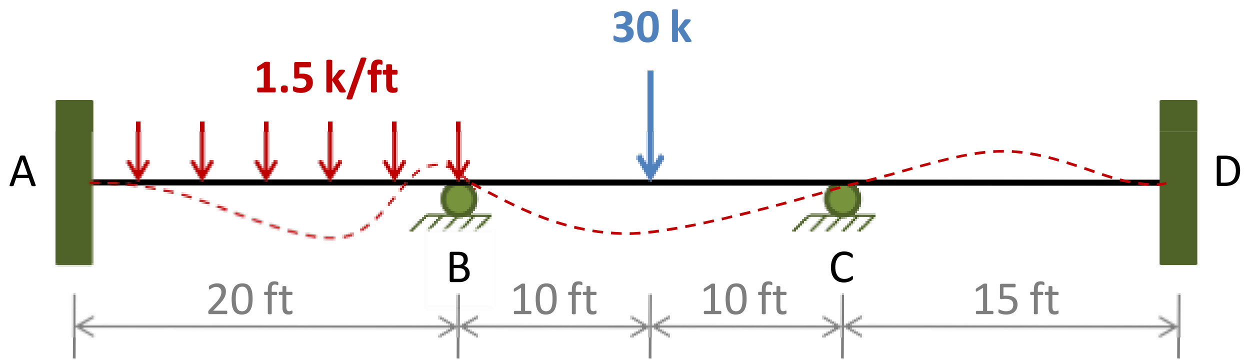
Joint B is then reclamped.

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1	+32.2	+42.8	+24.1
3.Balance joint B and carryover	-10.3	-20.6	-20.6	-10.3		
4.Balance joint C and carryover			+2.2	+4.4	+5.9	+2.9

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1	+32.2	+42.8	+24.1
3.Balance joint B and carryover	-10.3	-20.6	-20.6	-10.3		
4.Balance joint C and carryover			+2.2	+4.4	+5.9	+2.9
5.Balance joint B and carryover	-0.6	-1.1	-1.1	-0.6		

It can be seen from line 5 of the MD Table that the UM at joint C has now been reduced to only -0.6 k-ft.

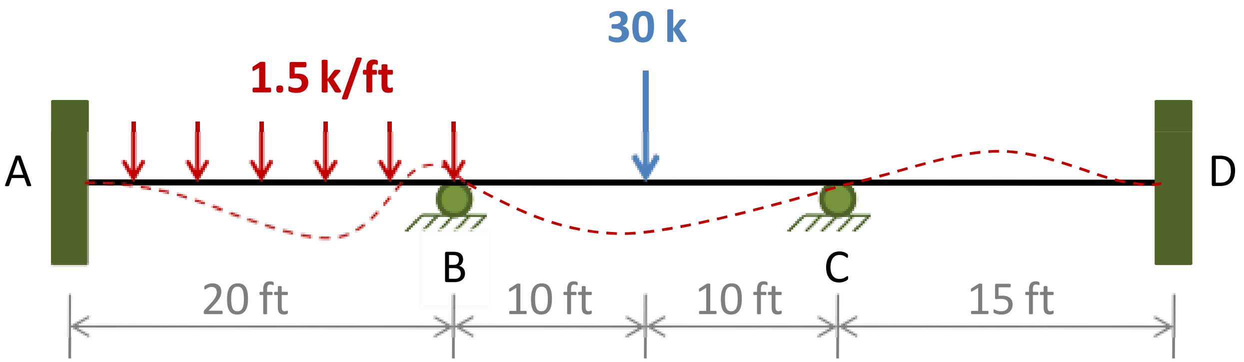
Another balancing of joint C produces an even smaller unbalanced moment of +0.2 k-ft at joint B, as shown on line 6 of the MD Table.

Since the DM induced by this unbalancing moment are negligibly small, we end the moment distribution process.

The final member end moments are obtained by algebraically summing the entries in each column of the MD Table.

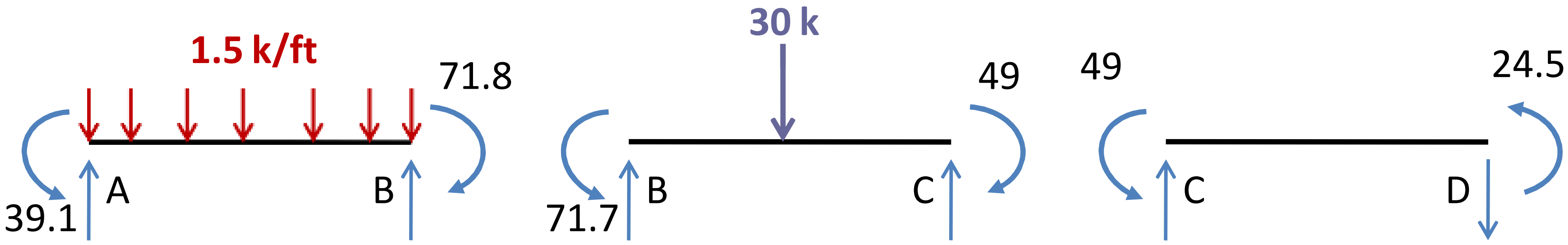
The final Moments are recorded on line 8 of The MD Table.

EI = constant
E = 29,000 ksi
I = 500 in^4



Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance joint C and carryover			+16.1	+32.2	+42.8	+24.1
3.Balance joint B and carryover	-10.3	-20.6	-20.6	-10.3		
4.Balance joint C and carryover			+2.2	+4.4	+5.9	+2.9
5.Balance joint B and carryover	-0.6	-1.1	-1.1	-0.6		
6.Balance joint C and carryover			+0.2	+0.3	+0.3	+0.2
7.Balance joint B		-0.1	-0.1			
8.Final Moments	+39.1	-71.8	+71.7	-49	+49	+24.5

The final moments are shown on the free body diagrams of members in Fig.



With the MEM known, member end shears and support reactions can now be determined by considering the equilibrium of members and joints.

SFD and BMD are same to those which are drawn in Slope Deflection Method for the same beam.

Practical Application of the MDM

The foregoing approach provides the clearer insight into the basic concept of the MDM.

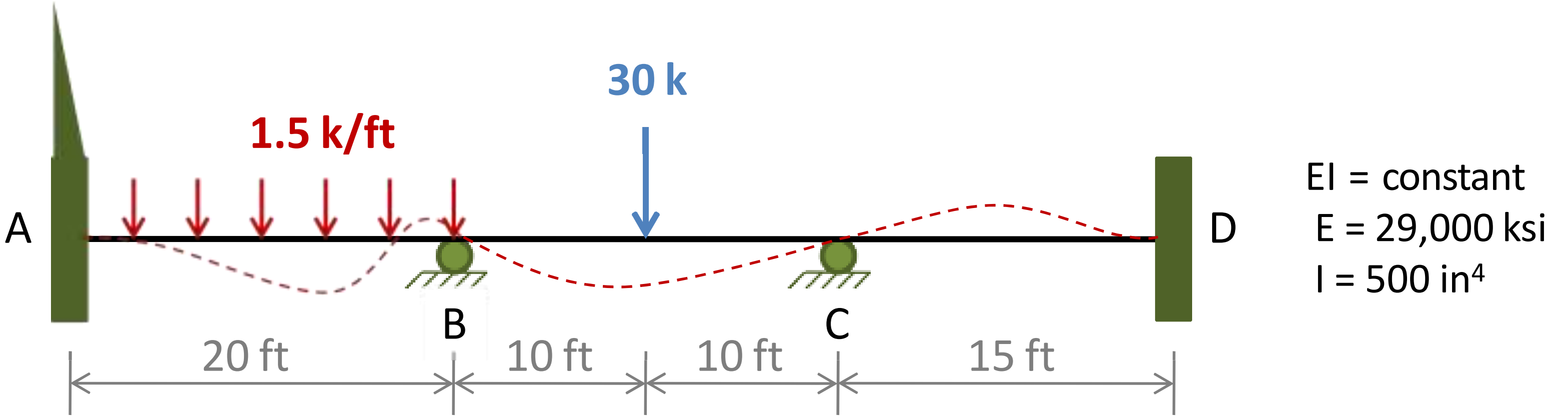
From a practical point of view, it is usually more convenient to use an alternative approach in which all the joints of the structure that are free to rotate are balanced simultaneously in the same step.

All the COMs that are induced at the far ends of the members are then computed simultaneously in the following step.

The process of balancing of joints and COMs is repeated until the UMs at the joints are negligibly small.

Practical Application of the MDM

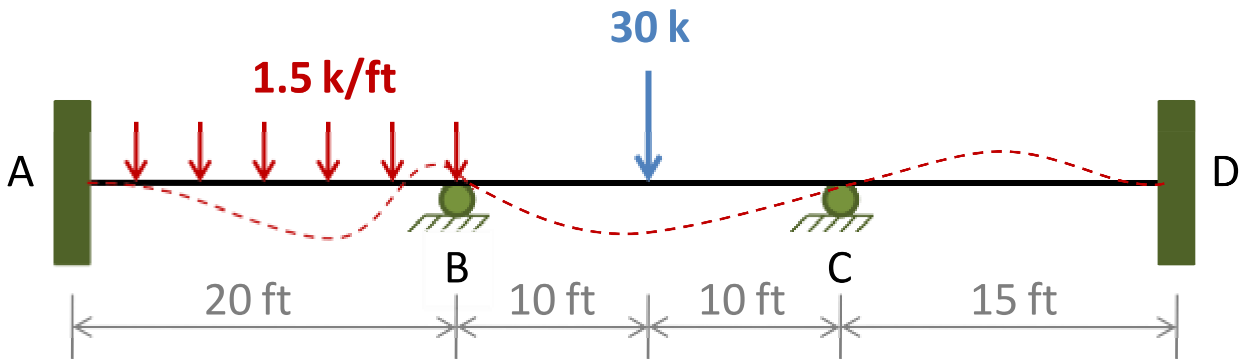
Consider again the three span continuous beam shown in figure.



The MD Table used for carrying out the computations is shown in the next slide.

The previously computed distribution factors and FEMs are recorded on the top and the first line, respectively of the table.

EI = constant
E = 29,000 ksi
I = 500 in^4



	AB	BA	BC	CB	CD	DC
Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		

The MD process is started by balancing joints B and C.

From line 1 of the MD Table we can see that the UM at joint B is

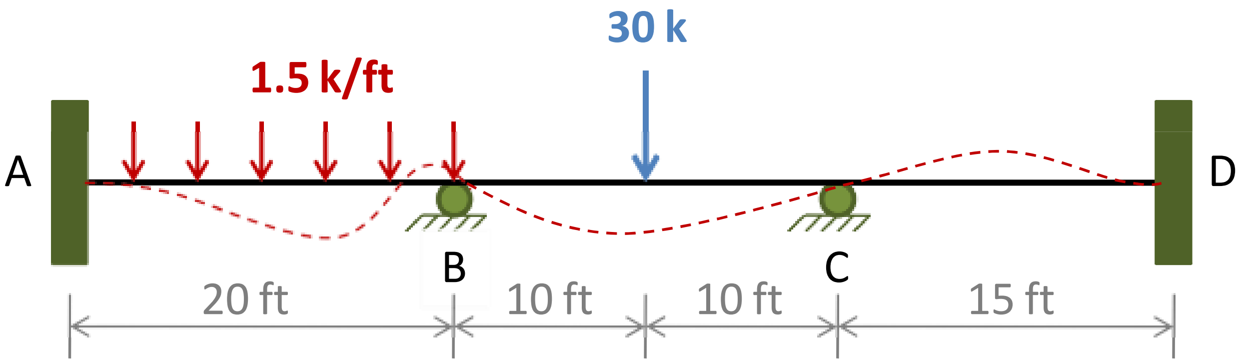
$$UM_B = -50 + 75 = +25 \text{ k} - \text{ft}$$

The balancing of joint B induces DMs at ends B of members AB and BC, which can be evaluated by multiplying the negative of the UM by the distribution factor.

$$DM_{BA} = 0.5(-25) = -12.5 \text{ k} - \text{ft}$$

$$DM_{BC} = 0.5(-25) = -12.5 \text{ k} - \text{ft}$$

EI = constant
E = 29,000 ksi
I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5			

Joint C is then balanced in a similar manner.

From line 1 of the MD Table, we can see that the UM at joint C is

$$UM_C = -75 \text{ k} - \text{ft}$$

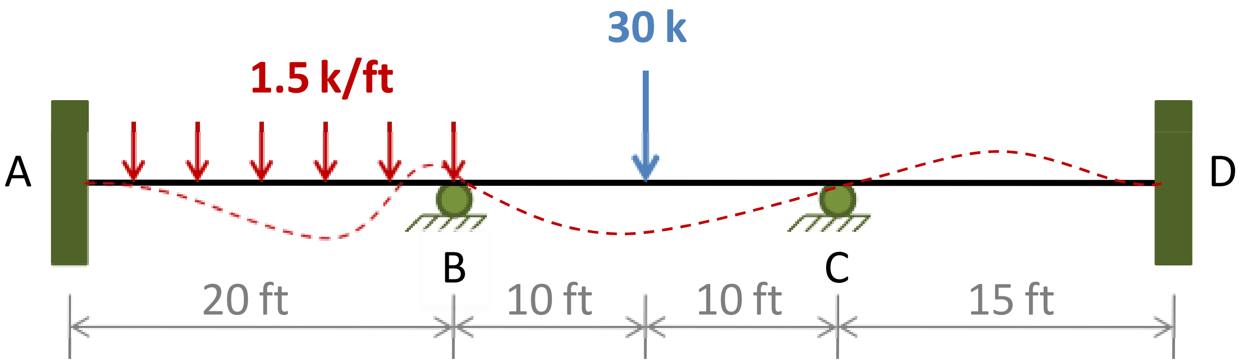
The balancing of joint C induces the following DMs at ends C of members BC and CD, respectively

$$DM_{CB} = 0.429(+75) = +32.2 \text{ k} - \text{ft}$$

$$DM_{CD} = 0.571(+75) = +42.8 \text{ k} - \text{ft}$$

The four DMs are recorded on line 2 on the MD Table, and a line is drawn beneath them, across the entire width of the table, to indicate that all the joints are now balanced.

EI = constant
E = 29,000 ksi
I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	

In the next step of analysis, the COMs that develops at the far ends of the members are computed by multiplying the distributed moments by the COFs.

$$COM_{AB} = \frac{1}{2}\left(DM_{BA}\right) = \frac{1}{2}\left(-12.5\right) = -6.3 \ \text{k} - \text{ft}$$

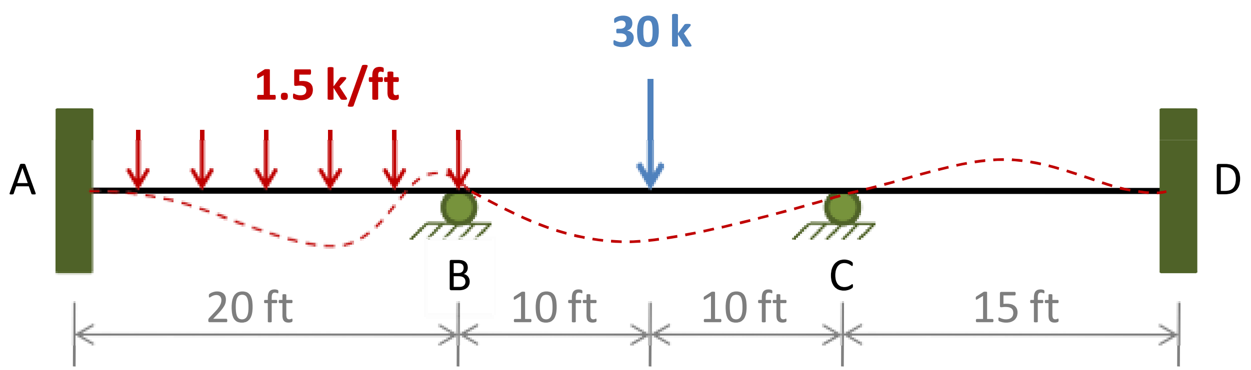
$$COM_{CB} = \frac{1}{2}\left(DM_{BC}\right) = \frac{1}{2}\left(-12.5\right) = -6.3 \ \text{k} - \text{ft}$$

$$COM_{BC} = \frac{1}{2}\left(DM_{CB}\right) = \frac{1}{2}\left(+32.2\right) = +16.1 \ \text{k} - \text{ft}$$

$$COM_{DC} = \frac{1}{2}\left(DM_{CD}\right) = \frac{1}{2}\left(+42.8\right) = +21.4 \ \text{k} - \text{ft}$$

These COMs are recorded on the line 3 of the MD Table, with an inclined arrow pointing from each DM to its COM in the next slide.

EI = constant
E = 29,000 ksi
I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	
3.Carryover	-6.3		+16.1	-6.3		+21.4

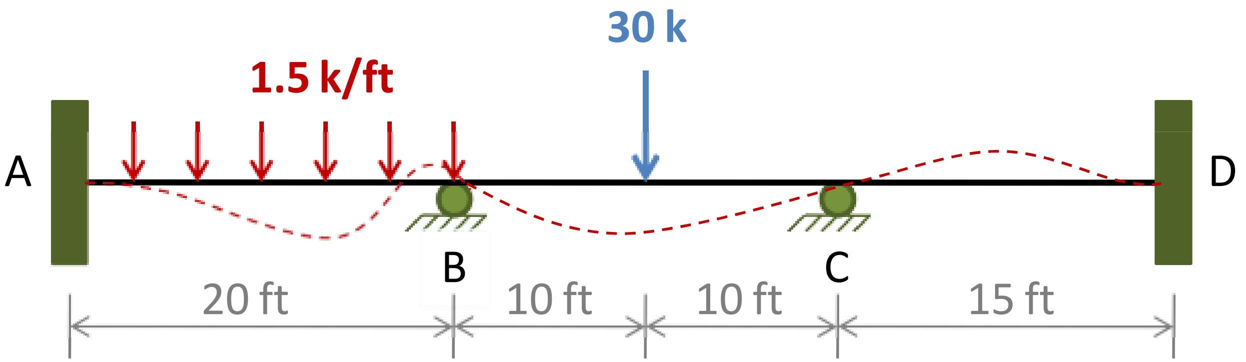
We can see from line 3 of MD Table that, due to the carryover effects, there are now +16.1 k-ft and -6.3 k-ft unbalanced moments at joints B and C, respectively.

Thus these joints are balanced again, and the DMs thus obtained are recorded on the line 4 of the MD Table.

One-half of the DMs are then carried over to the far ends of the members (line 5), and the process is continues until the UMs are negligibly small.

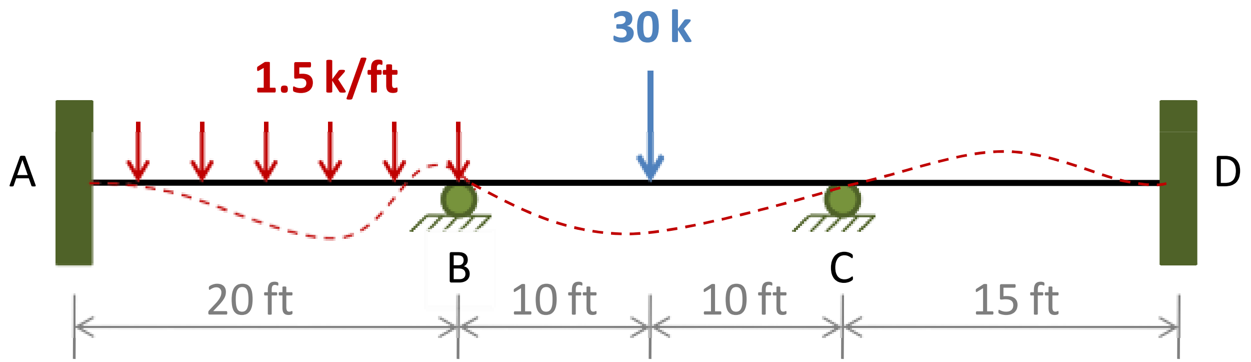
The final MEMs, obtained by algebraically summing the entries in each column of the MD Table, are recorded on line 11 of the table.

EI = constant
E = 29,000 ksi
I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	
3.Carryover	-6.3		+16.1	-6.3		+21.4
4.Balance Joints		-8.1	-8.1	+2.7	+3.6	
5.Carryover	-4.1		+1.4	-4.1		+1.8
6.Balance Joints		-0.7	-0.7	+1.8	+2.3	
7.Carryover	-0.4		+0.9	-0.4		+1.2
8.Balance Joints		-0.5	-0.5	+0.2	+0.2	
9.Carryover	-0.3		+0.1	-0.3		+0.1
10.Balance Joints		-0.05	-0.05	+0.1	+0.2	
11.Final Moments	**+38.9**	**-71.8**	**+71.7**	**-49.1**	**+49.1**	**+24.5**

EI = constant
E = 29,000 ksi
I = 500 in^4



Member Ends	AB	BA	BC	CB	CD	DC
Distribution Factors		0.5	0.5	0.429	0.571	
1.Fixed-end Moments	+50	-50	+75	-75		
2.Balance Joints		-12.5	-12.5	+32.2	+42.8	
3.Carryover	-6.3		+16.1	-6.3		+21.4
4.Balance Joints		-8.1	-8.1	+2.7	+3.6	
5.Carryover	-4.1		+1.4	-4.1		+1.8
6.Balance Joints		-0.7	-0.7	+1.8	+2.3	
7.Carryover	-0.4		+0.9	-0.4		+1.2
8.Balance Joints		-0.5	-0.5	+0.2	+0.2	
9.Carryover	-0.3		+0.1	-0.3		+0.1
10.Balance Joints		-0.05	-0.05	+0.1	+0.2	
11.Final Moments	**+38.9**	**-71.8**	**+71.7**	**-49.1**	**+49.1**	**+24.5**

Flow Chart for MDM

Calculate Distribution Factors, $DF = \dfrac{K}{\sum K}$

↓

Calculate Fixed End Moments

↓

Balance the Moments at All Joints Free to Rotate

↓

Evaluate UMs and then Find DMs

↓

Find Carryover Moments

↓

Repeat the Above Two Steps Until the UMs are Negligibly Small

↓

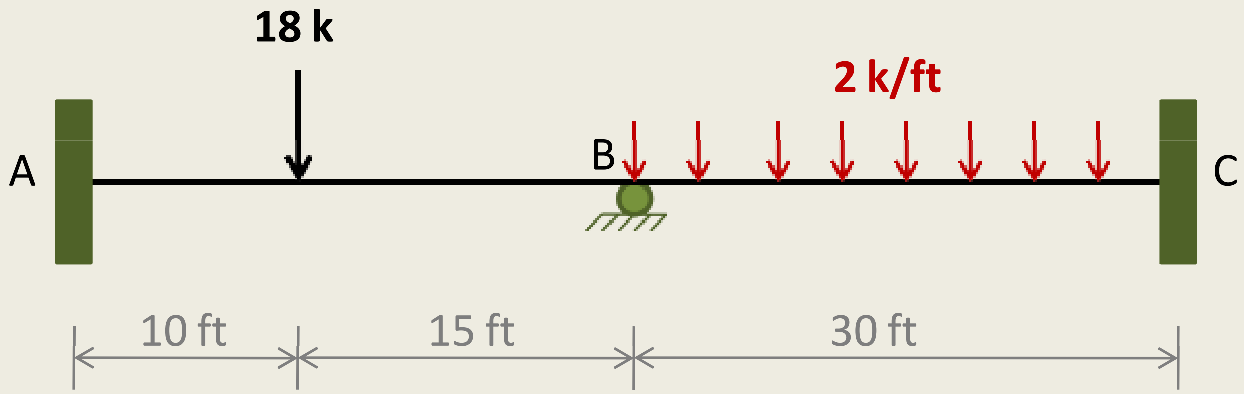
Determine the Final End Moments

↓

Compute Member End Shears, Determine Support Reactions, and draw SFD & BMD

Example 1

- Determine the reactions and draw the shear and bending moment diagrams for the two-span continuous beam shown in Figure.

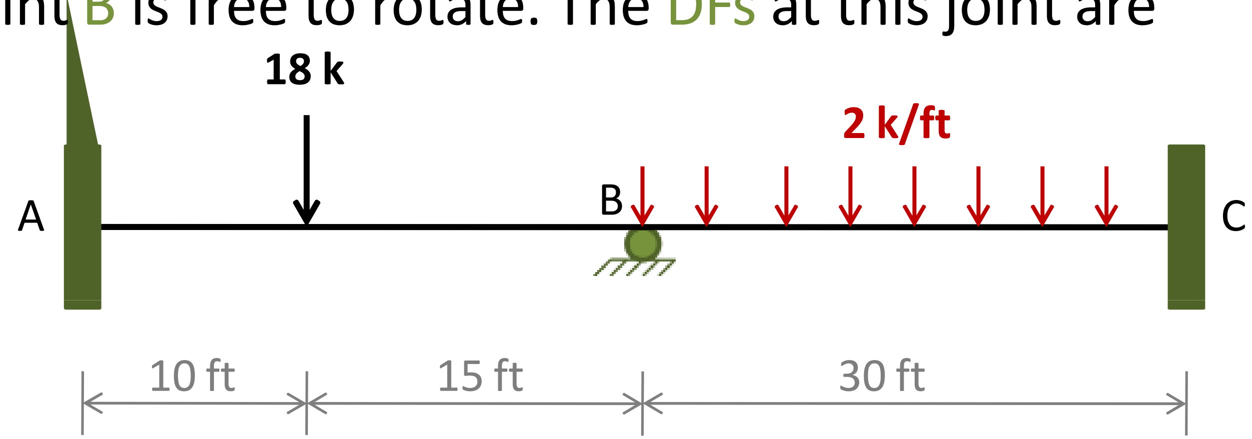


EI = constant

Solution

1.Distribution Factors

Only joint B is free to rotate. The DFs at this joint are

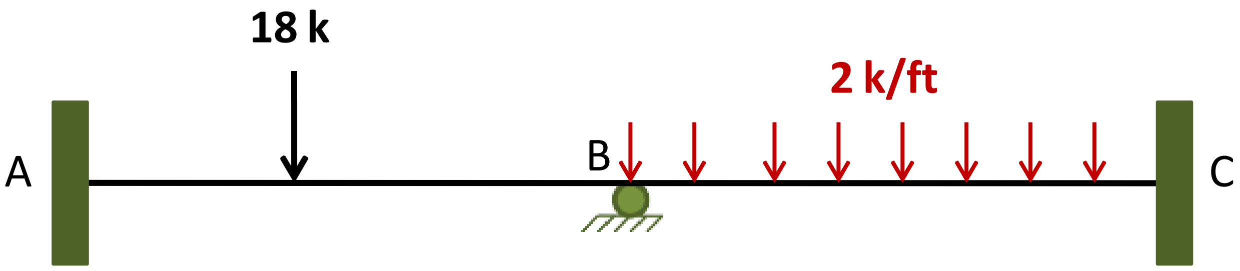


$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{I/25}{(I/25) + (I/30)} = 0.545$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{I/30}{(I/25) + (I/30)} = 0.455$$

$$DF_{BA} + DF_{BC} = 0.545 + 0.455 = 1$$ Checks

EI = constant

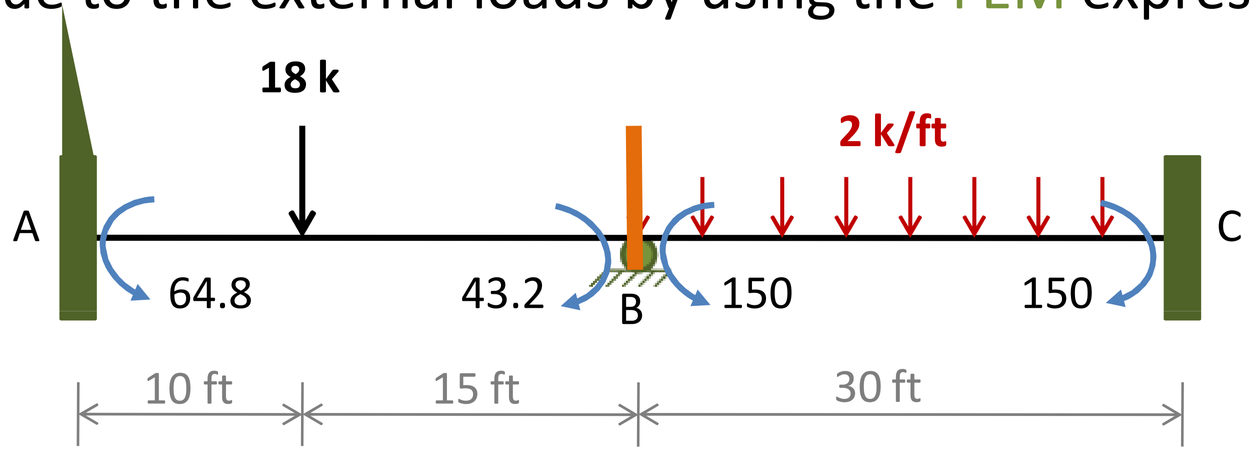


Distribution Factors

0.545	0.455

2.Fixed-End Moments (FEMs)

Assuming that joint B is clamped against rotation, we calculate the FEMs due to the external loads by using the FEM expressions



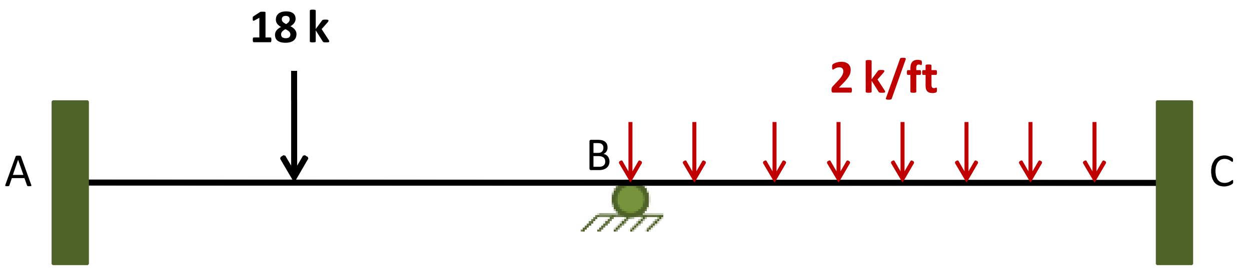
$$FEM_{AB} = \frac{18(10)(15)^2}{(25)^2} = 64.8\ \text{k}-\text{ft} \circlearrowleft \qquad \text{or} \qquad +64.8\ \text{k-ft}$$

$$FEM_{BA} = \frac{18(10)^2(15)}{(25)^2} = 43.2\ \text{k}-\text{ft} \circlearrowright \qquad \text{or} \qquad -43.2\ \text{k-ft}$$

$$FEM_{BC} = \frac{2(30)^2}{12} = 150\ \text{k}-\text{ft} \circlearrowleft \qquad \text{or} \qquad +150\ \text{k-ft}$$

$$FEM_{CB} = \frac{2(30)^2}{12} = 150\ \text{k}-\text{ft} \circlearrowright \qquad \text{or} \qquad -150\ \text{k-ft}$$

EI = constant



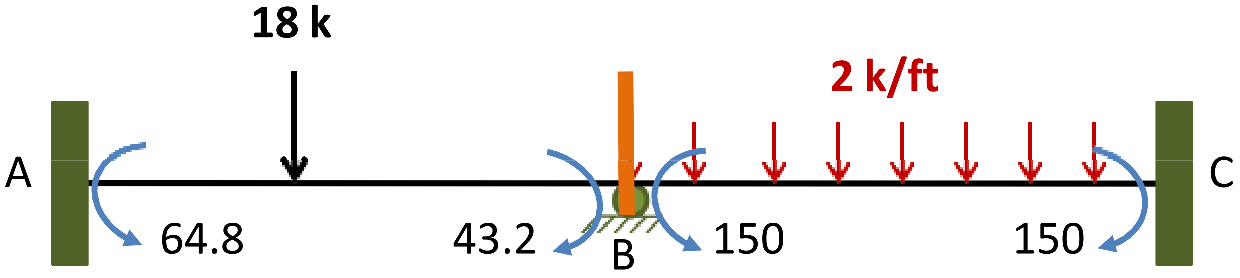
Distribution Factors

1.Fixed-end Moments

AB	BA	BC	CB
	0.545	0.455	
+64.8	-43.2	+150	-150

3.Moment Distribution

Since Joint B is actually not clamped, we release the joint and determine the unbalanced moment (UM) acting on it by summing the moments at ends B of members AB and BC



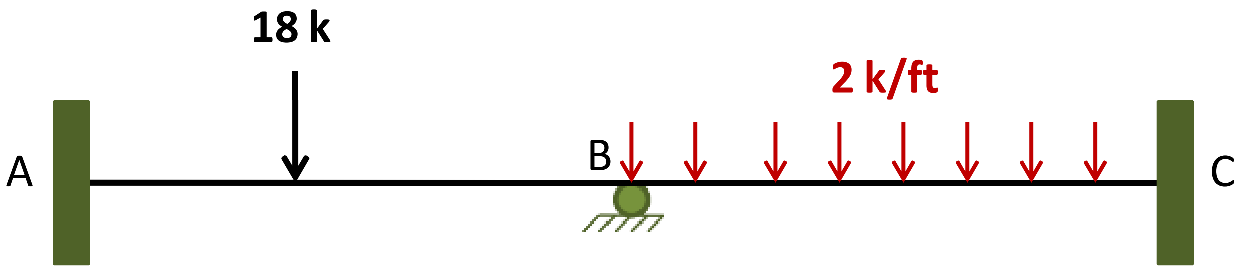
$$UM_B = -43.2 + 150 = +106.8 \ \ \text{k} - \text{ft}$$

The DMs due to these UMs at end B of member AB and BC are determined by multiplying the negative of the UM by the DF

$$DM_{BA} = DF_{BA} \quad UM_B \ = 0.545 \quad 106.8 = \quad 58.2 \ \ \text{k} \quad \text{ft}$$

$$DM_{BC} = DF_{BC}\left(-UM_B\right) = 0.455\left(-106.8\right) = -48.6 \ \ \text{k} - \text{ft}$$

EI = constant



	AB	BA	BC	CB
Distribution Factors		0.545	0.455	
1.Fixed-end Moments	+64.8	-43.2	+150	-150
2.Balance Joint B		-58.2	-48.6	

3.Moment Distribution

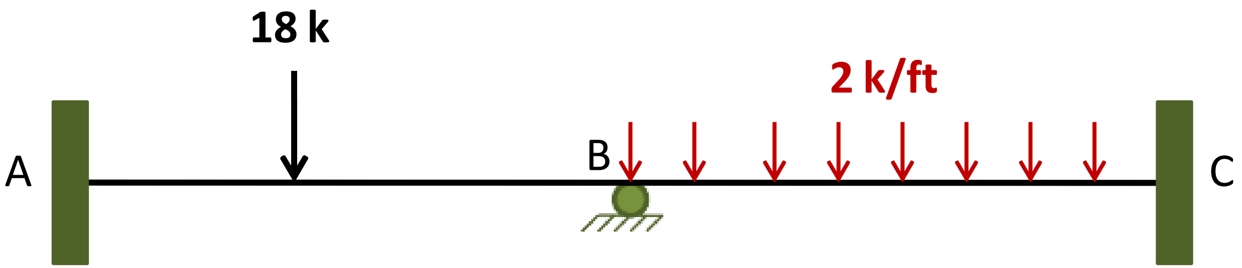
The COMs at the far ends A and C of members AB and BC, respectively, are then computed as

$$COM_{AB} = \frac{1}{2}\left(DM_{BA}\right) = \frac{1}{2}\left(-58.2\right) = -29.1 \text{ k} - \text{ft}$$

$$COM_{CB} = \frac{1}{2}\left(DM_{BC}\right) = \frac{1}{2}\left(-48.6\right) = -24.3 \text{ k} - \text{ft}$$

Joint B is the only joint of the structure that is free to rotate, and because it has been balanced, we end the moment distribution process.

EI = constant



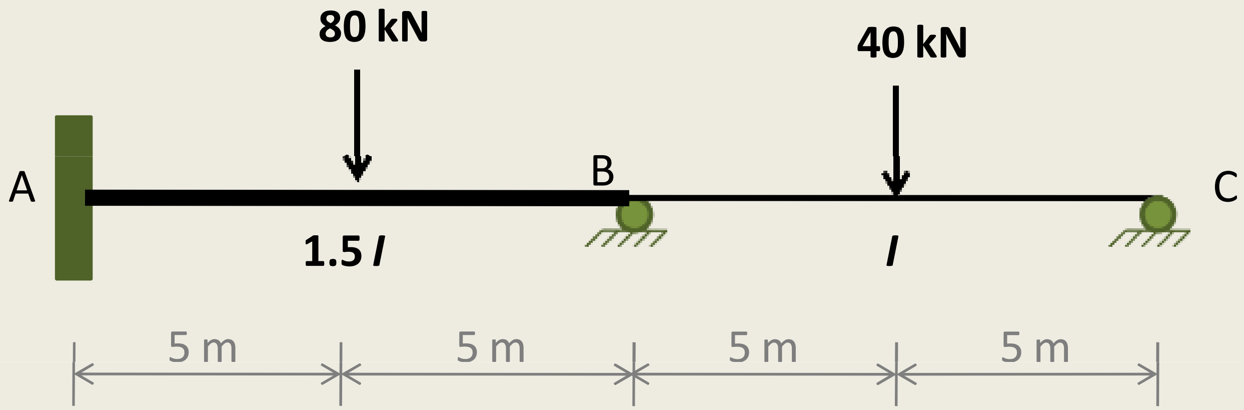
	AB	BA	BC	CB
Distribution Factors		0.545	0.455	
1.Fixed-end Moments	+64.8	-43.2	+150	-150
2.Balance Joint B		-58.2	-48.6	
3.Carryover	-29.1			-24.3
4.Final Moments	+35.7	-101.4	+101.4	-174.3

Member End Shears, Support Reactions, SFD & BMD

See Example 1 in Slope-Deflection Method

Example 2

- Determine the reactions and draw the shear and bending moment diagrams for the two-span continuous beam shown in Figure.

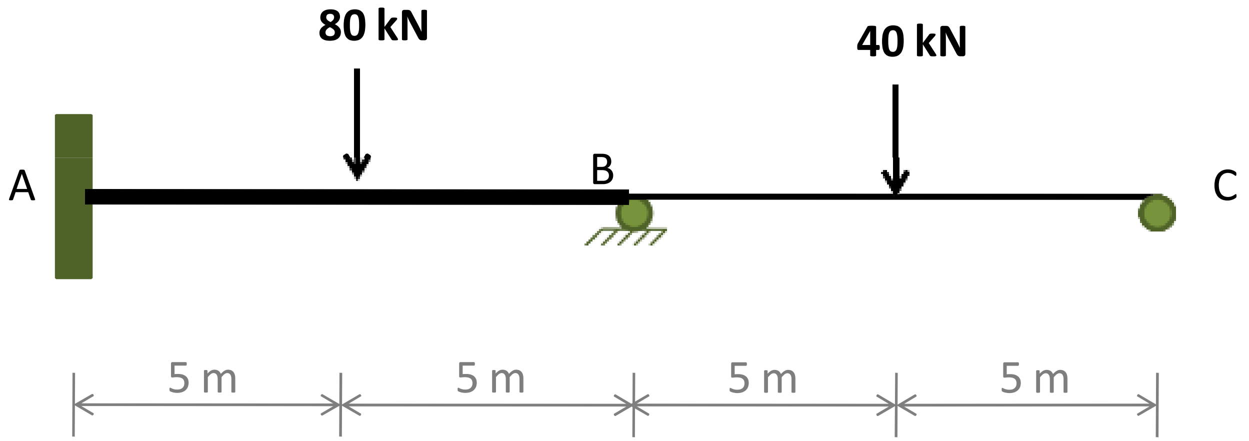


E = constant

Solution

1. Distribution Factors

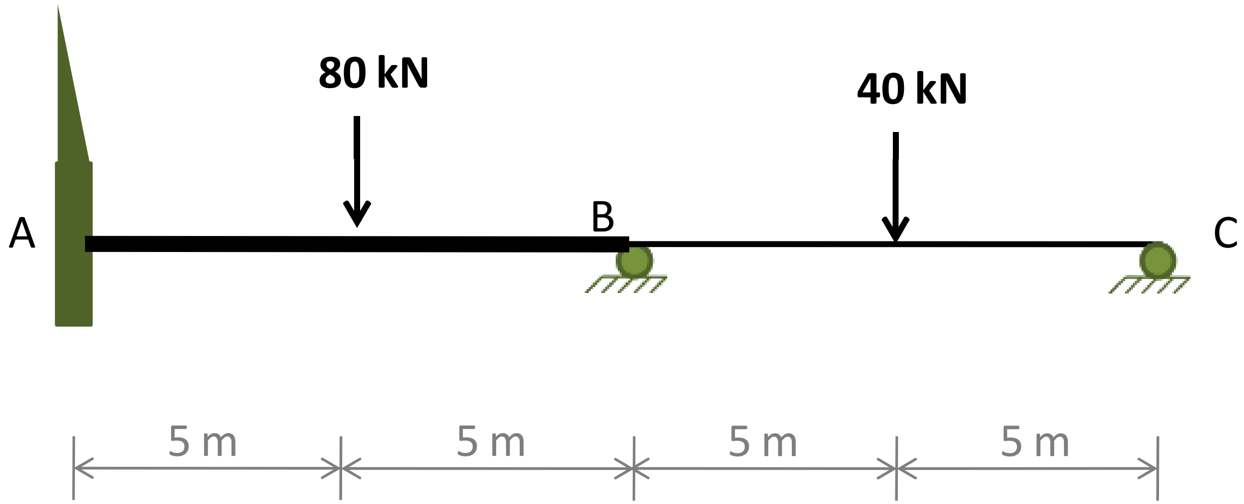
Joints B and C of the continuous beam are free to rotate. The DFs at joint B are



$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{1.5I/10}{(1.5I/10) + (I/10)} = 0.6$$

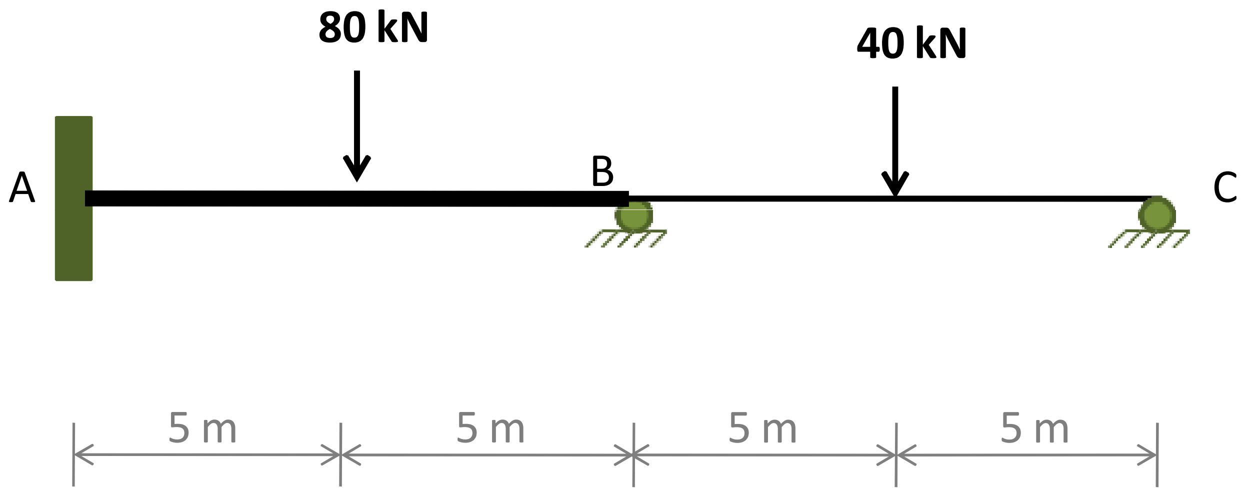
$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{I/10}{(1.5I/10) + (I/10)} = 0.4$$

Similarly, at joint C,



$$DF_{CB} = \frac{K_{CB}}{K_{CB}} = \frac{0.1I}{0.1I} = 1$$

2. Fixed-End Moments



$$FEM_{AB} = \frac{+80\ 10}{8} = +100\ \text{kN.m}\ \curvearrowleft$$

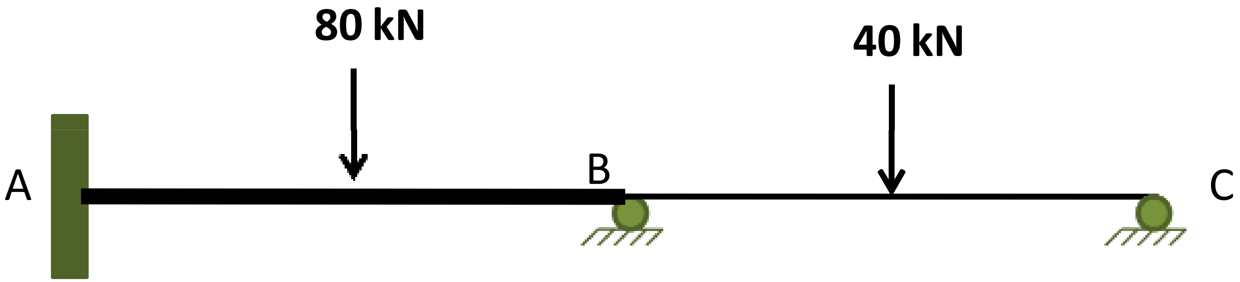
$$FEM_{BA} = -100\ \text{kN.m}\ \curvearrowright$$

$$FEM_{AB} = \frac{+40(10)}{8} = +50\ \text{kN.m}\ \curvearrowleft$$

$$FEM_{BA} = -50\ \text{kN.m}\ \curvearrowright$$

MD TABLE

E = constant



	AB	BA	BC	CB
Distribution Factors		0.6	0.4	1.0
1.Fixed-end Moments	+100	-100	+50	-50

3. Moment Distribution

After recording the DFs and the FEMs in the MD Table, we begin the MD process by balancing joints B and C.

The UM at joint B is equal to -100+50=-50 kN.m. Thus DMs at the ends B of members AB and BC are

$$DM_{BA} = DF_{BA}\left(-UM_{B}\right) = 0.6\left(+50\right) = +30 \quad \text{kN.m}$$

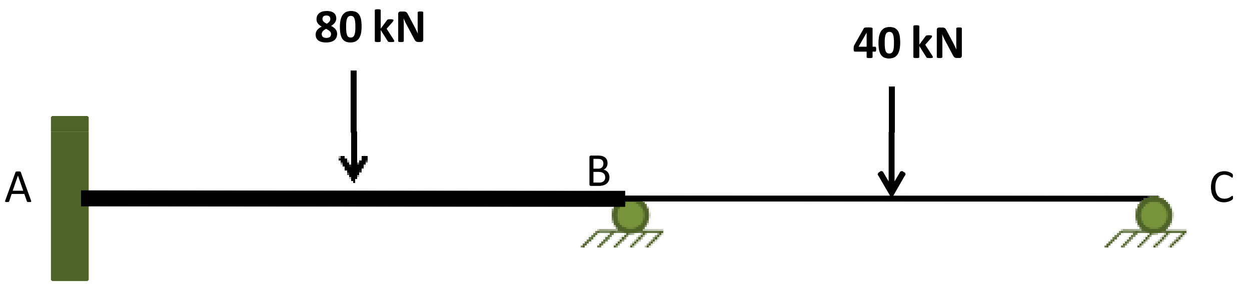
$$DM_{BC} = DF_{BC}\left(-UM_{B}\right) = 0.4\left(+50\right) = +20 \quad \text{kN.m}$$

Similarly, the UM at joint C is -50 kN.m, we determine the DM at end C of member BC to be

$$DM_{CB} = DF_{CB}\left(-UM_{C}\right) = 1\left(+50\right) = +50 \quad \text{kN.m}$$

MD TABLE

E = constant



Distribution Factors

1.Fixed-end Moments

2.Balance Joints B and C

AB	BA	BC	CB
	0.6	0.4	1.0
+100	-100	+50	-50
	+30	+20	+50

3. Moment Distribution

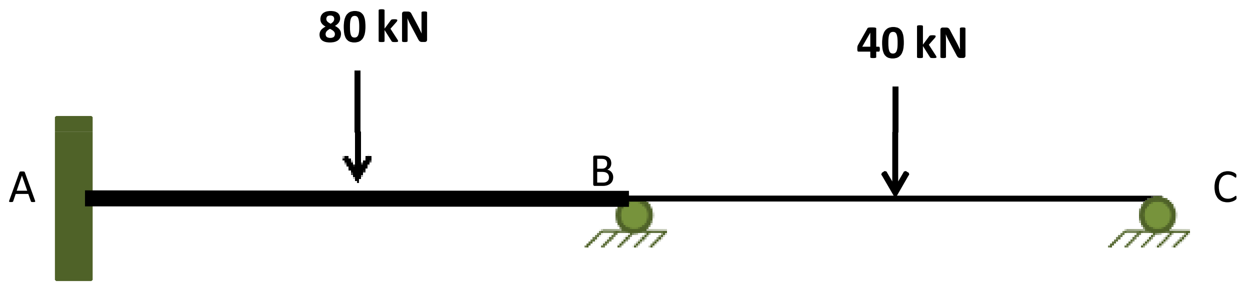
One-half of these DMs are then carried over to the far ends of the members.

This process is repeated, until the UMs are negligibly small.

4. Final Moments

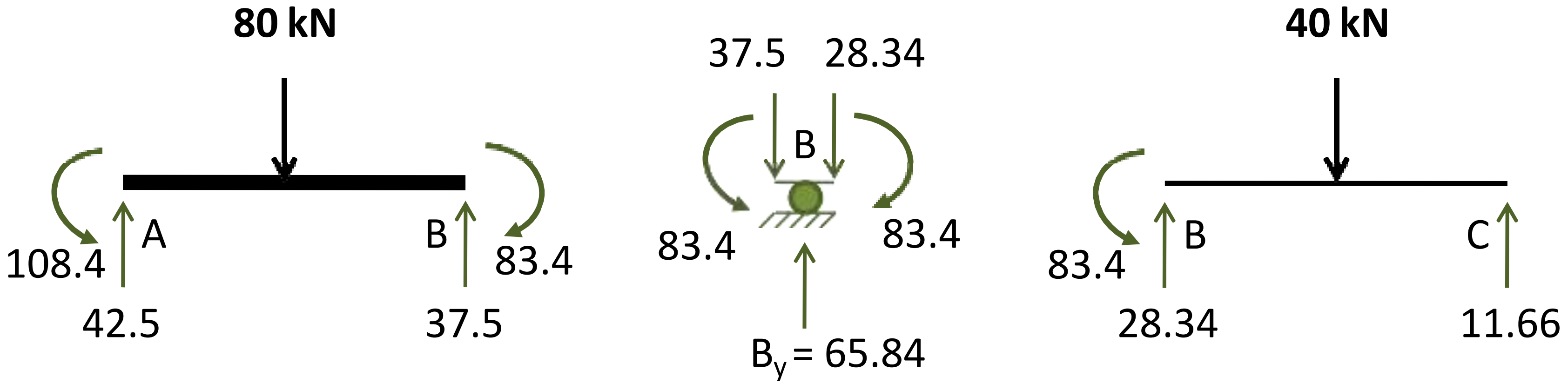
The final MEMs, obtained by summing the moments in each column of the MD Table, are recorded on the last line of the table.

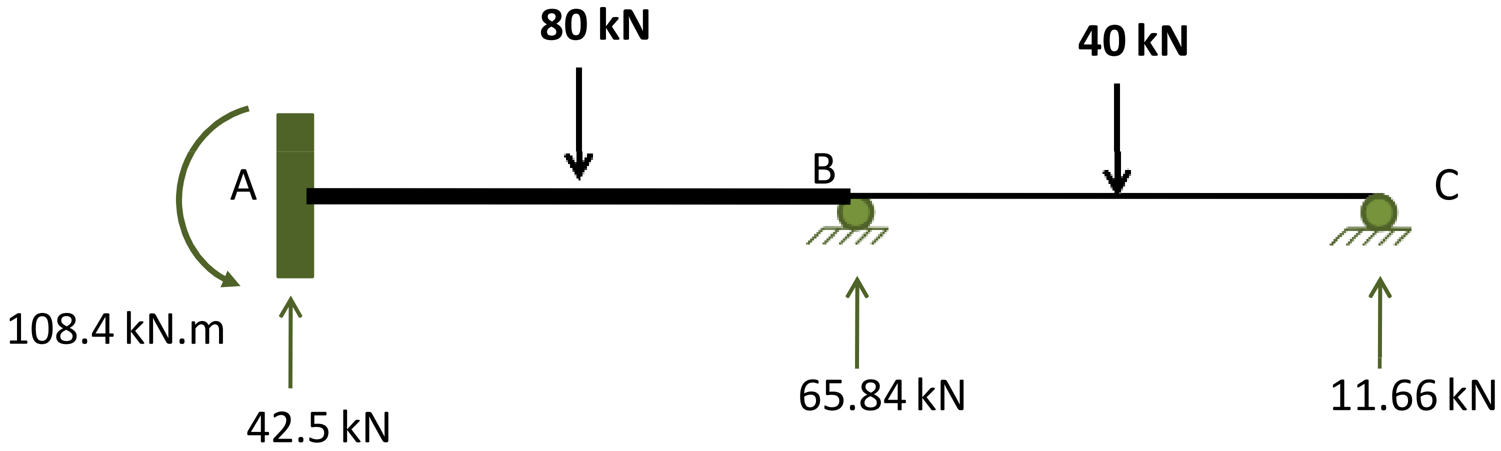
MD TABLE

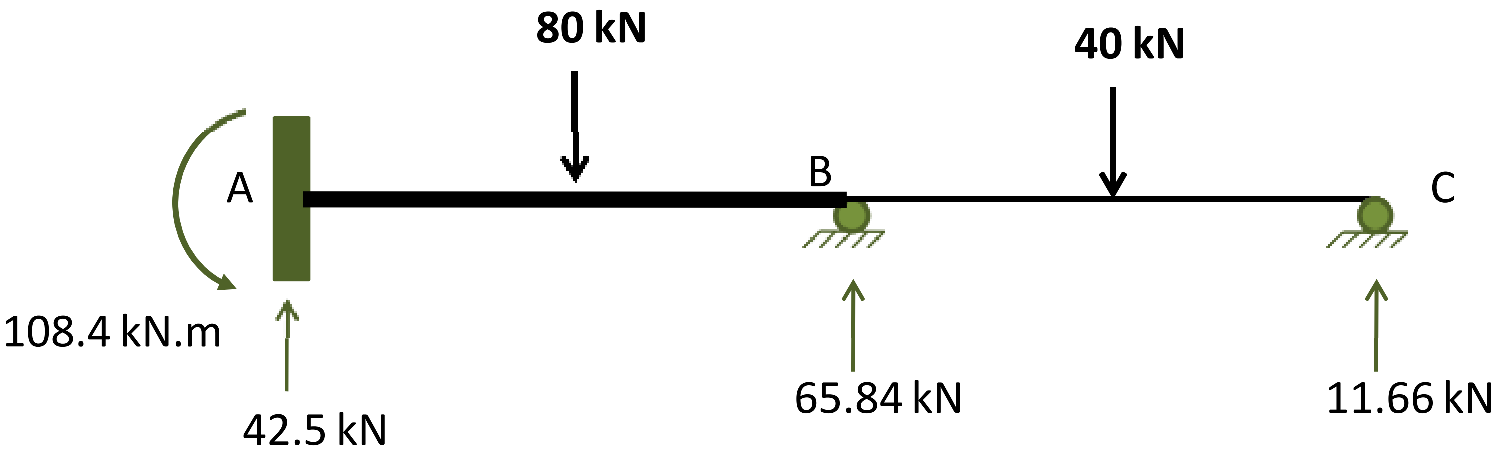


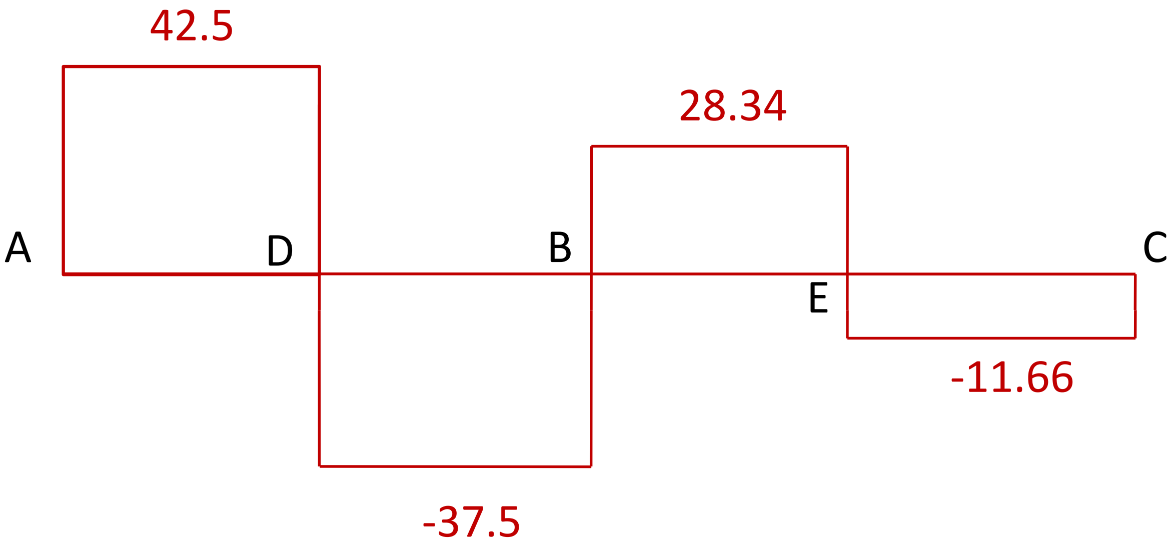
E = constant

	AB	BA	BC	CB
Distribution Factors		0.6	0.4	1.0
1.Fixed-end Moments	+100	-100	+50	-50
2.Balance Joints B and C		+30	+20	+50
3.Carryover	+15		+25	+10
4.Balance Joints B and C		-15	-10	-10
5.Carryover	-7.5		-5	-5
6.Balance Joints B and C		+3	+2	+5
7.Carryover	+1.5		+2.5	+1
8.Balance Joints B and C		-1.5	-1	-1
9.Carryover	-0.8		-0.5	-0.5
10.Balance Joints B and C		+0.3	+0.2	+0.5
11.Carryover	+0.2		+0.3	+0.1
12.Balance Joints B and C		-0.2	-0.1	-0.1
13. Final Moments	**+108.4**	**-83.4**	**+83.4**	0

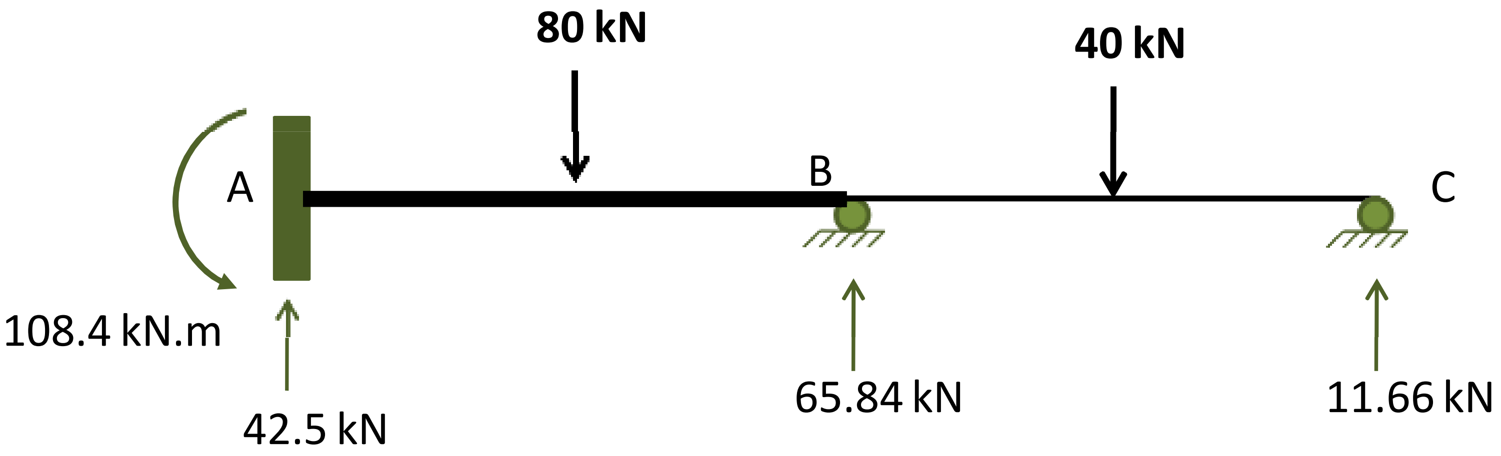

80 kN
108.4
A
42.5
B
83.4
37.5
37.5
28.34
B
83.4
83.4
B_y = 65.84
40 kN
83.4
B
28.34
C
11.66

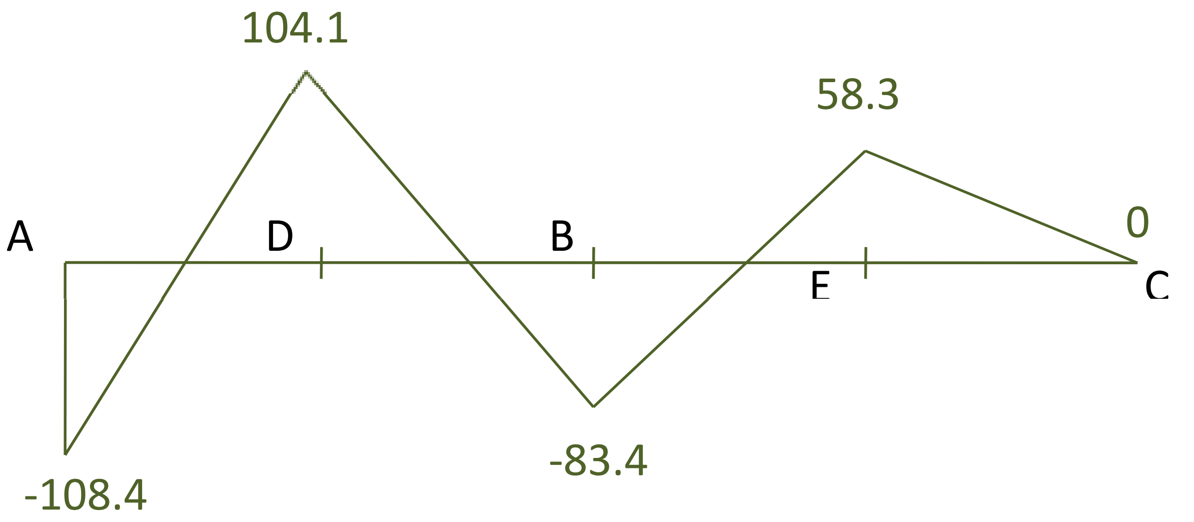

80 kN
40 kN
A
B
C
108.4 kN.m
42.5 kN
65.84 kN
11.66 kN





Shear Force Diagram (kN)

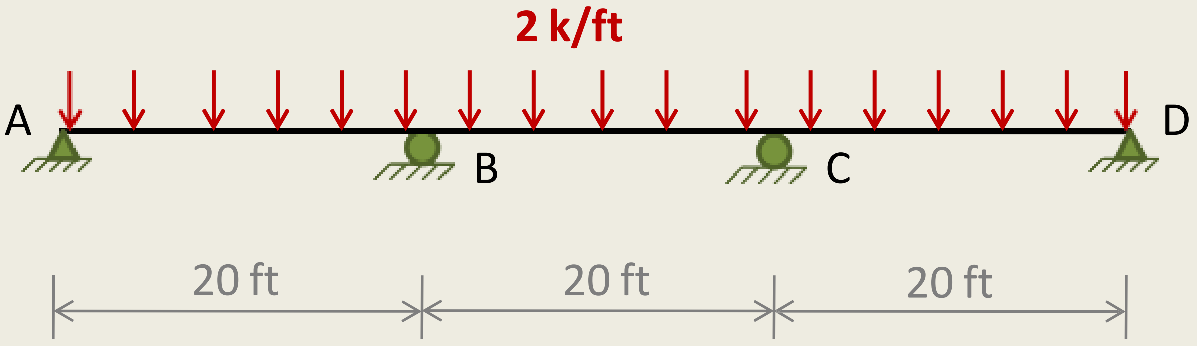




Bending Moment Diagram (kN . m)

Example 3

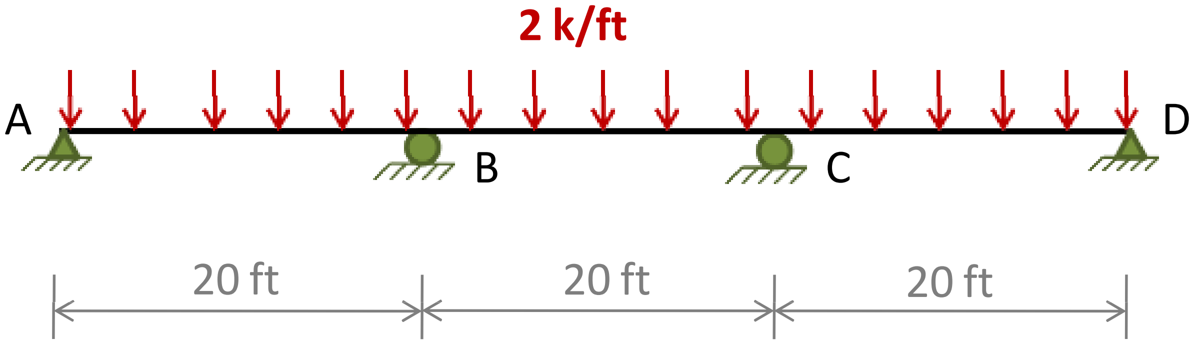
- Determine the member end moments and reactions for the three-span continuous beam shown, due to the uniformly distributed load and due to the support settlements of 5/8 in. at B, and 1.5 in. at C, and ¾ in. at D.



EI = 29,000 ksi
I = 7,800 in.^{4}

Solution

1. Distribution Factors



At Joint A

$$DF_{AB} = 1$$

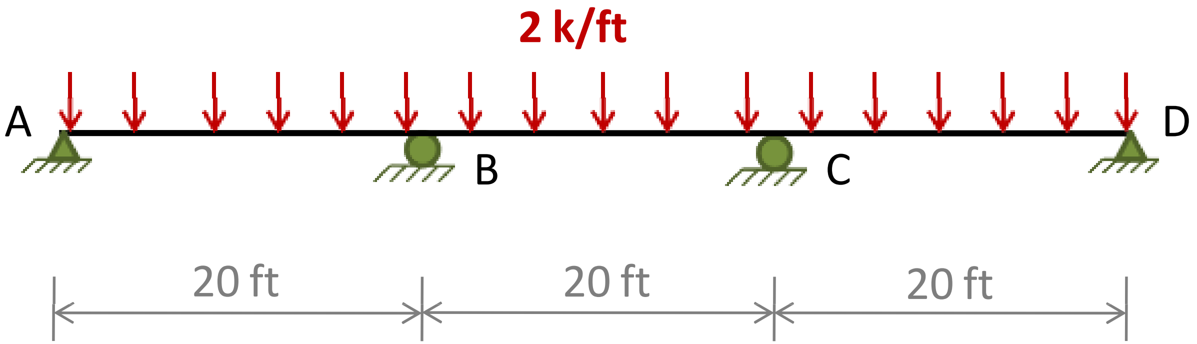
At Joint B

$$DF_{BA} = \frac{3I/80}{(3I/80)+(I/20)} = 0.429$$

$$DF_{BC} = \frac{I/20}{(3I/80)+(I/20)} = 0.571$$

Solution

1. Distribution Factors



At Joint C

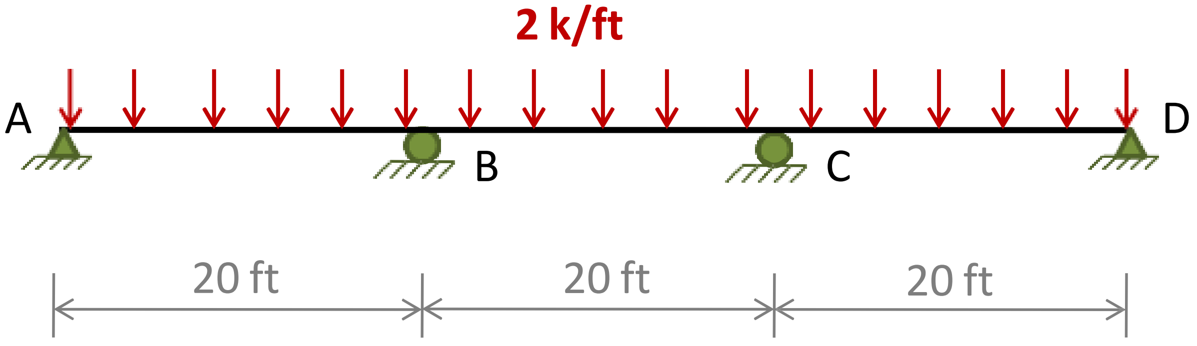
$$DF_{CB} = \frac{I/20}{(3I/80)+(I/20)} = 0.571$$

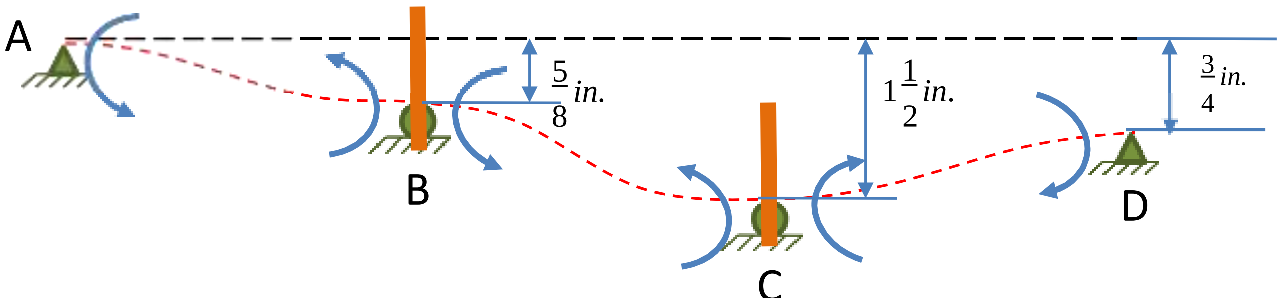
$$DF_{CD} = \frac{3I/80}{(3I/80)+(I/20)} = 0.429$$

At Joint D

$$DF_{DC} = 1$$

2. Fixed-End Moments



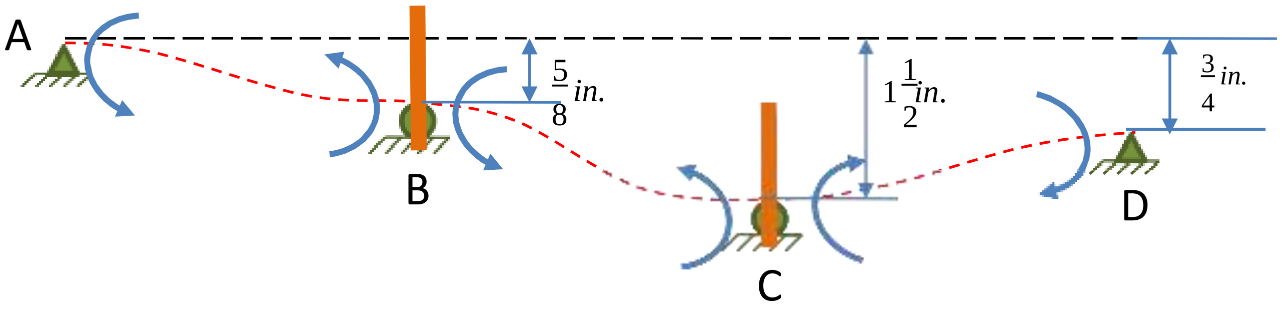


$$\Delta_{AB} = \tfrac{5}{8}\,in.$$

$$\Delta_{BC} = 1\tfrac{1}{2} - \tfrac{5}{8} = \tfrac{7}{8}\,in.$$

$$\Delta_{BC} = 1\tfrac{1}{2} - \tfrac{3}{4} = \tfrac{3}{4}\,in.$$

2. Fixed-End Moments

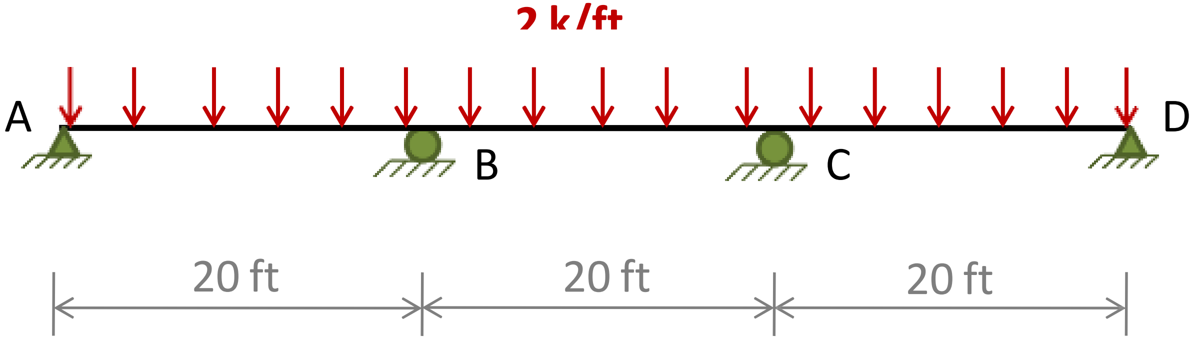


$$FEM_{AB} = FEM_{BA} = +\frac{6EI\Delta}{L^2} = +\frac{6(29{,}000)(7{,}800)\left(\frac{5}{8}\right)}{(20)^2(12)^3} = +1{,}227.2\ \text{k - ft}$$

$$FEM_{BC} = FEM_{CB} = +\frac{6EI\Delta}{L^2} = +\frac{6(29{,}000)(7{,}800)\left(\frac{7}{8}\right)}{(20)^2(12)^3} = +1{,}718.1\ \text{k - ft}$$

$$FEM_{CD} = FEM_{DC} = -\frac{6EI\Delta}{L^2} = +\frac{6(29{,}000)(7{,}800)\left(\frac{3}{4}\right)}{(20)^2(12)^3} = -1{,}472.7\ \text{k - ft}$$

2. Fixed-End Moments



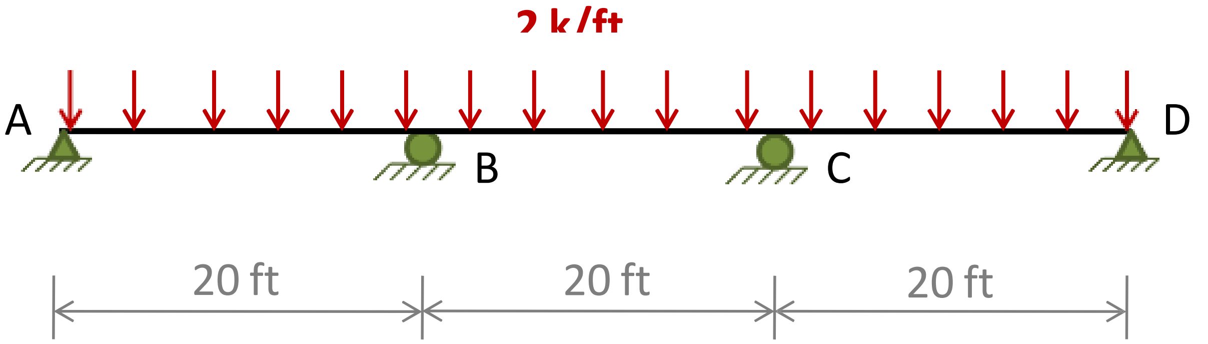
The FEMs due to the 2 k/ft external load are

$$FEM_{AB} = FEM_{BC} = FEM_{CD} = +\frac{2(20)^2}{12} = +66.7\ \mathrm{k\text{-}ft}$$

$$FEM_{BA} = FEM_{CB} = FEM_{DC} = -\frac{2(20)^2}{12} = -66.7\ \mathrm{k\text{-}ft}$$

Thus the FEMs due to the combined effect of the external load and the support settlements are

2. Fixed-End Moments



$$FEM_{AB} = +1{,}293.9 \text{ k - ft}$$

$$FEM_{BA} = +1{,}160.5 \text{ k - ft}$$

$$FEM_{BC} = +1{,}784.8 \text{ k - ft}$$

$$FEM_{CB} = +1{,}651.4 \text{ k - ft}$$

$$FEM_{CD} = -1{,}406 \text{ k - ft}$$

$$FEM_{DC} = -1{,}539.4 \text{ k - ft}$$

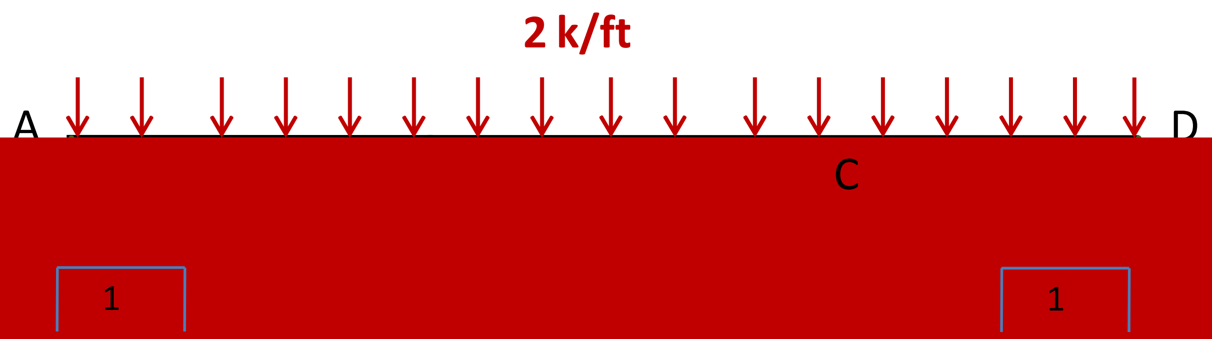
3. Moment Distribution

The MD is carried out in the usual manner, as shown in the MD Table.

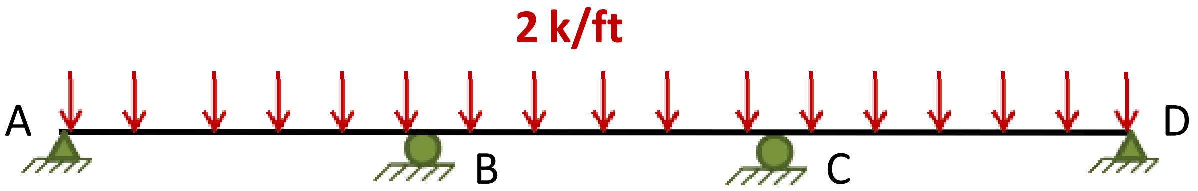
Note that the joints A and D at the simple end supports are balanced only once and that no moments are carried over to these joints.

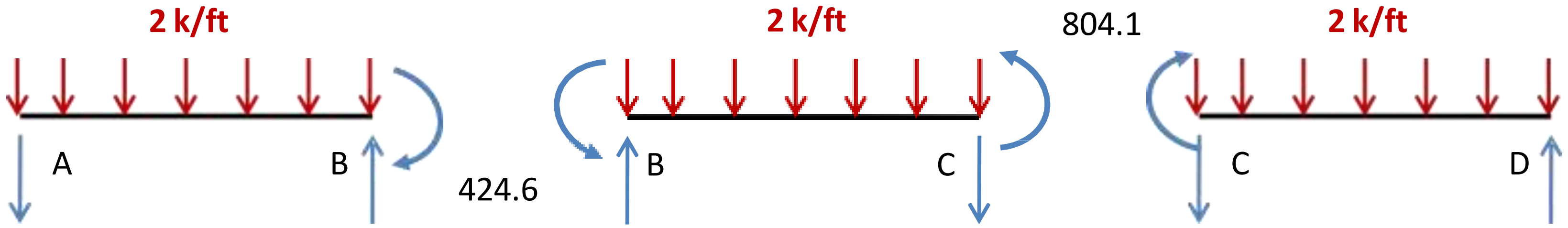
4. Final Moments

See the MD Table and Figure on next slides.



1.Fixed-end Moments	+1293.9	+1160.5	+1784.8	+1651.4	-1406	-1539.4
2.Balance Joints	-1293.9	-1263.5	-1681.8	-140.1	-105.3	+1539.4
3.Carryover		-647	-70.1	-840.9	+769.7	
4.Balance Joints		+307.6	+409.5	+40.7	+30.5	
5.Carryover			+20.4	+204.8		
6.Balance Joints		-8.8	-11.6	-116.9	-87.9	
7.Carryover			-58.5	-5.8		
8.Balance Joints		+25.1	+33.4	+3.3	+2.5	
9.Carryover			+1.7	+16.7		
10.Balance Joints		-0.7	-1.0	-9.5	-7.2	
11.Carryover			-4.8	-0.5		
12.Balance Joints		+2.1	+2.7	+0.3	+0.2	
13.Carryover			+0.2	+1.4		
14.Balance Joints		-0.1	-0.1	-0.8	-0.6	
13.Carryover			-0.4			
14.Balance Joints		+0.2	+0.2			
11.Final Moments	**0**	**-426.6**	**+426.6**	**+804.1**	**-804.1**	**0**


2 k/ft
A
B
C
D


2 k/ft
A
B
424.6
2 k/ft
B
C
804.1
2 k/ft
C
D

Concept of fixed end moments

Obtained using unit load method

Derivation of the Slope-Deflection Equation

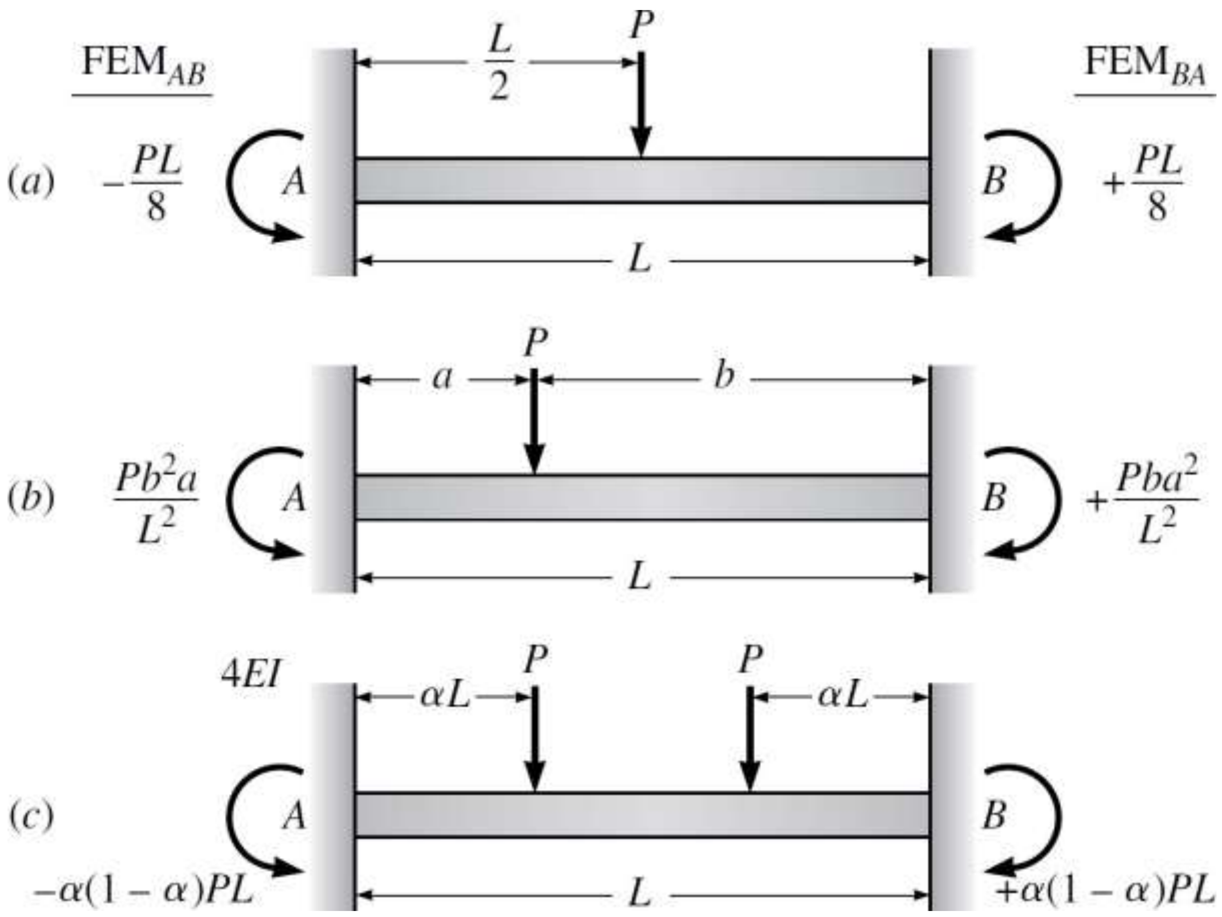


Figure 12.5 Fixed-end moments

Derivation of the Slope-Deflection Equation

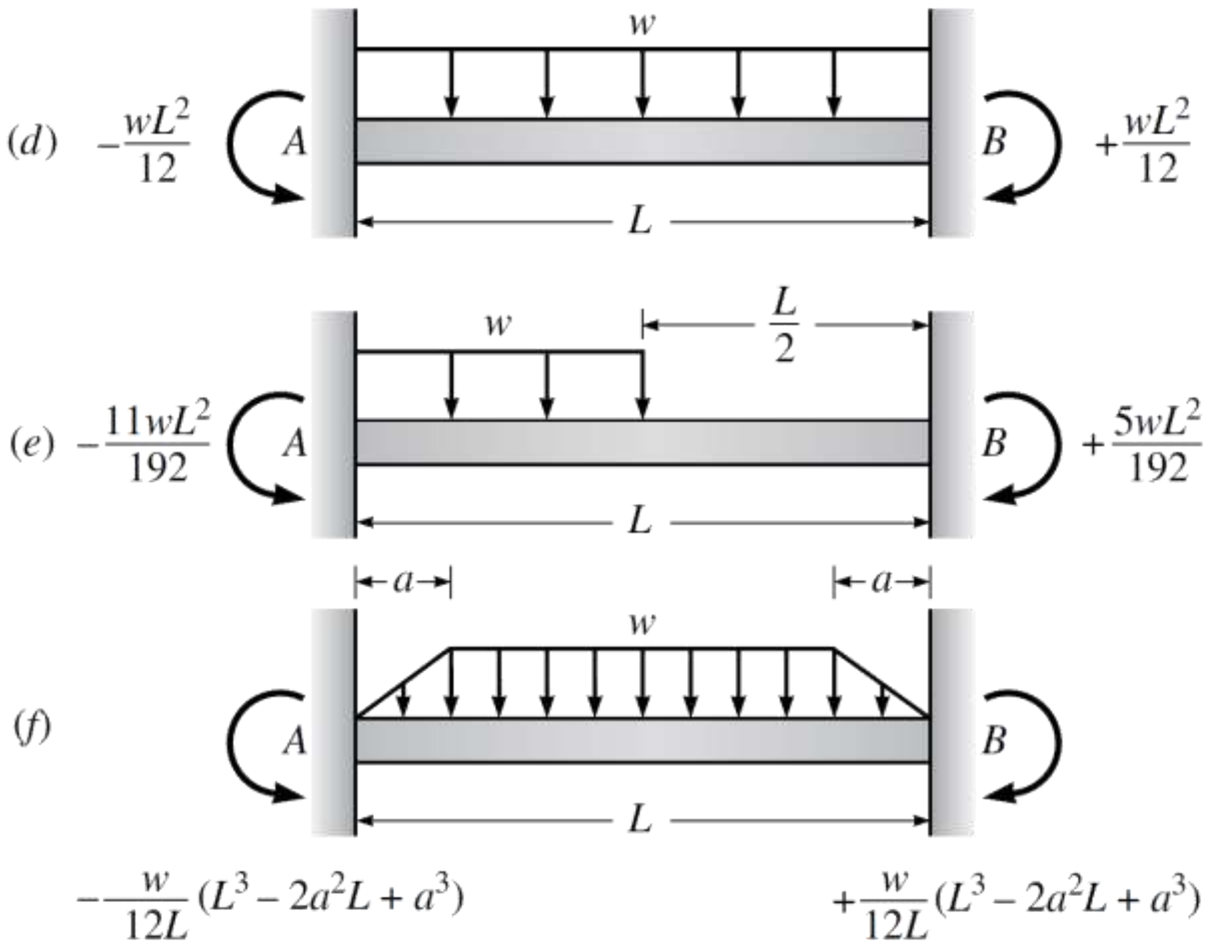


Figure 12.5 Fixed-end moments (continued)

Derivation of the Slope-Deflection Equation

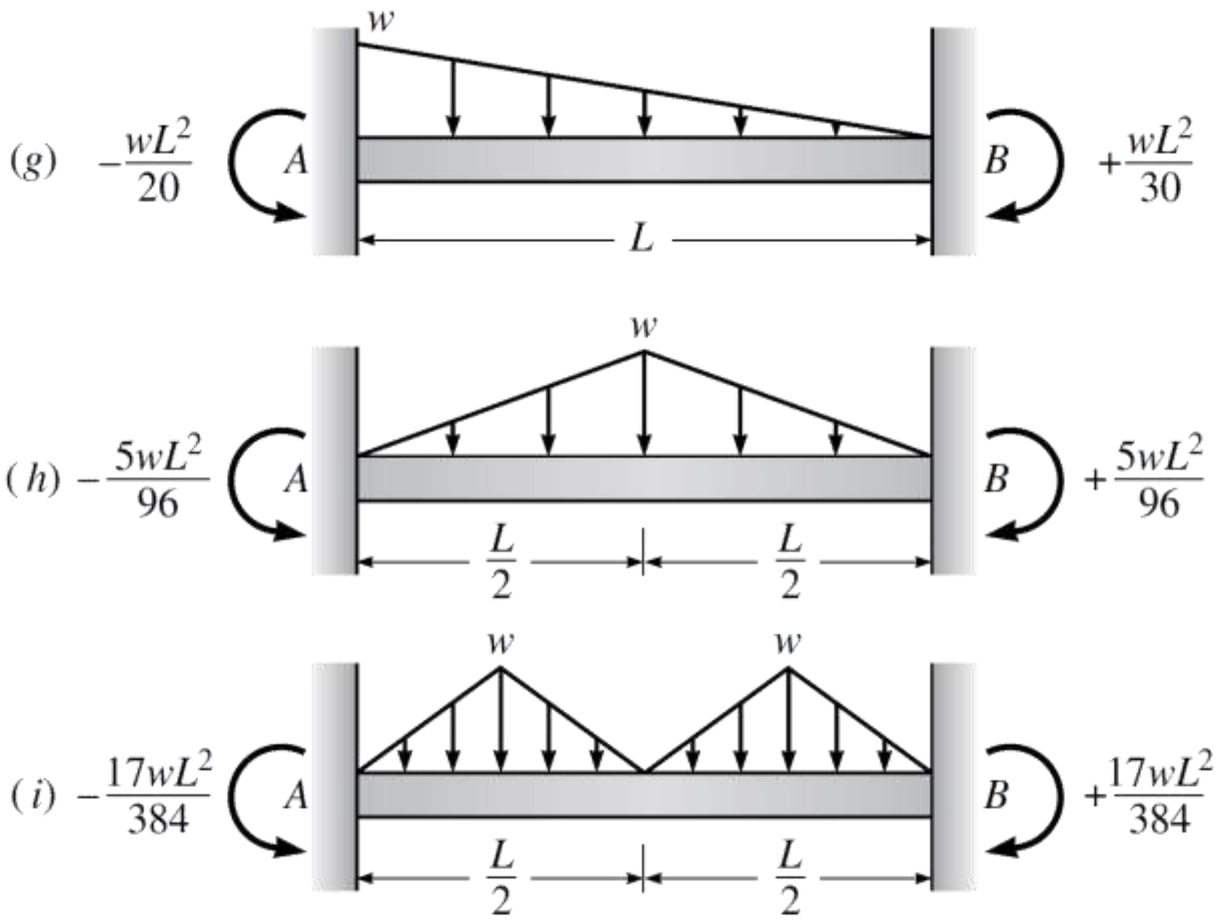


Figure 12.5 Fixed-end moments (continued)

Derivation of the Slope-Deflection Equation

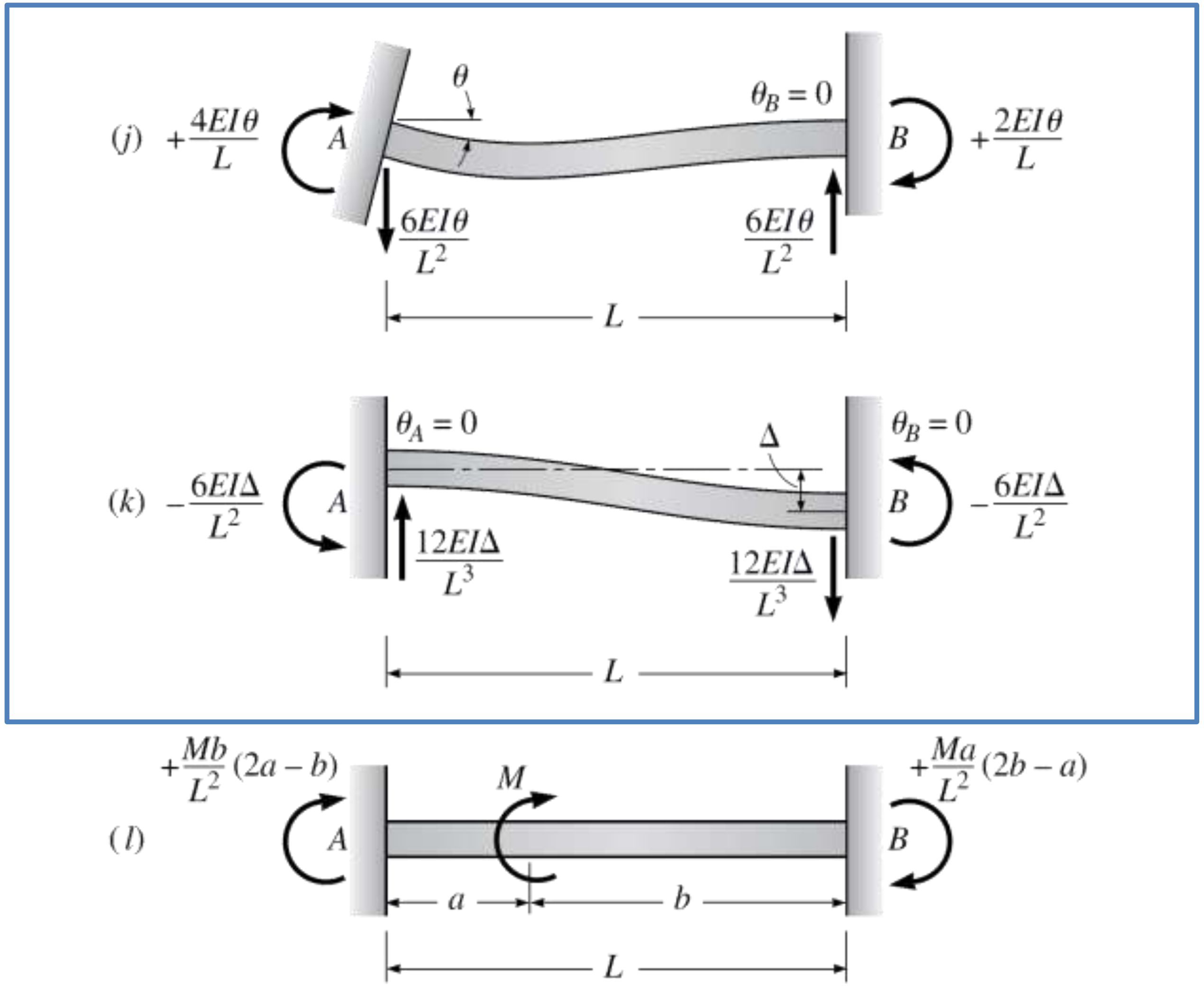
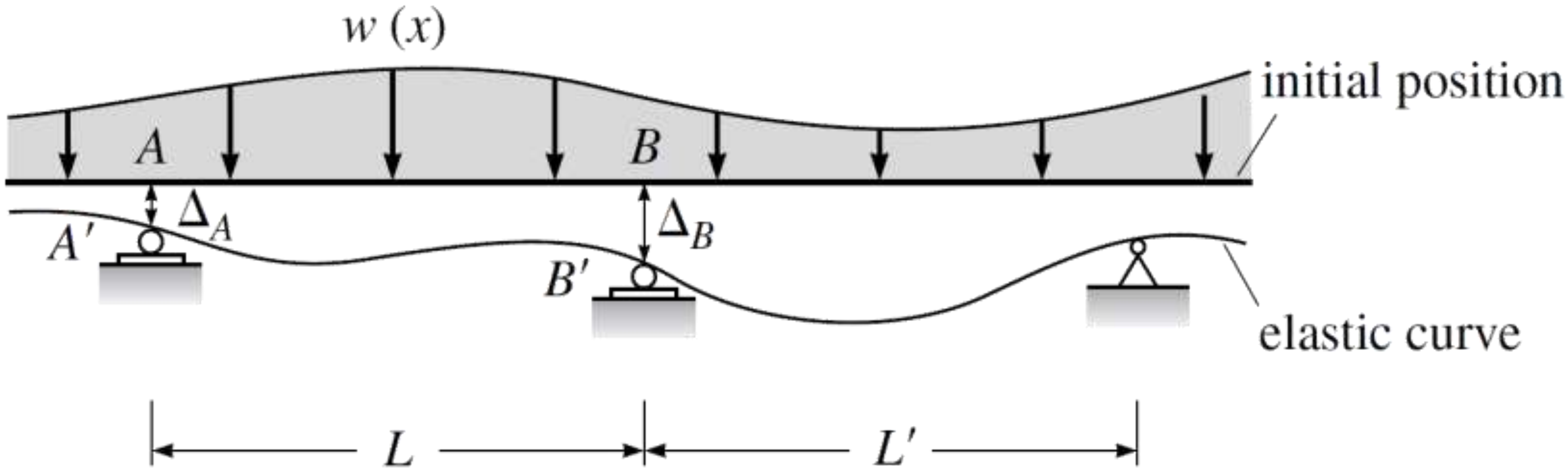


Figure 12.5 Fixed-end moments (continued)

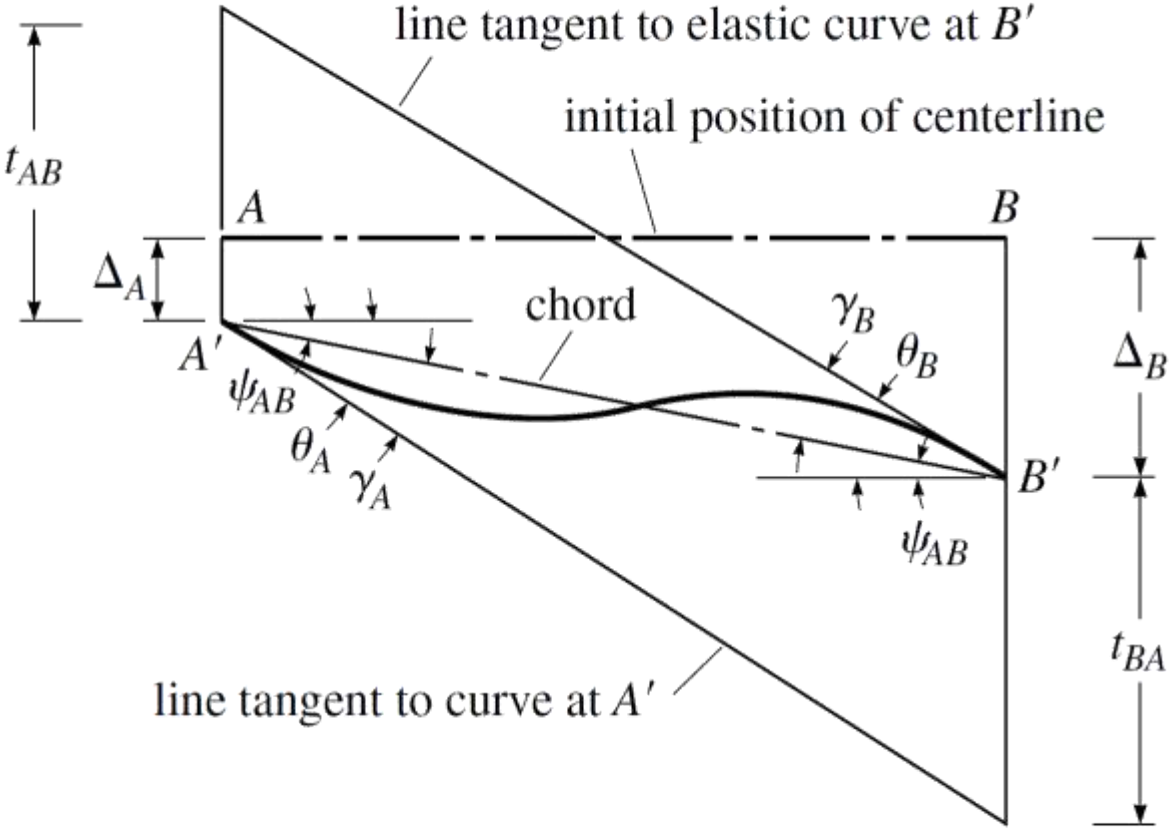
Derivation of the Slope-Deflection Equation



Continuous beam whose supports settle under load

Figure 12.2

§12.3 Derivation of the Slope-Deflection Equation



$$\psi_{AB} = \frac{\Delta_B - \Delta_A}{L}$$

Deformations of member *AB* plotted to an exaggerated vertical scale

Derivation of the Slope-Deflection Equation

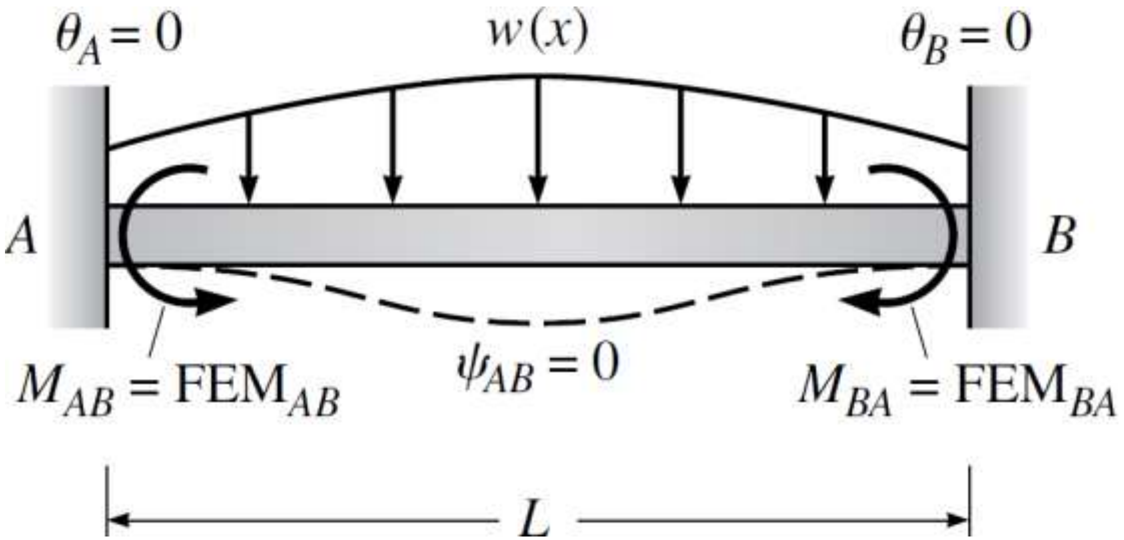
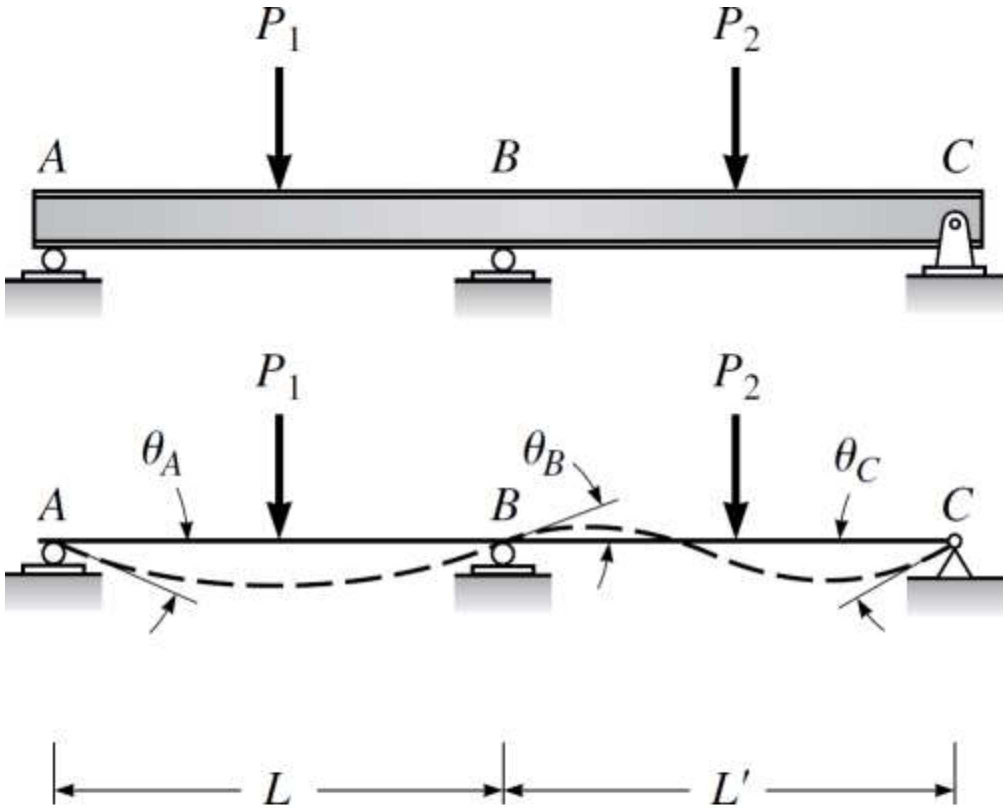


Figure 12.4

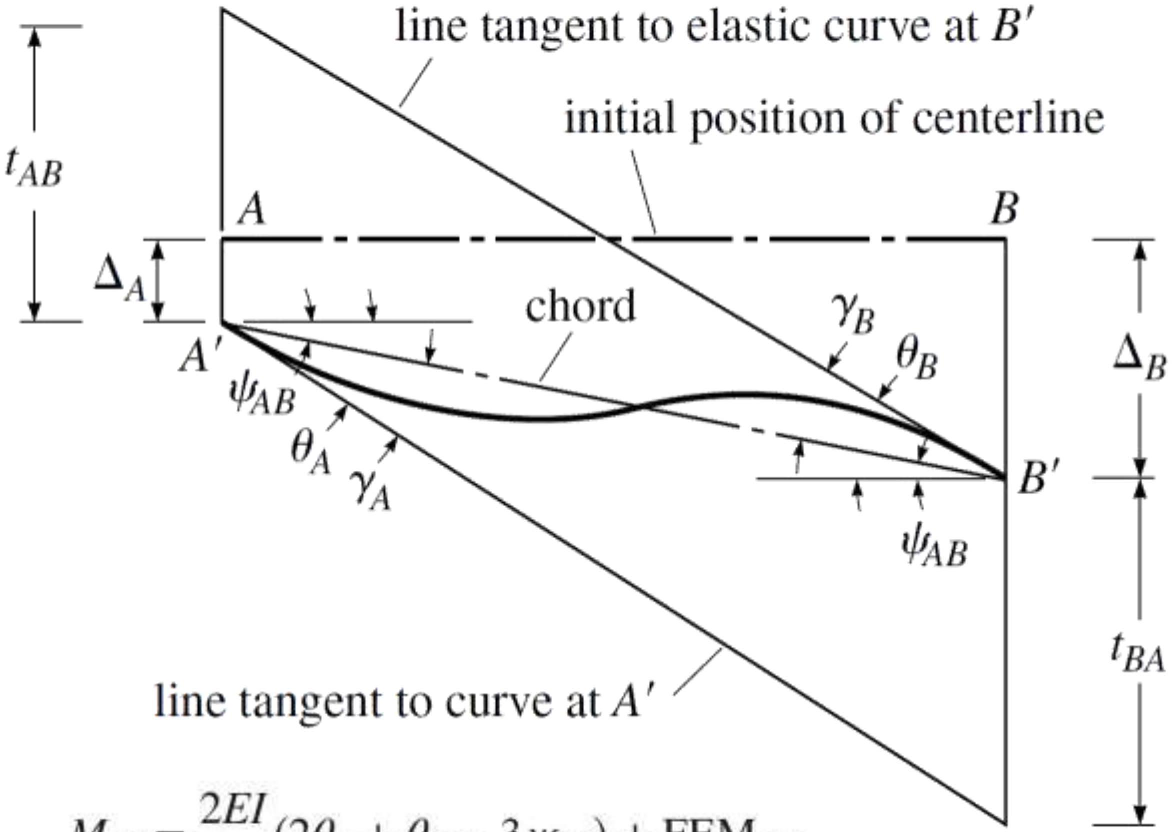
Illustration of the Slope-Deflection Method



Continuous beam with applied loads
(deflected shape shown by dashed line)

Figure 12.4

§12.3 Derivation of the Slope-Deflection Equation



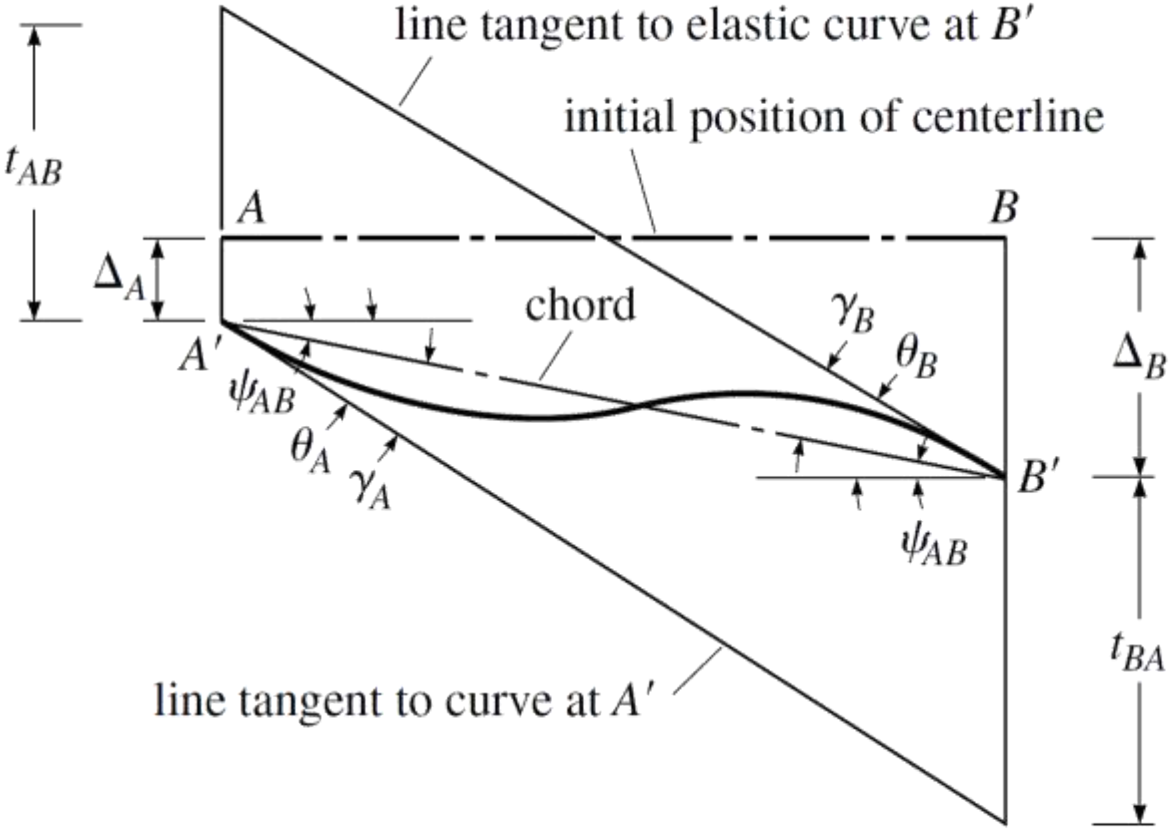
$$\psi_{AB} = \frac{\Delta_B - \Delta_A}{L}$$

Deformations of member *AB* plotted to an exaggerated vertical scale

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi_{AB}) + \mathrm{FEM}_{AB}$$

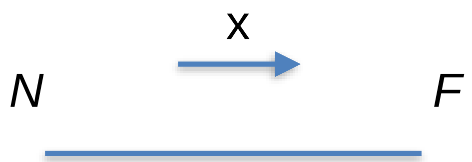
$$M_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi_{AB}) + \mathrm{FEM}_{BA}$$

§12.3 Derivation of the Slope-Deflection Equation



$$\psi_{AB} = \frac{\Delta_B - \Delta_A}{L}$$

Deformations of member *AB* plotted to an exaggerated vertical scale



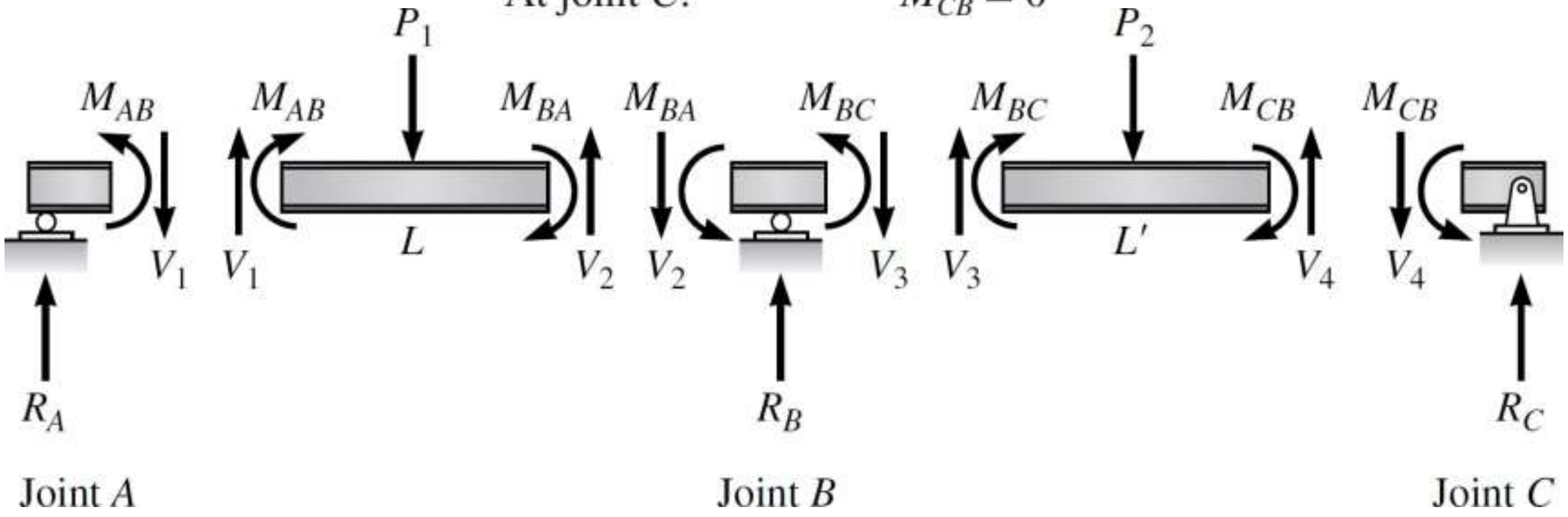
$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF}$$

Illustration of the Slope-Deflection Method

At joint A: $M_{AB} = 0$

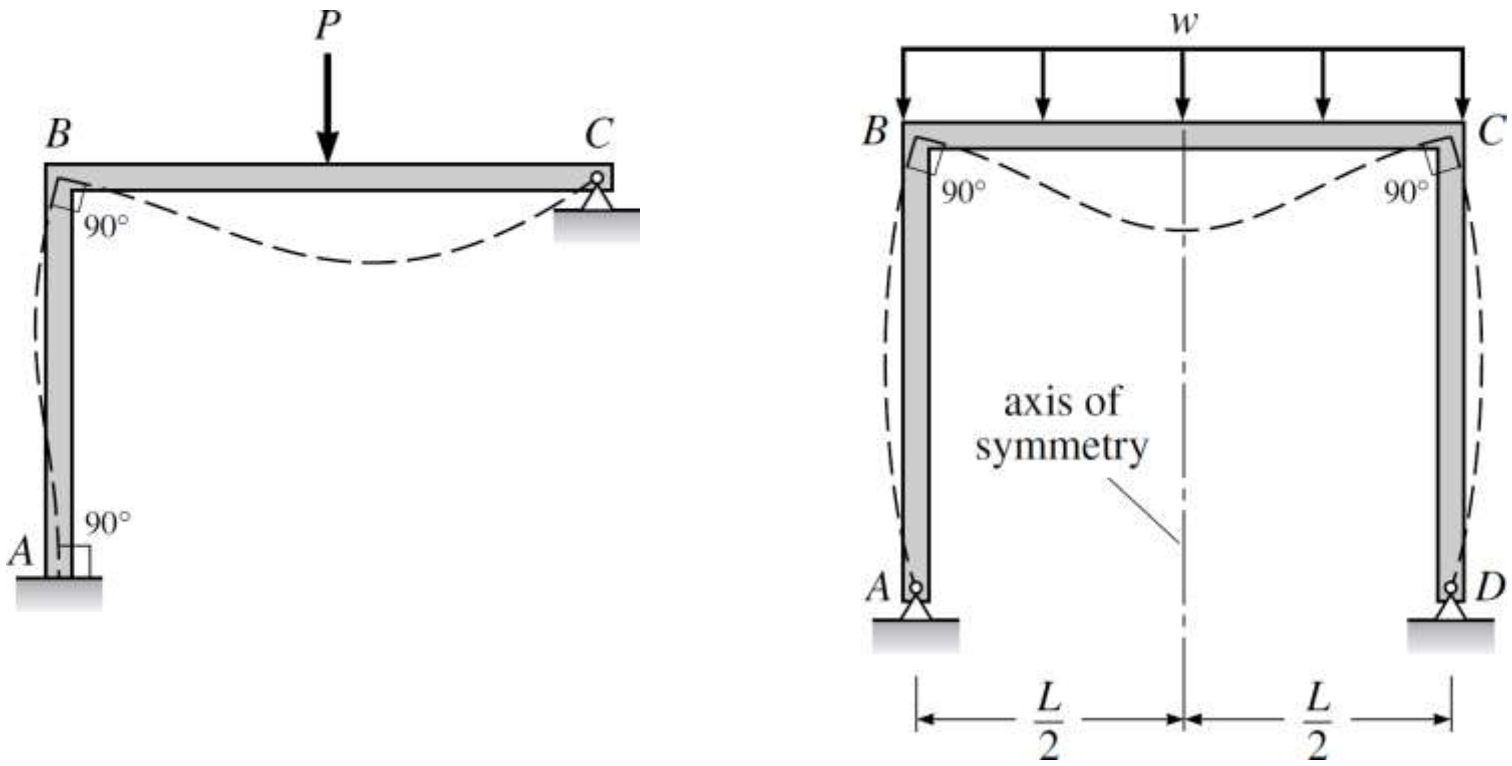
At joint B: $M_{BA} + M_{BC} = 0$

At joint C: $M_{CB} = 0$



Free bodies of joints and beams (sign convention: **Clockwise moment on the end of a member is positive**)

Analysis of Structures by the Slope-Deflection Method

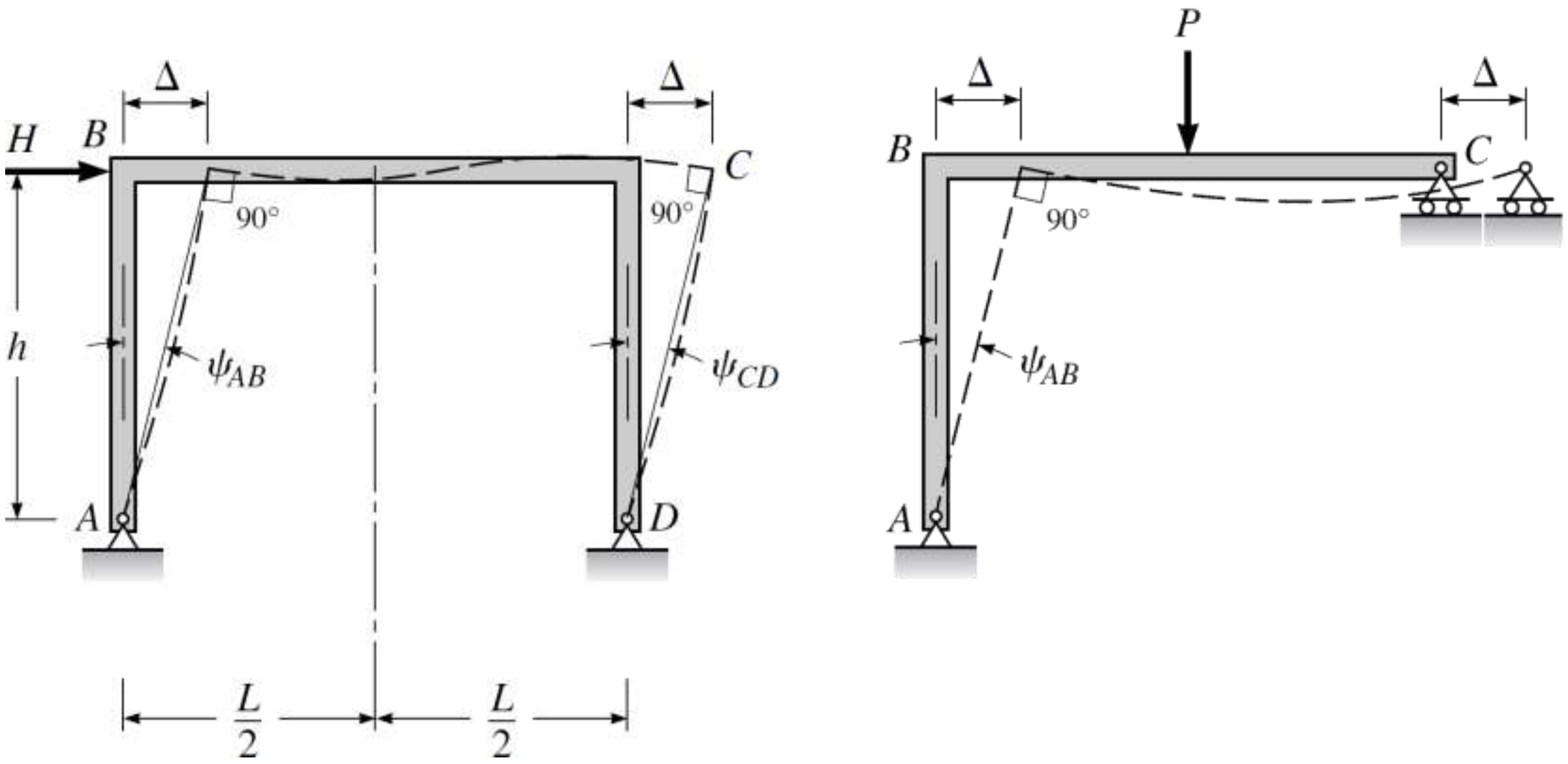


All joints restrained against displacement; all chord rotations ψ equal zero

Due to symmetry of structure and loading, joints free to rotate but not translate; chord rotations equal zero

Figure 12.7

Analysis of Structures by the Slope-Deflection Method

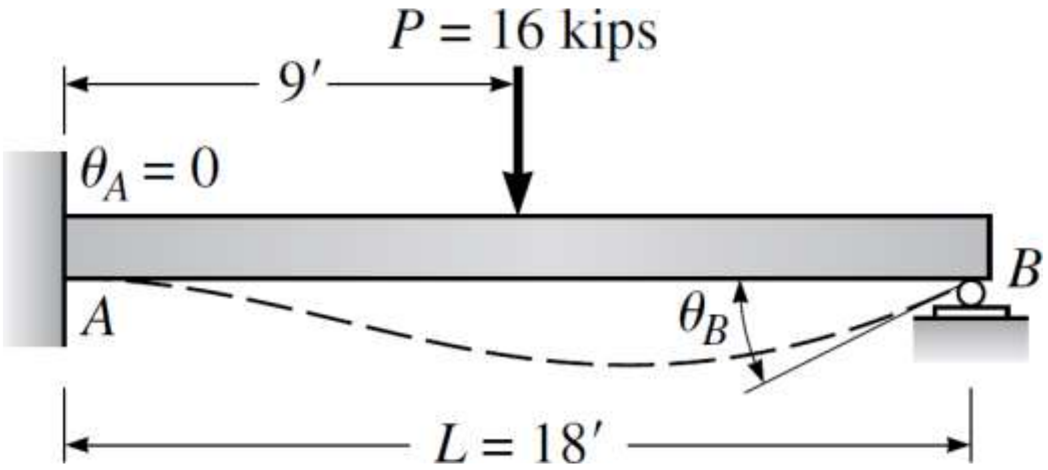


Unbraced frames with chord rotations

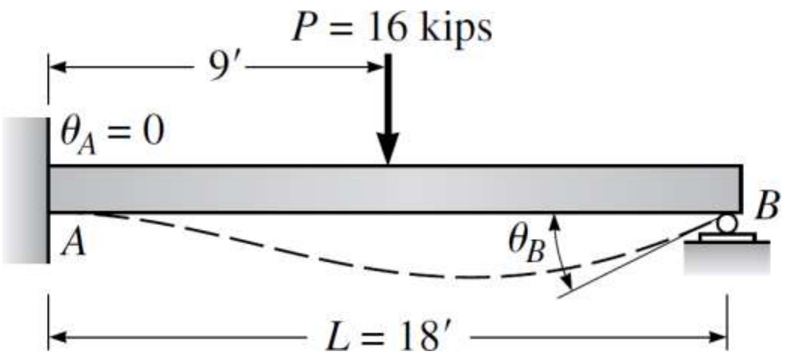
Figure 12.7 (continued)

Example 12.2

Using the slope-deflection method, determine the member end moments in the indeterminate beam shown in Figure 12.8*a*. The beam, which behaves elastically, carries a concentrated load at midspan. After the end moments are determined, draw the shear and moment curves. If $I = 240$ in^4 and $E = 30{,}000$ kips/in^2, compute the magnitude of the slope at joint *B*.

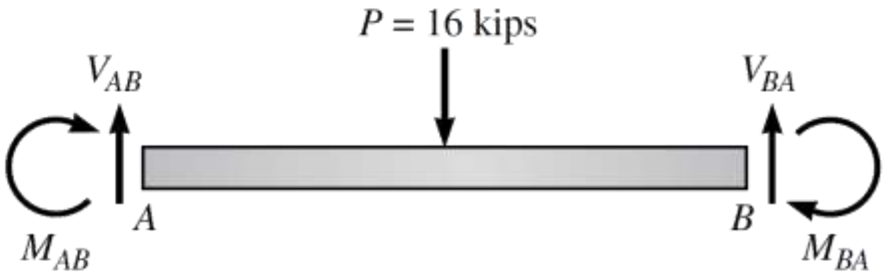


Example 12.2 Solution



- Since joint *A* is fixed against rotation, $\theta_A = 0$; therefore, the only unknown displacement is $\theta_{B.}$ Using the slope-deflection equation

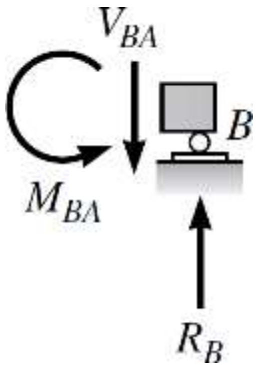
$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF}$$



- The member end moments are:

$$M_{AB} = \frac{2EI}{L}(\theta_B) - \frac{PL}{8}$$

$$M_{BA} = \frac{2EI}{L}(2\theta_B) + \frac{PL}{8}$$



- To determine θ_B, write the equation of moment equilibrium at joint *B*

$$\circlearrowright + \quad \Sigma M_B = 0$$

$$M_{BA} = 0$$

Example 12.2 Solution (continued)

- Substituting the value of M_{BA} and solving for θ_B give

$$\frac{4EI}{L}\theta_B + \frac{PL}{8} = 0$$

$$\theta_B = -\frac{PL^2}{32EI}$$

where the minus sign indicates both that the *B* end of member *AB* and joint *B* rotate in the counterclockwise direction

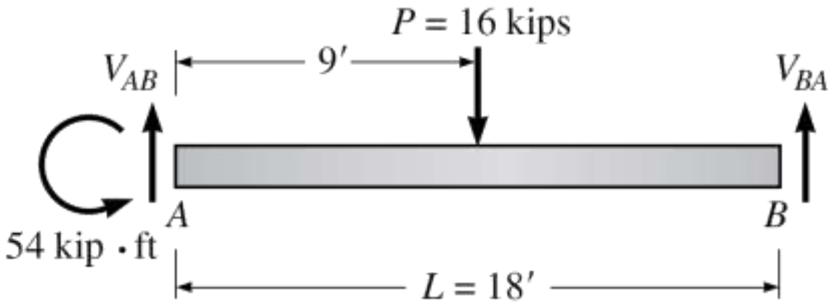
- To determine the member end moments,

$$M_{AB} = \frac{2EI}{L}\left(\frac{-PL^2}{32EI}\right) - \frac{PL}{8} = -\frac{3PL}{16} = -54 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

$$M_{BA} = \frac{4EI}{L}\left(\frac{-PL^2}{32EI}\right) + \frac{PL}{8} = 0$$

Example 12.2 Solution (continued)

- To complete the analysis, apply the equations of statics to a free body of member *AB*



Free body used to compute end shears

$$\circlearrowright^{+} \quad \Sigma M_A = 0$$

$$0 = (16\text{ kips})(9\text{ ft}) - V_{BA}(18\text{ ft}) - 54\text{ kip}\cdot\text{ft}$$

$$V_{BA} = 5\text{ kips}$$

$$\overset{+}{\uparrow} \quad \Sigma F_y = 0$$

$$0 = V_{BA} + V_{AB} - 16$$

$$V_{AB} = 11\text{ kips}$$

- To evaluate θ_B, express all variables in units of inches and kips.

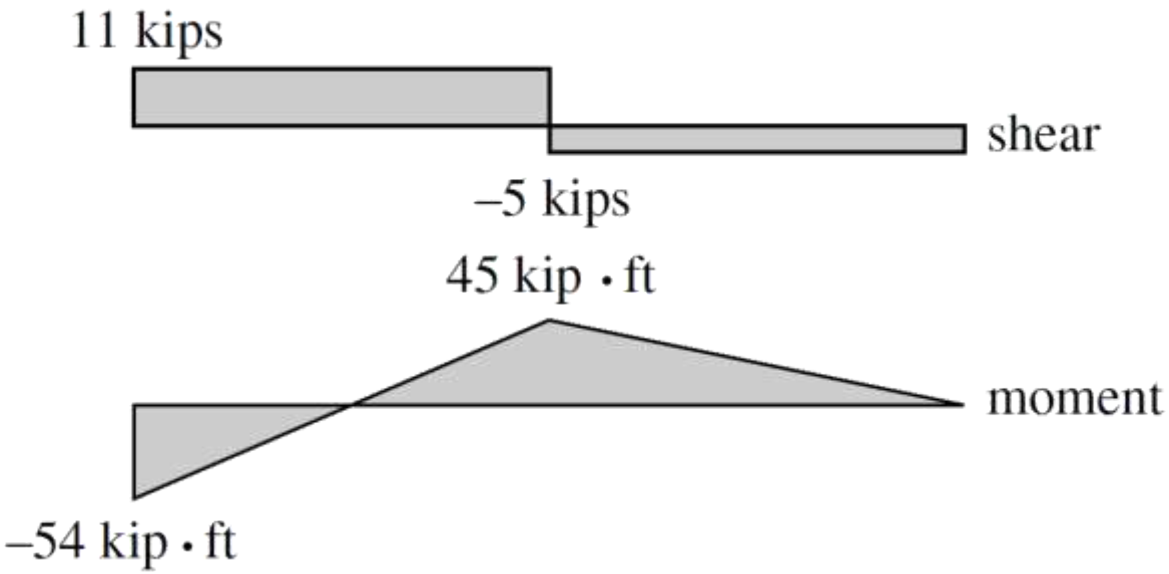
$$\theta_B = -\frac{PL^2}{32EI} = -\frac{16(18 \times 12)^2}{32(30{,}000)240} = -0.0032\text{ rad}$$

Example 12.2 Solution (continued)

- Expressing θ_B in degrees

$$\frac{2\pi \text{ rad}}{360^\circ} = \frac{-0.0032}{\theta_B}$$

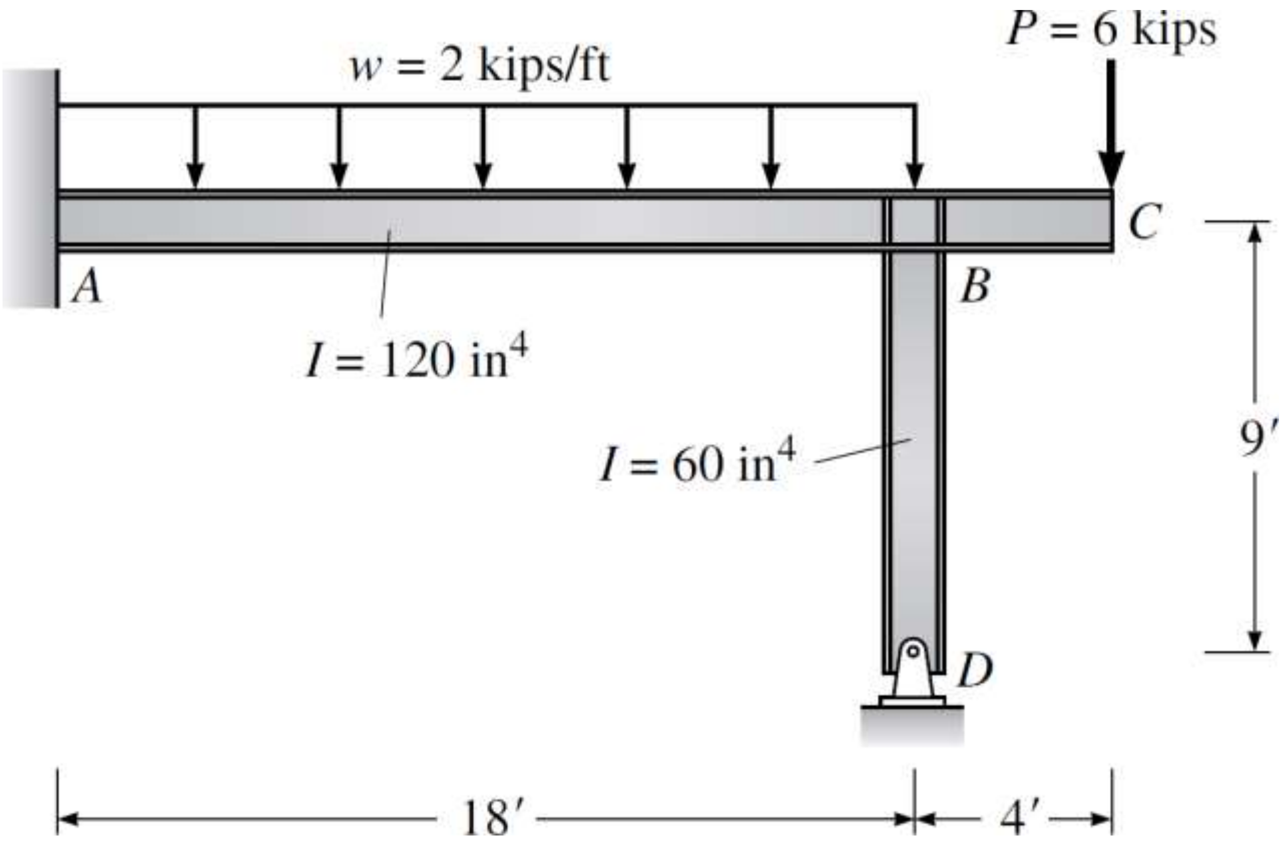
$$\theta_B = -0.183^\circ \qquad \textbf{Ans.}$$



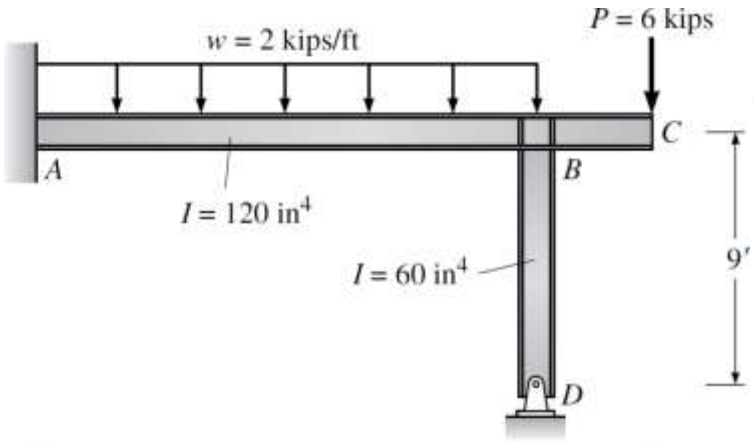
Shear and moment curves

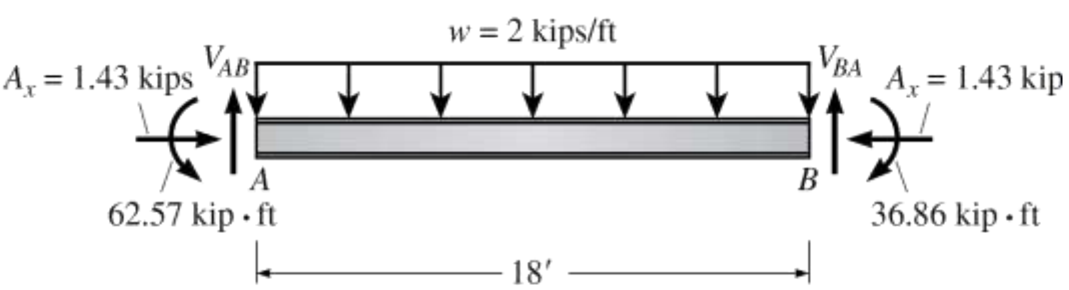
Example 12.3

Using the slope-deflection method, determine the member end moments in the braced frame shown in Figure 12.9*a*. Also compute the reactions at support *D*, and draw the shear and moment curves for members *AB* and *BD*.



Example 12.3 Solution





- Use the slope-deflection equation

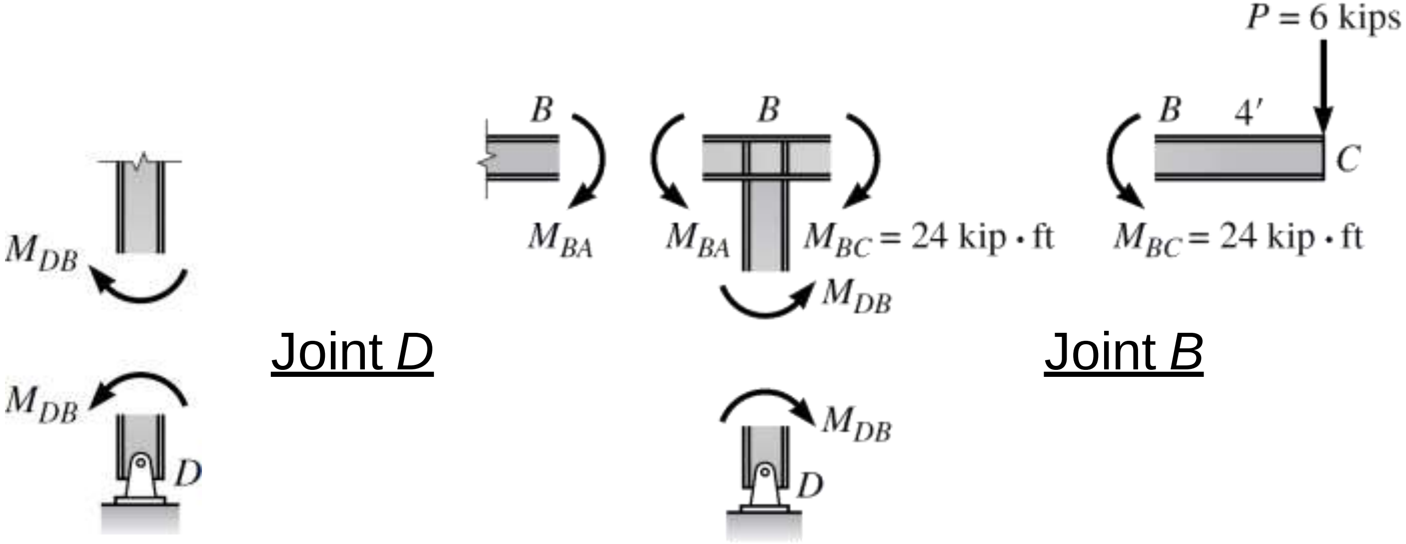
$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF}$$

- The fixed-end moments produced by the uniform load on member *AB*

$$\text{FEM}_{AB} = -\frac{wL^2}{12}$$

$$\text{FEM}_{BA} = +\frac{wL^2}{12}$$

Example 12.3 Solution (continued)



Joint *D*

Joint *B*

- Express the member end moments as

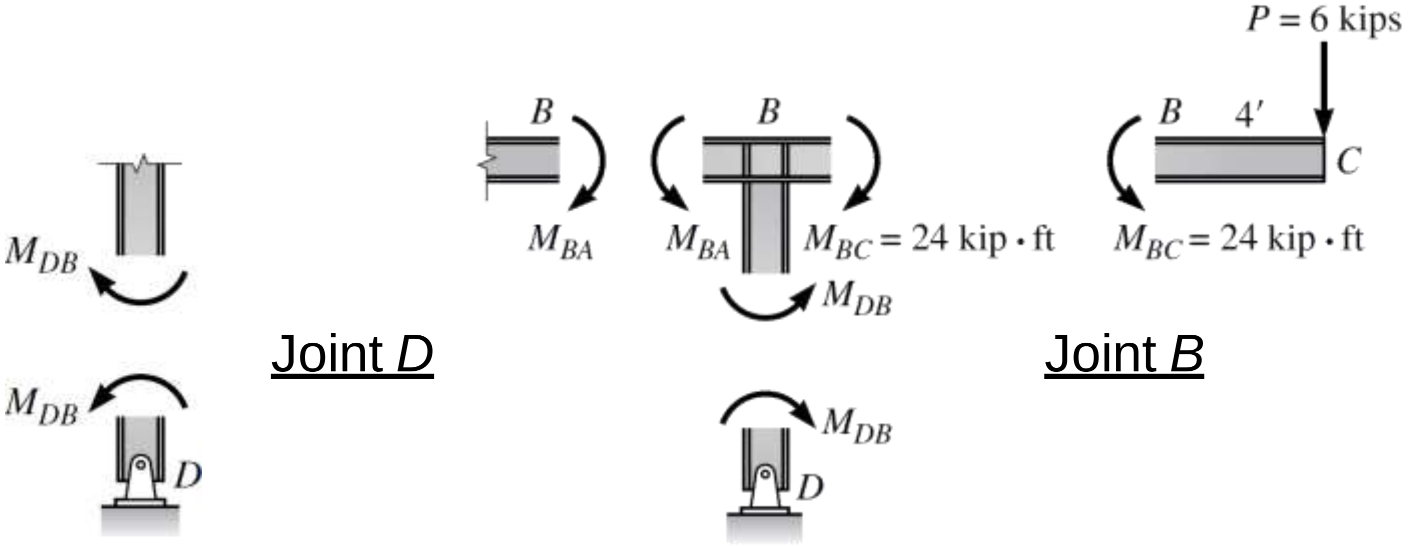
$$M_{AB} = \frac{2E(120)}{18(12)}(\theta_B) - \frac{2(18)^2(12)}{12} = 1.11E\theta_B - 648$$

$$M_{BA} = \frac{2E(120)}{18(12)}(2\theta_B) + \frac{2(18)^2(12)}{12} = 2.22E\theta_B + 648$$

$$M_{BD} = \frac{2E(60)}{9(12)}(2\theta_B + \theta_D) = 2.22E\theta_B + 1.11E\theta_D$$

$$M_{DB} = \frac{2E(60)}{9(12)}(2\theta_D + \theta_B) = 2.22E\theta_D + 1.11E\theta_B$$

Example 12.3 Solution (continued)



Joint *D*

Joint *B*

- To solve for the unknown joint displacements θ_B and θ_D, write equilibrium equations at joints *D* and *B*.

At joint *D* (see Fig. 12.9*b*): $\overset{+}{\circlearrowleft} \quad \Sigma M_D = 0$

$$M_{DB} = 0$$

At joint *B* (see Fig. 12.9*c*): $\overset{+}{\circlearrowleft} \quad \Sigma M_B = 0$

$$M_{BA} + M_{BD} - 24(12) = 0$$

Example 12.3 Solution (continued)

- Express the moments in terms of displacements; write the equilibrium equations as

$$\text{At joint } D: \qquad 2.22E\theta_D + 1.11E\theta_B = 0$$

$$\text{At joint } B: (2.22E\theta_B + 648) + (2.22E\theta_B + 1.11E\theta_D) - 288 = 0$$

- Solving equations simultaneously gives

$$\theta_D = \frac{46.33}{E}$$

$$\theta_B = -\frac{92.66}{E}$$

Example 12.3 Solution (continued)

- To establish the values of the member end moments, the values of θ_B and θ_D are substituted

$$M_{AB} = 1.11E\left(-\frac{92.66}{E}\right) - 648$$

$$= -750.85 \text{ kip} \cdot \text{in} = -62.57 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

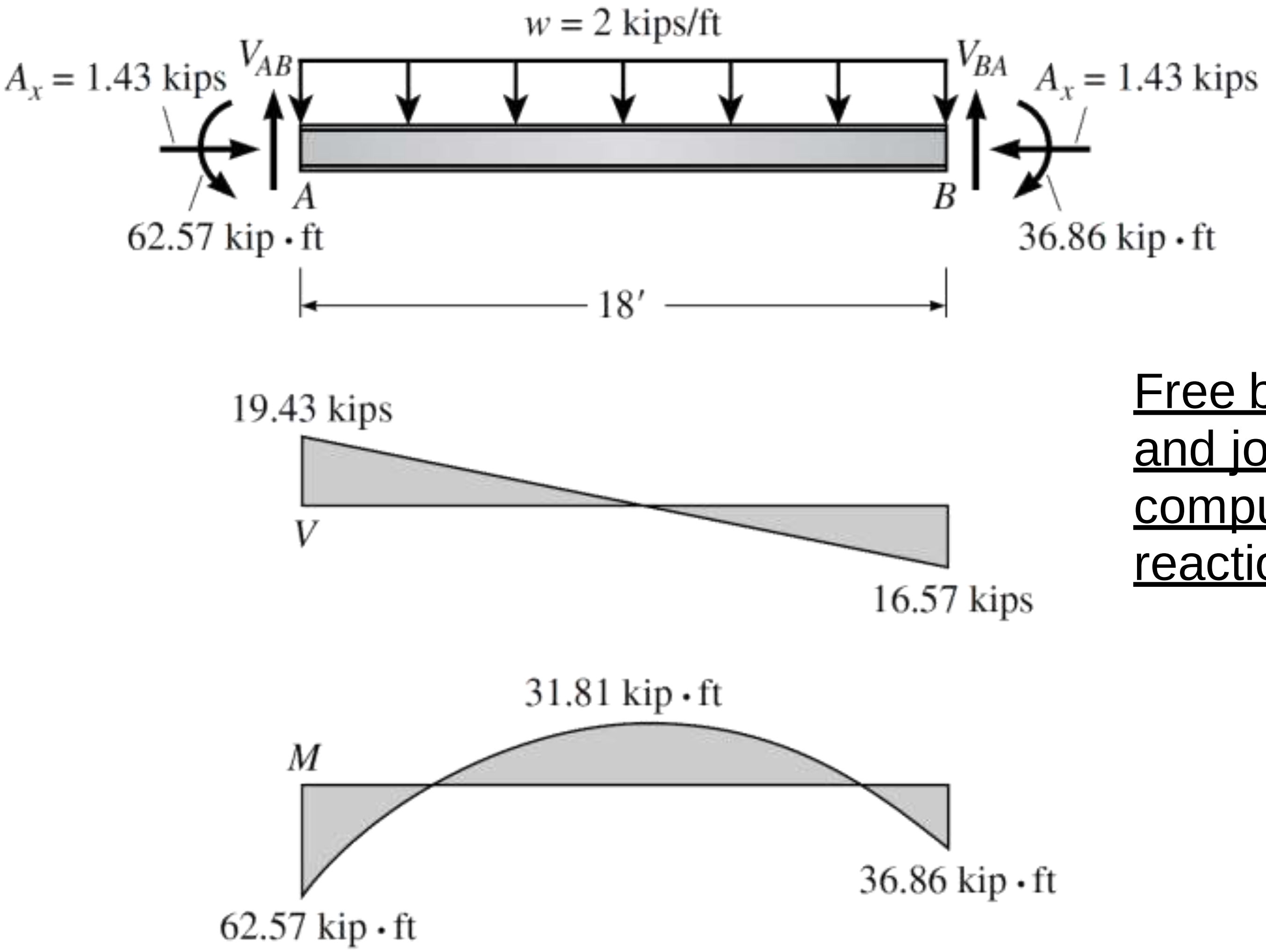
$$M_{BA} = 2.22E\left(-\frac{92.66}{E}\right) + 648$$

$$= 442.29 \text{ kip} \cdot \text{in} = +36.86 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

$$M_{BD} = 2.22E\left(-\frac{92.66}{E}\right) + 1.11E\left(\frac{46.33}{E}\right)$$

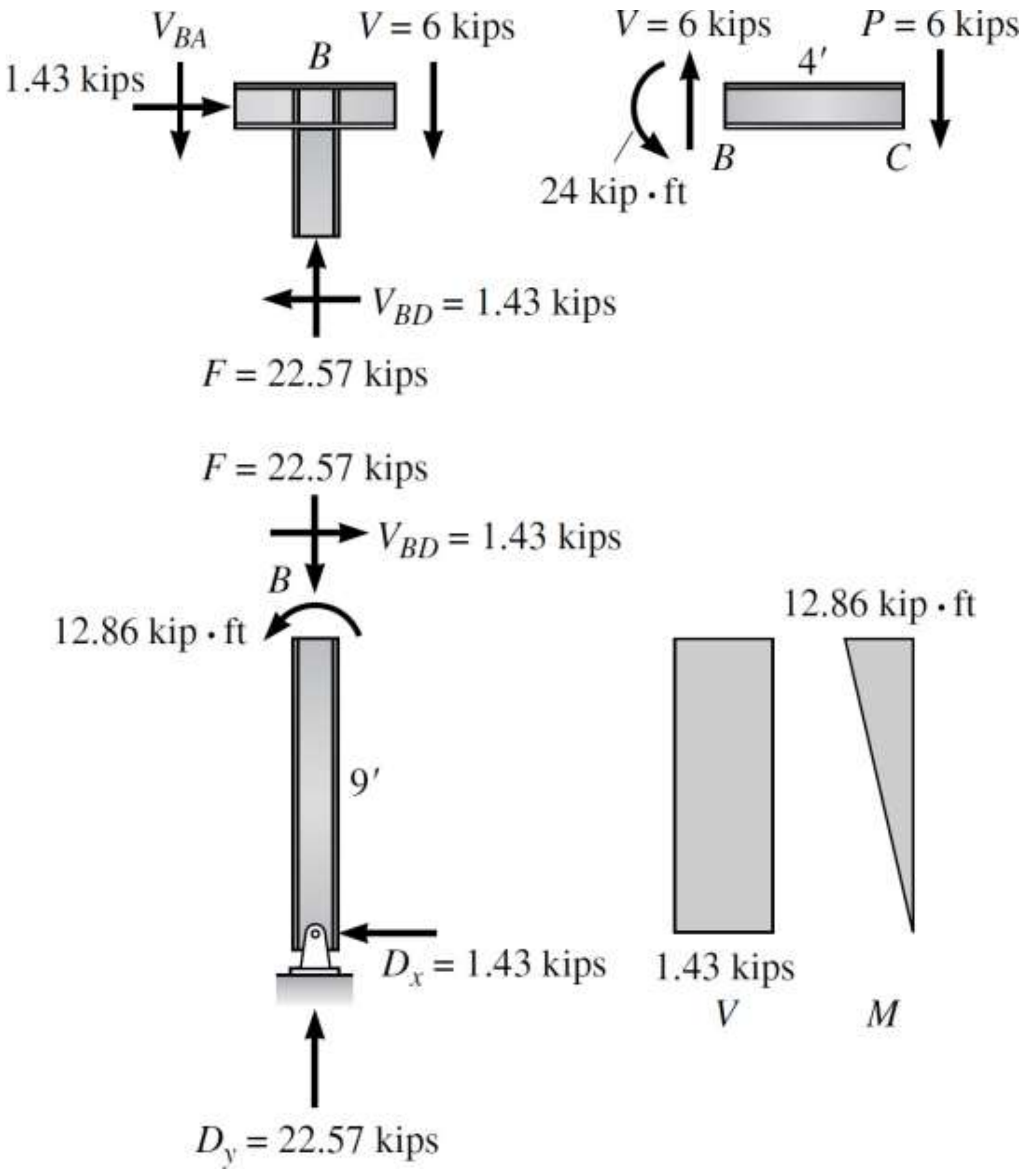
$$= -154.28 \text{ kip} \cdot \text{in} = -12.86 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

Example 12.3 Solution (continued)



Free bodies of members and joints used to compute shears and reactions

Example 12.3 Solution (continued)

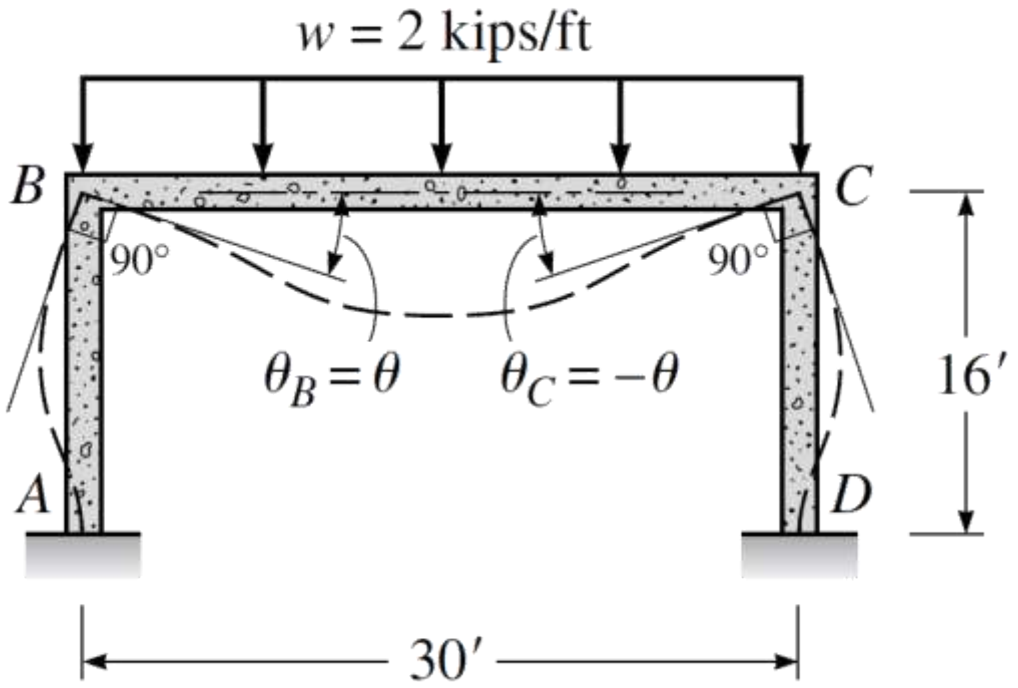


Free bodies of members and joints used to compute shears and reactions

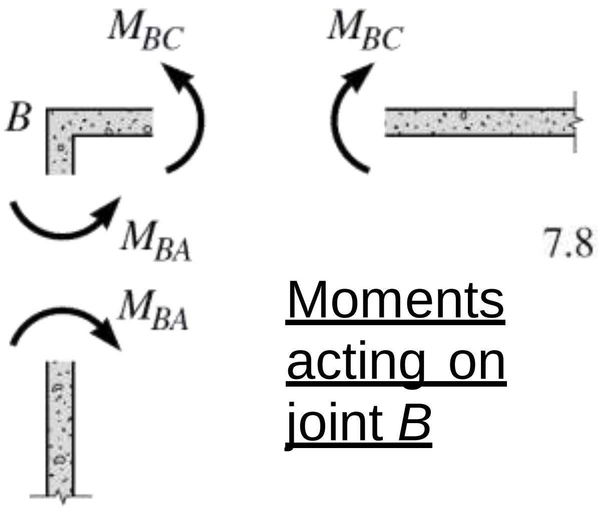
Example 12.4

Use of Symmetry to Simplify the Analysis of a Symmetric Structure with a Symmetric Load

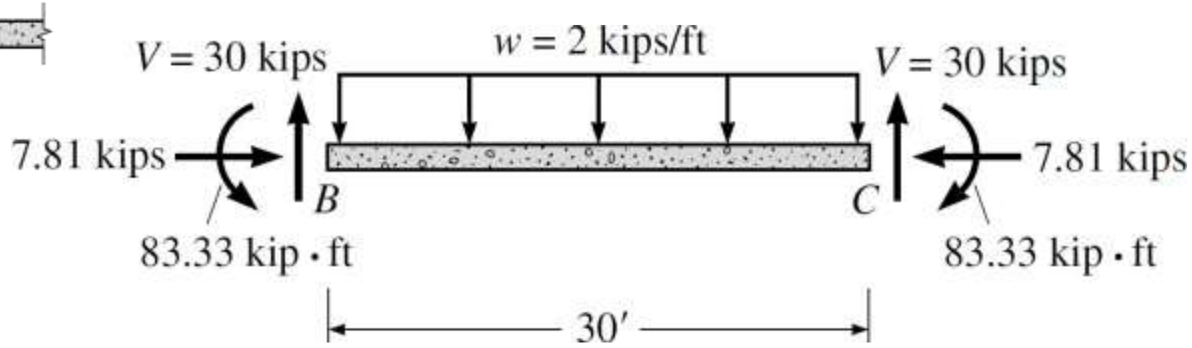
Determine the reactions and draw the shear and moment curves for the columns and girder of the rigid frame shown in Figure 12.10*a.* Given: $I_{AB} = I_{CD} = 120$ in^4, $I_{BC} = 360$ in^4, and E is constant for all members.



Example 12.4 Solution



Moments acting on joint *B*



- Expressing member end moments with Equation 12.16, reading the value of fixed-end moment for member *BC* from Figure 12.5*d*, and substituting $\theta_B = \theta$ and $\theta_C = -\theta$,

$$M_{AB} = \frac{2E(120)}{16(12)}(\theta_B) = 1.25E\theta_B$$

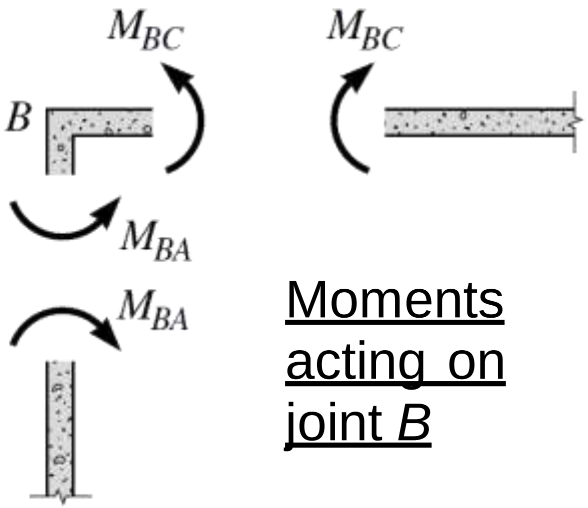
$$M_{BA} = \frac{2E(120)}{16(12)}(2\theta_B) = 2.50E\theta_B$$

$$M_{BC} = \frac{2E(360)}{30(12)}(2\theta_B + \theta_C) - \frac{wL^2}{12}$$

$$= 2E[2\theta + (-\theta)] - \frac{2(30)^2(12)}{12}$$

$$= 2E\theta - 1800$$

Example 12.4 Solution (continued)



Moments acting on joint *B*

- Writing the equilibrium equation at joint *B* yields

$$M_{BA} + M_{BC} = 0$$

- Substituting Equations 2 and 3 into Equation 4 and solving for *θ* produce

$$2.5E\theta + 2.0E\theta - 1800 = 0$$

$$\theta = \frac{400}{E}$$

Example 12.4 Solution (continued)

- Substituting the value of θ given by Equation 5 into Equations 1, 2, and 3 gives

$$M_{AB} = 1.25E\left(\frac{400}{E}\right)$$

$$= 500 \text{ kip} \cdot \text{in} = 41.67 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

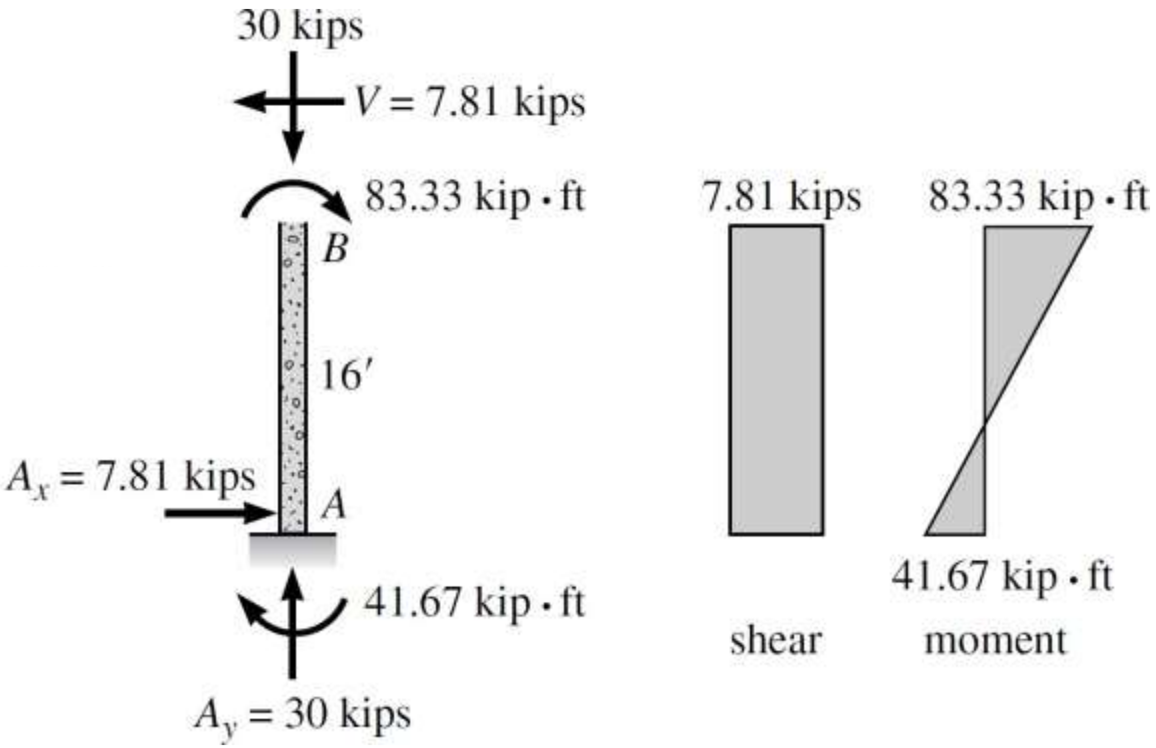
$$M_{BA} = 2.5E\left(\frac{400}{E}\right)$$

$$= 1000 \text{ kip} \cdot \text{in} = 83.33 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

$$M_{BC} = 2E\left(\frac{400}{E}\right) - 1800$$

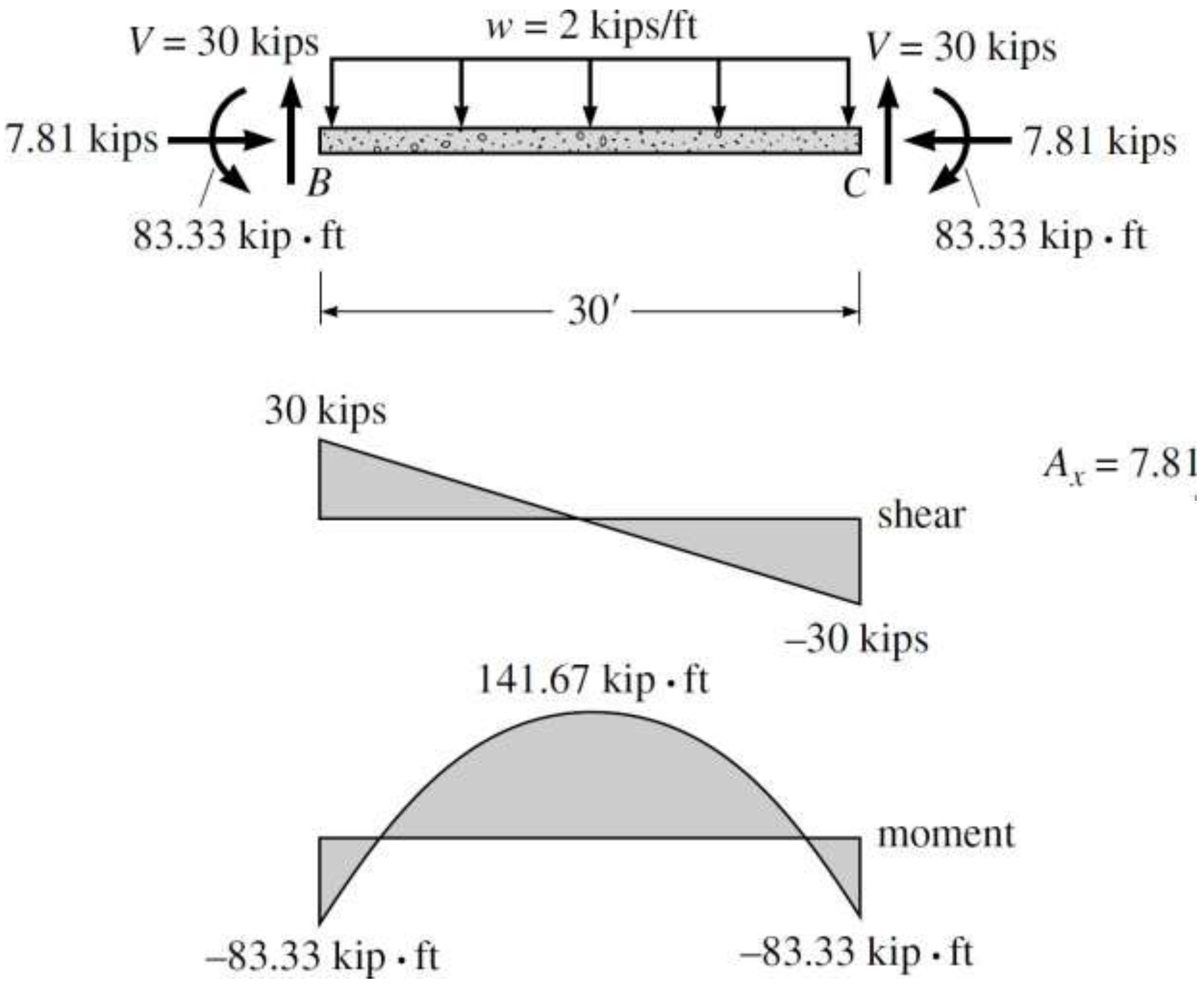
$$= -1000 \text{ kip} \cdot \text{in} = -83.33 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

Example 12.4 Solution (continued)



Free bodies of girder *BC* and column *AB* used to compute shears; final shear and moment curves also shown

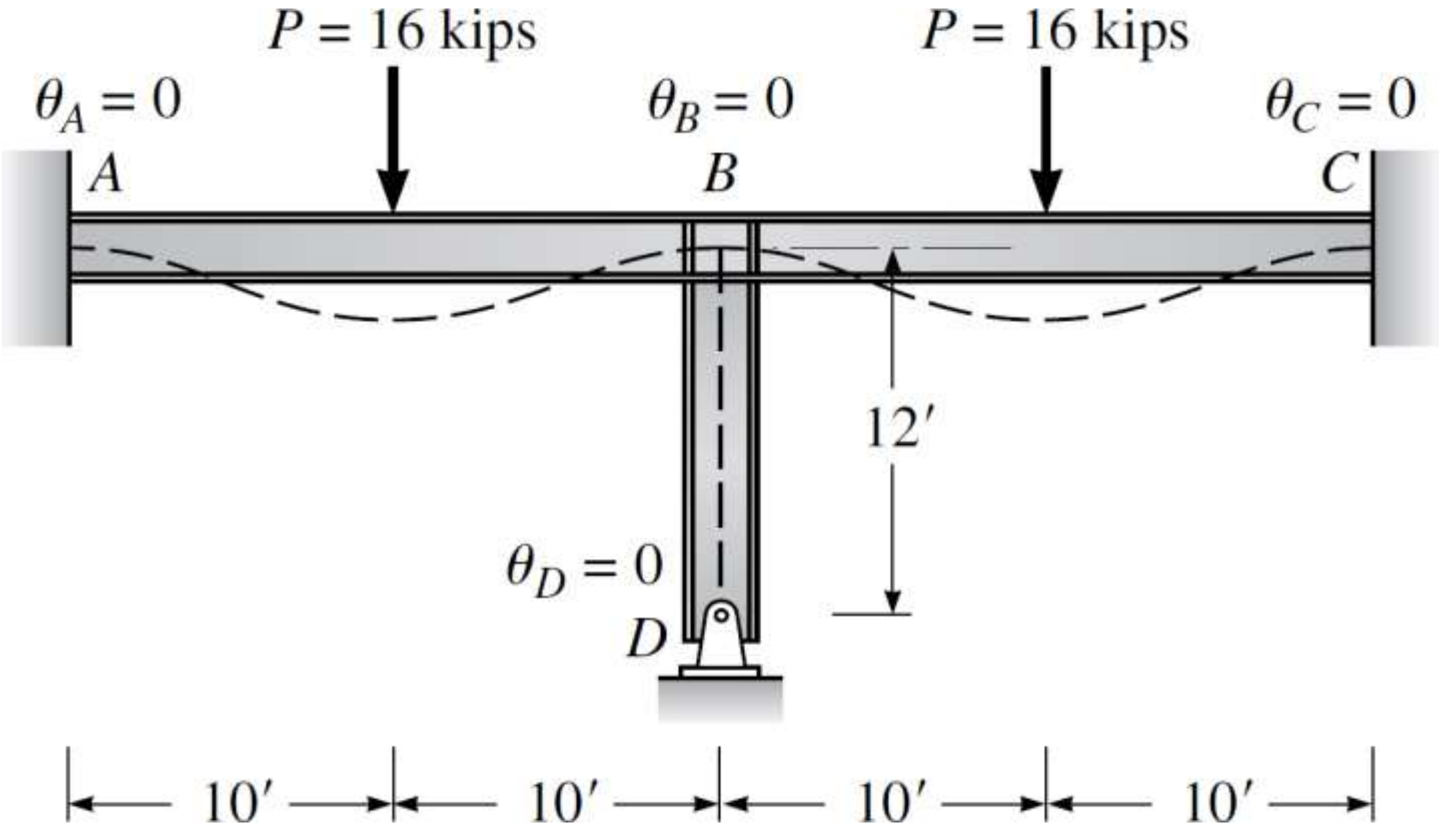
Example 12.4 Solution (continued)



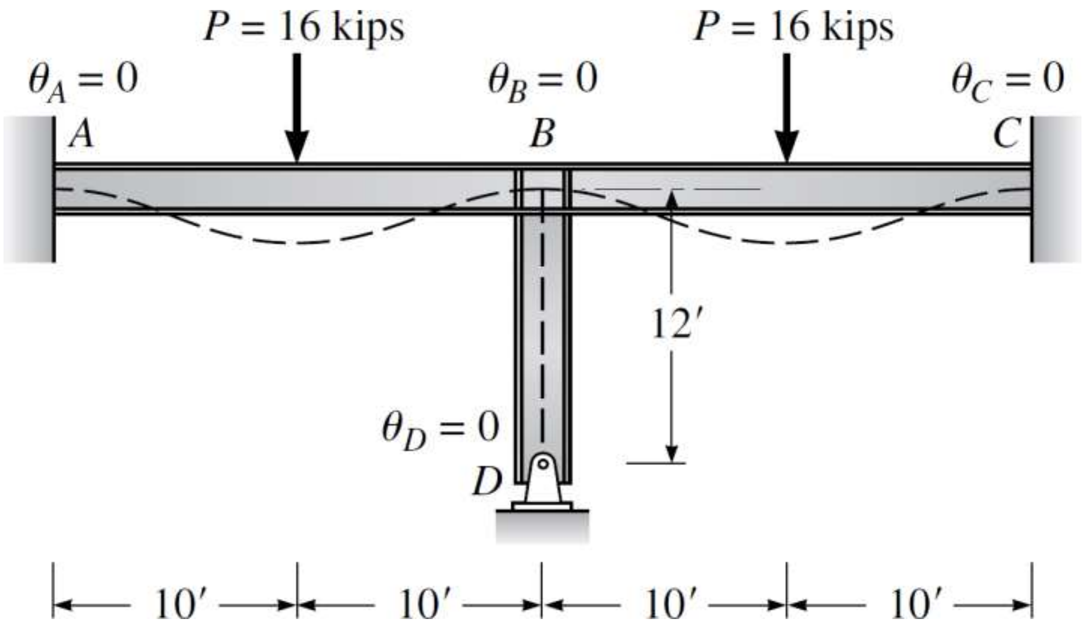
Free bodies of girder *BC* and column *AB* used to compute shears; final shear and moment curves also shown

Example 12.5

Using symmetry to simplify the slope-deflection analysis of the frame in Figure 12.11*a*, determine the reactions at supports *A* and *D*. *EI* is constant for all members.



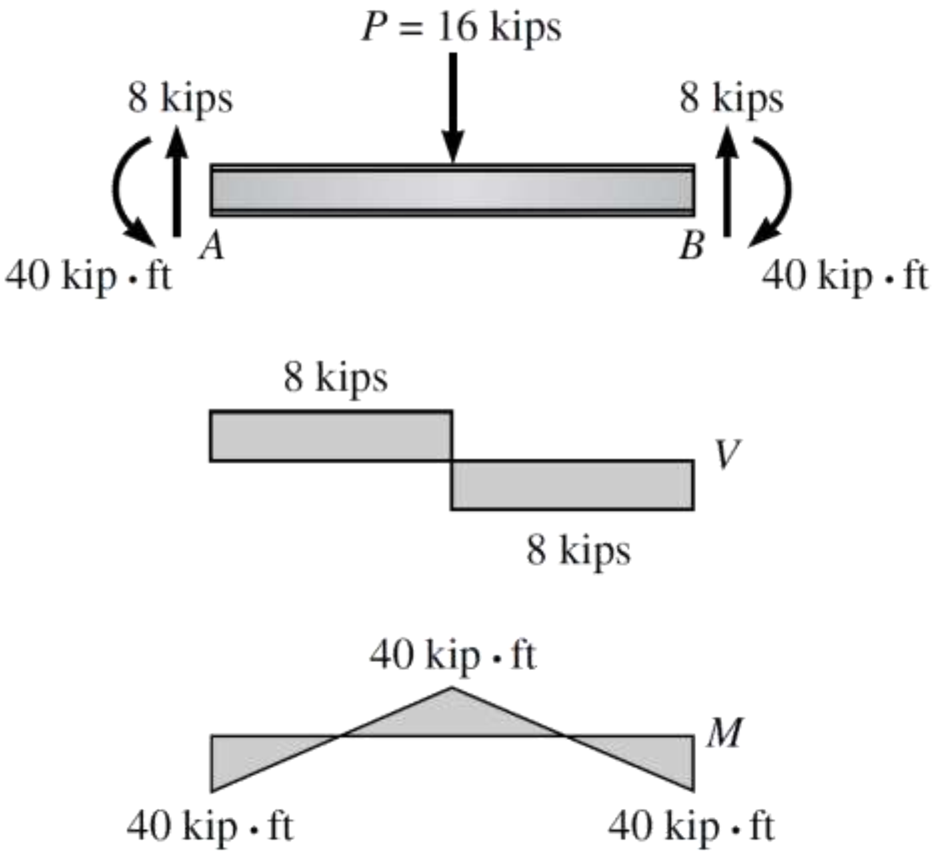
Example 12.5 Solution

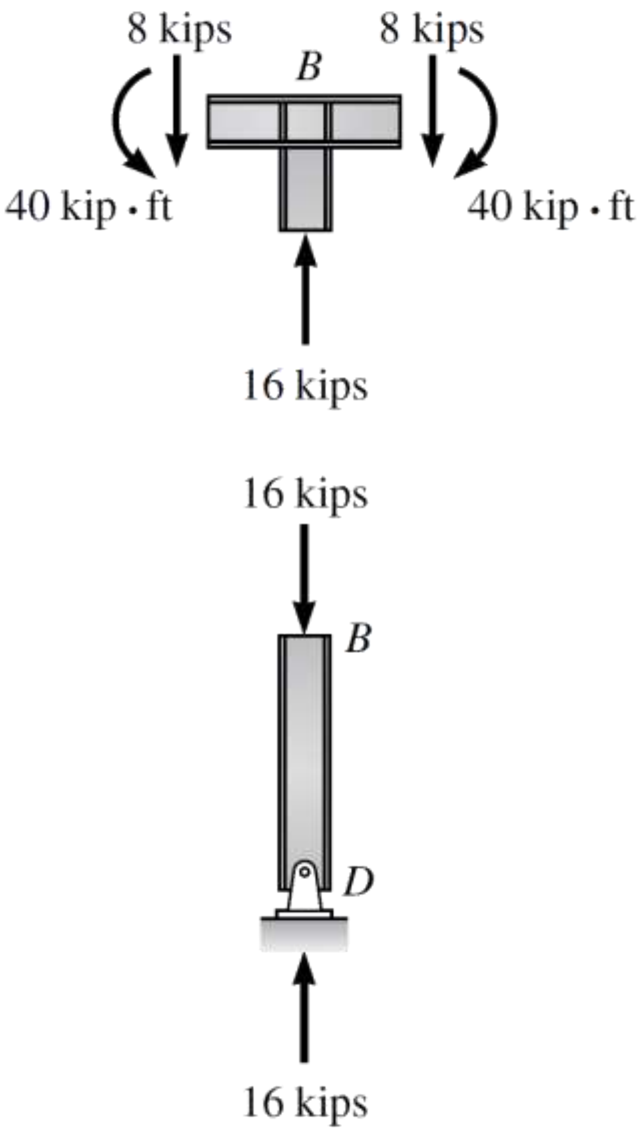


- Since all joint and chord rotations are zero, the member end moments at each end of beams *AB* and *BC* are equal to the fixed-end moments *PL*/8 given by Figure 12.5*a*:

$$\text{FEM} = \pm\frac{PL}{8} = \frac{16(20)}{8} = \pm 40 \text{ kip}\cdot\text{ft} \qquad \textbf{Ans.}$$

Example 12.5 Solution (continued)

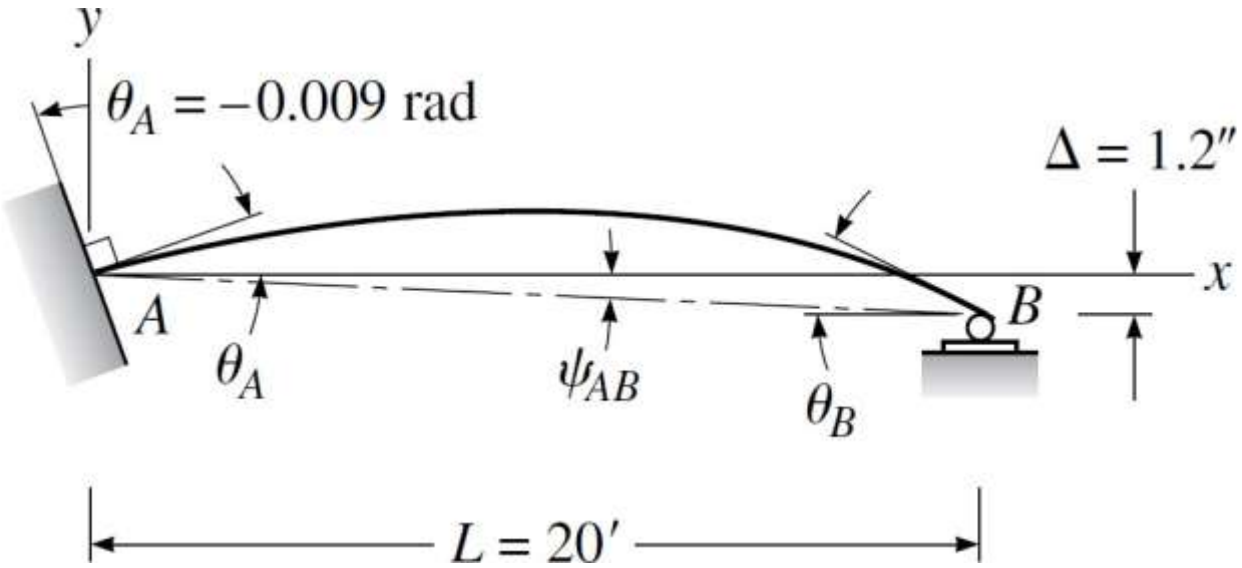




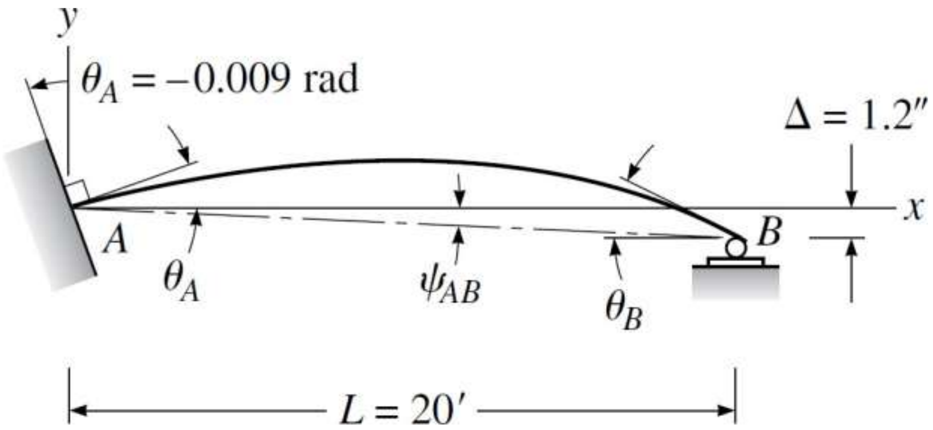
Free body of beam *AB*, joint *B*, and column *BD*. Final shear and moment diagrams for beam *AB*.

Example 12.6

Determine the reactions and draw the shear and moment curves for the beam in Figure 12.12. The support at *A* has been accidentally constructed with a slope that makes an angle of 0.009 rad with the vertical *y*-axis through support *A*, and *B* has been constructed 1.2 in below its intended position. Given: *EI* is constant, *I* = 360 in^4, and *E* = 29,000 kips/in^2.



Example 12.6 Solution



- $\theta_A = -0.009$ rad. The settlement of support *B* relative to support *A* produces a clockwise chord rotation

$$\psi_{AB} = \frac{\Delta}{L} = \frac{1.2}{20(12)} = 0.005 \text{ radians}$$

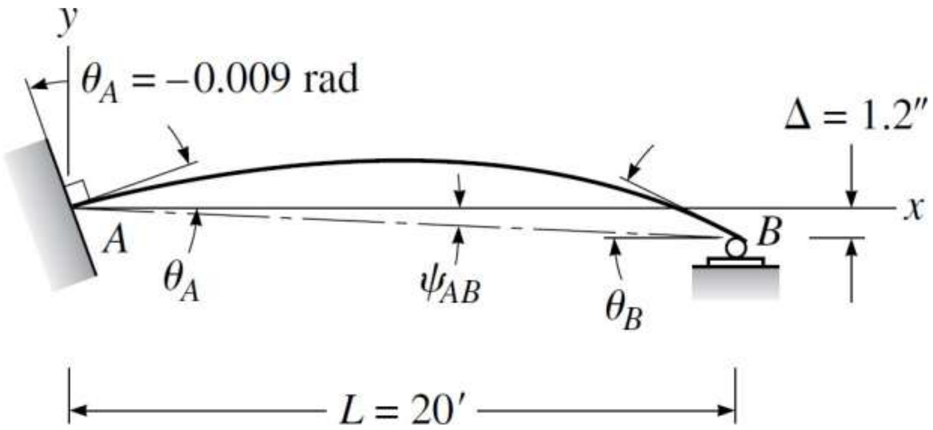
- Angle θ_B is the only unknown displacement. Expressing member end moments with the slope-deflection equation

$$M_{AB} = \frac{2EI_{AB}}{L_{AB}}(2\theta_A + \theta_B - 3\psi_{AB}) + \text{FEM}_{AB}$$

$$M_{AB} = \frac{2E(360)}{20(12)}[2(-0.009) + \theta_B - 3(0.005)]$$

$$M_{BA} = \frac{2E(360)}{20(12)}[2\theta_B + (-0.009) - 3(0.005)]$$

Example 12.6 Solution (continued)



- Writing the equilibrium equation at joint *B* yields

$$\overset{+}{\circlearrowleft} \quad \Sigma M_B = 0$$

$$M_{BA} = 0$$

- Substituting Equation 2 into Equation 3 and solving for θ_B yield

$$3E(2\theta_B - 0.009 - 0.015) = 0$$

$$\theta_B = 0.012 \text{ radians}$$

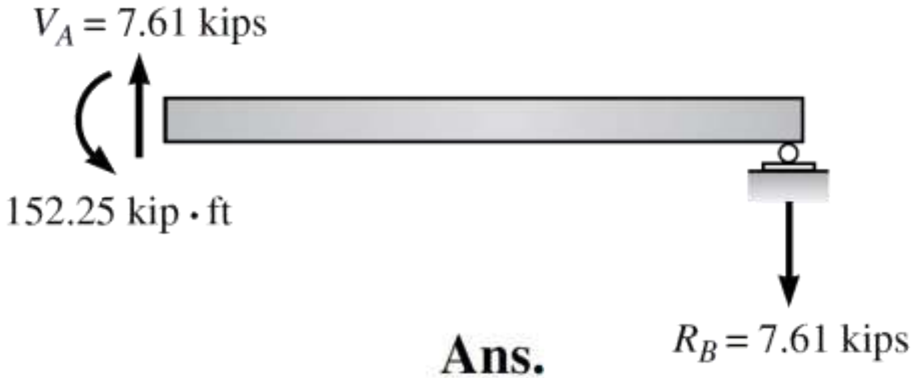
Example 12.6 Solution (continued)

- To evaluate M_{AB}, substitute θ_B into Equation 1:

$$M_{AB} = 3(29{,}000)[2(-0.009) + 0.012 - 3(0.005)]$$

$$= -1827 \text{ kip} \cdot \text{in} = -152.25 \text{ kip} \cdot \text{ft}$$

- Complete the analysis by using the equations of statics to compute the reaction at *B* and the shear at *A*.



$$\circlearrowright^{+} \quad \Sigma M_A = 0$$

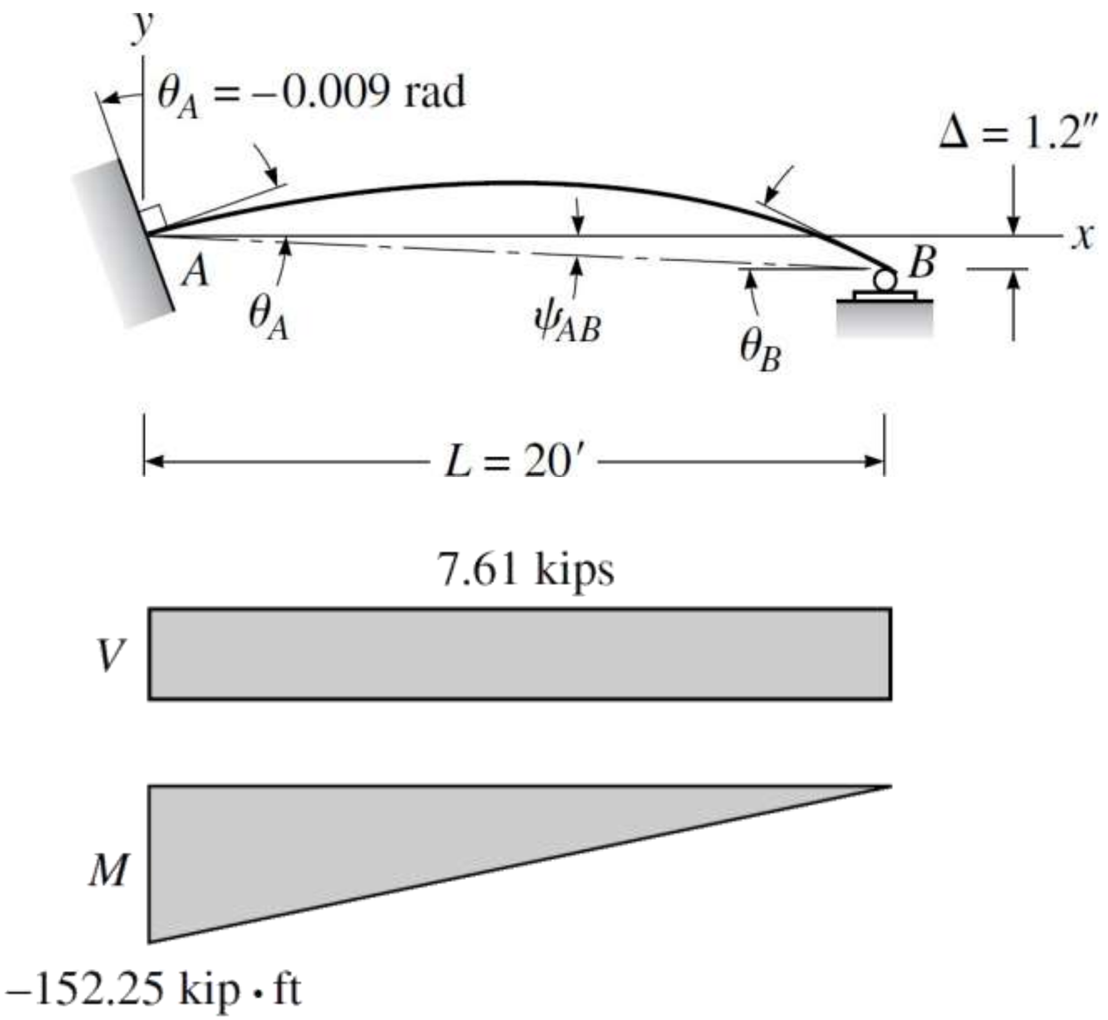
$$0 = R_B(20) - 152.25$$

$$R_B = 7.61 \text{ kips}$$ **Ans.**

$$\overset{+}{\uparrow} \quad \Sigma F_y = 0$$

$$V_A = 7.61 \text{ kips}$$ **Ans.**

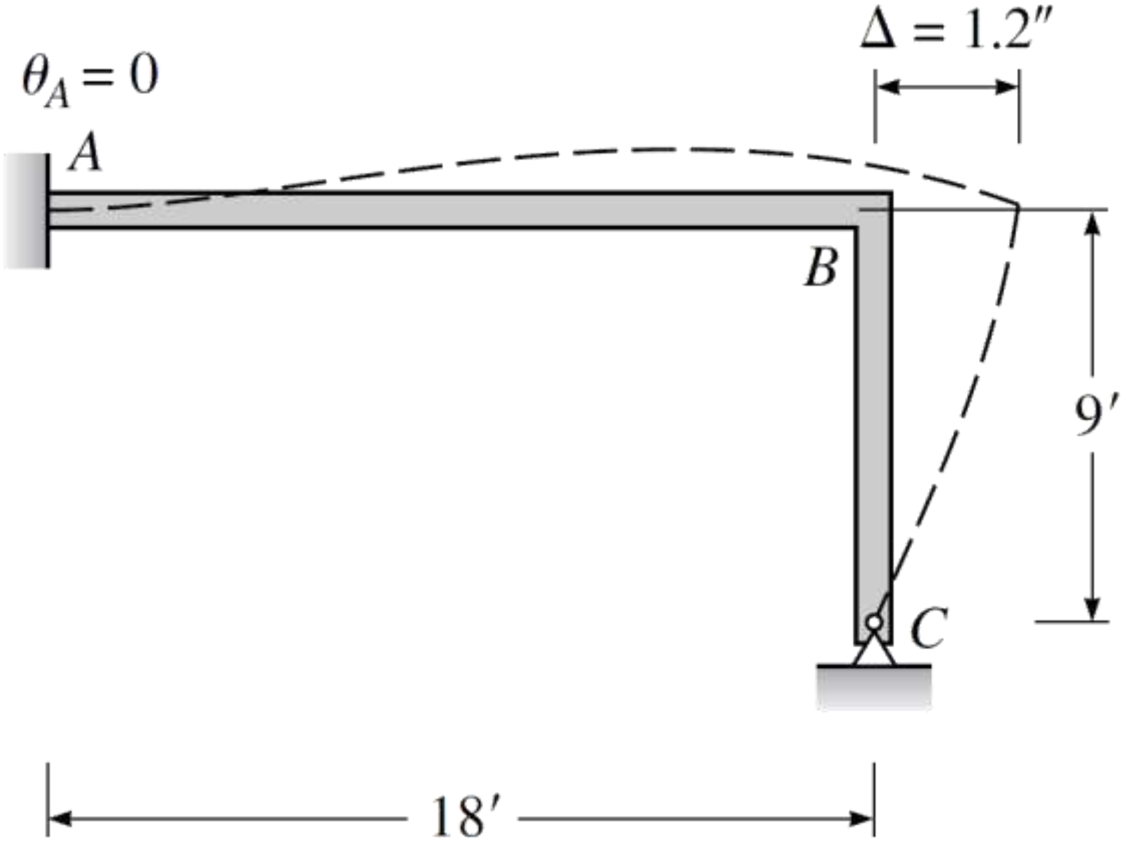
Example 12.6 Solution (continued)



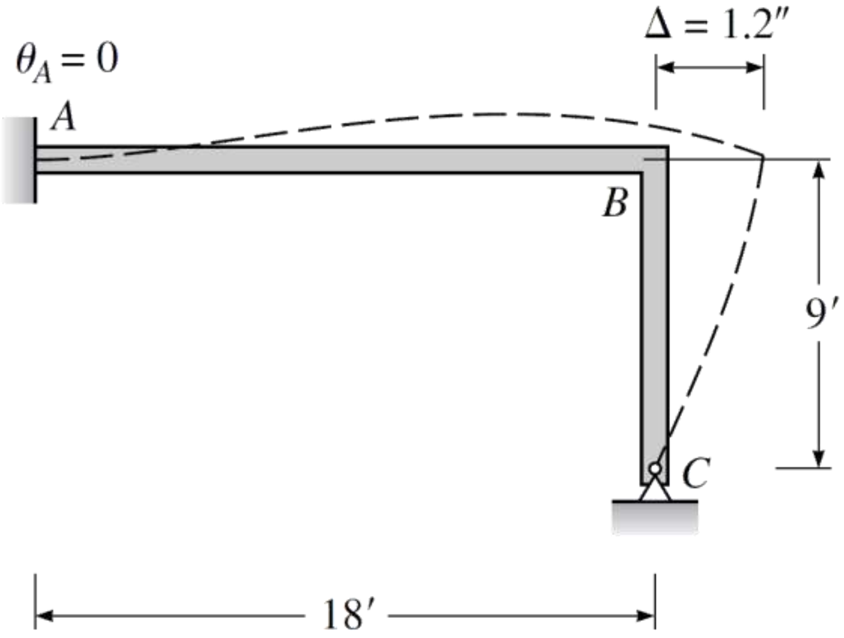
Shear and moment curves

Example 12.7

Although the supports are constructed in their correct position, girder *AB* of the frame shown in Figure 12.13 is fabricated 1.2 in too long. Determine the reactions created when the frame is connected into the supports. Given: *EI* is a constant for all members, $I = 240\ \text{in}^4$, and $E = 29{,}000\ \text{kips/in}^2$.



Example 12.7 Solution



- The chord rotation ψ_{BC} of column *BC* equals

$$\psi_{BC} = \frac{\Delta}{L} = \frac{1.2}{9(12)} = \frac{1}{90} \text{ rad}$$

- Since the ends of girder *AB* are at the same level, $\psi_{AB} = 0$. The unknown displacements are θ_B and θ_C

Example 12.7 Solution (continued)

- Using the slope-deflection equation (Equation 12.16), express member end moments in terms of the unknown displacements. Because no loads are applied to the members, all fixed-end moments equal zero.

$$M_{AB} = \frac{2E(240)}{18(12)}(\theta_B) = 2.222E\theta_B$$

$$M_{BA} = \frac{2E(240)}{18(12)}(2\theta_B) = 4.444E\theta_B$$

$$M_{BC} = \frac{2E(240)}{9(12)}\left[2\theta_B + \theta_C - 3\left(\frac{1}{90}\right)\right]$$

$$= 8.889E\theta_B + 4.444E\theta_C - 0.1481E$$

$$M_{CB} = \frac{2E(240)}{9(12)}\left[2\theta_C + \theta_B - 3\left(\frac{1}{90}\right)\right]$$

$$= 8.889E\theta_C + 4.444E\theta_B - 0.1481E$$

Example 12.7 Solution (continued)

- Writing equilibrium equations gives

$$\text{Joint } C: \qquad M_{CB} = 0$$

$$\text{Joint } B: \qquad M_{BA} + M_{BC} = 0$$

- Substituting and solving for θ_B and θ_C yield

$$8.889E\theta_C + 4.444E\theta_B - 0.1481E = 0$$

$$4.444E\theta_B + 8.889E\theta_B + 4.444E\theta_C - 0.1481E = 0$$

$$\theta_B = 0.00666 \text{ rad}$$

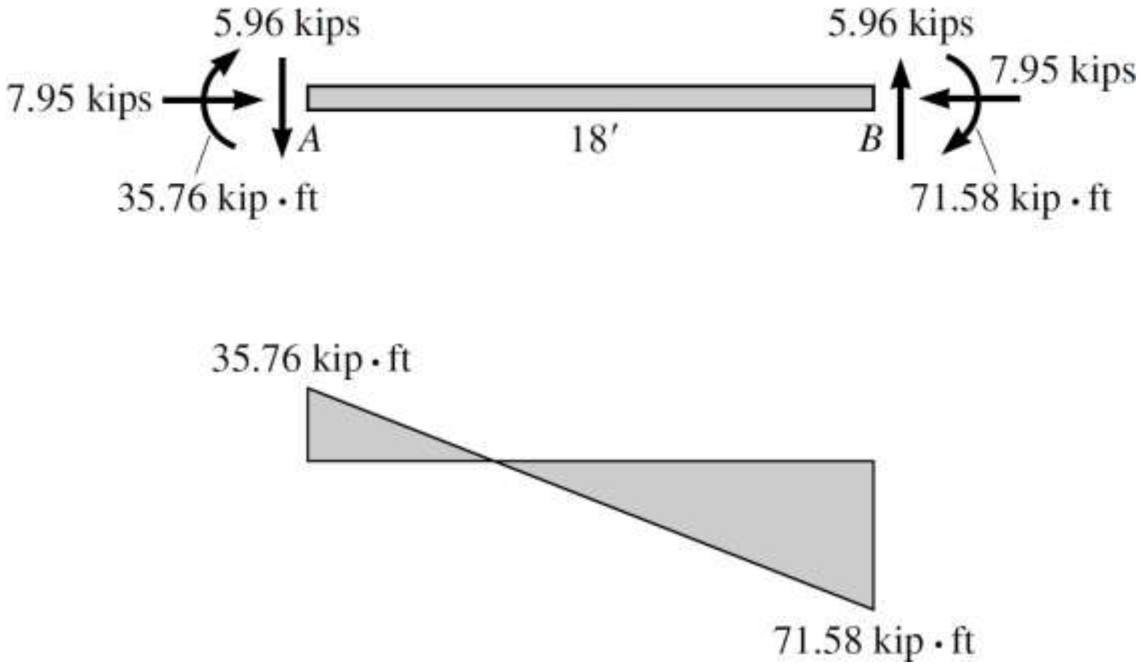
$$\theta_C = 0.01332 \text{ rad}$$

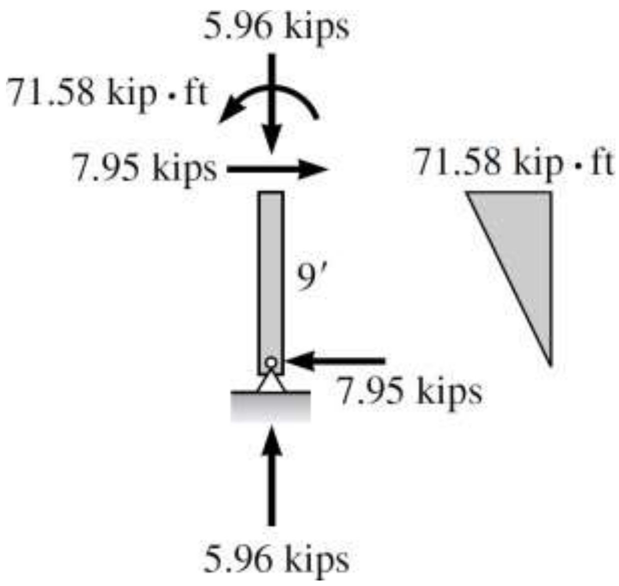
- Substituting θ_C and θ_B into Equations 1 to 3 produces

$$M_{AB} = 35.76 \text{ kip} \cdot \text{ft} \qquad M_{BA} = 71.58 \text{ kip} \cdot \text{ft}$$

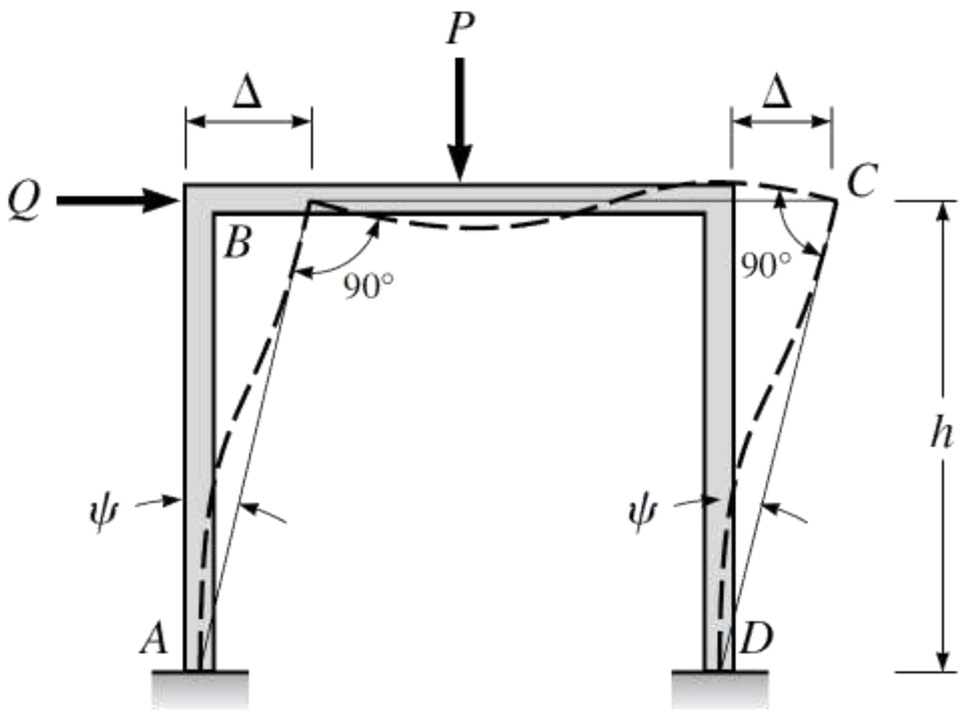
$$M_{BC} = -71.58 \text{ kip} \cdot \text{ft} \qquad M_{CB} = 0 \qquad \textbf{Ans.}$$

Example 12.7 Solution (continued)





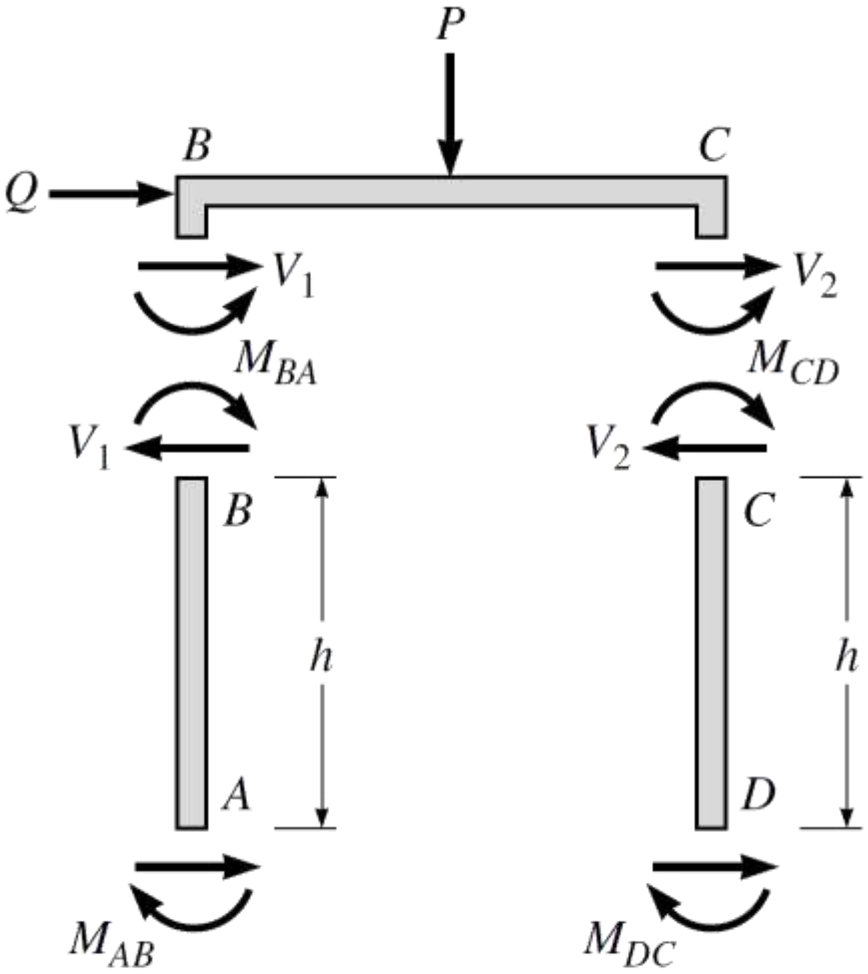
§12.5 Analysis of Structures That Are Free to Sidesway



Unbraced frame, deflected shape shown to an exaggerated scale by dashed lines, column chords rotate through a clockwise angle *ψ*

Figure 12.14

§12.5 Analysis of Structures That Are Free to Sidesway

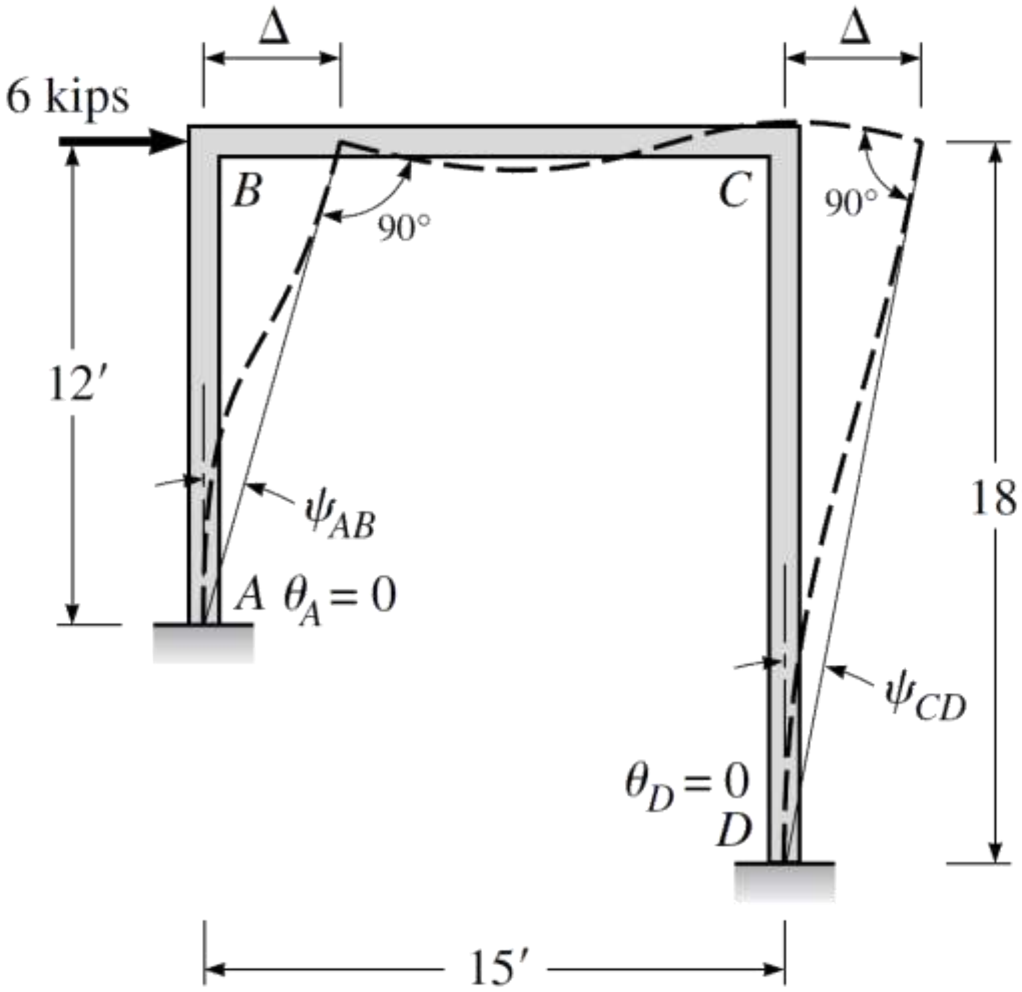


Free-body diagrams of columns and girders; unknown moments shown in the positive sense, that is, clockwise on ends of members

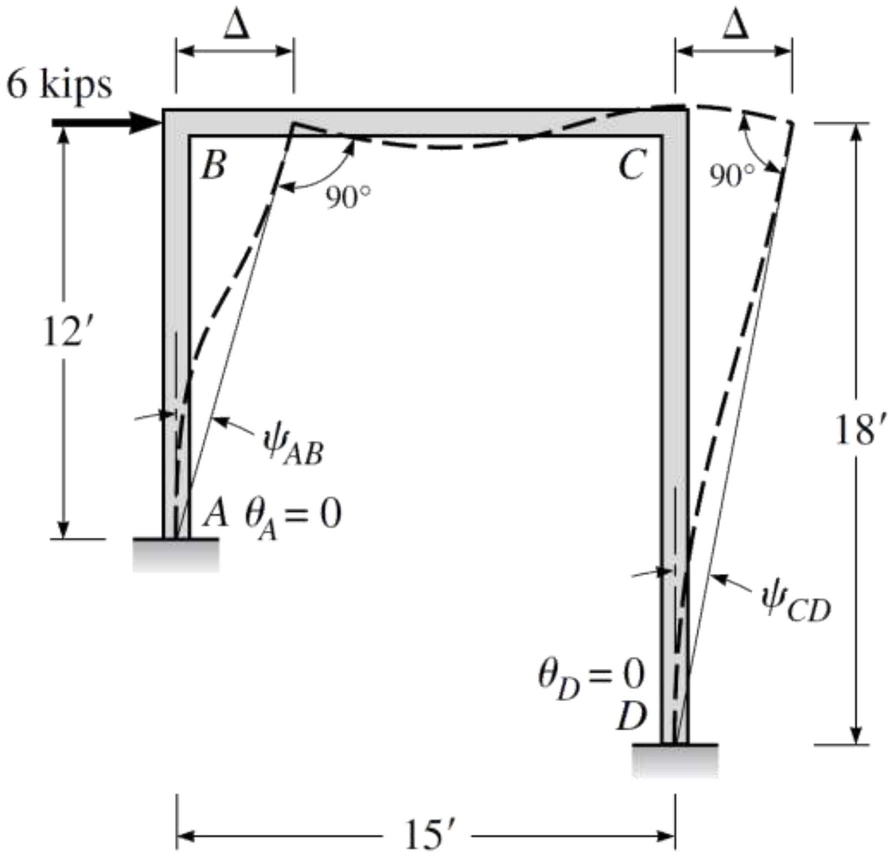
Figure 12.14 (continued)

Example 12.8

Analyze the frame in Figure 12.15*a* by the slope-deflection method. *E* is constant for all members; I_{AB} = 240 in^4, I_{BC} = 600 in^4, and I_{CD} = 360 in^4.



Example 12.8 Solution



- Identify the unknown displacements θ_B, θ_C, and Δ. Express the chord rotations ψ_{AB} and ψ_{CD} in terms of Δ:

$$\psi_{AB} = \frac{\Delta}{12} \qquad \text{and} \qquad \psi_{CD} = \frac{\Delta}{18}$$

$$\text{so} \qquad \psi_{AB} = 1.5\psi_{CD}$$

- Compute the relative bending stiffness of all members.

$$K_{AB} = \frac{EI}{L} = \frac{240E}{12} = 20E$$

$$K_{BC} = \frac{EI}{L} = \frac{600E}{15} = 40E$$

$$K_{CD} = \frac{EI}{L} = \frac{360E}{18} = 20E$$

Example 12.8 Solution (continued)

- Set 20*E* = *K*, then

$$K_{AB} = K \qquad K_{BC} = 2K \qquad K_{CD} = K$$

- Express member end moments in terms of displacements: $M_{NF} = (2EI/L)(2\theta_N + \theta_F - 3\psi_{NF}) + FEM_{NF}$. Since no loads are applied to members between joints, all $FEM_{NF} = 0$.

$$M_{AB} = 2K_{AB}(\theta_B - 3\psi_{AB}) \qquad M_{CB} = 2K_{BC}(2\theta_C + \theta_B)$$

$$M_{BA} = 2K_{AB}(2\theta_B - 3\psi_{AB}) \qquad M_{CD} = 2K_{CD}(2\theta_C - 3\psi_{CD})$$

$$M_{BC} = 2K_{BC}(2\theta_B + \theta_C) \qquad M_{DC} = 2K_{CD}(\theta_C - 3\psi_{CD})$$

- Use Equations 1 to express ψ_{AB} in terms of ψ_{CD}, and use Equations 2 to express all stiffness in terms of the parameter *K*.

$$M_{AB} = 2K(\theta_B - 4.5\psi_{CD}) \qquad M_{CB} = 4K(2\theta_C + \theta_B)$$

$$M_{BA} = 2K(2\theta_B - 4.5\psi_{CD}) \qquad M_{CD} = 2K(2\theta_C - 3\psi_{CD})$$

$$M_{BC} = 4K(2\theta_B + \theta_C) \qquad M_{DC} = 2K(\theta_C - 3\psi_{CD})$$

Example 12.8 Solution (continued)

- The equilibrium equations are:

Joint B: $$M_{BA} + M_{BC} = 0$$

Joint C: $$M_{CB} + M_{CD} = 0$$

Shear equation (see Eq. 12.21): $$\frac{M_{BA} + M_{AB}}{12} + \frac{M_{CD} + M_{DC}}{18} + 6 = 0$$

- Substitute Equations 4 into Equations 5, 6, and 7 and combine terms.

$$12\theta_B + 4\theta_C - 9\psi_{CD} = 0$$

$$4\theta_B + 12\theta_C - 6\psi_{CD} = 0$$

$$9\theta_B + 6\theta_C - 39\psi_{CD} = -\frac{108}{K}$$

Example 12.8 Solution (continued)

- Solving the equations simultaneously gives

$$\theta_B = \frac{2.257}{K} \qquad \theta_C = \frac{0.97}{K} \qquad \psi_{CD} = \frac{3.44}{K}$$

Also, $$\psi_{AB} = 1.5\psi_{CD} = \frac{5.16}{K}$$

Since all angles are positive, all joint rotations and the sidesway angles are clockwise.

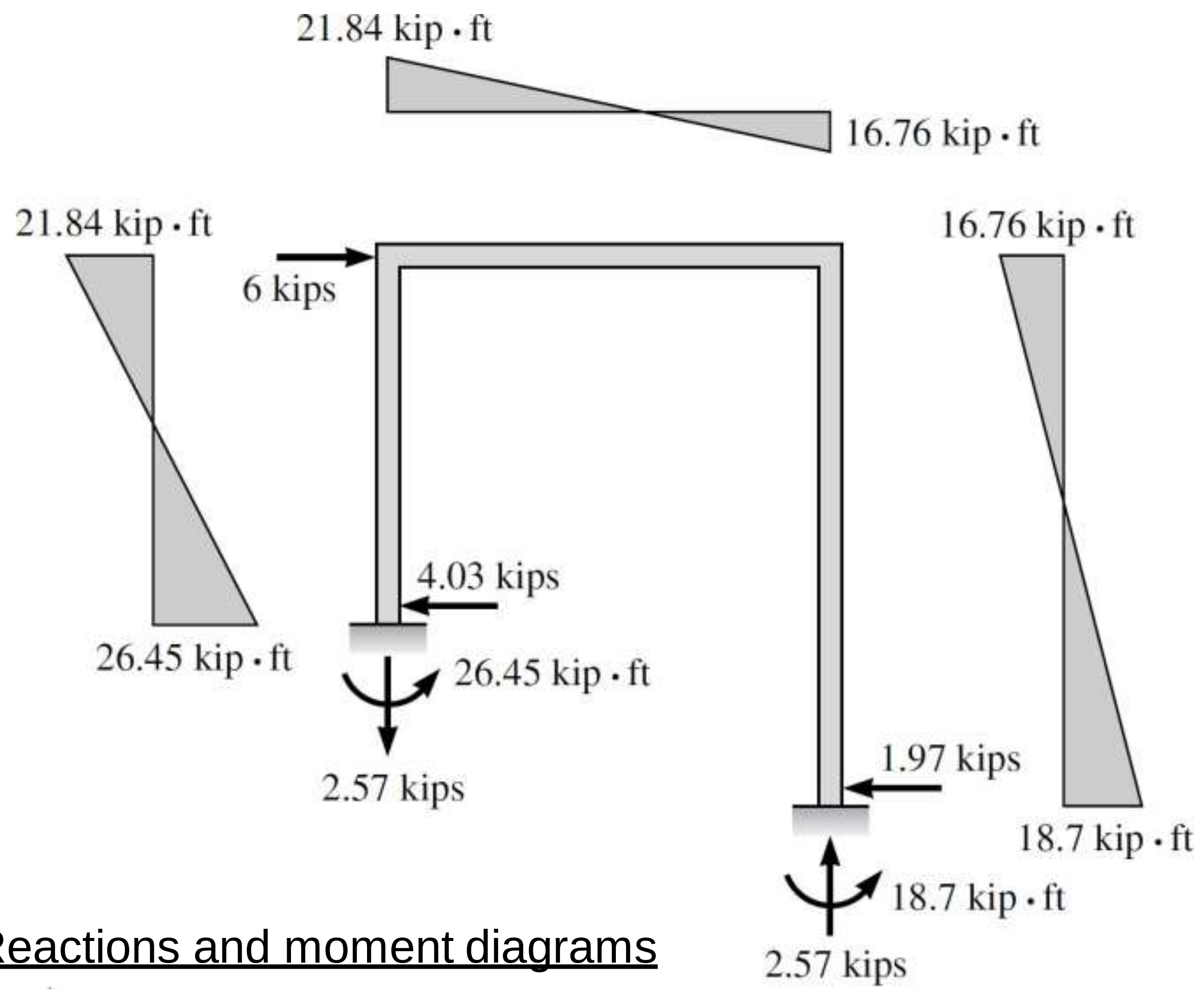
- Substituting the values of displacement above into Equations 4, establish the member end moments.

$$M_{AB} = -26.45 \text{ kip}\cdot\text{ft} \qquad M_{BA} = -21.84 \text{ kip}\cdot\text{ft}$$

$$M_{BC} = 21.84 \text{ kip}\cdot\text{ft} \qquad M_{CB} = 16.78 \text{ kip}\cdot\text{ft}$$

$$M_{CD} = -16.76 \text{ kip}\cdot\text{ft} \qquad M_{DC} = -18.7 \text{ kip}\cdot\text{ft} \qquad \textbf{Ans.}$$

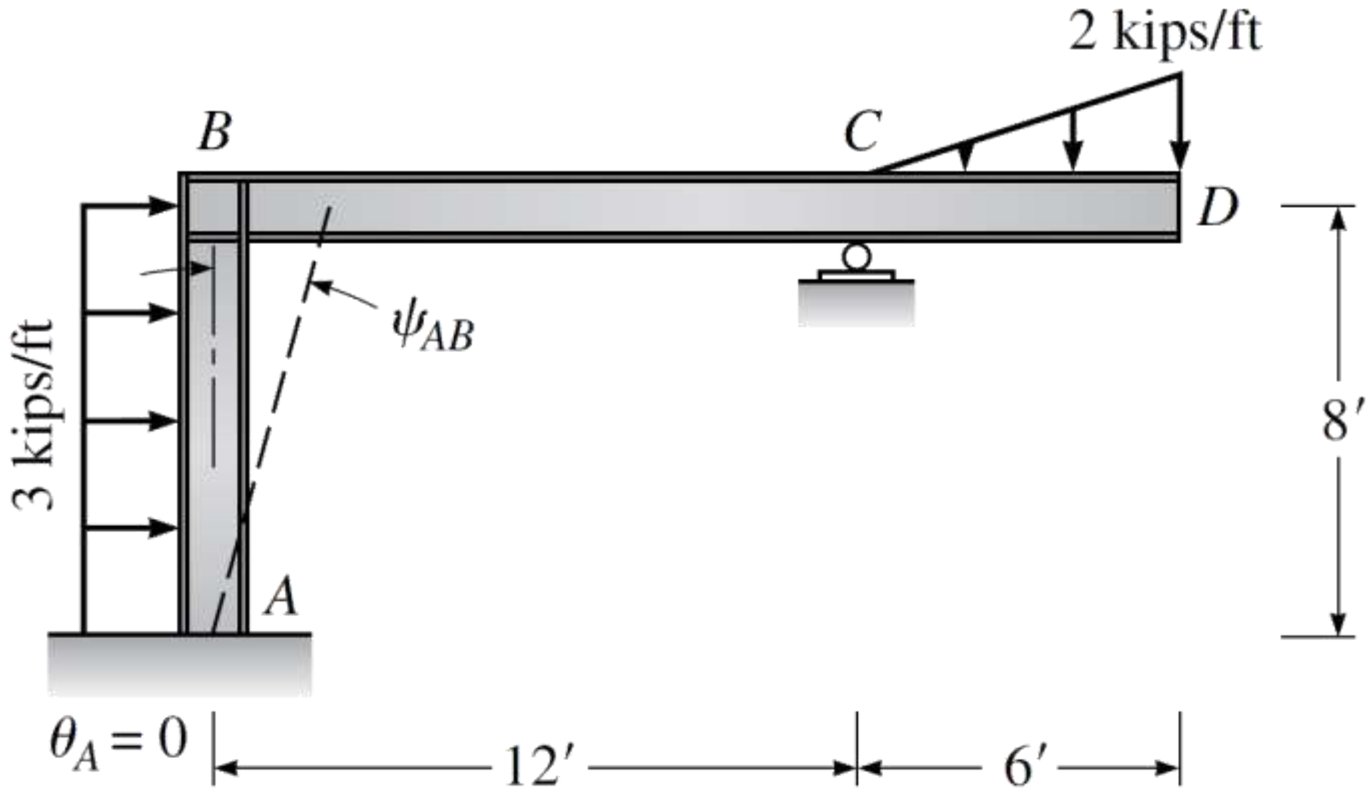
Example 12.8 Solution (continued)



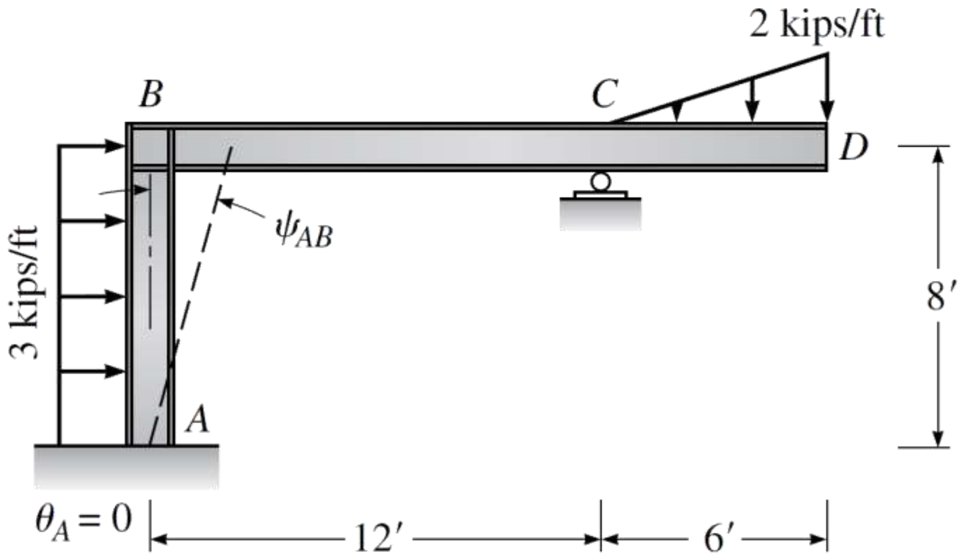
Reactions and moment diagrams

Example 12.9

Analyze the frame in Figure 12.16*a* by the slope-deflection method.
Given: *EI* is constant for all members.



Example 12.9 Solution



- Express member end moments in terms of displacements with Equation 12.16 (all units in kip-feet).

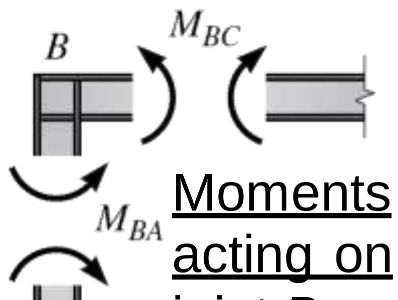
$$M_{AB} = \frac{2EI}{8}(\theta_B - 3\psi_{AB}) - \frac{3(8)^2}{12}$$

$$M_{BA} = \frac{2EI}{8}(2\theta_B - 3\psi_{AB}) + \frac{3(8)^2}{12}$$

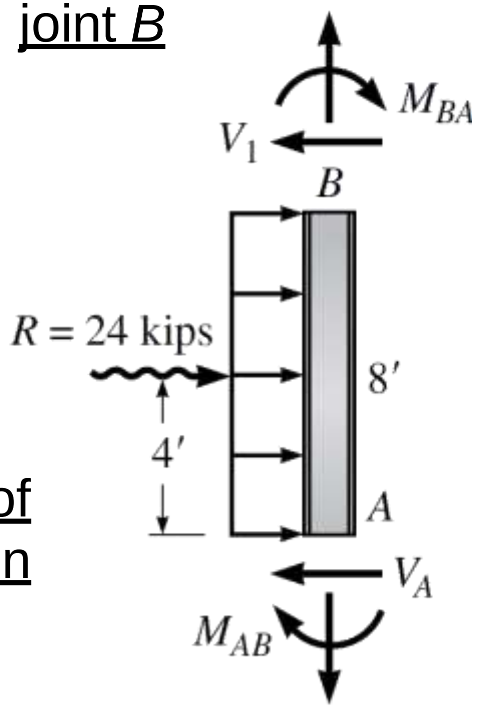
$$M_{BC} = \frac{2EI}{12}(2\theta_B + \theta_C)$$

$$M_{CB} = \frac{2EI}{12}(2\theta_C + \theta_B)$$

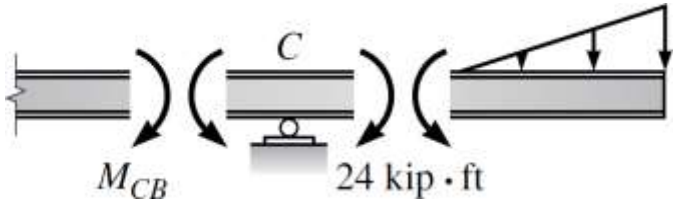
Example 12.9 Solution (continued)



Moments acting on joint *B*



Free body of column *AB*



Moments acting on joint *C*

- Write the joint equilibrium equations at *B* and *C*. Joint *B*:

$$\overset{+}{\circlearrowleft} \quad \Sigma M_B = 0: \qquad M_{BA} + M_{BC} = 0$$

- Joint C:

$$\overset{+}{\circlearrowleft} \quad \Sigma M_C = 0: \qquad M_{CB} - 24 = 0$$

- Shear equation:

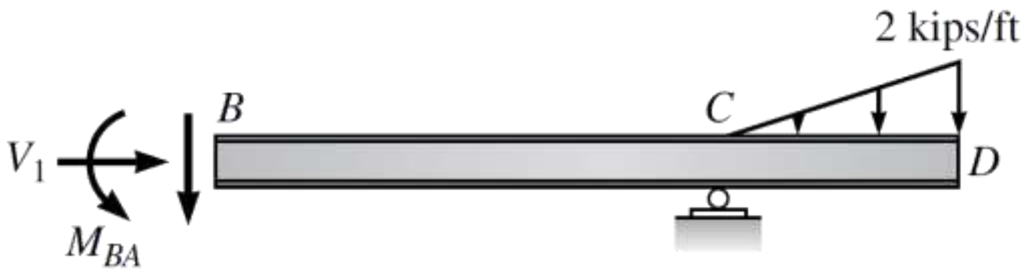
$$\circlearrowright^{+} \quad \Sigma M_A = 0$$

$$M_{BA} + M_{AB} + 24(4) - V_1(8) = 0$$

- Solving for V_1 gives

$$V_1 = \frac{M_{BA} + M_{AB} + 96}{8}$$

Example 12.9 Solution (continued)



Free body of girder used to establish third equilibrium equation

- Isolate the girder and consider equilibrium in the horizontal direction.

$$\xrightarrow{+} \quad \Sigma F_x = 0: \quad \text{therefore} \quad V_1 = 0$$

- Substitute Equation 4*a* into Equation 4*b*:

$$M_{BA} + M_{AB} + 96 = 0$$

- Express equilibrium equations in terms of displacements by substituting Equations 1 into Equations 2, 3, and 4. Collecting terms and simplifying,

$$10\theta_B - 2\theta_C - 9\psi_{AB} = -\frac{192}{EI}$$

$$\theta_B - 2\theta_C = \frac{144}{EI}$$

$$3\theta_B - 6\psi_{AB} = -\frac{384}{EI}$$

Example 12.9 Solution (continued)

- Solution of the equations

$$\theta_B = \frac{53.33}{EI} \qquad \theta_C = \frac{45.33}{EI} \qquad \psi_{AB} = \frac{90.66}{EI}$$

- Establish the values of member end moments by substituting the values of θ_B, θ_C, and ψ_{AB} into Equations 1.

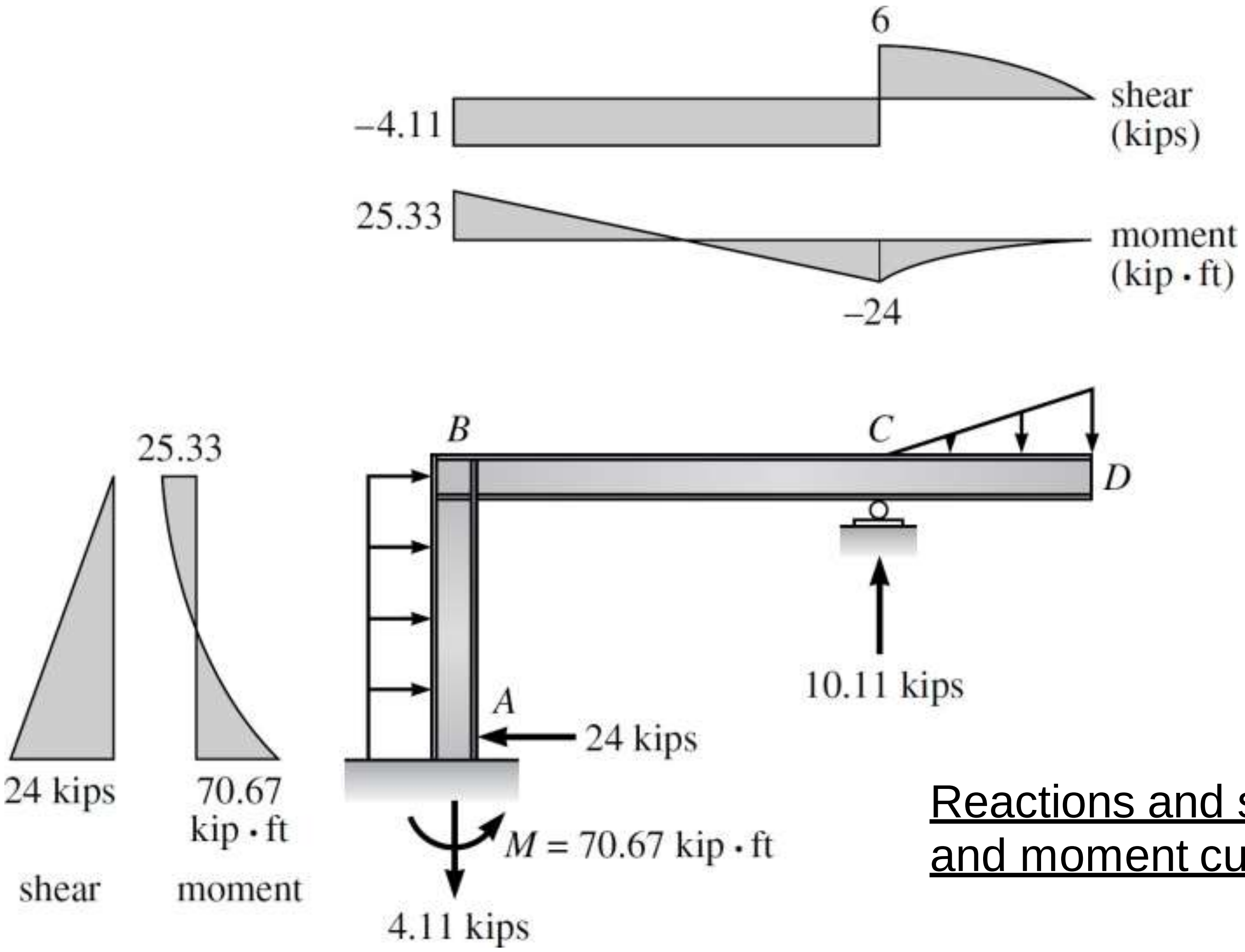
$$M_{AB} = \frac{2EI}{8}\left[\frac{53.33}{EI} - \frac{(3)(90.66)}{EI}\right] - 16 = -70.67 \text{ kip} \cdot \text{ft}$$

$$M_{BA} = \frac{2EI}{8}\left[\frac{(2)(53.33)}{EI} - \frac{(3)(90.66)}{EI}\right] + 16 = -25.33 \text{ kip} \cdot \text{ft}$$

$$M_{BC} = \frac{2EI}{12}\left[\frac{(2)(53.33)}{EI} + \frac{45.33}{EI}\right] = 25.33 \text{ kip} \cdot \text{ft}$$

$$M_{CB} = \frac{2EI}{12}\left[\frac{(2)(45.33)}{EI} + \frac{53.33}{EI}\right] = 24 \text{ kip} \cdot \text{ft} \qquad \textbf{Ans.}$$

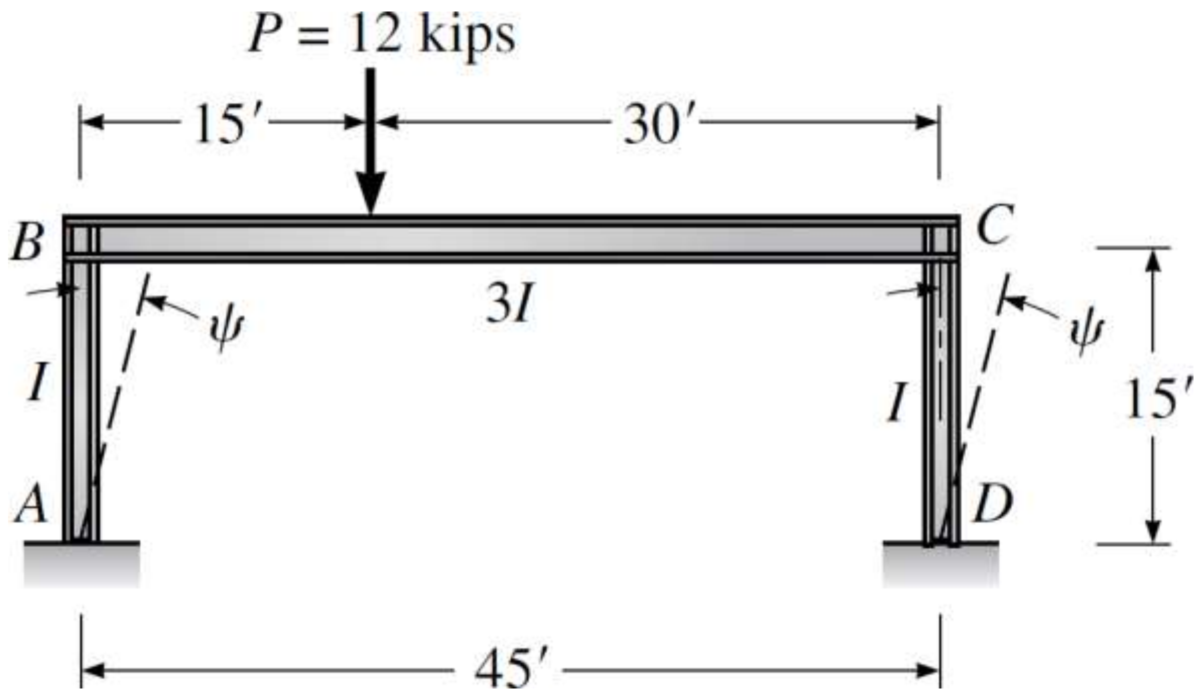
Example 12.9 Solution (continued)



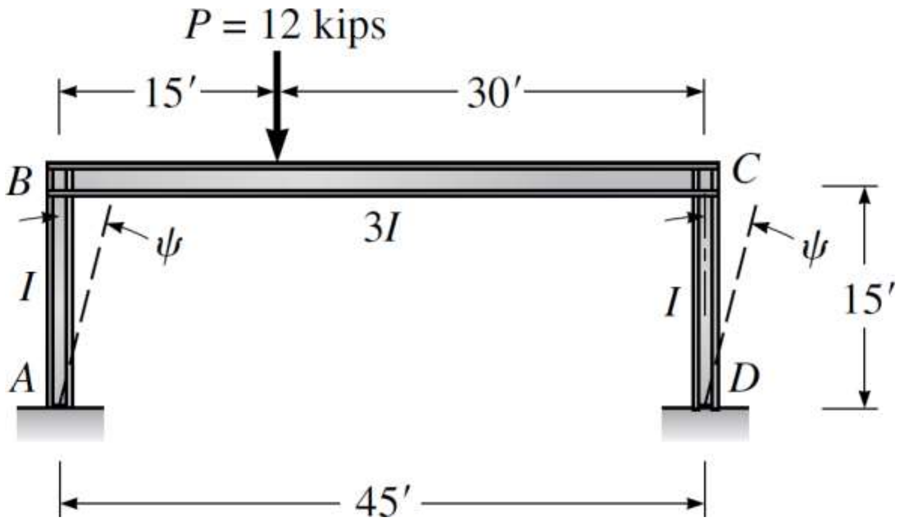
Reactions and shear and moment curves

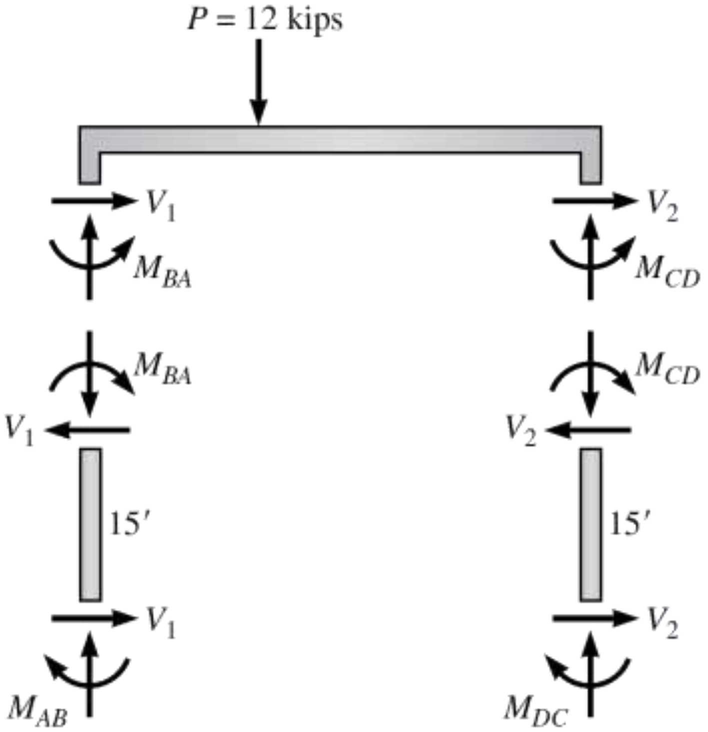
Example 12.10

Analyze the frame in Figure 12.17*a* by the slope-deflection method. Determine the reactions, draw the moment curves for the members, and sketch the deflected shape. If I = 240 in^4 and E = 30,000 kips/in^2, determine the horizontal displacement of joint *B*.



Example 12.10 Solution





- Express member end moments in terms of displacements with the slope-deflection equation.

$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF} \qquad (12.16)$$

$$\text{FEM}_{BC} = -\frac{Pb^2a}{L^2} = \frac{12(30)^2(15)}{(45)^2} = -80 \text{ kip}\cdot\text{ft} \qquad \text{FEM}_{CD} = \frac{Pa^2b}{L^2} = \frac{12(15)^2(30)}{(45)^2} = 40 \text{ kip}\cdot\text{ft}$$

Example 12.10 Solution (continued)

- To simplify slope-deflection expressions, set *EI*/15 = *K*.

$$M_{AB} = \frac{2EI}{15}(\theta_B - 3\psi) \qquad = 2K(\theta_B - 3\psi)$$

$$M_{BA} = \frac{2EI}{15}(2\theta_B - 3\psi) \qquad = 2K(2\theta_B - 3\psi)$$

$$M_{BC} = \frac{2EI}{45}(2\theta_B + \theta_C) - 80 = \frac{2}{3}K(2\theta_B + \theta_C) - 80$$

$$M_{CB} = \frac{2EI}{45}(2\theta_C + \theta_B) + 40 = \frac{2}{3}K(2\theta_C + \theta_B) + 40$$

$$M_{CD} = \frac{2EI}{15}(2\theta_C - 3\psi) \qquad = 2K(\theta_C - 3\psi)$$

$$M_{DC} = \frac{2EI}{15}(\theta_C - 3\psi) \qquad = 2K(\theta_C - 3\psi)$$

Example 12.10 Solution (continued)

- The equilibrium equations are:

Joint B: $$M_{BA} + M_{BC} = 0$$

Joint C: $$M_{CB} + M_{CD} = 0$$

- Shear equation:

$$\xrightarrow{}+ \quad \Sigma F_x = 0 \qquad V_1 + V_2 = 0$$

where $$V_1 = \frac{M_{BA} + M_{AB}}{15} \qquad V_2 = \frac{M_{CD} + M_{DC}}{15}$$

- Substituting V_1 and V_2 given by Equations 4*b* into 4*a* gives

$$M_{BA} + M_{AB} + M_{CD} + M_{DC} = 0$$

Alternatively, set Q = 0 in Equation 12.21 to produce Equation 4.

Example 12.10 Solution (continued)

- Express equilibrium equations in terms of displacements by substituting Equations 1 into Equations 2, 3, and 4. Combining terms and simplifying give

$$8K\theta_B + K\theta_C - 9K\psi = 120$$

$$2K\theta_B + 16K\theta_C - 3K\psi = -120$$

$$K\theta_B + K\theta_C - 4K\psi = 0$$

- Solving the equations simultaneously,

$$\theta_B = \frac{410}{21K} \qquad \theta_C = -\frac{130}{21K} \qquad \psi = \frac{10}{3K}$$

- Substituting the values of the θ_B, θ_C, and ψ into Equations 1,

$$\begin{aligned} M_{AB} &= 19.05 \text{ kip}\cdot\text{ft} & M_{BA} &= 58.1 \text{ kip}\cdot\text{ft} & & \\ M_{CD} &= -44.76 \text{ kip}\cdot\text{ft} & M_{DC} &= -32.38 \text{ kip}\cdot\text{ft} & & (6) \\ M_{BC} &= -58.1 \text{ kip}\cdot\text{ft} & M_{CB} &= 44.76 \text{ kip}\cdot\text{ft} & & \textbf{Ans.} \end{aligned}$$

Example 12.10 Solution (continued)

- Compute the horizontal displacement of joint *B*. Use Equation 1 for M_{AB}. Express all variables in units of inches and kips.

$$M_{AB} = \frac{2EI}{15(12)}(\theta_B - 3\psi)$$

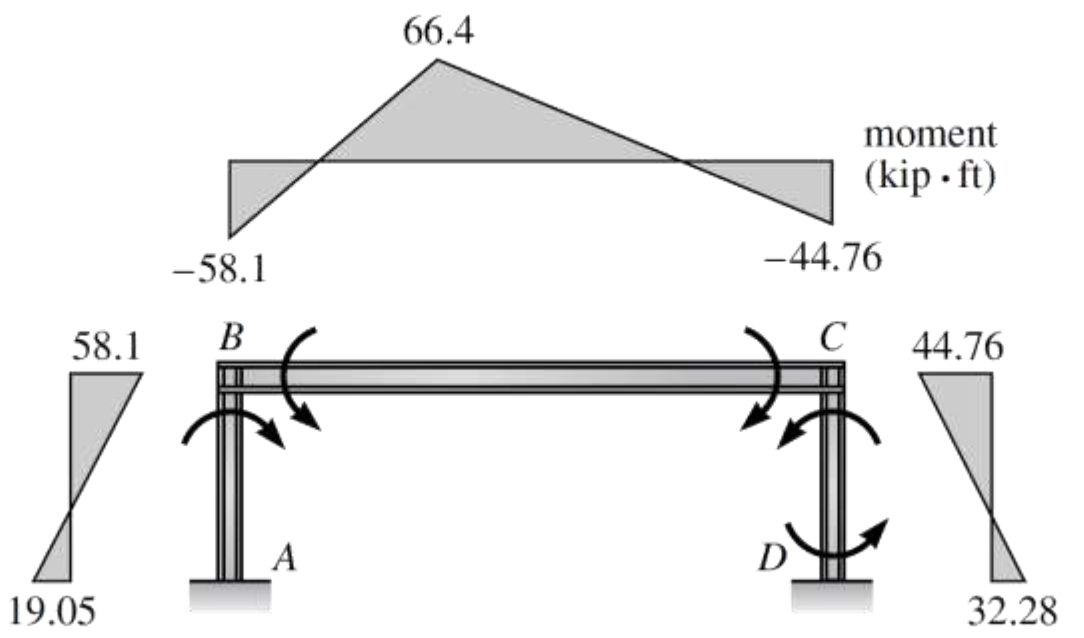
- From the values in Equation 5 (p. 485), $\theta_B = 5.86\psi$; substituting into Equation 7,

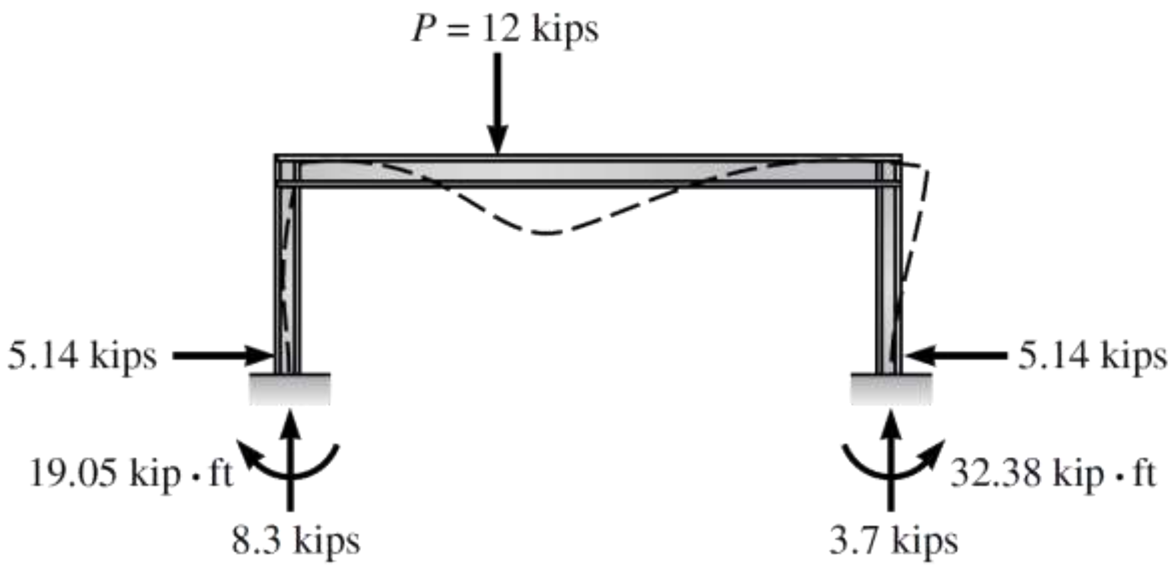
$$19.05(12) = \frac{2(30{,}000)(240)}{15(12)}(5.86\psi - 3\psi)$$

$$\psi = 0.000999 \text{ rad}$$

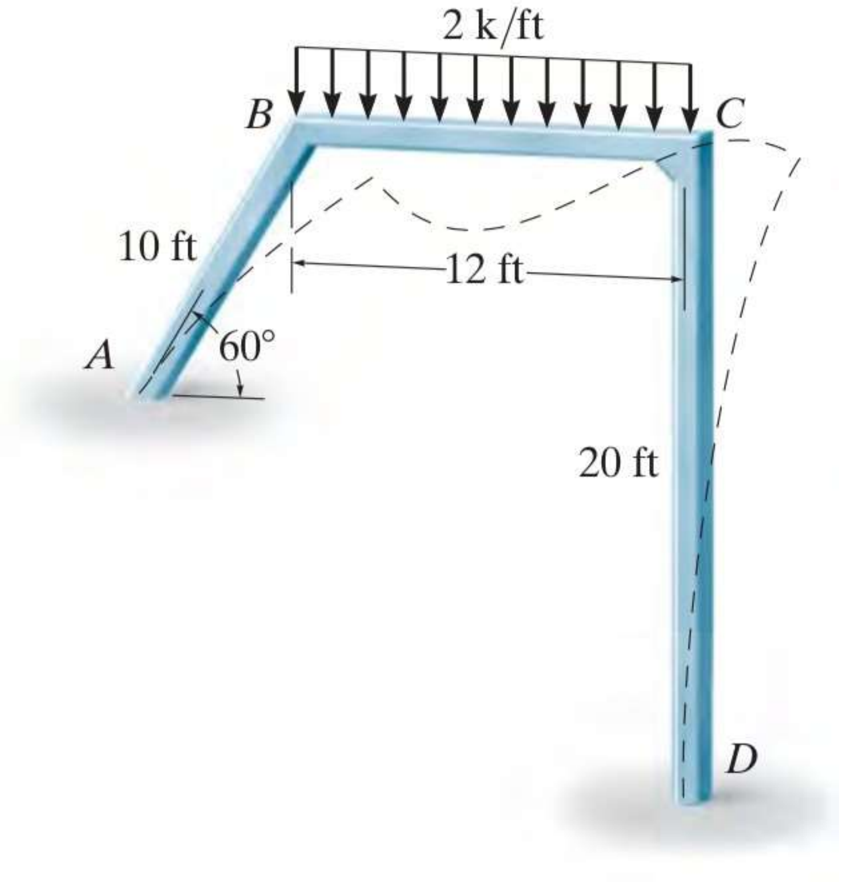
$$\psi = \frac{\Delta}{L} \qquad \Delta = \psi L = 0.000999(15 \times 12) = 0.18 \text{ in} \qquad \textbf{Ans.}$$

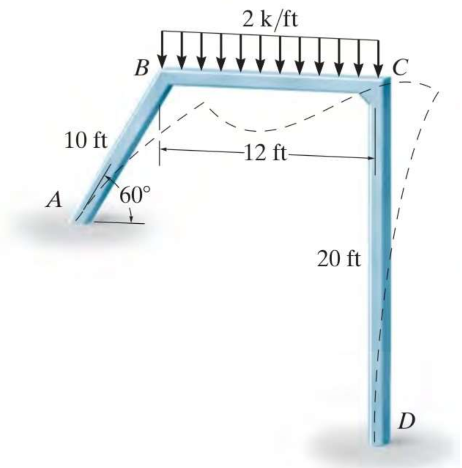
Example 12.10 Solution (continued)

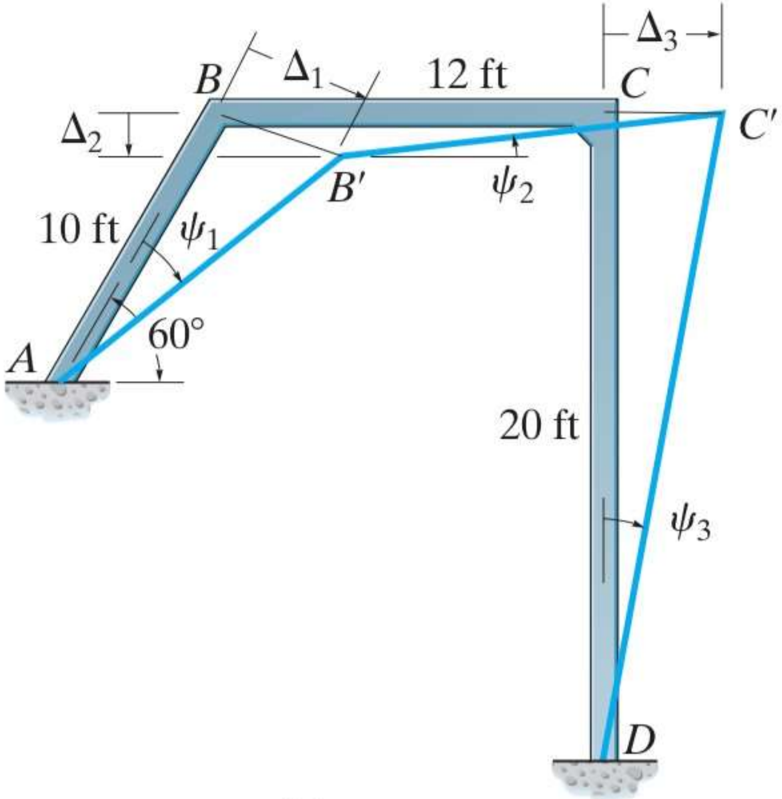




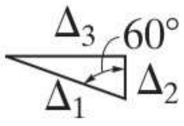
Determine the moments at each joint of the frame shown in Fig. 11–22*a*. *EI* is constant for each member.




2 k/ft
B
C
10 ft
12 ft
A
60°
20 ft
D


Δ_3
Δ_1
12 ft
B
C
Δ_2
C′
B′
ψ_2
10 ft
ψ_1
60°
A
20 ft
ψ_3
D

(b)


Δ_3
60°
Δ_2
Δ_1

(c)

$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{2(12)^2}{12} = -24 \text{ k} \cdot \text{ft}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{2(12)^2}{12} = 24 \text{ k} \cdot \text{ft}$$

$$\psi_1 = \frac{\Delta_1}{10} \qquad \psi_2 = -\frac{\Delta_2}{12} \qquad \psi_3 = \frac{\Delta_3}{20}$$

As shown in Fig. 11–22*c*, the three displacements can be related. For example, $\Delta_2 = 0.5\Delta_1$ and $\Delta_3 = 0.866\Delta_1$. Thus, from the above equations we have

$$\psi_2 = -0.417\psi_1 \qquad \psi_3 = 0.433\psi_1$$

$$M_{AB} = 2E\left(\frac{I}{10}\right)[2(0) + \theta_B - 3\psi_1] + 0 \tag{1}$$

$$M_{BA} = 2E\left(\frac{I}{10}\right)[2\theta_B + 0 - 3\psi_1] + 0 \tag{2}$$

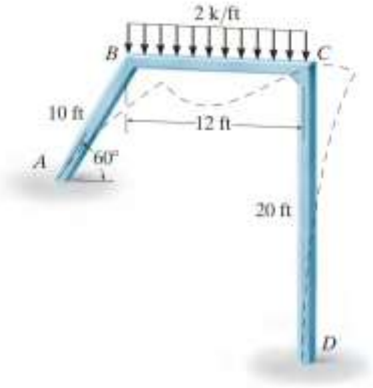
$$M_{BC} = 2E\left(\frac{I}{12}\right)[2\theta_B + \theta_C - 3(-0.417\psi_1)] - 24 \tag{3}$$

$$M_{CB} = 2E\left(\frac{I}{12}\right)[2\theta_C + \theta_B - 3(-0.417\psi_1)] + 24 \tag{4}$$

$$M_{CD} = 2E\left(\frac{I}{20}\right)[2\theta_C + 0 - 3(0.433\psi_1)] + 0 \tag{5}$$

$$M_{DC} = 2E\left(\frac{I}{20}\right)[2(0) + \theta_C - 3(0.433\psi_1)] + 0 \tag{6}$$

Equations of Equilibrium. Moment equilibrium at joints B and C yields



$$M_{BA} + M_{BC} = 0 \tag{7}$$

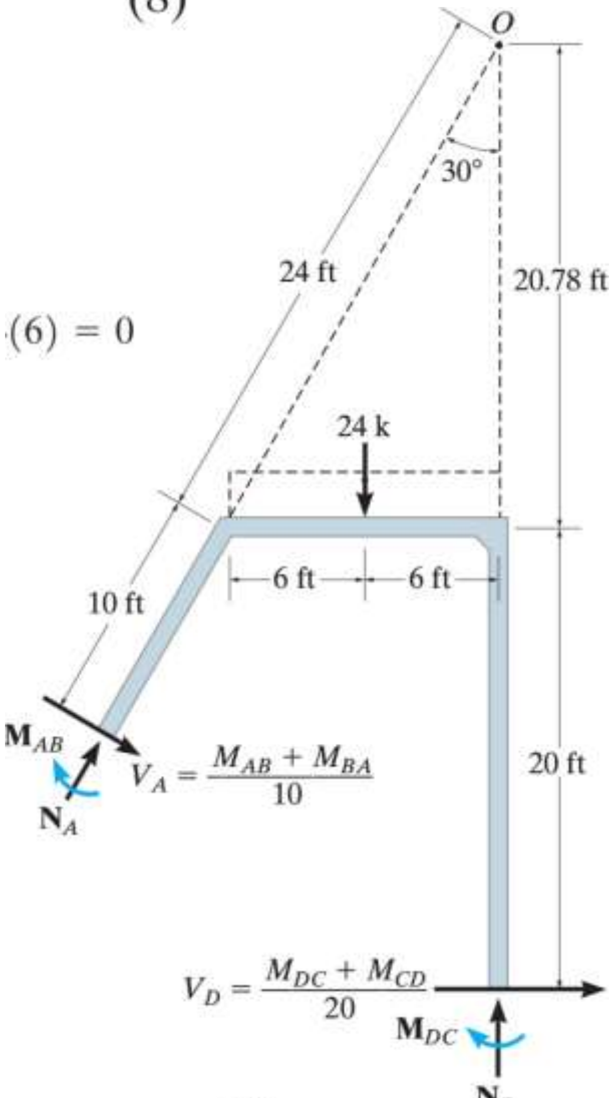
$$M_{CD} + M_{CB} = 0 \tag{8}$$

$(6) = 0$

$\circlearrowleft + \Sigma M_O = 0;$

$$M_{AB} + M_{DC} - \left(\frac{M_{AB} + M_{BA}}{10}\right)(34) - \left(\frac{M_{DC} + M_{CD}}{20}\right)(40.78) - 24(6) = 0$$

$$-2.4M_{AB} - 3.4M_{BA} - 2.04M_{CD} - 1.04M_{DC} - 144 = 0 \tag{9}$$



(d)

$$0.733\theta_B + 0.167\theta_C - 0.392\psi_1 = \frac{24}{EI}$$

$$0.167\theta_B + 0.533\theta_C + 0.0784\psi_1 = -\frac{24}{EI}$$

$$-1.840\theta_B - 0.512\theta_C + 3.880\psi_1 = \frac{144}{EI}$$

Solving these equations simultaneously yields

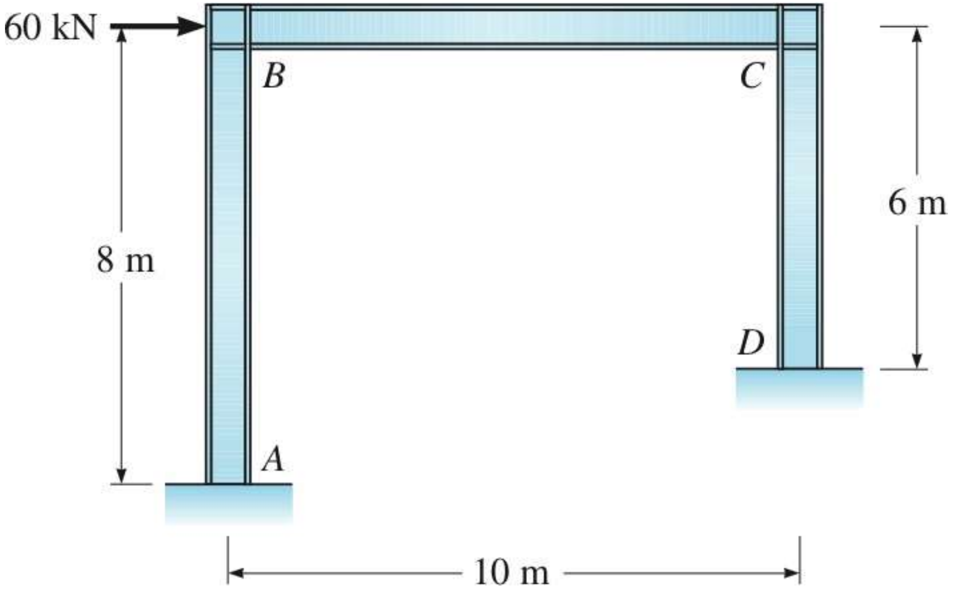
$$EI\theta_B = 87.67 \qquad EI\theta_C = -82.3 \qquad EI\psi_1 = 67.83$$

Substituting these values into Eqs. (1)–(6), we have

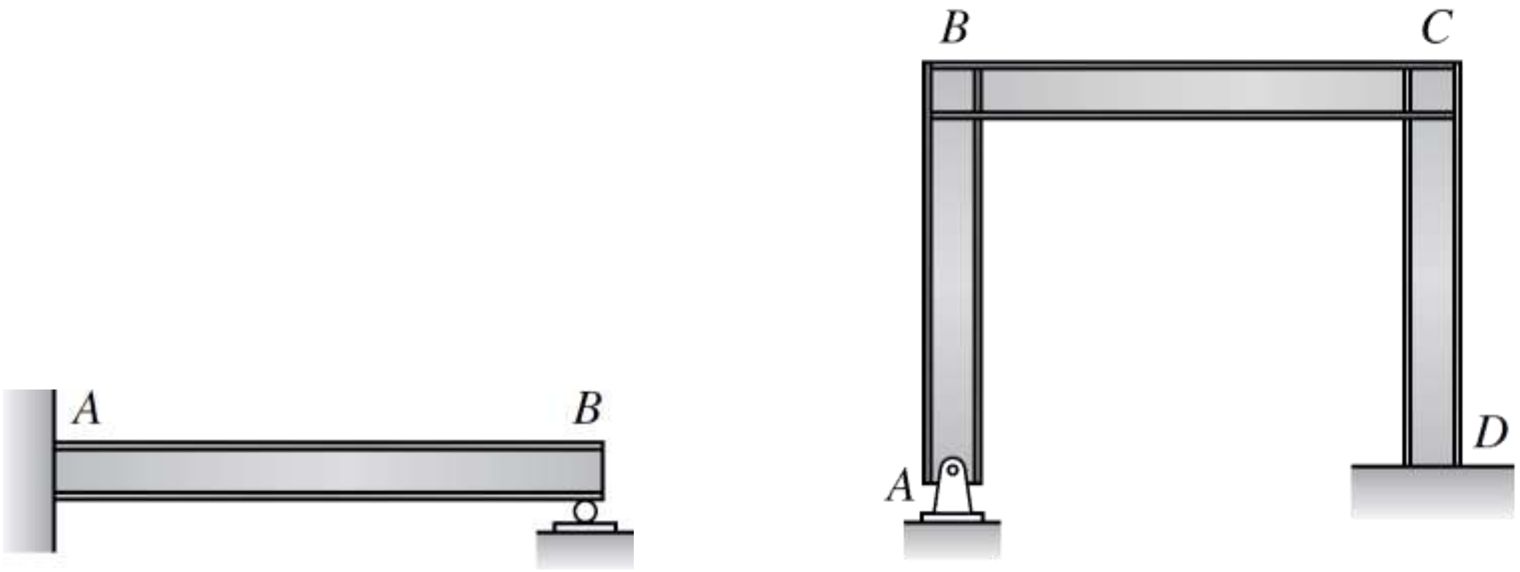
$M_{AB} = -23.2\text{ k}\cdot\text{ft}$ $M_{BC} = 5.63\text{ k}\cdot\text{ft}$ $M_{CD} = -25.3\text{ k}\cdot\text{ft}$ *Ans.*

$M_{BA} = -5.63\text{ k}\cdot\text{ft}$ $M_{CB} = 25.3\text{ k}\cdot\text{ft}$ $M_{DC} = -17.0\text{ k}\cdot\text{ft}$ *Ans.*

Determine all relations at points *A* and *D* in Figure shown. *EI* is constant.



§12.6 Kinematic Indeterminacy

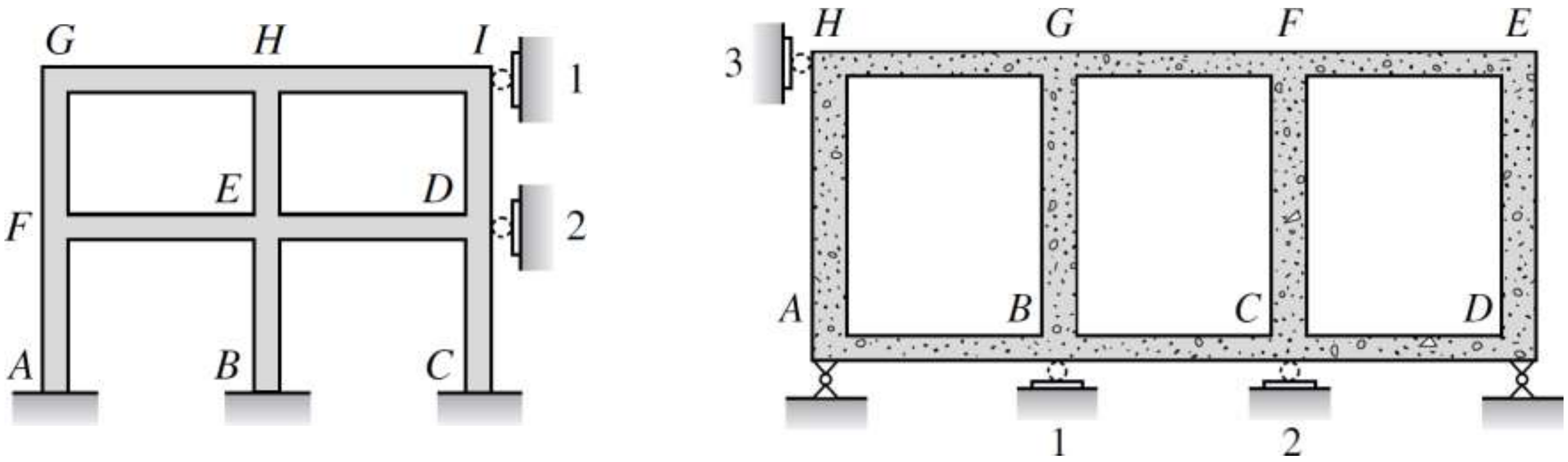


Indeterminate first degree, neglecting axial deformations

Indeterminate fourth degree

Figure 12.18 Evaluating degree of kinematic indeterminacy

§12.6 Kinematic Indeterminacy



Indeterminate eighth degree, imaginary rollers added at points 1 and 2

Indeterminate eleventh degree, imaginary rollers added at points 1, 2, and 3

Figure 12.18 Evaluating degree of kinematic indeterminacy (continued)

Figure 10.17: Evaluating degree of kinematic indeterminacy: (*a*) indeterminate first degree, neglecting axial deformations; (*b*) indeterminate fourth degree; (*c*) indeterminate eighth degree, imaginary rollers added at points 1 and 2; (*d*) indeterminate eleventh degree, imaginary rollers added at points 1, 2, and 3.