(20A01301) Advanced Strength of Materials

- **PREPARED BY**

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COURSE OUTCOMES

CO 1	Use the appropriate method to determine slope and beam deflection for different beam sections.
CO 2	To analyze the structural sections subjected to torsion.
CO 3	Analyze the crippling load and equivalent length for various types of columns of different end conditions.
CO 4	Calculate the strain energy, stress distribution & deformation in springs
CO 5	Calculate the stresses and strains associated with thick-wall cylindrical pressure vessels and rotating disks.

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UNIT - I Deflection of Beams

Uniform bending – slope, deflection and radius of curvature – Differential equation for elastic line of a beam – Double integration and Macaulay's methods. Determination of slope and deflection for cantilever and simply supported beams under point loads, U.D.L. uniformly varying load-Mohr's theorems – Moment area method – application to simply supported and overhanging beams- analysis of propped cantilever beams under UDL and point loads.

UNIT - II Torsion

Torsion: Theory of pure torsion – Assumptions and Derivation of Torsion formula for circular shaft – Torsional moment of resistance – Polar section modulus – power transmission through shafts –Combined bending and torsion –. Springs -Types of springs – deflection of close coiled helical springs under axial pull and axial couple – Carriage or leaf springs.

UNIT – III Columns and Struts

Introduction – classification of columns – Axially loaded compression members – Euler's crippling load theory – derivation of Euler's critical load formulae for various end conditions – Equivalent length – Slenderness ratio – Euler's critical stress – Limitations of Euler's theory – Rankine – Gordon formula – eccentric loading and Secant formula – Prof. Perry's formula.

UNIT - IV Springs

Axial load and torque on helical springs - stresses and deformations - strain energy - compound springs - leaf springs.

UNIT - V Thin and Thick Cylinders

Introduction - Thin Cylindrical shells - hoop stress - longitudinal stresses - Lame's theory - Design of thin & thick cylindrical shells- Wire wound thin cylinders - Compound cylinders - Shrink fit - compound cylinders

Textbooks:

1. Bansal R. K, "Strength of Materials", Laxmi Publications, 2010.

2. B. C. Punmia Strength of Materials by.- Laxmi publications.

Reference Books:

1. Schaum’s outline series Strength of Materials, Mc Graw hill International Editions.

2. L.S. Srinath, Strength of Materials, Macmillan India Ltd., New Delhi

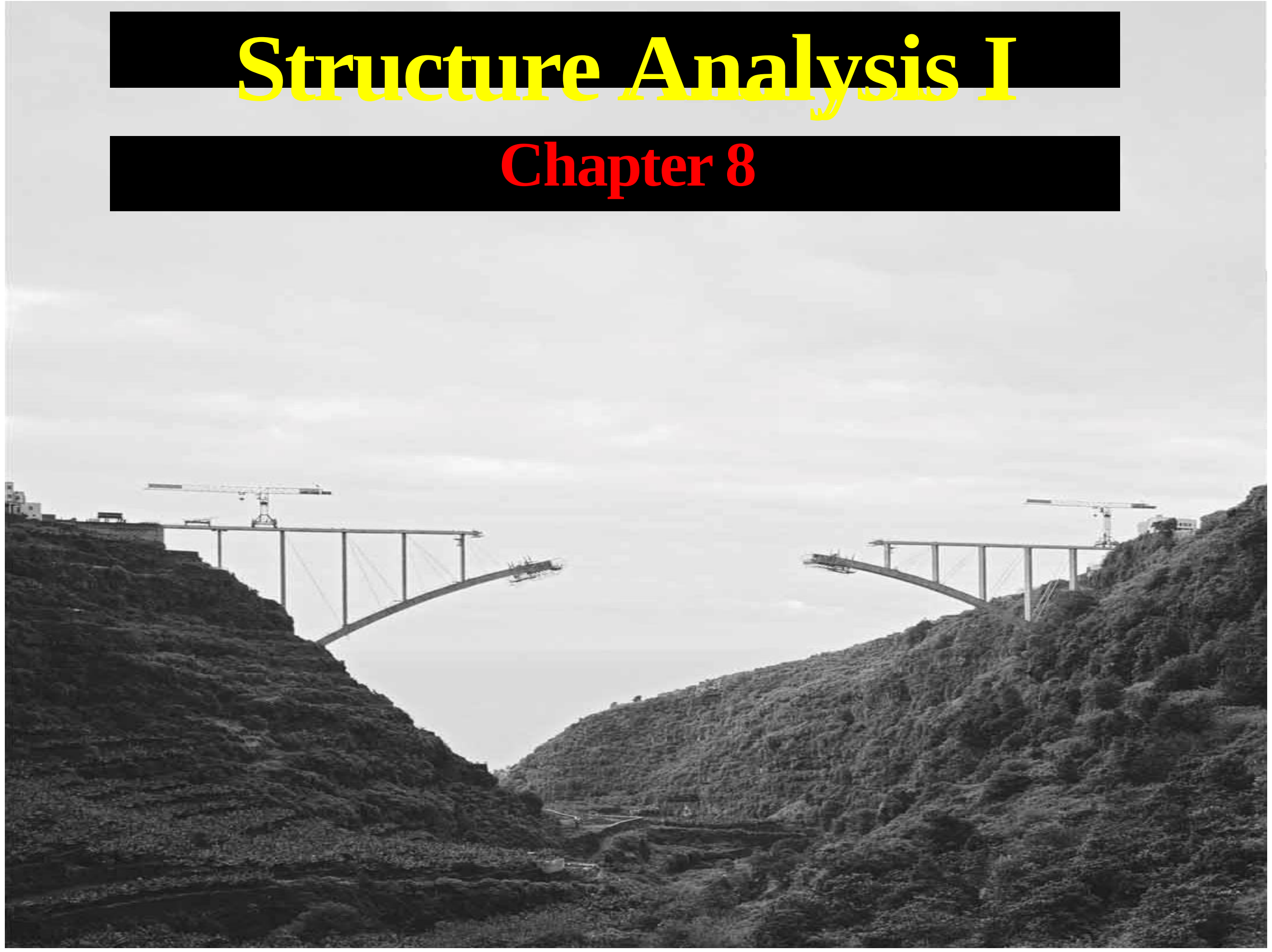
3. Gere J.M. and Goodno B.J. “Strength of Materials” Indian Edition (4th reprint), Cengage Learning India Private Ltd., 2009.

4. R.S.Khurmi and N.Khurmi, “Strength of Materials (Mechanics of Solids)”, S Chand And Company Limited, Ramnagar, New Delhi-110 055

5. B. S. Basavarajaiah and P. Mahadevappa, “Strength of Materials” 3rd Edition 2010, in SI UNITs, Universities Press Pvt Ltd, Hyderabad.

Structure Analysis I

Chapter 8



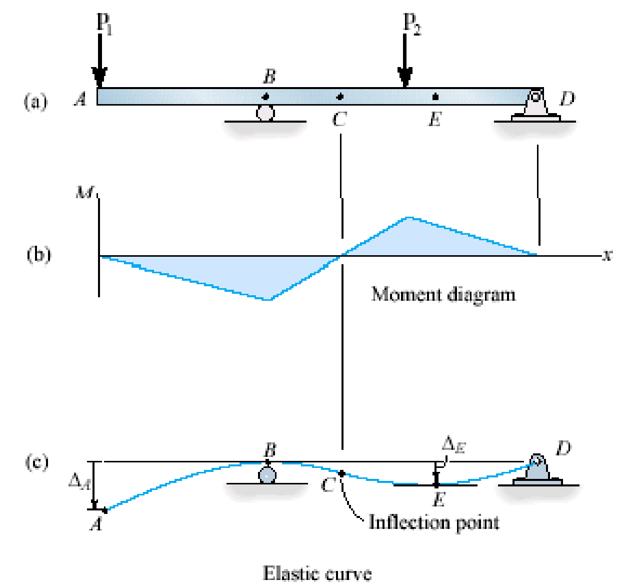
Deflections

Introduction

- Calculation of deflections is an important part of structural analysis
- Excessive beam deflection can be seen as a mode of failure.
  - Extensive glass breakage in tall buildings can be attributed to excessive deflections
  - Large deflections in buildings are unsightly (and unnerving) and can cause cracks in ceilings and walls.
  - Deflections are limited to prevent undesirable vibrations

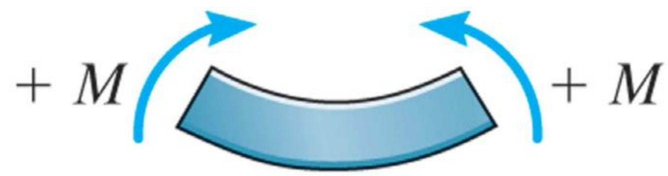
Beam Deflection

- Bending changes the initially straight longitudinal axis of the beam into a curve that is called the **Deflection Curve** or **Elastic Curve**



Beam Deflection

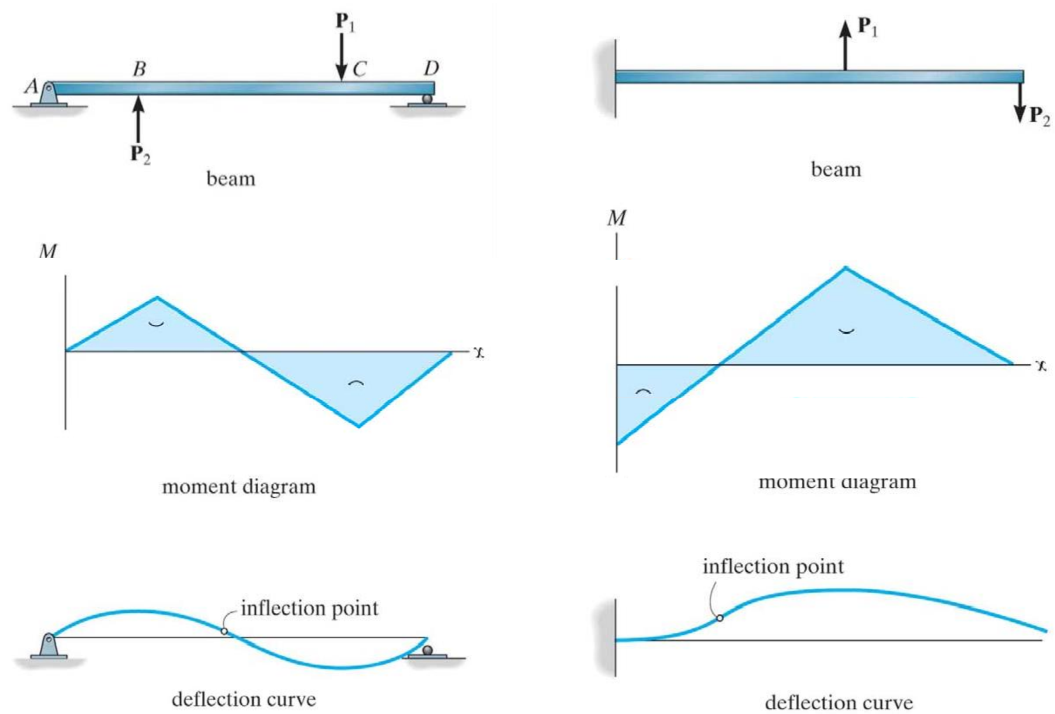
- To determine the deflection curve:
  - Draw shear and moment diagram for the beam
  - Directly under the moment diagram draw a line for the beam and label all supports
  - At the supports displacement is zero
  - Where the moment is negative, the deflection curve is concave downward.
  - Where the moment is positive the deflection curve is concave upward
  - Where the two curve meet is the Inflection Point



positive moment,
concave upward

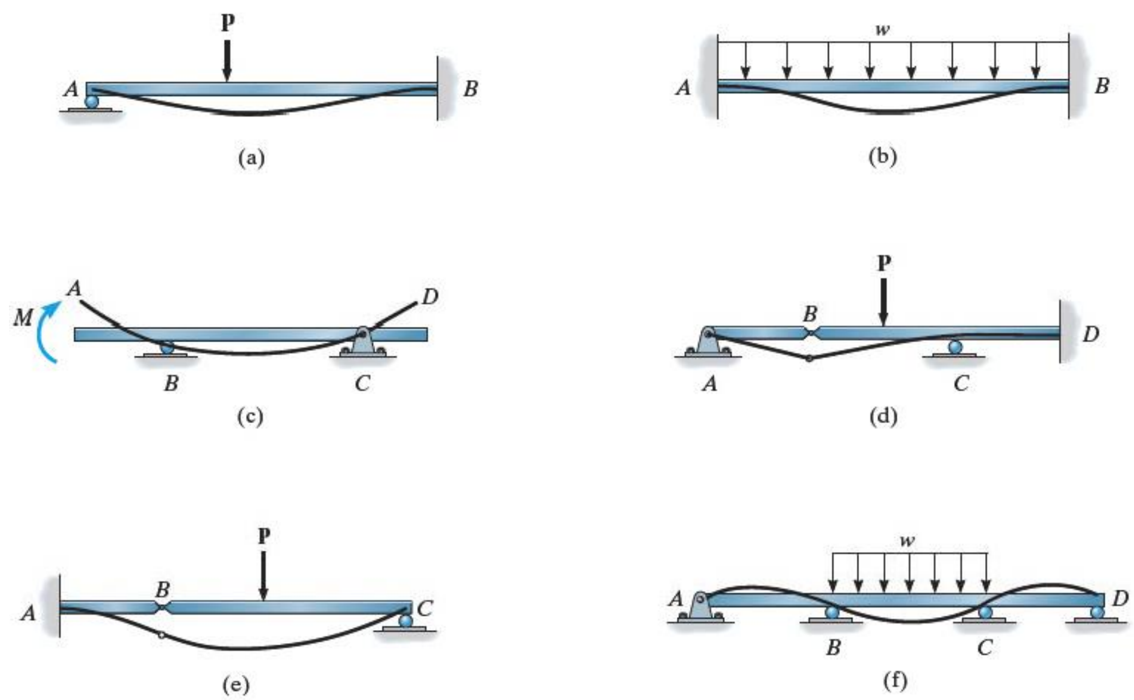
negative moment,
concave downward

Deflected Shape



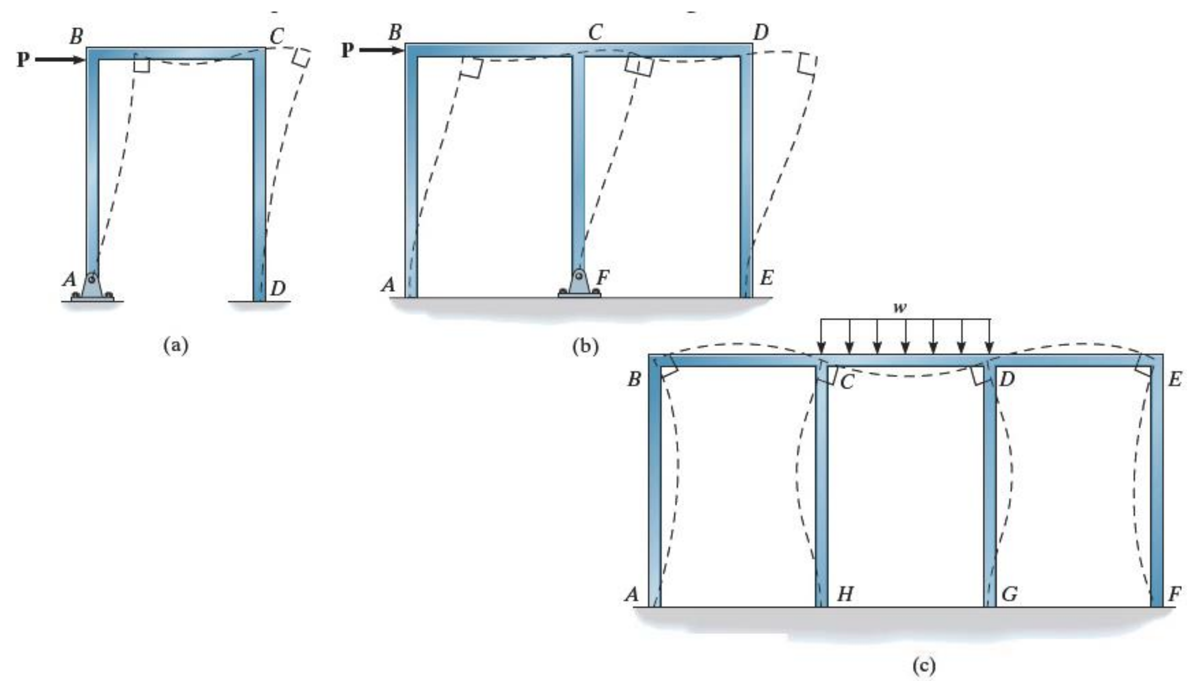
Example 1

Draw the deflected shape for each of the beams shown



Example 2

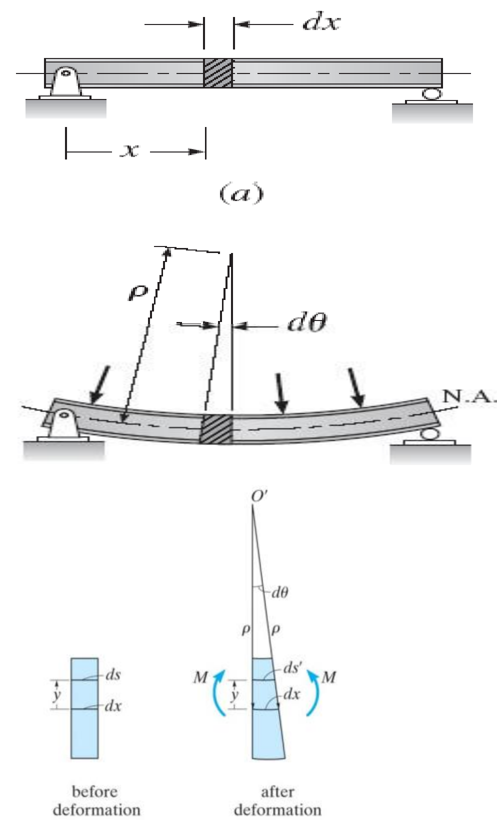
Draw the deflected shape for each of the frames shown



Double Integration Method

Elastic-Beam Theory

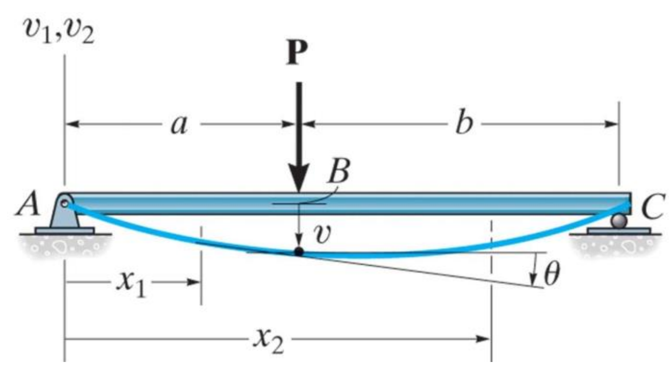
- Consider a differential element of a beam subjected to pure bending.
- The radius of curvature ρ is measured from the center of curvature to the neutral axis
- Since the NA is unstretched, the $dx = \rho d\theta$



Elastic-Beam Theory

- Applying Hooke's law and the Flexure formula, we obtain:

$$\frac{1}{\rho} = \frac{M}{EI}$$



Elastic-Beam Theory

- The product *EI* is referred to as the flexural rigidity.
- Since $dx = \rho d\theta$, then

$$d\theta = \frac{M}{EI} dx \quad (Slope)$$

- In most calculus books

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{\frac{3}{2}}}$$

$$\frac{M}{EI} = \frac{d^2v/dx^2}{\left[1+(dv/dx)^2\right]^{\frac{3}{2}}} \qquad (exact \quad solution)$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

The Double Integration Method

Relate Moments to Deflections

$$\frac{d^2 v}{dx^2} = \frac{M}{EI}$$

$$\theta(x) = \frac{dv}{dx} = \int \frac{M(x)}{EI(x)} dx$$

$$v(x) = \iint \frac{M \quad x}{EI(x)} dx^2$$

Do Not

Integration Constants

Use Boundary Conditions to Evaluate Integration Constants

Assumptions and Limitations

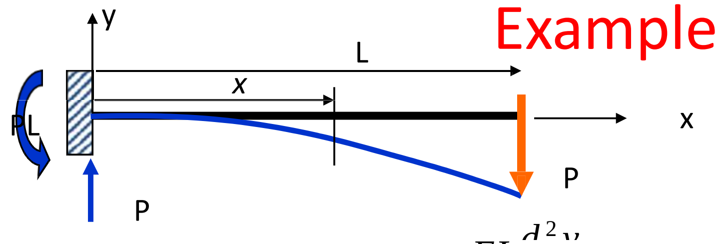
- ❖ Deflections caused by shearing action negligibly small compared to bending

- ❖ Deflections are small compared to the cross-sectional dimensions of the beam

- ❖ All portions of the beam are acting in the elastic range

- ❖ Beam is straight prior to the application of loads

Examples



$$M = -PL + Px$$

$$EI\frac{d^2y}{dx^2} = M$$

@ x $$EI\frac{d^2y}{dx^2} = -PL + Px$$

Integrating once $$EI\frac{dy}{dx} = -PLx + P\frac{x^2}{2} + c_1$$

@ x = 0 $$\frac{dy}{dx} = 0 \Rightarrow EI(0) = -PL(0) + P\frac{(0)^2}{2} + c_1 \Rightarrow c_1 = 0$$

Integrating twice $$EIy = -\frac{PLx^2}{2} + P\frac{x^3}{6} + c_2$$

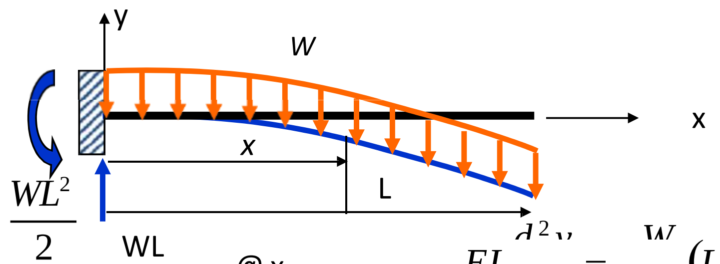
@ x = 0 $$y = 0 \Rightarrow EI(0) = -\frac{PL}{2}(0)^2 + P\frac{(0)^3}{6} + c_2 \Rightarrow c_2 = 0$$

$$\boxed{\Delta_{max} = \frac{PL^3}{3EI}}$$

@ x = L y = y_{max}

$$\boxed{EIy = -\frac{PLx^2}{2} + P\frac{x^3}{6}}$$

$$\boxed{EIy_{max} = -\frac{PL\,L^2}{2} + P\frac{L^3}{6} = -\frac{PL^3}{6} \Rightarrow y_{max} = -\frac{PL^3}{3EI}}$$



$$M = -\frac{W}{2}(L-x)^2$$

$$EI\frac{d^2y}{dx^2} = M$$

@ x

$$EI\frac{d^2y}{dx^2} = -\frac{W}{2}(L-x)^2$$

Integrating once

$$EI\frac{dy}{dx} = \frac{W}{2}\frac{(L-x)^3}{3} + c_1$$

@ x = 0 $\quad \frac{dy}{dx} = 0 \Rightarrow EI(0) = \frac{W}{2}\frac{(L-0)^3}{3} + c_1 \Rightarrow \boxed{c_1 = -\frac{WL^3}{6}}$

$$\therefore \boxed{EI\frac{dy}{dx} = \frac{W}{6}(L-x)^3 - \frac{WL^3}{6}}$$

Integrating twice $EIy = -\frac{W}{6}\frac{(L-x)^4}{4} - \frac{WL^3}{6}x + c_2$

@ x = 0 $\quad y = 0 \;\Rightarrow\; EI(0) = -\frac{W}{6}\frac{(L-0)^4}{4} - \frac{WL^3}{6}(0) + c_2 \;\Rightarrow\; c_2 = \frac{WL^4}{24}$

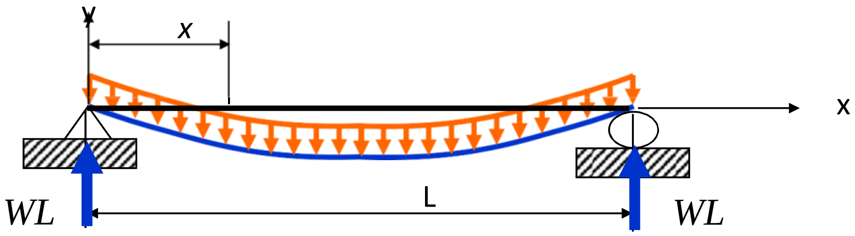
$$\boxed{EIy = -\frac{W}{24}(L-x)^4 - \frac{WL^3}{6}x + \frac{WL^4}{24}}$$

Max. occurs @ x = L

$$\boxed{EIy_{\max} = -\frac{W\,L^4}{6} + \frac{WL^4}{24} = -\frac{WL^4}{8} \;\Rightarrow\; y_{\max} = -\frac{WL^4}{8EI}}$$

$$\boxed{\Delta_{\max} = \frac{WL^4}{8EI}}$$

Example



$\frac{WL}{2}$ $\frac{WL}{2}$

$$M = \frac{WL}{2}x - Wx\frac{x}{2}$$

$$EI\frac{d^2v}{dx^2} = \frac{WL}{2}x - W\frac{x^2}{2}$$

Integrating $$EI\frac{dy}{dx} = \frac{WL}{2}\frac{x^2}{2} - \frac{W}{2}\frac{x^3}{3} + c_1$$

Since the beam is symmetric $@\ x = \frac{L}{2}$ $\frac{dy}{dx} = 0$

$$@\ x = \frac{L}{2} \qquad EI(0) = \frac{WL}{2}\frac{\left(\frac{L}{2}\right)^2}{2} - \frac{W}{2}\frac{\left(\frac{L}{2}\right)^3}{3} + c_1 \Rightarrow \boxed{c_1 = -\frac{WL^3}{24}}$$

$$\therefore \quad \boxed{EI\frac{dy}{dx} = \frac{WL}{4}x^2 - \frac{W}{6}x^3 - \frac{WL^3}{24}}$$

Integrating $$EIv = \frac{WL}{4}\frac{x^3}{3} - \frac{W}{6}\frac{x^4}{4} - \frac{WL^3}{24}x + c_2$$

4

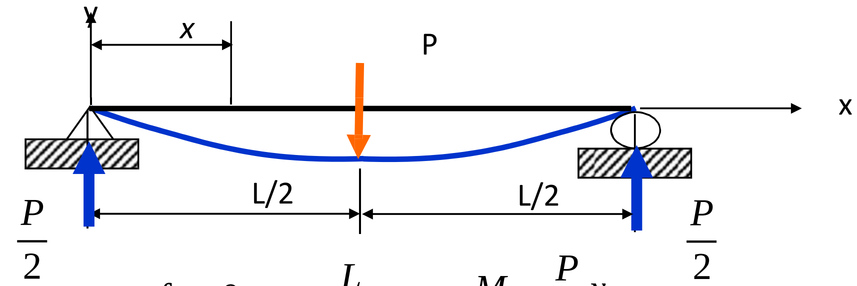
@ x = 0 y = 0 $$\Rightarrow EI(0) = \frac{WL}{4}\frac{(0)^3}{3} - \frac{W}{6}\frac{(0)^4}{4} - \frac{WL^3}{24}(0) + c_2 \quad \Rightarrow \boxed{c_2 = 0}$$

$$\boxed{EIv = \frac{WL}{12}x^3 - \frac{W}{24}x^4 - \frac{WL^3}{24}x}$$

Max. occurs @ x = L /2 $$\boxed{EIy_{\max} = \frac{5WL^4}{384}}$$

$$\boxed{\Delta_{\max} = \frac{5WL^4}{384EI}}$$

Example



$$\text{for } 0 < x < \frac{L}{2} \qquad M = \frac{P}{2}x$$

$$EI\frac{d^2v}{dx^2} = \frac{P}{2}x \qquad \text{for } 0 < x < \frac{L}{2}$$

Integrating $$EI\frac{dy}{dx} = \frac{P}{2}\frac{x^2}{2} + c_1$$

Since the beam is symmetric $@\ x = \frac{L}{2} \qquad \frac{dy}{dx} = 0$

$$@\ x = \frac{L}{2} \qquad EI(0) = \frac{P}{2}\frac{\left(\frac{L}{2}\right)^2}{2} + c_1 \Rightarrow \boxed{c_1 = -\frac{PL^2}{16}}$$

$$\boxed{\therefore \quad EI\frac{dy}{dx} = \frac{P}{4}x^2 - \frac{PL^2}{16}}$$

Integrating $$EIy = \frac{P}{4}\frac{x^3}{3} - \frac{PL^2}{16}x + c_2$$

@ x = 0 y = 0 $$\Rightarrow EI(0) = \frac{P}{4}\frac{(0)^3}{3} - \frac{PL^2}{16}(0) + c_2 \quad \Rightarrow \boxed{c_2 = 0}$$

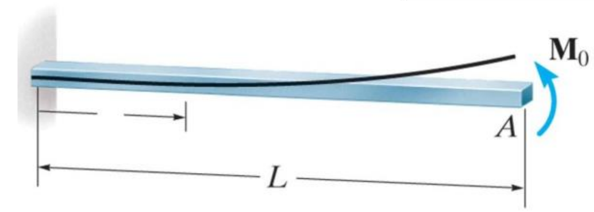
$$\boxed{EIy = \frac{P}{12}x^3 - \frac{PL^2}{16}x}$$

Max. occurs @ x = L /2 $$\boxed{EIy_{\max} = \frac{PL^3}{48}}$$

$$\boxed{\Delta_{\max} = \frac{PL^3}{48EI}}$$

Example

Slope and Elastic Curve. Applying Eq. 8–4 and integrating twice yields



$$EI\frac{d^2v}{dx^2} = M_0$$

$$EI\frac{dv}{dx} = M_0x + C_1$$

$$EIv = \frac{M_0x^2}{2} + C_1x + C_2 \qquad (3)$$

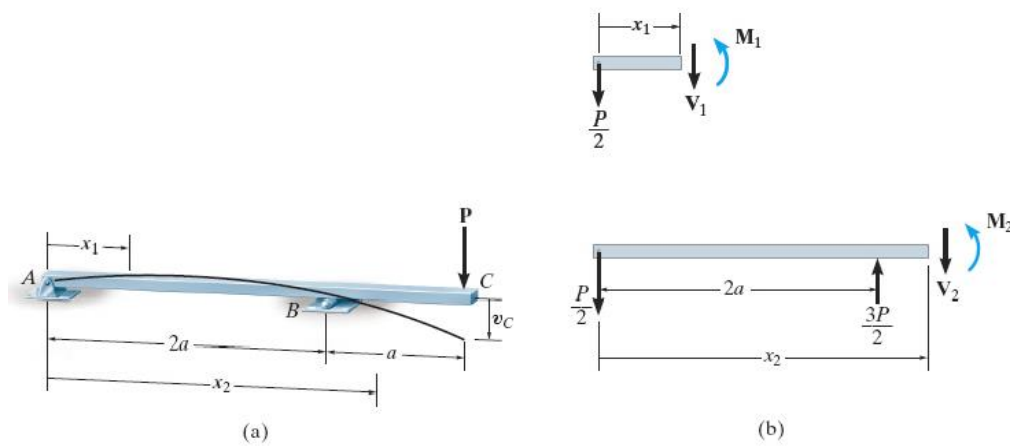
Using the boundary conditions $dv/dx = 0$ at $x = 0$ and $v = 0$ at $x = 0$, then $C_1 = C_2 = 0$. Substituting these results into Eqs. (2) and (3) with $\theta = dv/dx$, we get

$$\theta = \frac{M_0x}{EI}$$

$$v = \frac{M_0x^2}{2EI} \qquad \textit{Ans.}$$

Example 5

The beam in Fig. 8–12*a* is subjected to a load **P** at its end. Determine the displacement at *C*. *EI* is constant.



$$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$$

$$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$$

$$= Px_2 - 3Pa \qquad 2a \le x_2 \le 3a$$

Slope and Elastic Curve. Applying Eq. 8–4,

for x_1,
$$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

For x_2,
$$EI\frac{d^2v_2}{dx_2^2} = Px_2 - 3Pa$$

$$EI\frac{dv_2}{dx_2} = \frac{P}{2}{x_2}^2 - 3Pax_2 + C_3 \qquad (3)$$

$$EIv_2 = \frac{P}{6}{x_2}^3 - \frac{3}{2}Pa{x_2}^2 + C_3x_2 + C_4 \qquad (4)$$

$$v_1 = 0 \text{ at } x_1 = 0; \qquad 0 = 0 + 0 + C_2$$

$$v_1 = 0 \text{ at } x_1 = 2a; \qquad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$$

$$v_2 = 0 \text{ at } x_2 = 2a; \qquad 0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$$

$$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}; \qquad -\frac{P}{4}(2a)^2 + C_1 = \frac{P}{2}(2a)^2 - 3Pa(2a) + C_3$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{10}{3}Pa^2 \qquad C_4 = -2Pa^3$$

Substituting C_3 and C_4 into Eq. (4) gives

$$v_2 = \frac{P}{6EI}x_2^{\,3} - \frac{3}{2}\frac{Pa}{EI}x_2^{\,2} + \frac{10Pa^2}{3EI}x_2 - \frac{2Pa^3}{EI}$$

The displacement at C is determined by setting $x_2 = 3a$. We get

$$v_C = -\frac{Pa^3}{EI} \qquad \textit{Ans.}$$

Moment-Area Theorems

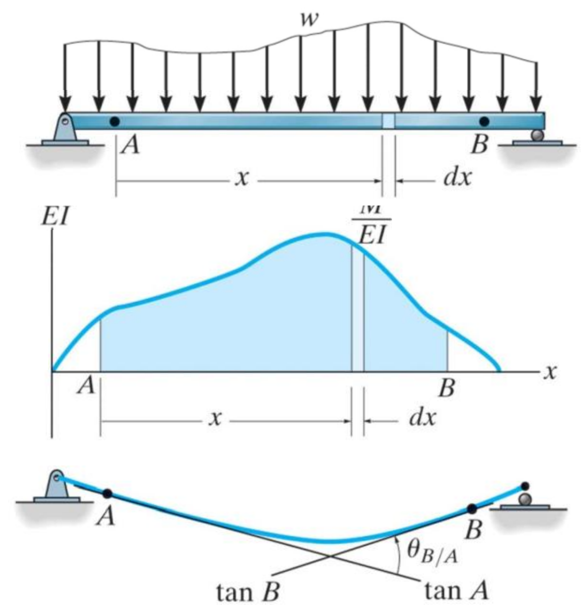
Moment-Area Theorems

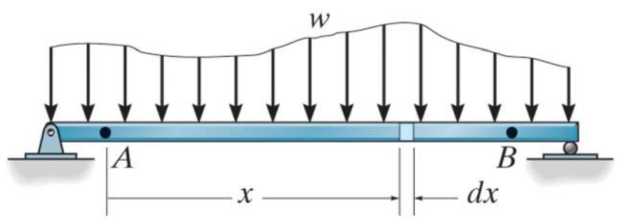
Theorem 1: The change in slope between any two points on the elastic curve equal to the area of the bending moment diagram between these two points, divided by the product EI.

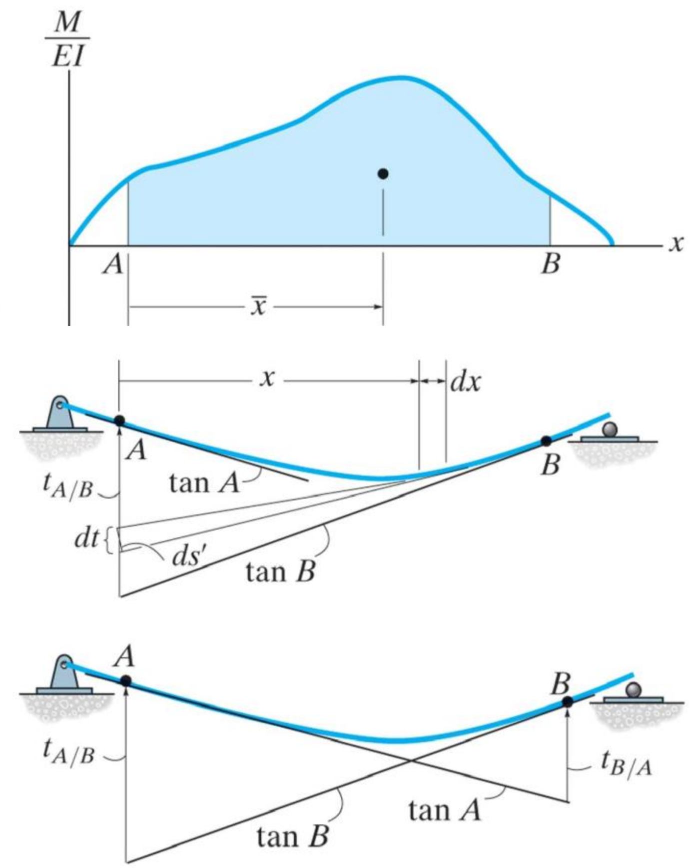
$$\frac{d^2v}{dx^2} = \frac{M}{EI} \quad \Rightarrow \quad \theta = \frac{dv}{dx}$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \quad \Rightarrow d\theta = \left(\frac{M}{EI}\right)dx$$

$$\theta_{B/A} = \int_A^B \frac{M}{EI}dx$$







$$dt = x\,d\theta$$

$$d\theta = \left(\frac{M}{EI}\right)dx$$

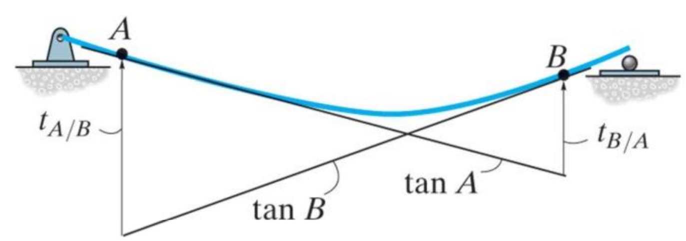
$$t_{B/A} = \int_A^B x\,\frac{M}{EI}\,dx = \bar{x}\int_A^B \frac{M}{EI}\,dx$$

Moment-Area Theorems

Theorem 2: The vertical distance of point A on a elastic curve from the tangent drawn to the curve at B is equal to the moment of the area under the M/EI diagram between two points (A and B) about point A .

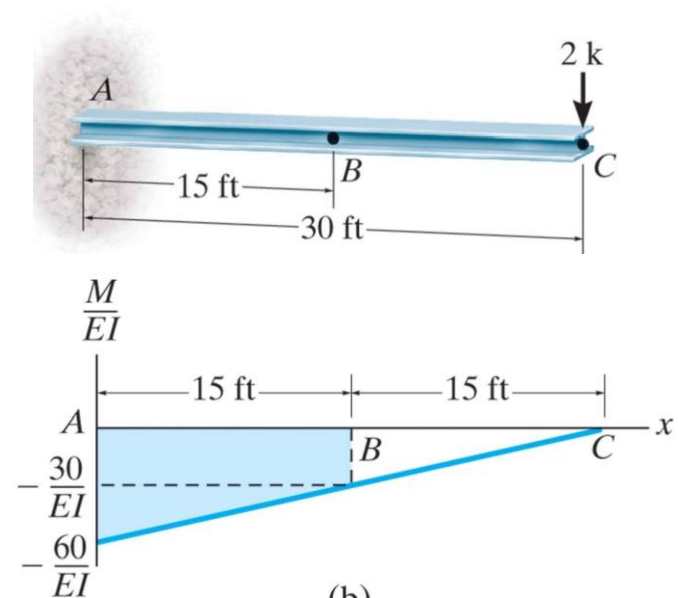
$$t_{A/B} = \int_A^B x \frac{M}{EI} dx$$

$$t_{A/B} = \bar{x} \int_A^B \frac{M}{EI} dx$$



Example 1

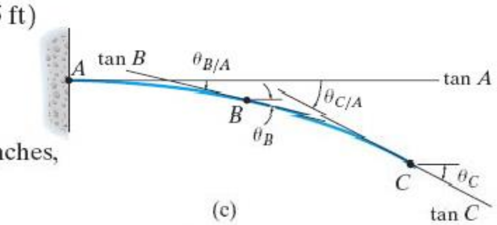
Determine the slope at points B and C of the beam shown in Fig. 8–15a. Take $E = 29(10^3)$ ksi and $I = 600\ \text{in}^4$.



(b)

$$\theta_B = \theta_{B/A} = -\left(\frac{30\text{ k}\cdot\text{ft}}{EI}\right)(15\text{ ft}) - \frac{1}{2}\left(\frac{60\text{ k}\cdot\text{ft}}{EI} - \frac{30\text{ k}\cdot\text{ft}}{EI}\right)(15\text{ ft})$$

$$= -\frac{675\text{ k}\cdot\text{ft}^2}{EI}$$



(c)

Fig. 8–15

Substituting numerical data for E and I, and converting feet to inches, we have

$$\theta_B = \frac{-675\text{ k}\cdot\text{ft}^2(144\text{ in}^2/1\text{ ft}^2)}{29(10^3)\text{ k/in}^2(600\text{ in}^4)}$$

$$= -0.00559\text{ rad} \qquad \textit{Ans.}$$

The *negative sign* indicates that the angle is measured clockwise from A, Fig. 8–15*c*.

In a similar manner, the area under the M/EI diagram between points A and C equals $\theta_{C/A}$. We have

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{60\text{ k}\cdot\text{ft}}{EI}\right)(30\text{ ft}) = -\frac{900\text{ k}\cdot\text{ft}^2}{EI}$$

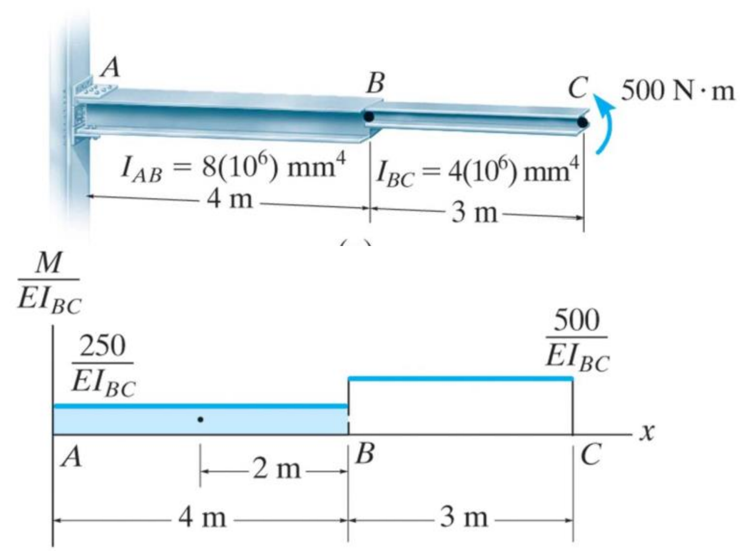
Substituting numerical values for EI, we have

$$\theta_C = \frac{-900\text{ k}\cdot\text{ft}^2(144\text{ in}^2/\text{ft}^2)}{29(10^3)\text{ k/in}^2(600\text{ in}^4)}$$

$$= -0.00745\text{ rad} \qquad \textit{Ans.}$$

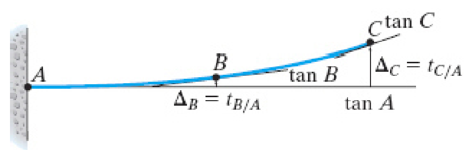
Example

Determine the deflection at points B and C of the beam shown in Fig. 8–16a. Values for the moment of inertia of each segment are indicated in the figure. Take $E = 200$ GPa.



$$\Delta_B = t_{B/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](2\ \text{m}) = \frac{2000\ \text{N}\cdot\text{m}^3}{EI_{BC}}$$

Substituting the numerical data yields



(c)

Fig. 8–16

$$\Delta_B = \frac{2000\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)\ \text{mm}^4(1\ \text{m}^4/(10^3)^4\ \text{mm}^4)]}$$
$$= 0.0025\ \text{m} = 2.5\ \text{mm}. \qquad \textit{Ans.}$$

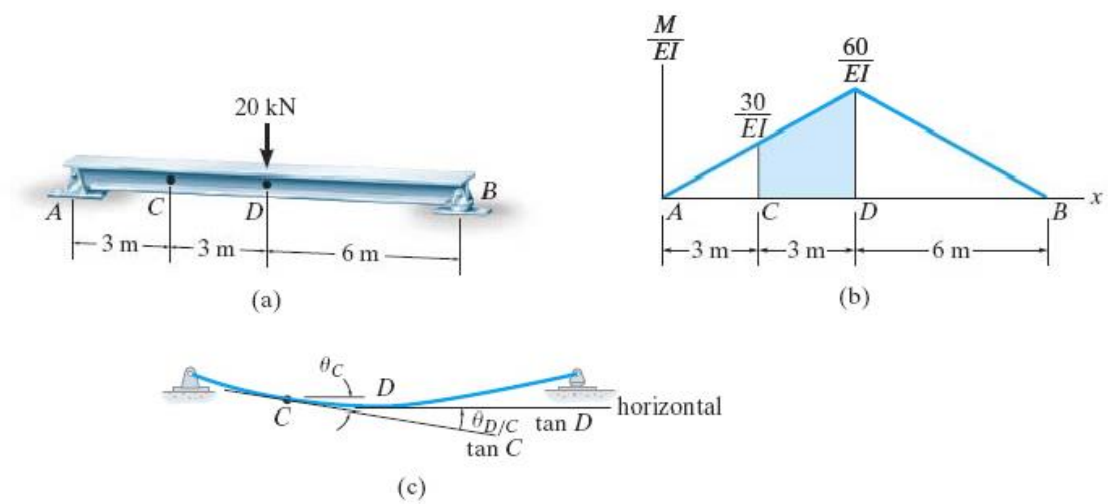
Likewise, for $t_{C/A}$ we must compute the moment of the entire M/EI_{BC} diagram from A to C about point C. We have

$$\Delta_C = t_{C/A} = \left[\frac{250\ \text{N}\cdot\text{m}}{EI_{BC}}(4\ \text{m})\right](5\ \text{m}) + \left[\frac{500\ \text{N}\cdot\text{m}}{EI_{BC}}(3\ \text{m})\right](1.5\ \text{m})$$
$$= \frac{7250\ \text{N}\cdot\text{m}^3}{EI_{BC}} = \frac{7250\ \text{N}\cdot\text{m}^3}{[200(10^9)\ \text{N/m}^2][4(10^6)(10^{-12})\ \text{m}^4]}$$
$$= 0.00906\ \text{m} = 9.06\ \text{mm} \qquad \textit{Ans.}$$

Since both answers are *positive*, they indicate that points B and C lie *above* the tangent at A.

Example 3

Determine the slope at point C of the beam in Fig. 8–17*a*. $E = 200$ GPa, $I = 6(10^6)$ mm^4.



$$\theta_C = \theta_{D/C}$$

Moment-Area Theorem. Using Theorem 1, $\theta_{D/C}$ is equal to the shaded area under the *M/EI* diagram between points C and D. We have

$$\theta_C = \theta_{D/C} = 3\text{ m}\left(\frac{30\text{ kN}\cdot\text{m}}{EI}\right) + \frac{1}{2}(3\text{ m})\left(\frac{60\text{ kN}\cdot\text{m}}{EI} - \frac{30\text{ kN}\cdot\text{m}}{EI}\right)$$

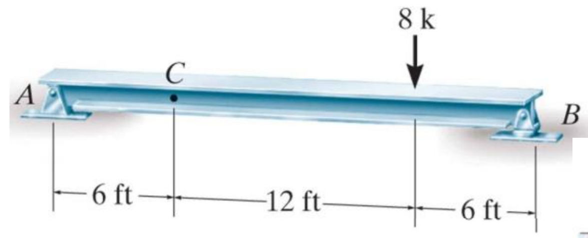
$$= \frac{135\text{ kN}\cdot\text{m}^2}{EI}$$

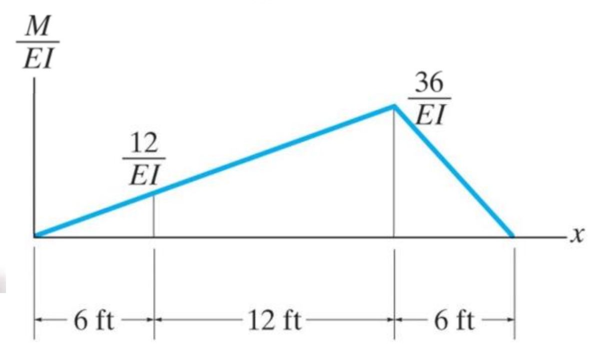
Thus,

$$\theta_C = \frac{135\text{ kN}\cdot\text{m}^2}{[200(10^6)\text{ kN/m}^2][6(10^6)(10^{-12})\text{ m}^4]} = 0.112\text{ rad} \qquad \textit{Ans.}$$

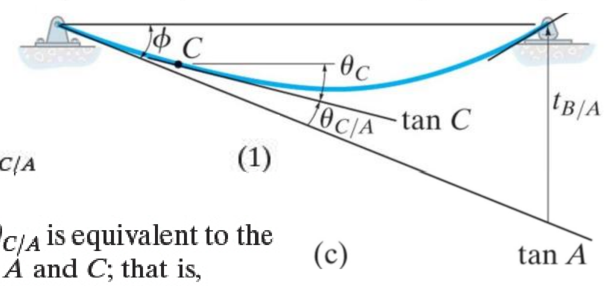
Example

Determine the slope at point C of the beam in Fig. 8–18*a*. $E = 29(10^3)$ ksi, $I = 600\text{ in}^4$.





an B



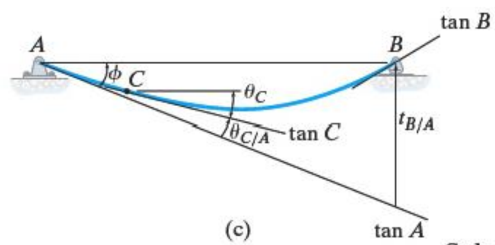
(c)

$$\theta_C = \phi - \theta_{C/A} = \frac{t_{B/A}}{24} - \theta_{C/A} \qquad (1)$$

Moment-Area Theorems. Using Theorem 1, $\theta_{C/A}$ is equivalent to the area under the M/EI diagram between points A and C; that is,

$$\theta_{C/A} = \frac{1}{2}(6\text{ ft})\left(\frac{12\text{ k}\cdot\text{ft}}{EI}\right) = \frac{36\text{ k}\cdot\text{ft}^2}{EI}$$

Applying Theorem 2, $t_{B/A}$ is equivalent to the moment of the area under the M/EI diagram between B and A about point B, since this is the point where the tangential deviation is to be determined. We have



(c)

Fig. 8–18

$$t_{B/A} = \left[6\text{ ft} + \frac{1}{3}(18\text{ ft})\right]\left[\frac{1}{2}(18\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right]$$
$$+\frac{2}{3}(6\text{ ft})\left[\frac{1}{2}(6\text{ ft})\left(\frac{36\text{ k}\cdot\text{ft}}{EI}\right)\right]$$
$$= \frac{4320\text{ k}\cdot\text{ft}^3}{EI}$$

Substituting these results into Eq. 1, we have

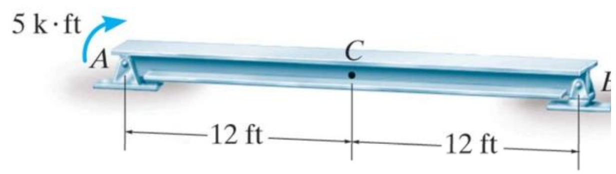
$$\theta_C = \frac{4320\text{ k}\cdot\text{ft}^3}{(24\text{ ft})\,EI} - \frac{36\text{ k}\cdot\text{ft}^2}{EI} = \frac{144\text{ k}\cdot\text{ft}^2}{EI}$$

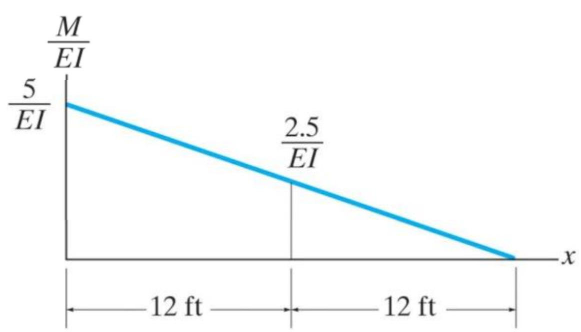
so that

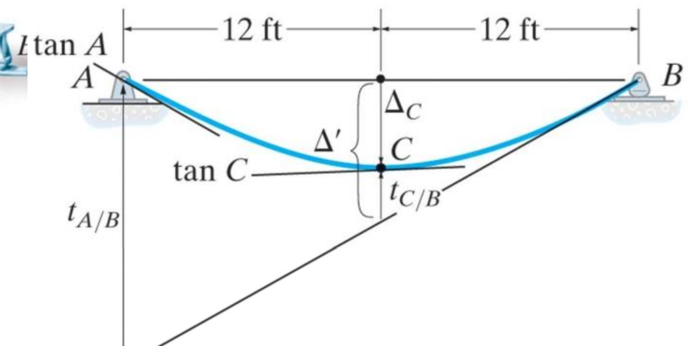
$$\theta_C = \frac{144\text{ k}\cdot\text{ft}^2}{29(10^3)\text{ k/in}^2(144\text{ in}^2/\text{ft}^2)\ 600\text{ in}^4(1\text{ ft}^4/(12)^4\text{ in}^4)}$$
$$= 0.00119\text{ rad}$$ ***Ans.***

Example 5

Determine the deflection at C of the beam shown in Fig. 8–19*a*. Take $E = 29(10^3)$ ksi, $I = 21$ in^4.







(c)

$$\Delta_C = \frac{t_{A/B}}{2} - t_{C/B}$$

Moment-Area Theorem. We will apply Theorem 2 to determine $t_{A/B}$ and $t_{C/B}$. Here $t_{A/B}$ is the moment of the M/EI diagram between A and B about point A,

$$t_{A/B} = \left[\frac{1}{3}(24\text{ ft})\right]\left[\frac{1}{2}(24\text{ ft})\left(\frac{5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{480\text{ k}\cdot\text{ft}^3}{EI}$$

and $t_{C/B}$ is the moment of the M/EI diagram between C and B about C.

$$t_{C/B} = \left[\frac{1}{3}(12\text{ ft})\right]\left[\frac{1}{2}(12\text{ ft})\left(\frac{2.5\text{ k}\cdot\text{ft}}{EI}\right)\right] = \frac{60\text{ k}\cdot\text{ft}^3}{EI}$$

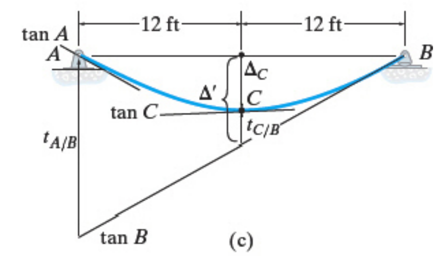


Fig. 8–19

Substituting these results into Eq. (1) yields

$$\Delta_C = \frac{1}{2}\left(\frac{480\text{ k}\cdot\text{ft}^3}{EI}\right) - \frac{60\text{ k}\cdot\text{ft}^3}{EI} = \frac{180\text{ k}\cdot\text{ft}^3}{EI}$$

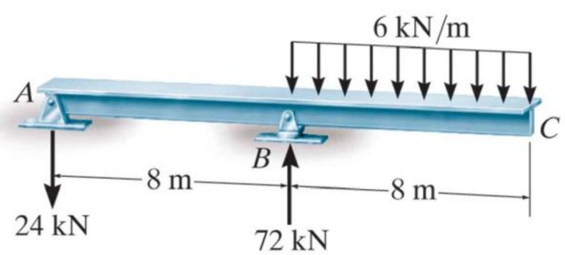
Working in units of kips and inches, we have

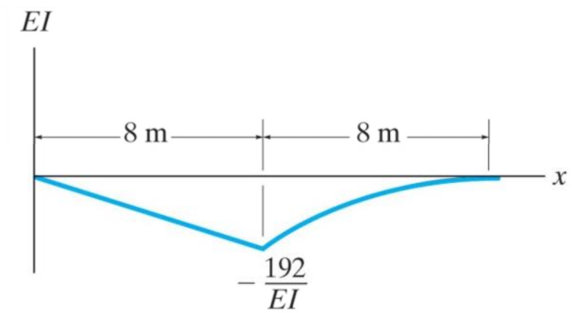
$$\Delta_C = \frac{180\text{ k}\cdot\text{ft}^3(1728\text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(21\text{ in}^4)}$$
$$= 0.511\text{ in.}$$

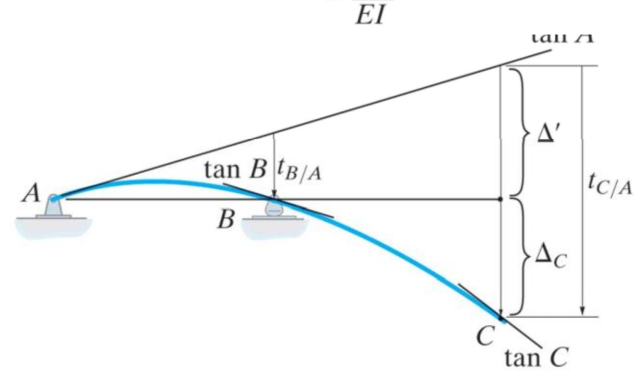
Ans.

Example 6

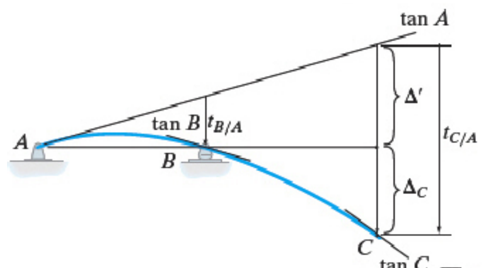
Determine the deflection at point C of the beam shown in Fig. 8–20*a*. $E = 200$ GPa, $I = 250(10^6)$ mm^4.







$$\Delta_C = t_{C/A} - 2t_{B/A}$$



(c)

Fig. 8–20

$$t_{C/A} = \left[\frac{3}{4}(8\text{ m})\right]\left[\frac{1}{3}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right]$$
$$+\left[\frac{1}{3}(8\text{ m}) + 8\text{ m}\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right]$$
$$= -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI}$$

The moment of the M/EI diagram between A and B about point B gives

$$t_{B/A} = \left[\frac{1}{3}(8\text{ m})\right]\left[\frac{1}{2}(8\text{ m})\left(-\frac{192\text{ kN}\cdot\text{m}}{EI}\right)\right] = -\frac{2048\text{ kN}\cdot\text{m}^3}{EI}$$

Why are these terms negative? Substituting the results into Eq. (1) yields

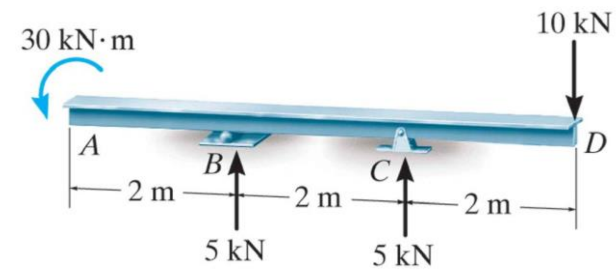
$$\Delta_C = -\frac{11\,264\text{ kN}\cdot\text{m}^3}{EI} - 2\left(-\frac{2048\text{ kN}\cdot\text{m}^3}{EI}\right)$$
$$= -\frac{7168\text{ kN}\cdot\text{m}^3}{EI}$$

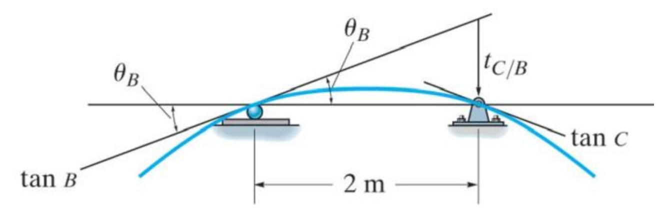
Thus,

$$\Delta_C = \frac{-7168\text{ kN}\cdot\text{m}^3}{[200(10^6)\text{ kN/m}^2][250(10^6)(10^{-12})\text{ m}^4]}$$
$$= -0.143\text{ m}$$ ***Ans.***

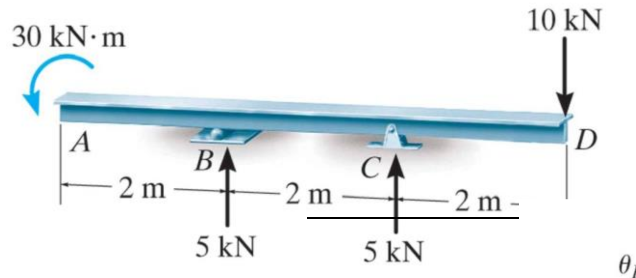
Example 7

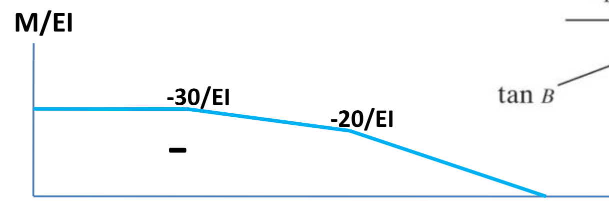
Determine the slope at the roller B of the double overhang beam shown in Fig. 8–21a. Take $E = 200$ GPa, $I = 18(10^6)$ mm^4.

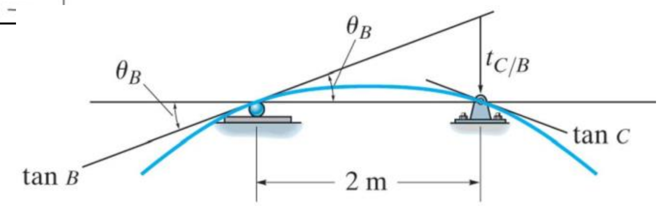




$$\theta_B = \frac{t_{C/B}}{2\text{ m}}$$





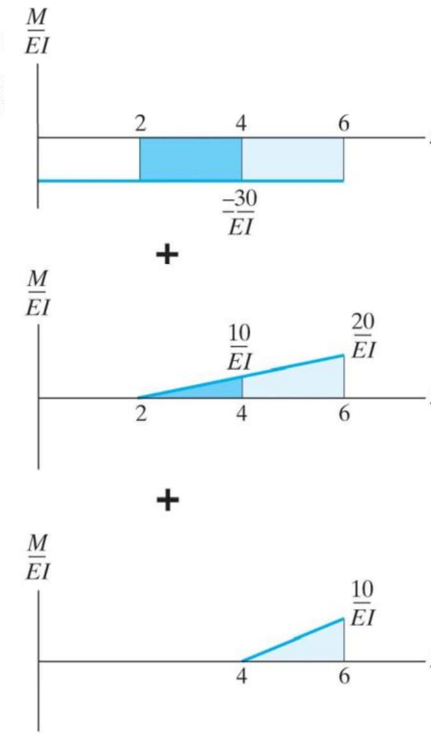


$$t_{C/B} = \left(-\frac{20}{EI}\cdot 2\right)\cdot(1) + \frac{1}{2}\cdot\left(-\frac{10}{EI}\cdot 2\right)\cdot\left(\frac{2}{3}\cdot 2\right)$$

$$= -\frac{53.33}{EI} kN \cdot m^3 = 0.00741 \ rad$$

Another Solution

Moment Area Theorem. To determine $t_{B/C}$ we apply the moment area theorem by finding the moment of the M/EI diagram between BC about point C. This only involves the shaded area under two of the diagrams in Fig. 8–21b. Thus,



$$t_{C/B} = (1\text{ m})\left[(2\text{ m})\left(\frac{-30\text{ kN}\cdot\text{m}}{EI}\right)\right] + \left(\frac{2\text{ m}}{3}\right)\left[\frac{1}{2}(2\text{ m})\left(\frac{10\text{ kN}\cdot\text{m}}{EI}\right)\right]$$

$$= \frac{53.33\text{ kN}\cdot\text{m}^3}{EI}$$

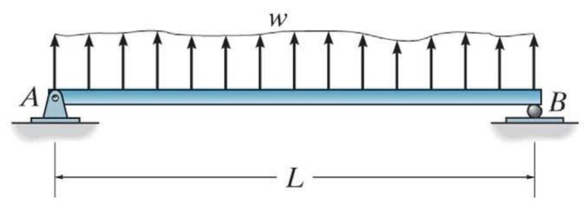
Substituting into Eq. (1),

$$\theta_B = \frac{53.33\text{ kN}\cdot\text{m}^3}{(2\text{ m})[200(10^6)\text{ kN/m}^3][18(10^6)(10^{-12})\text{ m}^4]}$$

$$= 0.00741\text{ rad}$$

Conjugate-Beam Method

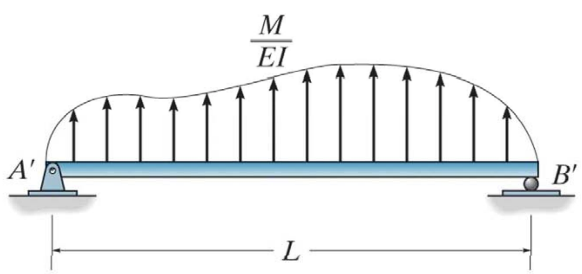
Conjugate-Beam Method



real beam

$$\frac{dV}{dx} = w \qquad \frac{d^2M}{dx^2} = w$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \frac{d^2v}{dx^2} = \frac{M}{EI}$$



conjugate beam

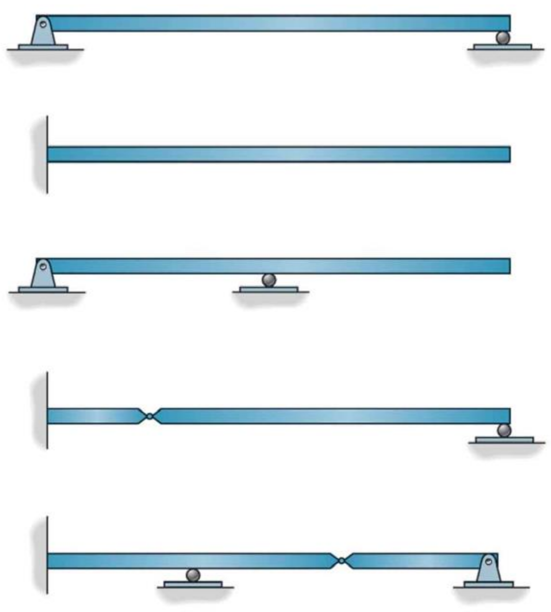
Integrating

$$V = \int w\,dx \qquad M = \int\left[\int w\,dx\right]dx$$

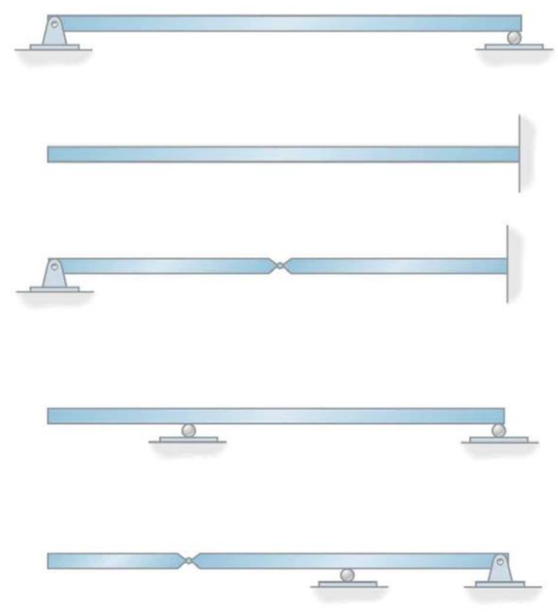
$$\theta = \int\left(\frac{M}{EI}\right)dx \qquad v = \int\left[\int\left(\frac{M}{EI}\right)dx\right]dx$$

Conjugate-Beam Supports

	Real Beam		Conjugate Beam	
1)	θ $\Delta = 0$	pin	V $M = 0$	pin
2)	θ $\Delta = 0$	roller	V $M = 0$	roller
3)	$\theta = 0$ $\Delta = 0$	fixed	$V = 0$ $M = 0$	free
4)	θ Δ	free	V M	fixed
5)	θ $\Delta = 0$	internal pin	V $M = 0$	hinge
6)	θ $\Delta = 0$	internal roller	V $M = 0$	hinge
7)	θ Δ	hinge	V M	internal roller



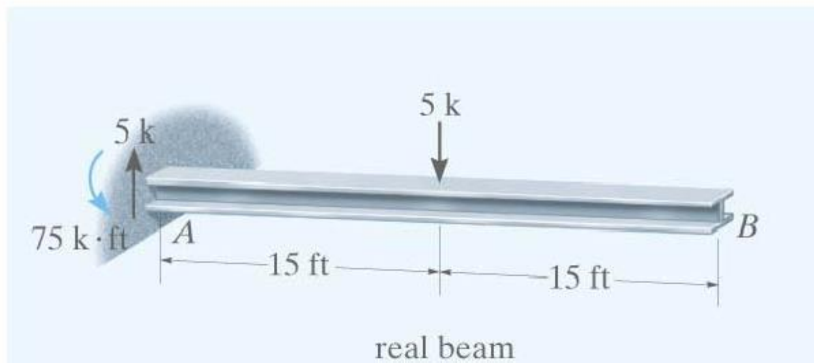
real beam

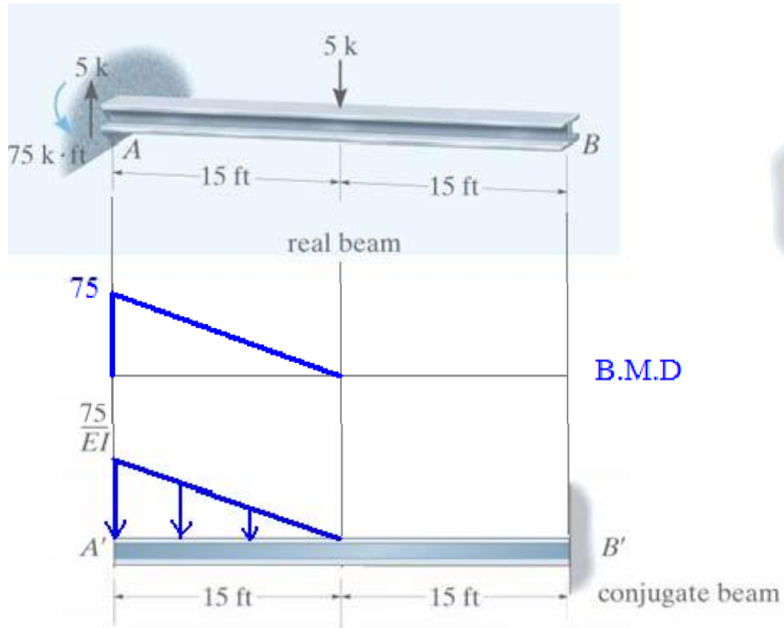


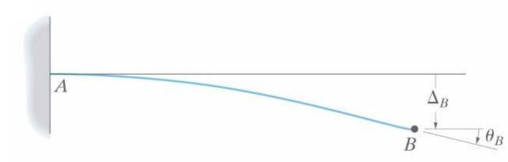
conjugate beam

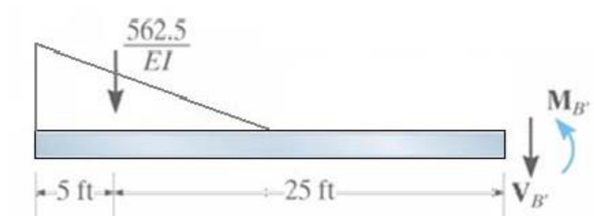
Example 1

Find the Max. deflection *Take E=200Gpa, I=60(10^6)*







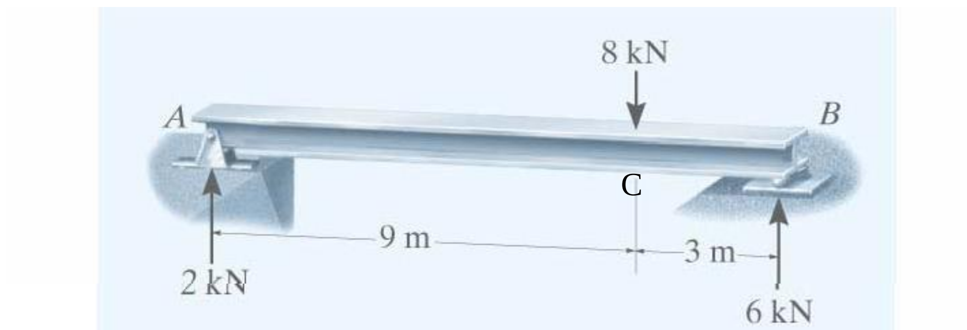


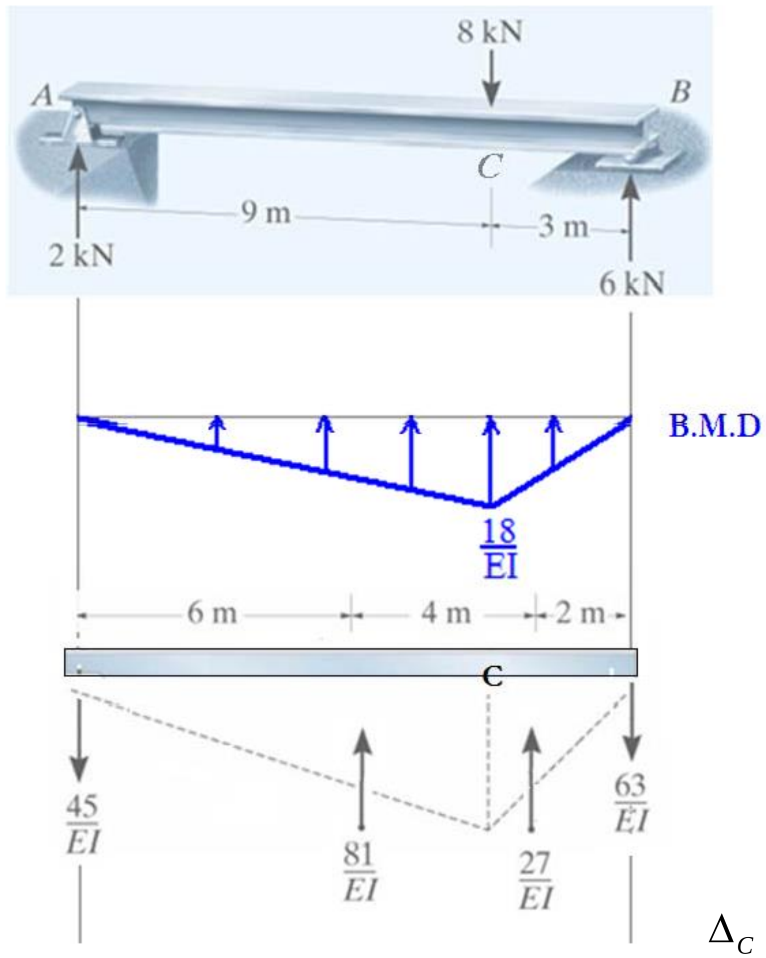
$$\theta_B = V_{B'} = -\frac{562.5}{EI}$$

$$\Delta_B = M_{B'} = \frac{562.5}{EI}(25) = \frac{-14062.5}{EI}$$

Example 2

Find the deflection at Point C

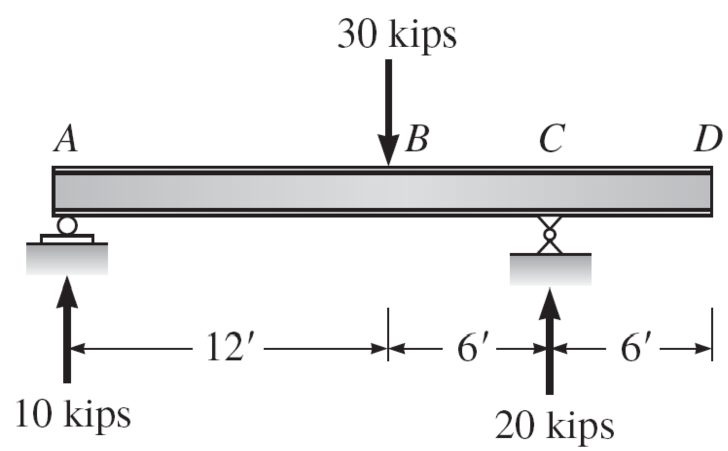


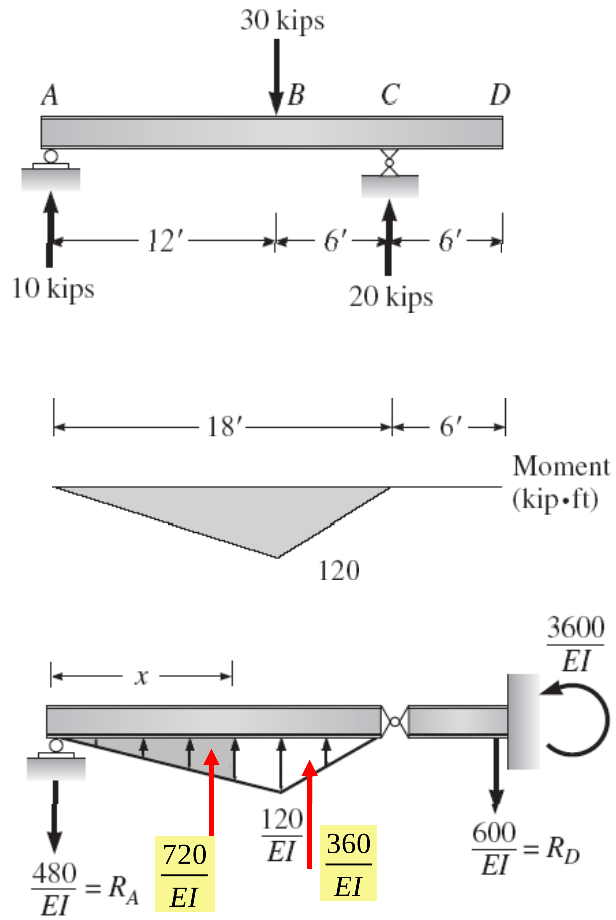


$$\Delta_C = M_{C'} = \frac{27}{EI}(1) - \frac{63}{EI}(3) = \frac{-162}{EI}$$

Example

Find the deflection at Point D





$$\circlearrowright^{+} \quad \Sigma M_{\text{hinge}} = 0$$

$$-18R_A + \frac{720(10)}{EI} + \frac{360(4)}{EI} = 0$$

$$R_A = \frac{480}{EI}$$

$$+\uparrow \quad \Sigma F_y = 0$$

$$\frac{720}{EI} + \frac{360}{EI} - \frac{480}{EI} - R_D = 0$$

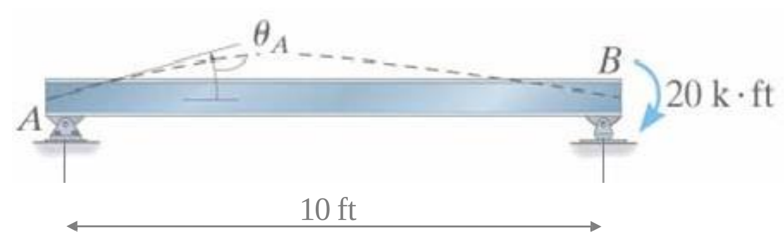
$$R_D = \frac{600}{EI}$$

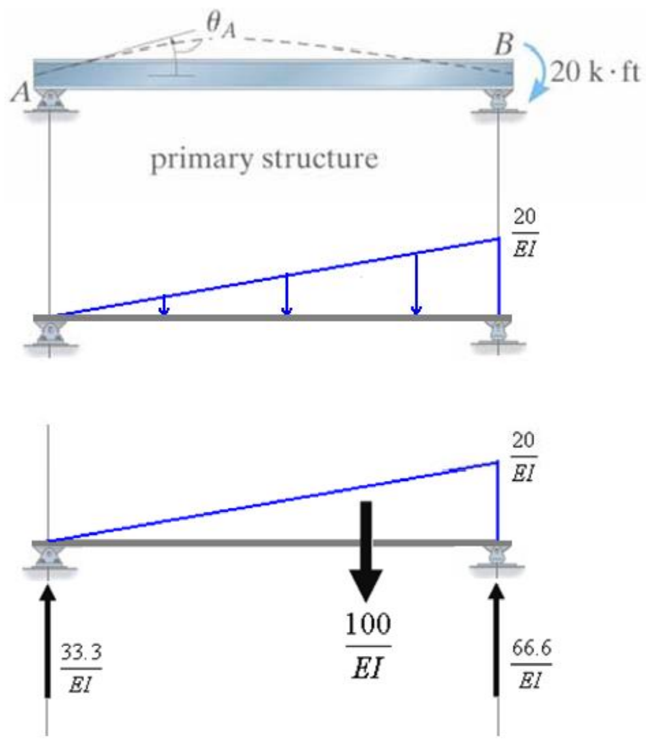
$$M_{D'} = \frac{600}{EI}(6) = \frac{3600}{EI}$$

$$\boxed{\Delta_D = M_{D'} = \frac{3600}{EI}}$$

Example 4

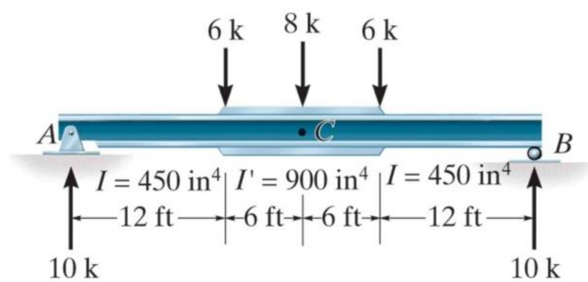
Find the Rotation at A

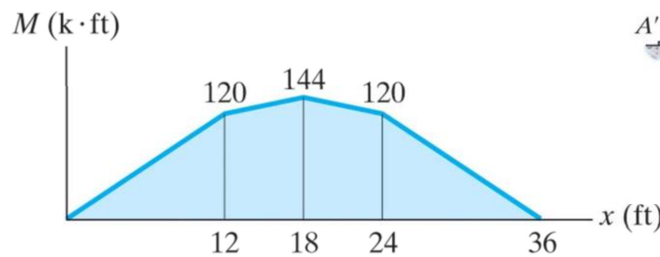



θ_A
B
A
20 k · ft
primary structure
$\frac{20}{EI}$
$\frac{20}{EI}$
$\frac{33.3}{EI}$
$\frac{100}{EI}$
$\frac{66.6}{EI}$

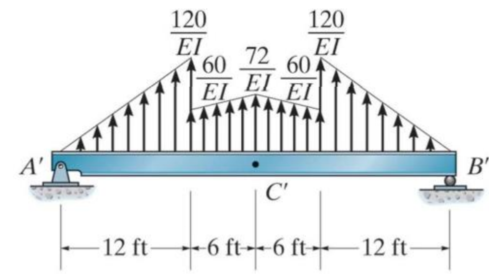
$$_A = \frac{33.3}{EI}$$

Example 5





moment diagram



conjugate beam

EXAMPLE 8-14 (Continued)

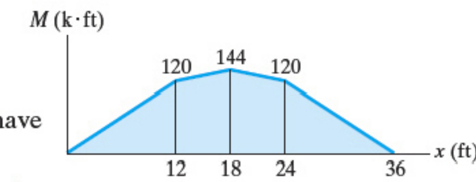
$$\circlearrowleft + \Sigma M_{C'} = 0; \quad \frac{1116}{EI}(18) - \frac{720}{EI}(10) - \frac{360}{EI}(3) - \frac{36}{EI}(2) + M_{C'} = 0$$

$$M_{C'} = -\frac{11\,736 \text{ k}\cdot\text{ft}^3}{EI}$$

Substituting the numerical data for EI and converting units, we have

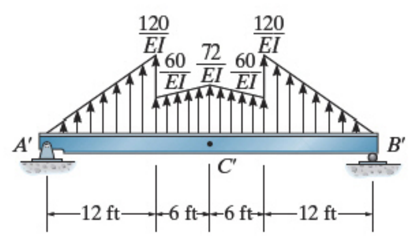
$$\Delta_C = M_{C'} = -\frac{11\,736 \text{ k}\cdot\text{ft}^3(1728 \text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(450 \text{ in}^4)} = -1.55 \text{ in.} \quad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.



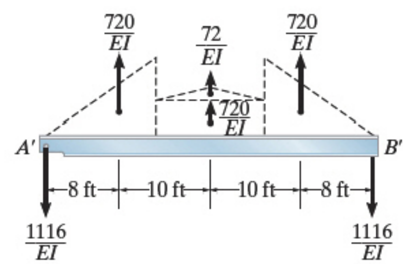
moment diagram

(b)



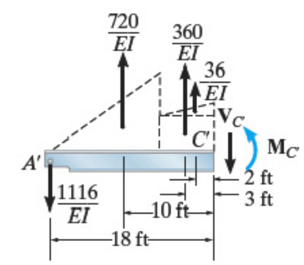
conjugate beam

(c)



external reactions

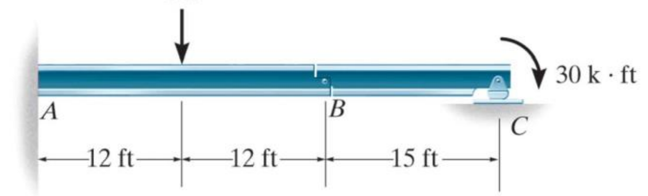
(d)



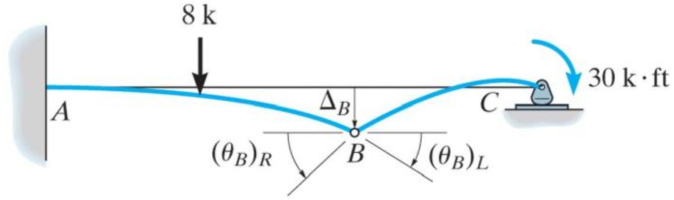
internal reactions

(e)

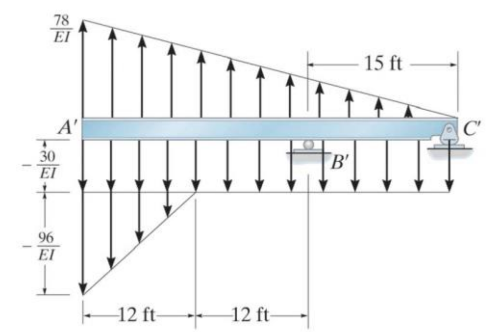
Example 6



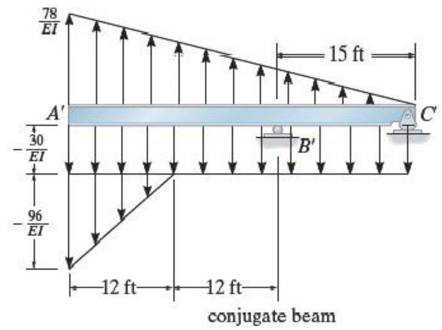
real beam



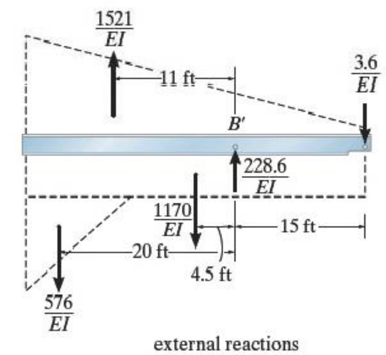
elastic curve



conjugate beam

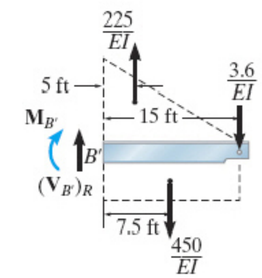


(c)



(d)

Equilibrium. The external reactions at B' and C' are calculated first and the results are indicated in Fig. 8–27*d*. In order to determine $(\theta_B)_R$, the conjugate beam is sectioned just to the *right* of B' and the shear force $(V_{B'})_R$ is computed, Fig. 8–27*e*. Thus,



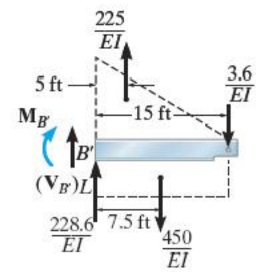
(e)

$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_R + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_R = (V_{B'})_R = \frac{228.6 \text{ k}\cdot\text{ft}^2}{EI}$$

$$= \frac{228.6 \text{ k}\cdot\text{ft}^2}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= 0.0378 \text{ rad} \qquad \textbf{\textit{Ans.}}$$



(f)

The internal moment at B' yields the displacement of the pin. Thus,

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad -M_{B'} + \frac{225}{EI}(5) - \frac{450}{EI}(7.5) - \frac{3.6}{EI}(15) = 0$$

$$\Delta_B = M_{B'} = -\frac{2304 \text{ k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-2304 \text{ k}\cdot\text{ft}^3}{[29(10^3)(144) \text{ k/ft}^2][30/(12)^4] \text{ ft}^4}$$

$$= -0.381 \text{ ft} = -4.58 \text{ in.} \qquad \textit{Ans.}$$

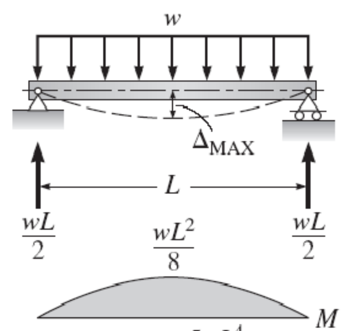
The slope $(\theta_B)_L$ can be found from a section of beam just to the *left* of B', Fig. 8–27*f*. Thus,

$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_L + \frac{228.6}{EI} + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

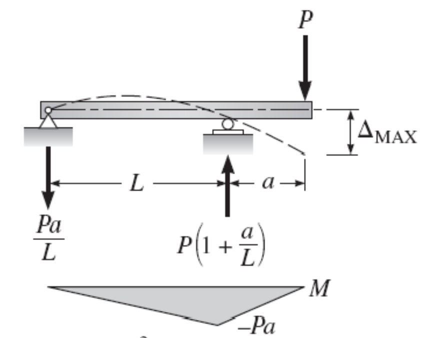
$$(\theta_B)_L = (V_{B'})_L = 0 \qquad \textit{Ans.}$$

Obviously, $\Delta_B = M_{B'}$ for this segment is the *same* as previously calculated, since the moment arms are only slightly different in Figs. 8–27*e* and 8–27*f*.

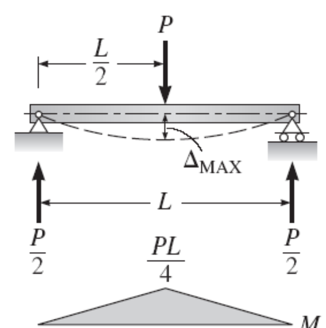
Moment Diagrams and Equations for Maximum Deflection



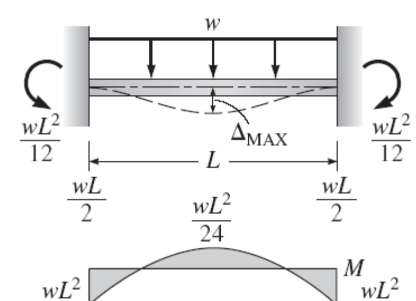
$$\Delta_{MAX} = \frac{5wL^4}{384EI}$$



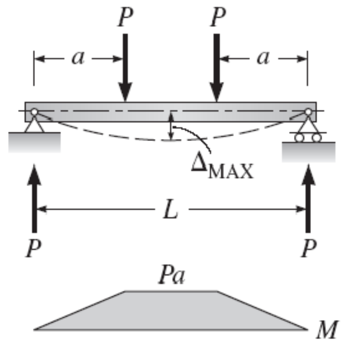
$$\Delta_{MAX} = \frac{Pa^2}{3EI}(L + a)$$



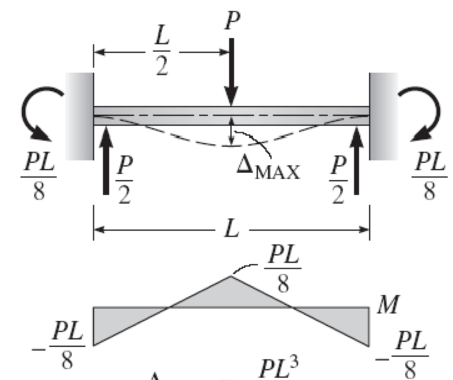
$$\Delta_{MAX} = \frac{PL^3}{48EI}$$



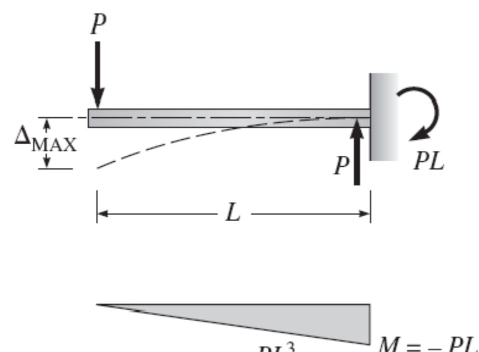
$$\Delta_{MAX} = \frac{wL^4}{384EI}$$



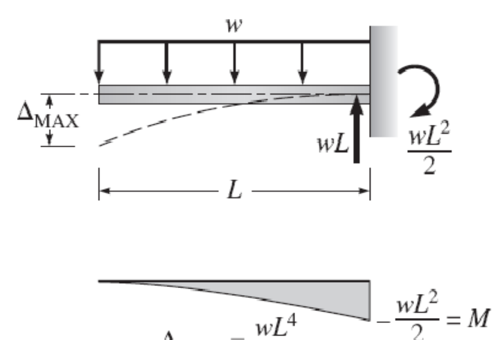
$$\Delta_{\text{MAX}} = \frac{Pa}{24EI}\,(3L^2 - 4a^2)$$



$$\Delta_{\text{MAX}} = \frac{PL^3}{192EI}$$



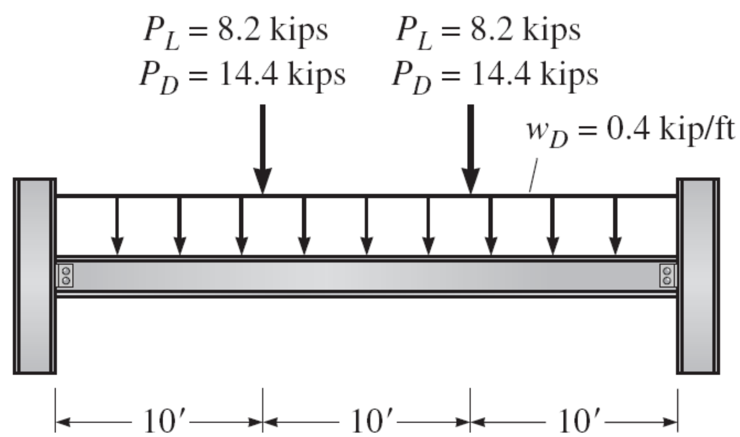
$$\Delta_{\text{MAX}} = \frac{PL^3}{3EI}$$



$$\Delta_{\text{MAX}} = \frac{wL^4}{8EI}$$

Example 4

Find the Maximum deflection for the following structure based on
The previous diagrams



(*a*) Dead load deflection produced by uniform load is

$$\Delta_{D1} = \frac{5wL^4}{384EI} = \frac{5(0.4)(30)^4(1728)}{384(30{,}000)(758)} = 0.32 \text{ in}$$

Dead load deflection produced by concentrated loads is

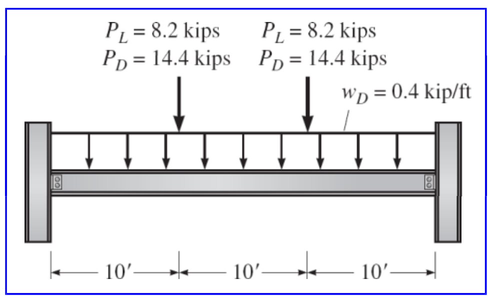
$$\Delta_{D2} = \frac{Pa(3L^2 - 4a^2)}{24EI} = \frac{14.4(10)[3(30)^2 - 4(10)^2](1728)}{24(30{,}000)(758)}$$

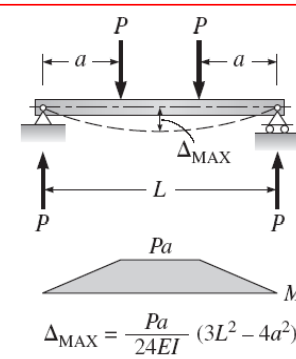
$$\Delta_{D2} = 1.05 \text{ in}$$

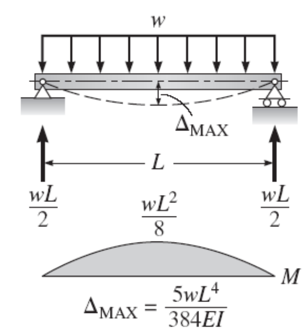
Total dead load deflection, $\Delta_{DT} = \Delta_{D1} + \Delta_{D2} = 0.32 + 1.05 = 1.37$ in

Live load deflection, $\Delta_L = \frac{Pa(3L^2 - 4a^2)}{24EI} = \frac{8.2(10)[3(30)^2 - 4(10)^2](1728)}{24(30{,}000)(758)}$

$$\Delta_L = 0.6 \text{ in}$$







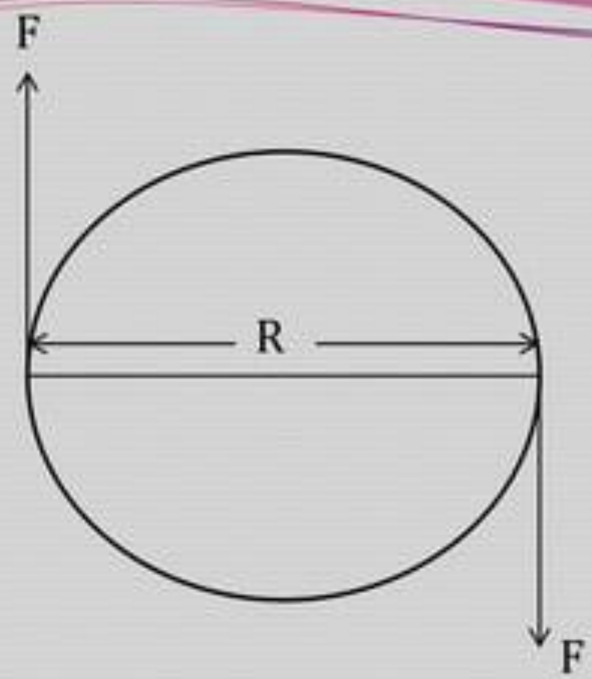
TORSION OF CIRCULAR SHAFT

- TORQUE OR TURNING MOMENT OR TWISTING MOMENT :-

- In factories and workshops, shafts is used to transmit energy from one end to other end.

- To transmit the energy, a turning force is applied either to the rim of a pulley, keyed to the shafts, or to any other suitable point at some distance from the axis of the shaft.

- The moment of couple acting on the shaft is called torque or turning moment or twisting moment.



Torque = turning force x diameter of shaft

$T = F \times 2R$

where :

T=Torque

F=Turning force

S=Radius of the shaft

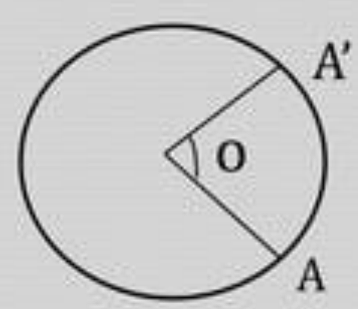
Unit of Torque(T) is N.mm or kN.mm

ANGLE OF TWIST (θ)

- When the shaft is subjected to Torque (T),point A on the surface of the shaft comes to A′ position. The angle AOA′ at the centre of the shaft is called the angle of twist.

- $\angle AOA' = \theta$ =Angle of twist

- Angle of twist is measured in radians.



SHEAR STRESS IN SHAFT:(τ)

- When a shaft is subjected to equals and opposite end couples, whose axes coincide with the axis of the shaft, the shaft is said to be in pure torsion and at any point in the section of the shaft stress will be induced.

- That stress is called shear stress in shaft.

STRENGTH OF SHAFTS

Maximum torque or power the shaft can transmit from one pulley to another, is called strength of shaft.

(a) For solid circular shafts:

Maximum torque (T)is given by :

$$T = \frac{\pi}{16} \times \tau \times D^3$$

where, D = dia. of the shaft

τ=shear stress in the shaft

(B) for hollow circular shaft

maximum torque (t) is given by.

$$T = \frac{\pi}{16} \times \tau \times \frac{D^4 - d^4}{D}$$

Where, d= outer dia of shaft
d=inner dia of shaft.

ASSUMPTION IN THE THEORY OF TORSION:

- The following assumptions are made while finding out shear stress in a circular shaft subjected to torsion.

1) The material of shaft is uniform throughout the length.
2) The twist along the shaft is uniform.
3) The shaft is of uniform circular section throughout the length.
4) Cross section of the shaft, which are plane before twist remain plain after twist.
5) All radii which are straight before twist remain straight after twist.

POLLAR MOMENT OF INERTIA : (J)

- The moment of inertia of a plane area, with respect to an axis perpendicular to the plane of the figure is called polar moment of inertia.
- As per the perpendicular axis theorem.

$$I_{ZZ} = I_{XX} + I_{YY} = J$$

$$= \frac{\pi}{64} \times D^4 + \frac{\pi}{64} \times D^4$$

$$J = \frac{\pi}{32} \times D^4$$

THEORY OF TORSION AND TORSION EQUATION

- Consider a shaft fixed at one end subjected to torque at the other end.

  Let T= Torque

  l= length of the shaft

  R=Radius of the shaft
- As a result of torque every cross-section of the shaft will be subjected to shear stress.
- Line CA on the surface of the shaft will be deform to CA' and OA to OA', as shown in figure.
- Let, ∠ACA'=shear strain
- ∠AOA'=angle of twist

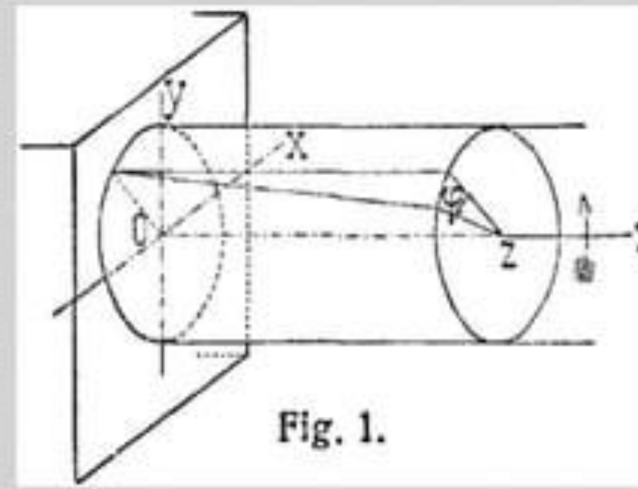


Fig. 1.

TORSION RIGIDITY

- Let twisting moment Produce a twist radians in length L.
  $\frac{T}{J}=\frac{C\theta}{L}$
- for given shaft the twist is therefore proportional to the twisting moment T.
- In a beam the bending moment produce deflection, in the same manner a torque produces a twist in shaft .
- The quantity CJ stands for the torque required to produce a twist of 1 radian per unit of the shaft.
- The quantity CJ corresponding to a similar EI, in expression for deflection of beams, EI is known as flexure rigidity.

EXAMPLE FOR SHAFT

EXAMPLE 1:-

Calculate diameter of shaft to transmit 10 KW at a speed of 15 Hz. The maximum shear stress should not exceed 60 mpa.

P = 10 kw

N = 15 Hz = 15 cycles/sec

= 15 x 60 rpm

= 900 rpm

τ= 60 Mpa

$$P = \frac{2\pi\, N\, T}{60}$$

$$10 \times 10^3 = \frac{2\pi \times 900 \times T}{60}$$

T = 106.10 N.m

now,

$$T = \frac{\pi}{16} \times \tau \times D^3$$

$$106.10 \times 10^3 = \frac{\pi}{16} \times 60 \times D^3$$

$$D^3 = 9006.0$$

$$\boxed{D = 20.80 \text{ mm}}$$

EXAMPLE 2:-

A haft of 60 mm diameter rotates with 180 rpm. If permissible shear stress is 100 Mpa, find torque and power in KW.

Solution:-

D = 60 mm

N = 180rpm

T = 100 Mpa

$$T = \frac{\pi}{16} \times \tau \times \mathrm{D}^3$$

$$= \frac{\pi}{16} \times 100 \times 60^3$$

T = 4240 N.mm

Now,

$$P = \frac{2\pi\ N\ T}{60}$$

$$P = \frac{2\pi \times 180 \times 4240}{60}$$

$$= 79922.11 \text{watt}$$

$$\boxed{P = 79.92\ \text{Kw}}$$

EXAMPLE NO:- 3

External and internal diameter of a propeller shaft are 400mm and 200mm respectively. Find maximum shear stress developed in the cross section when a twisting moment of 50kn.M is applied. Take modulus of rigidity $C = 0.8 \times 10^3 \text{ N/mm}^2$ .if span of shaft is 4m,also find twisting angle of shaft.

Solution:

D = 400mm

D = 200mm

T = 50 KN.m

L = 4m

$$T = \frac{\pi}{16} \times \tau \times \frac{D^4 - d^4}{D}$$

$$50 \times 10^6 = \frac{\pi}{16} \times \tau \times \frac{400^4 - 200^4}{400}$$

$$\tau = 4.24 \text{ N/mm}^2$$

Now,

$$\frac{\tau}{\mathrm{R}} = \frac{C\theta}{l}$$

$$R = \frac{D}{2} = \frac{400}{2} = 200mm$$

$$\therefore \boxed{\theta = 0.00106 \text{ radians}}$$

EXAMPLE-4 :-

Calculate the diameter of the shaft required to transmit 45 kw at 120 rpm. The maximum torque is likely to exceed the mean by 30% for a maximum permissible shear stress of 55 N/mm^2 .Calculate also the angle of twist for a length of 2m .

Solution :

P = 45 Kw

N = 120rpm

$\tau = 55 \ N/mm^2$

$$P = \frac{2\pi \times N \times T}{60}$$

$$45 \times 10^3 = \frac{2\pi \times 120 \times T}{60}$$

$\therefore$ T =3580.98 N.m

$$T_{max} = 1.30 \times T_{min}$$

= 1.30 x 3580.98

= 4655.28 N.m

= 465528 N.mm

$$T_{max} = \frac{\pi}{16} \times \tau \times D^3$$

$$4655.28 \times 10^3 = \frac{\pi}{16} \times 55 \times D^3$$

D = 75.54 mm

Now using the relation

$$\frac{\tau}{\mathrm{R}} = \frac{C\theta}{l}$$

$$\frac{80 \times 10^{3} \times \theta}{2000} = \frac{55}{37.77}$$

$$\boxed{\theta = 0.0364 \text{ radians}}$$

EXAMPLE 5 :-

A shaft has to transmit 105 kw power at 160 rpm. If the shear stress is not to exceed $65 N/mm^2$ & the twist in a length of 3.5 m must not to exceed 1 degree. Find suitable diameter. Take $P = 6266.72 \times 10^3$ N.mm

P = 105 kw

N = 160 rpm

$\tau = 65 N/mm^2$

L = 3500 mm

$G = 8 \times 10^4 N/mm^2$

Now,

$$P = \frac{2\pi N T}{60}$$

$$105 \times 10^3 = \frac{2\pi \times 160 \times T}{60}$$

$$T = 6266.72 \times 10^3 \text{ N.mm}$$

1) For strength

$$T = \frac{\pi}{16} \times \tau \times D^3$$

$$6266.72 \times 10^3 = \frac{\pi}{16} \times 65 \times D^3$$

$$\boxed{D = 78.89 \text{ mm}}$$

2) For stiffness

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\frac{6266.72 \times 10^3}{J} = \frac{8 \times 10^4 \times 0.0174}{3500}$$

$$\therefore \; J = 15.756 \times 10^6 \, mm^4$$

Now,

$$J = \frac{\pi}{32} \times D^4$$

$$15.756 \times 10^6 = \frac{\pi}{32} \times D^4$$

$$\boxed{D = 112.55 \text{ mm}}$$

EXAMOLE 6 :-

A solid shaft ABC is fixed at A and free at C and torque of 900 N.m is applied at B. The length of BC is 2m and that of BC is 1m. The diameter of AB is 40mm and that of BC is 20mm. If the shaft is made up of same material, find the angle of twisty in radius at the free end C.

Solution :

$$d_1 = 40\text{mm}$$

$$d_2 = 20\text{mm}$$

$$l_1 = 2000\text{mm}$$

$$l_2 = 1000\text{mm}$$

$$T_B = 900 \times 10^3 \text{ N.mm}$$

As per given data,

twist will occur in the shaft AB and there will be zero twist in shaft BC.

The torque $T = 600 \times 10^3$ N.mm will act only on part AB.

We know that,

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\frac{900\times10^3}{0.251\times10^6} = \frac{80\times10^3\times\theta}{2000}$$

$\boxed{\theta = 0.0896 \text{ radians}}$ twist at part B.

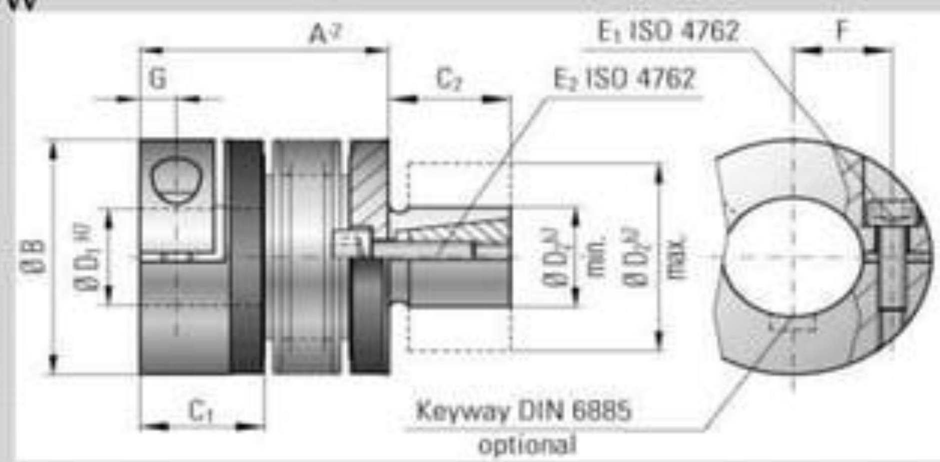
$$\theta_B = \theta_C = 0.0896 \text{radians}$$

$$J = \frac{\pi}{32}\times40^4$$

$$= 0.251\times10^6\,\text{mm}^4$$

SHAFT COUPLING

When length of shaft required is very large, due to non availability of a single shaft of required length, it becomes necessary to connect two shafts together. This is usually done by means of flanged coupling as shown below



- The flange of two shafts are joined together by bolts nuts or rivets and the torque is then transferred from one shaft to another through the couplings.

- As the torque is transferred through the bolts, will be subjected to shear stress. As the diameter of bolts is small, as compared to the diameter of the flange therefore shear stress is assumed to be uniform in the bolts.

1) Design of bolts :

We know that torque transmitted by the shaft,

$$T = \frac{\pi}{16} \times \tau \times D^3$$

Where, τ= shear stress in shaft

d = diameter of shaft

Now,

Torque rested by one bolt,

= (area x shear stress) x radius of bolt circle

Where

$$= \frac{\pi}{4} \times db^2 \times \tau_b \times R$$

$$= \frac{\pi}{4} \times db^2 \times \tau_b \times \frac{D}{2}$$

$$= \frac{\pi}{4} \times db^2 \times \tau_b \times D$$

$\therefore$ total torque resisted by n bolts

$$= n \times \tau \times db^2 \times \tau_b \times D \text{ -------------- } (2)$$

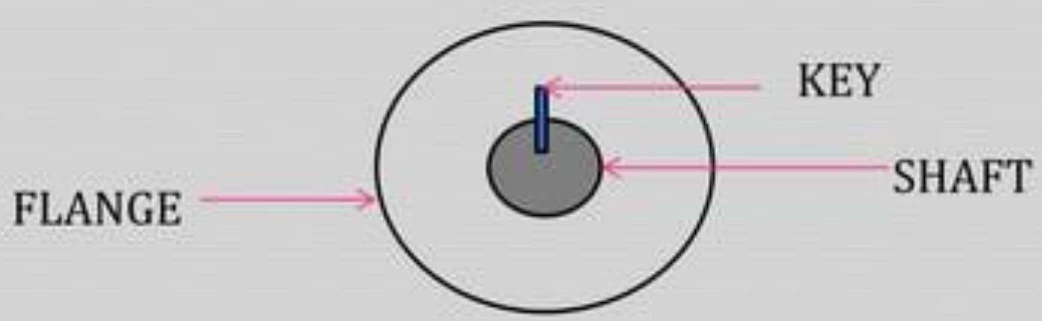
From equation (1) and (2)

$$\boxed{\frac{\pi}{16} \times \tau \times d^3 = n \times \frac{\pi}{8} \times db^2 \times \tau_b \times D}$$

DESIGN OF KEYS

A flange is attached to the shaft by means of a key. A rectangle notch is cut on the circumference of the shaft and a similar notch is cut on the inner side of the flange. The flange is then placed over the shaft in such a way that the two notches from a rectangular hole. A rectangular key is then inserted in to the hole and the flange is said to be keyed to the shaft.

Torque is transmitted by the shaft to flange through the key. Key is subjected to the shear stress.



EXAMPLE OF KEYS

EXAMPLE -7:-

A flanged coupling connecting two lengths of solid circular shaft has 6 nos of 20 mm diameter bolts equally spaced along a pitch circle of 240 mm diameter. Determine the shaft if the average shear stress in the bolts is to be the same as maximum shear stress in the shaft.

Solution :

n = 6 Nos

d = 20 mm

D = 240 mm

$$\tau_b = \tau$$

We know that,

Torque transmitted by shaft = Torque resisted by bolt

Now,

$$\frac{\pi}{16} \times \tau \times d^3 = n \times \frac{\tau}{8} \times db^2 \times \tau_b \times D$$

$$\therefore \frac{\pi}{16} \times \tau \times d^3 = 6 \times \frac{\pi}{8} \times 20^2 \times \tau \times 240$$

$$\therefore \boxed{d = 104.82 \text{mm dia of shafty}}$$

EXAMPLE -8 :-

The shaft each of 100 mm diameter are to be connected to the end by a bolted coupling. If the maximum shear stress in the shaft is 80 Mpa and in the bolts is 70 Mpa , find the number of 20 mm diameter bolts required for the coupling. Take diameter of bolt circle as 200 mm.

Solution :

$$d = 100\text{mm}$$

$$\tau = 80\text{N/mm}^2$$

$$\tau_b = 70\text{N/mm}^2$$

$$d_b = 20\text{mm}$$

$$D = 200\text{mm}$$

We know that,

Torque transmitted by the shaft = torque rested by the bolt

Now,

$$\frac{\pi}{16} \times \tau \times d^3 = n \times \frac{\pi}{8} \times db^2 \times \tau_b \times D$$

$$\frac{\pi}{16} \times 80 \times 100^3 = n \times \frac{\pi}{8} \times 20^2 \times 70 \times 200$$

$$\therefore n = 7.143 \text{ Nos.}$$

$$\text{Say } n = \boxed{8 \text{ Nos} \quad \text{Bolt}}$$

EXAMPLE OF DESIGN OF SHAFT

EXAMPLE – 9 :-

A shaft 100 in diameter is transmitted torque of 6000 N.m by means of key 200 mm long and 25 mm wide. Find the stress developed in shaft.

d= 100 mm

T = 6000 N.m

L = 200 mm

τ = shear stress in shaft

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\frac{6000 \times 10^3}{9.817 \times 10^6} = \frac{\tau}{50}$$

$$\boxed{\tau = 30.56 \text{N/mm}^2}$$

$$J = \frac{\pi}{32} \times d^4 = \frac{\pi}{32} \times 100^4$$

$$J = 9.817 \times 10^6$$

$$R = \frac{100}{2} = 50\text{mm}$$

Example -10 :-

Two shaft of diameter 50 mm are joined by a rigid flange coupling and transmit a torque in such a way that the shear stress in shaft does not exceed $100 N/mm^2$. If six bolt are used to join the flange and the bolt circle is 150 mm in diameter. Determine the diameter of the bolt if the permitted shear stress in the bolt is $\tau_b = 80 N/mm^2$.

Solution :

d= 50mm

$\tau = 100 N/mm^2$

$\tau_b = 80 N/mm^2$

n= 6 Nos

D = 150 mm

We know that,

Torque transmitted by the shaft = Torque resisted by bolts.

Now,

$$\frac{\pi}{16} \times \tau \times d^3 = n \times \frac{\tau}{8} \times db^2 \times \tau_b \times D$$

$$\frac{\pi}{16} \times 100 \times 50^3 = 6 \times \frac{\pi}{8} \times db^2 \times 80 \times 150$$

$\boxed{d_b = 9.32\text{mm}}$......dia of bolt.

TORSION VEDIO

THANK YOU

Module 6
Columns and Struts

Columns and Struts

- Any member subjected to axial compressive load is called a column or Strut.
- A vertical member subjected to axial compressive load – *COLUMN* (Eg: Pillars of a building)
- An inclined member subjected to axial compressive load - *STRUT*
- A strut may also be a horizontal member
- Load carrying capacity of a compression member depends not only on its cross sectional area, but also on its length and the manner in which the ends of a column are held.

- Equilibrium of a column – Stable, Unstable, Neutral.
- Critical or Crippling or Buckling load – Load at which buckling starts
- Column is said to have developed an elastic instability.

Classification of Columns

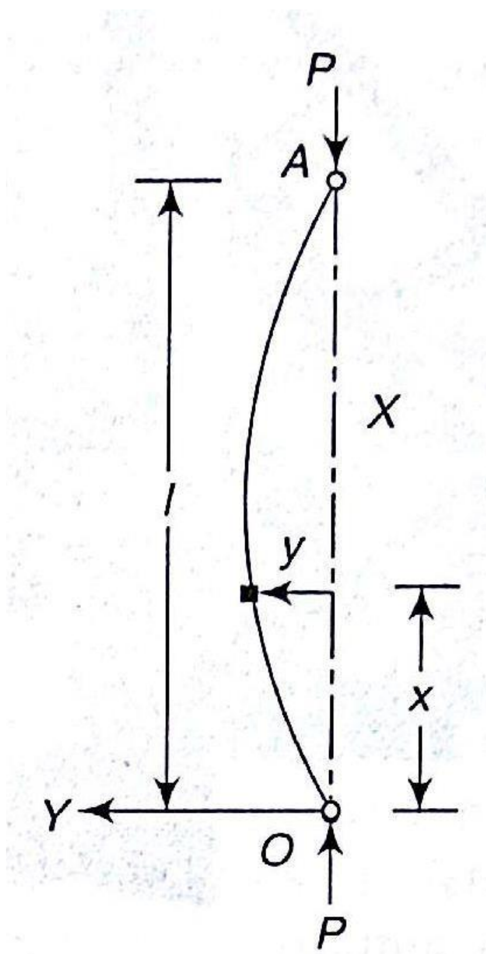
- According to nature of failure – short, medium and long columns
- 1. Short column – whose length is so related to its c/s area that *failure occurs mainly due to direct compressive stress* only and the role of bending stress is negligible
- 2. Medium Column - whose length is so related to its c/s area that *failure occurs by a combination of direct compressive stress and bending stress*
- 3. Long Column - whose length is so related to its c/s area that *failure occurs mainly due to bending stress* and the role of direct compressive stress is negligible

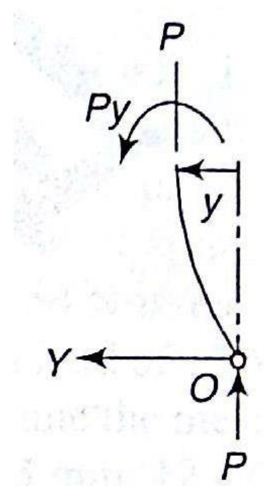
Euler's Theory

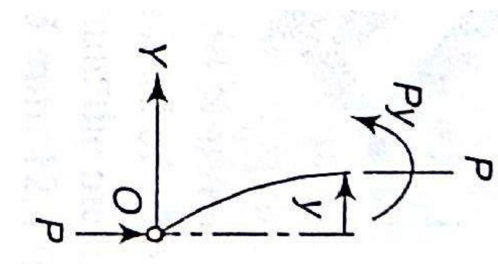
- Columns and struts which fail by buckling may be analyzed by Euler's theory
- <u>Assumptions made</u>

- the column is initially straight
- the cross-section is uniform throughout
- the line of thrust coincides exactly with the axis of the column
- the material is homogeneous and isotropic
- the shortening of column due to axial compression is negligible.

Case (i) Both Ends Hinged







$$EI\frac{d^2y}{dx^2} = M = -Py$$

$$EI \frac{d^2 y}{dx^2} = M = -Py$$

The equation can be written as $\frac{d^2 y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \frac{P}{EI}$

The solution is $y = A \sin \alpha x + B \cos \alpha x$

At $x = 0, y = 0, \therefore B = 0$

at $x = l, y = 0$ and thus $A \sin \alpha l = 0$

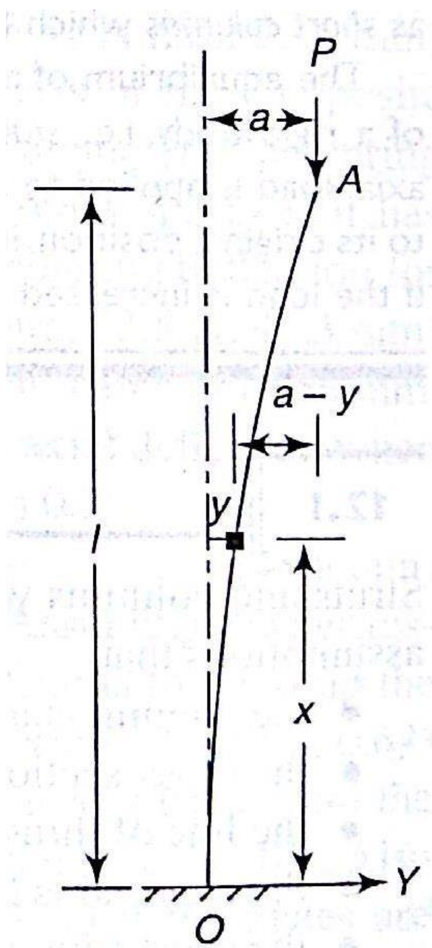
If $A = 0$, y is zero for all values of load and there is no bending.

$\therefore \quad \sin \alpha l = 0 \quad$ or $\quad \alpha l = \pi \qquad$ (considering the least value)

or $\quad \alpha = \pi / l$

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \frac{\pi^2 EI}{l^2}$

Case (ii) One end fixed other free



$$EI\frac{d^2y}{dx^2} = M = P(a - y) = Pa - Py$$

$$EI\frac{d^2y}{dx^2} = M = P(a - y) = Pa - Py$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{P \cdot a}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{P.a}{EI\alpha^2}$

$$= A\sin\alpha x + B\cos\alpha x + a$$

$$x = 0, y = 0, \ \therefore B = -a;$$

$$x = 0, \ \frac{dy}{dx} = 0$$

or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0 \quad$ or $\quad A = 0$

$$y = -a\cos\alpha x + a = a(1 - \cos\alpha x)$$

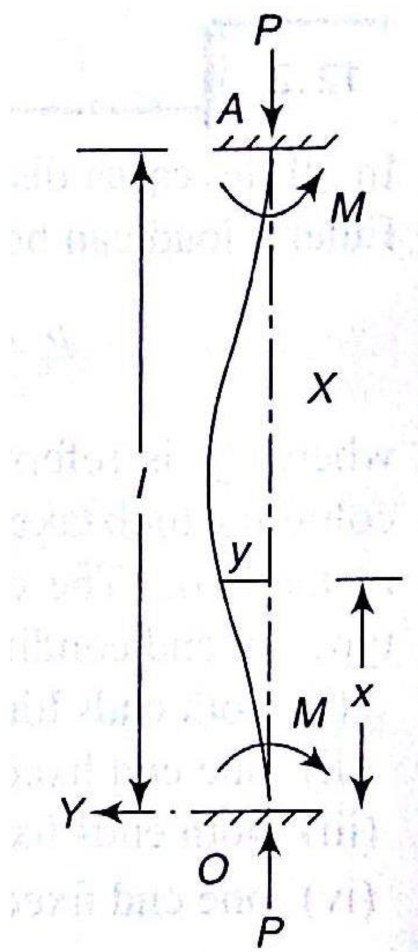
At $x = l, y = a, \therefore a = a(1 - \cos \alpha l)$

or $\cos \alpha l = 0$ or $\alpha l = \frac{\pi}{2}$ (considering the least value)

$$\alpha = \pi / 2l$$

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \frac{\pi^2 EI}{4l^2}$

Case (iii) Fixed at both ends



$$EI\,\frac{d^2y}{dx^2} = -Py + M$$

$$EI\frac{d^2y}{dx^2} = -Py + M$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{M}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{EI\alpha^2} = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{P}$

$$x = 0,\ y = 0,\ \therefore\ B = -\frac{M}{P};$$

$$x = 0,\ \frac{dy}{dx} = 0$$

or $A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0$ or $A = 0$

$$\therefore\ y = -\frac{M}{P}\cos\alpha x + \frac{M}{P} = \frac{M}{P}(1 - \cos\alpha x)$$

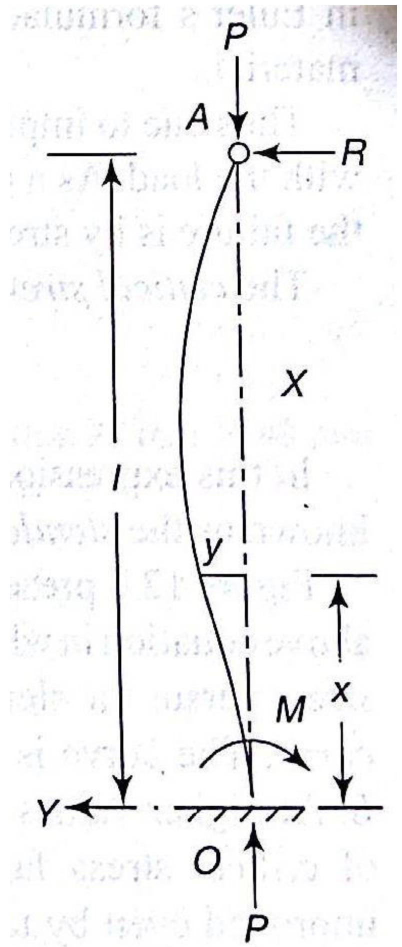
At $x = l, y = 0, \therefore \; 0 = \dfrac{M}{P}(1 - \cos \alpha l)$ or $\cos \alpha l = 1$

or $\alpha l = 2\pi$ (considering the least value)

or $\alpha = 2\pi / l$

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \dfrac{4\pi^2 EI}{l^2}$

Case (iv) One end fixed, other hinged



$$EI\frac{d^2y}{dx^2} = -Py + R(l-x)$$

$$EI\frac{d^2y}{dx^2} = -Py + R(l-x)$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{R(l-x)}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{R(l-x)}{EI\alpha^2}$

$$= A\sin\alpha x + B\cos\alpha x + \frac{R}{P}(l-x)$$

At $x = 0, y = 0, \therefore \; B = -\dfrac{Rl}{P}$;

At $x = 0, \dfrac{dy}{dx} = 0$

or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x - \dfrac{R}{P} = 0 \qquad$ or $\quad A = \dfrac{R}{P\alpha}$

$$\therefore \quad y = \frac{R}{P\alpha}\sin \alpha x - \frac{Rl}{P}\cos \alpha x + \frac{R}{P}(l - x)$$

At $x = l, y = 0, \therefore \; 0 = \dfrac{R}{P\alpha}\sin \alpha l - \dfrac{Rl}{P}\cos \alpha l$

or $\quad \tan \alpha l = \alpha l$

$\alpha l = 4.49$ rad (considering the least value)
$\alpha = 4.49 / l$

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \dfrac{4.49^2 EI}{l^2} = \dfrac{20.2EI}{l^2} \approx \dfrac{2\pi^2 EI}{l^2}$

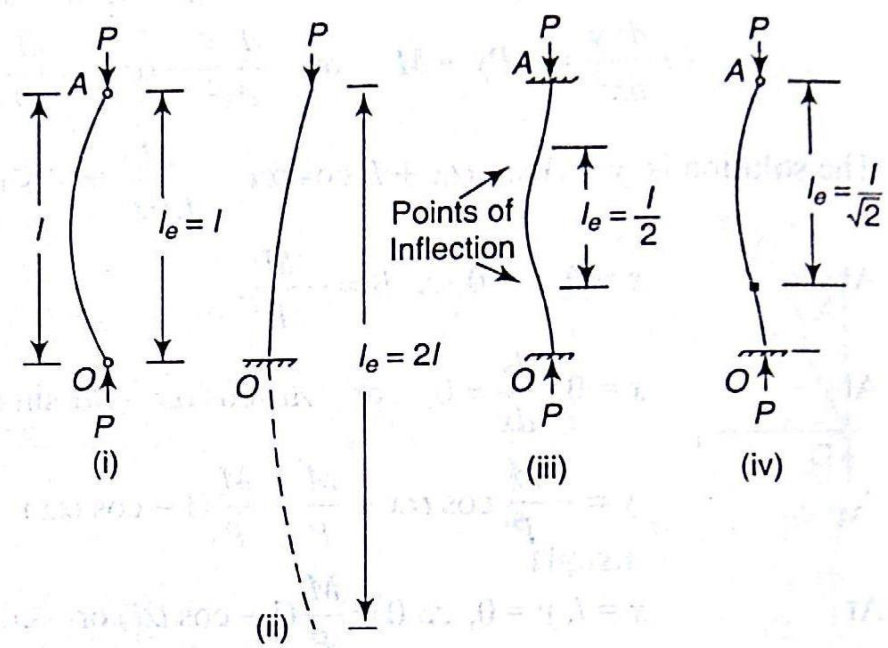
Equivalent Length (l_e)

Euler's load can be expressed as $P_e = \dfrac{\pi^2 EI}{l_e^2}$

where l_e^2 is referred as *equivalent length* of the column which takes into account the type of fixing of the ends.

The equivalent lengths for different types of end conditions are

(i) both ends hinged, $l_e = l$
(ii) one end fixed and the other free, $l_e = 2l$
(iii) both ends fixed, $l_e = l/2$
(iv) one end fixed, other hinged, $l_e = l/\sqrt{2}$



Limitations of Euler's Formula

- Assumption – Struts are initially perfectly straight and the load is exactly axial.
- There is always some eccentricity and initial curvature present.
- In practice a strut suffers a deflection before the Crippling load.

- Critical stress (σ_c) – average stress over the cross section

$$\sigma_c = \frac{P_e}{A} = \frac{\pi^2 EI}{Al_e^2}$$

$$= \frac{\pi^2 EAk^2}{Al_e^2}$$

$$\sigma_c = \frac{\pi^2 E}{(l_e/k)^2}$$

- l/k is known as **Slenderness Ratio**

Slenderness Ratio

- **Slenderness ratio** is the ratio of the length of a column and the radius of gyration of its cross section
- Slenderness Ratio = l/k

The Radius of Gyration $\mathbf{k_x}$ of an Area (A) about an axis (x) is defined as:

$$I_x = k_x^{\,2} A$$

$$\mathbf{k_x} = \sqrt{\frac{I_x}{A}}$$

Rankine's Formula
OR
Rankine-Gorden Formula

- Euler's formula is applicable to long columns only for which l/k ratio is larger than a particular value.
- Also doesn't take in to account the direct compressive stress.
- Thus for columns of medium length it doesn't provide accurate results.
- Rankine forwarded an empirical relation

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

where P = Rankine's crippling load

P_c = ultimate load for a strut = $\sigma_u \cdot A$, constant for a material

P_e = Eulerial load for a strut = $\pi^2 EI/l^2$

- For short columns, P_e is very large and therefore $1/P_e$ is small in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_c
- For long columns, P_e is very small and therefore $1/P_e$ is quite large in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_e

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \dfrac{\sigma_c A \cdot l^2}{\pi^2 EI}}$$

$$= \frac{\sigma_c \cdot A}{1 + \dfrac{\sigma_c \cdot A \cdot l^2}{\pi^2 EAk^2}}$$

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\dfrac{l}{k}\right)^2}$$

where σ_c is the crushing stress

a is the Rankine's constant $(\sigma_c / \pi^2 E)$

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$$

$$\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EI}}$$

$$= \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c \cdot A \cdot l^2}{\pi^2 EAk^2}}$$

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l}{k}\right)^2}$$

where σ_c is the crushing stress

a is the Rankine's constant $(\sigma_c / \pi^2 E)$

- A Factor of Safety may be considered for the value of σ_c in the above formula

- Rankine's formula for columns with other end conditions

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2}$$

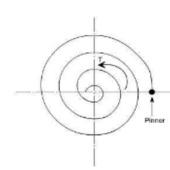
Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released.

Important types of springs are:

There are various types of springs such as

Helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is torsional shear stress due to twisting. They are both used in tension and compression.

Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.
In this the major stresses are tensile and compression due to bending.

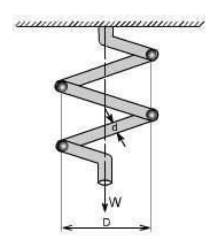


Uses of springs :

To apply forces and to control motions as in brakes and clutches.
To measure forces as in spring balance.
To store energy as in clock springs.
To reduce the effect of shock or impact loading as in carriage springs.
To change the vibrating characteristics of a member as inflexible mounting of motors.

<u>Derivation of the Formula :</u>

In order to derive a necessary formula which governs the behavior of springs, consider a closed coiled spring subjected to an axial load W.



Let
W = axial load
D = mean coil diameter d = diameter of spring wire
n = number of active coils
C = spring index = D / d For circular wires

l = length of spring wire G = modulus of rigidity
x = deflection of spring q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that
$x = D / 2 . \theta$

UNIT-5

COLUMNS & STRUTS

THIN AND THICK CYLINDERS

INTRODUCTION:

In many engineering applications, cylinders are frequently used for transporting or storing of liquids, gases or fluids.

Eg: Pipes, Boilers, storage tanks etc.

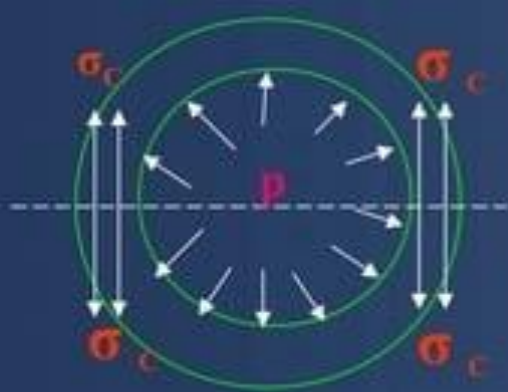
These cylinders are subjected to fluid pressures. When a cylinder is subjected to a internal pressure, at any point on the cylinder wall, three types of stresses are induced on three mutually perpendicular planes.

They are,

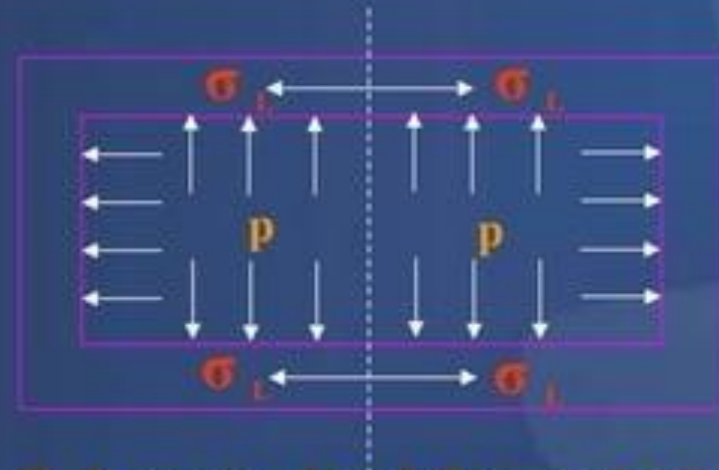
1. Hoop or Circumferential Stress (σ_C) – This is directed along the tangent to the circumference and tensile in nature. Thus, there will be increase in diameter.

2. Longitudinal Stress (σ_L) – This stress is directed along the length of the cylinder. This is also tensile in nature and tends to increase the length.

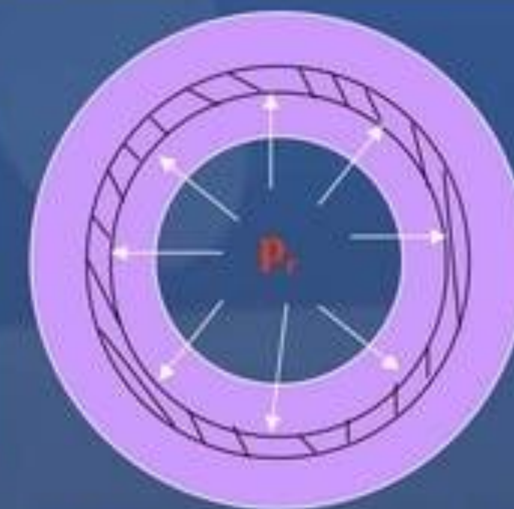
3. Radial pressure (p_r) – It is compressive in nature.
<u>Its magnitude is equal to fluid pressure on the inside wall and zero on the outer wall if it is open to atmosphere.</u>



1. Hoop Stress (σ_C)

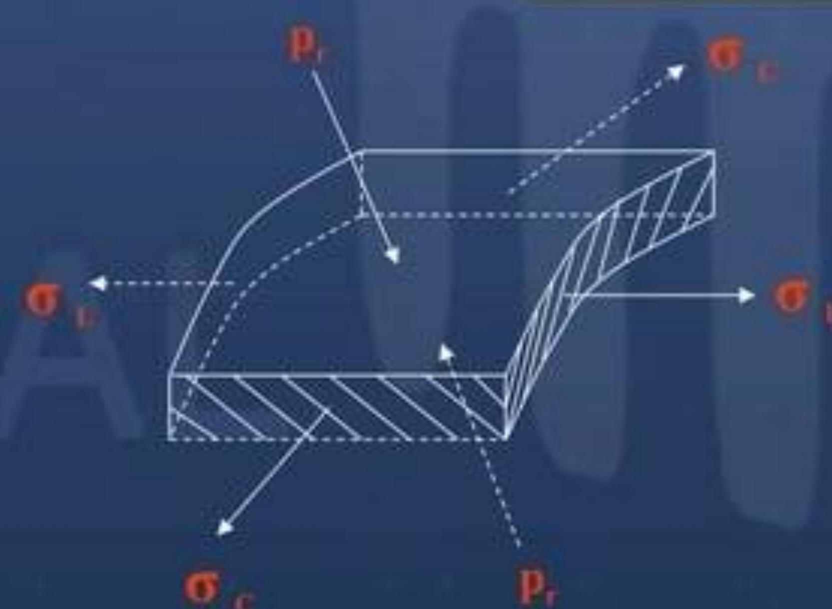


2. Longitudinal Stress (σ_L)



3. Radial Stress (p_r)

Element on the cylinder wall subjected to these three stresses



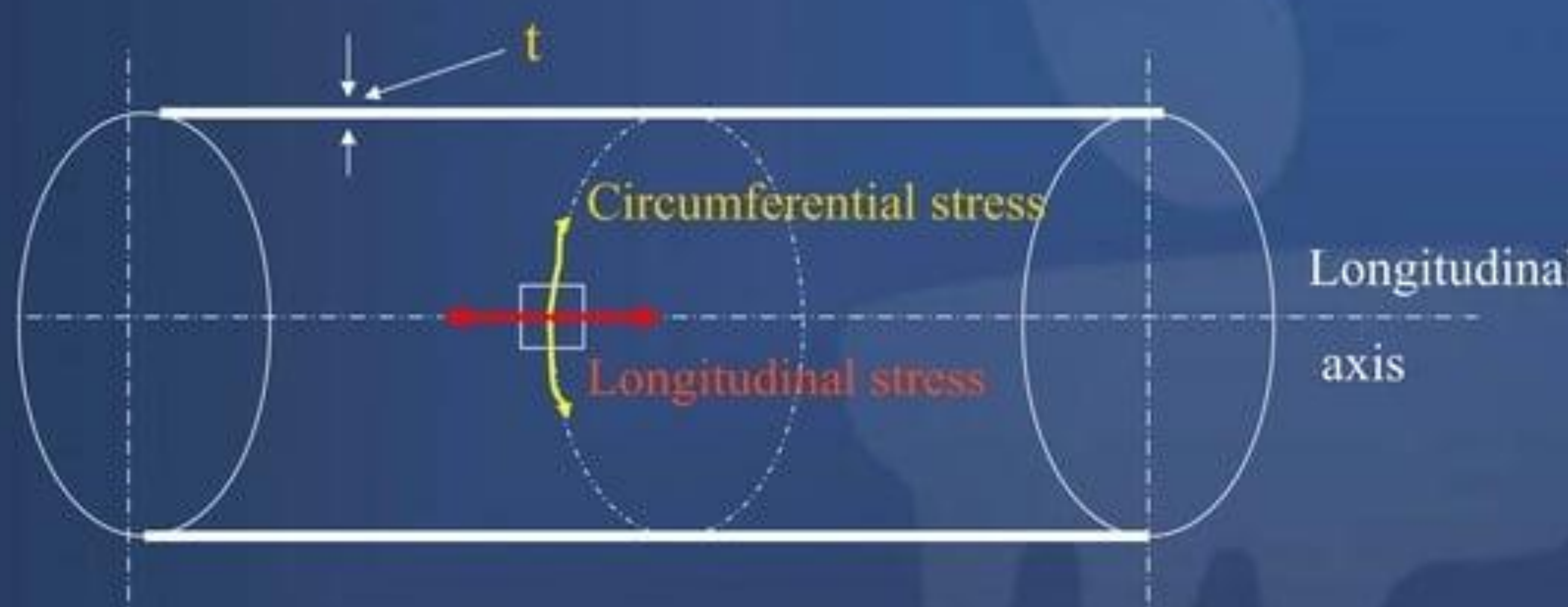
THIN CYLINDERS

INTRODUCTION:

A cylinder or spherical shell is considered to be thin when the metal thickness is small compared to internal diameter.

i. e., when the wall thickness, ‘t’ is equal to or less than ‘d/20’, where ‘d’ is the internal diameter of the cylinder or shell, we consider the cylinder or shell to be thin, otherwise thick.

Magnitude of radial pressure is very small compared to other two stresses in case of thin cylinders and hence neglected.

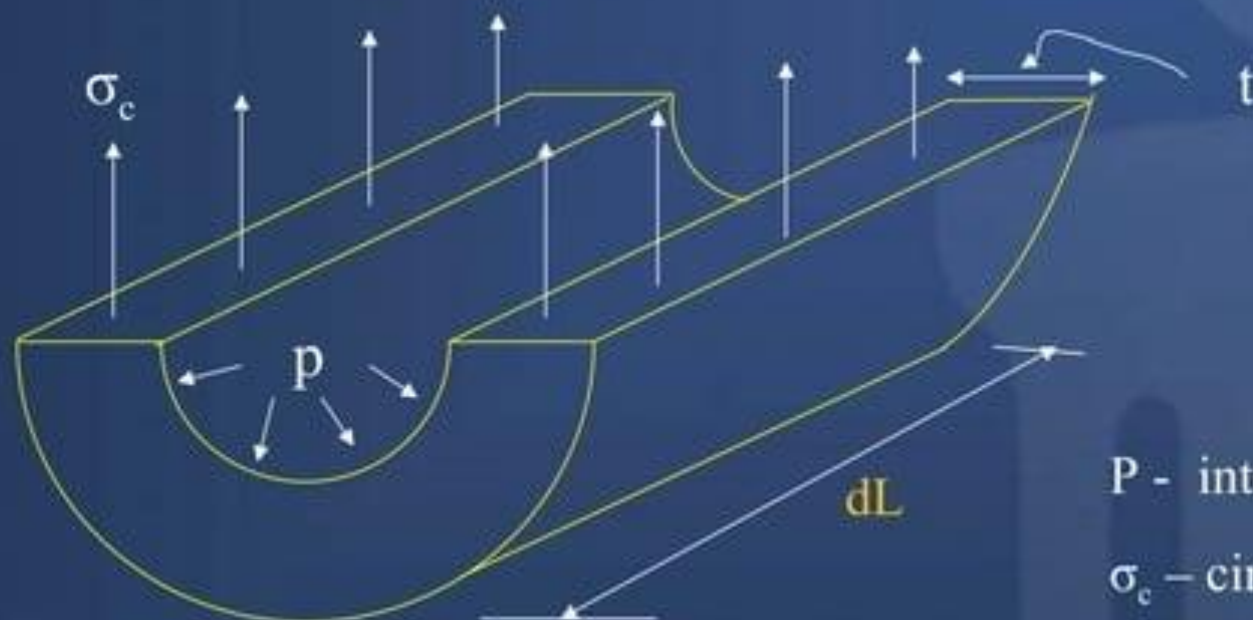


The stress acting along the circumference of the cylinder is called circumferential stresses whereas the stress acting along the length of the cylinder (i.e., in the longitudinal direction) is known as longitudinal stress

The bursting will take place if the force due to internal (fluid) pressure (acting vertically upwards and downwards) is more than the resisting force due to circumferential stress set up in the material.

P - internal pressure (stress)

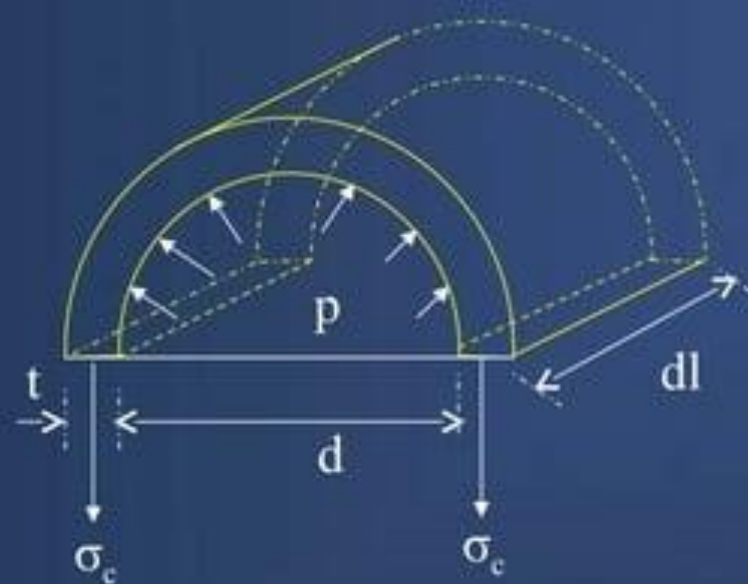
σ_c –circumferential stress



P - internal pressure (stress)

σ_c – circumferential stress

EVALUATION OF CIRCUMFERENTIAL or HOOP STRESS (σ_C):



Consider a thin cylinder closed at both ends and subjected to internal pressure ‘p’ as shown in the figure.

Let d=Internal diameter, t = Thickness of the wall

L = Length of the cylinder.

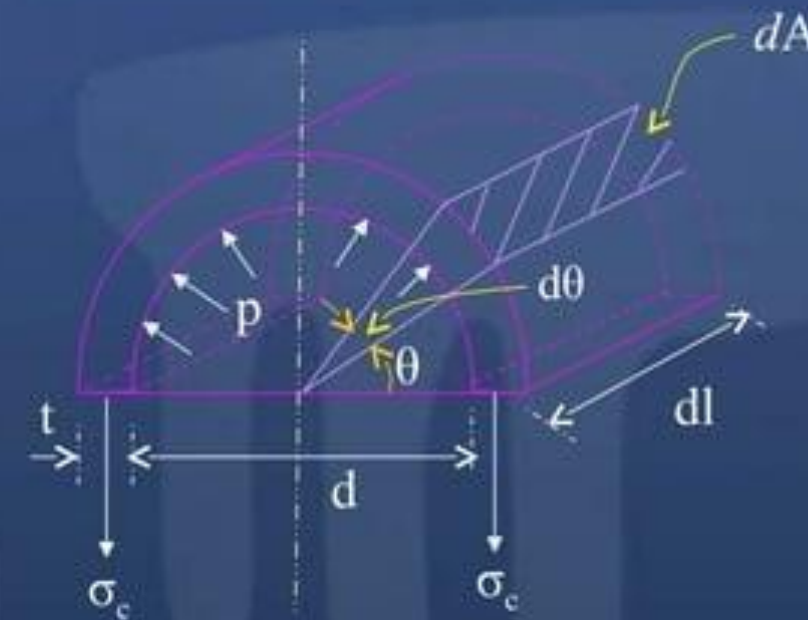
To determine the Bursting force across the diameter:

Consider a small length '*dl*' of the cylinder and an elementary area '*dA*' as shown in the figure.

Force on the elementary area,

$$dF = p \times dA = p \times r \times dl \times d\theta$$

$$= p \times \frac{d}{2} \times dl \times d\theta$$



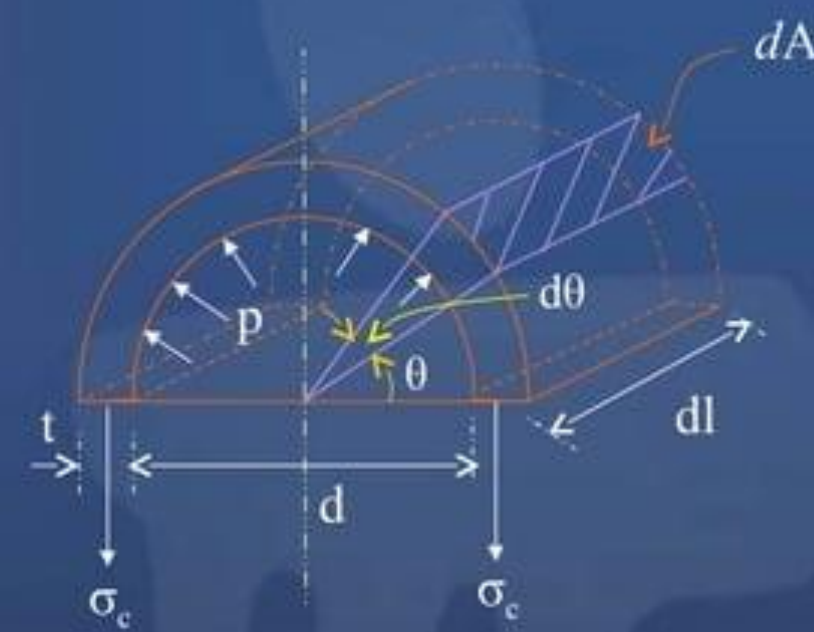
Horizontal component of this force

$$dF_x = p \times \frac{d}{2} \times dl \times \cos\theta \times d\theta$$

Vertical component of this force

$$dF_y = p \times \frac{d}{2} \times dl \times \sin\theta \times d\theta$$

The horizontal components cancel out when integrated over semi-circular portion as there will be another equal and opposite horizontal component on the other side of the vertical axis.



$$\therefore \text{Total diametrica l bursting force} = \int_0^{\pi} p \times \frac{d}{2} \times dl \times \sin\theta \times d\theta$$

$$= p \times \frac{d}{2} \times dl \times \left[-\cos\theta\right]_0^{\pi} = \underline{p \times d \times dl}$$

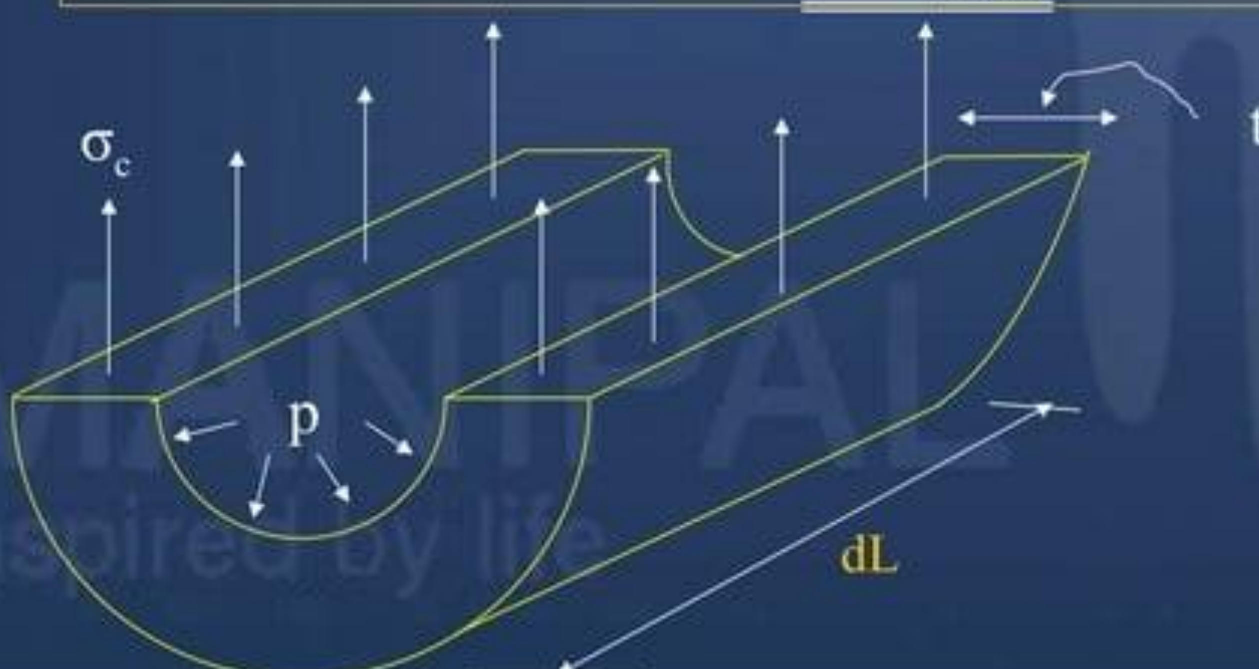
$$= p \times \text{projected area of the curved surface.}$$

$$\therefore \text{Resisting force (due to circumferential stress } \sigma_c) = 2\times\sigma_c\times t\times dl$$

Under equillibrium, Resisting force = Bursting force

$$\text{i.e., } 2\times\sigma_c\times t\times dl = p\times d\times dl$$

$$\boxed{\therefore \text{Circumferential stress, } \sigma_c = \frac{p\times d}{2\times t}\text{........................}(1)}$$



Assumed as rectangular

Force due to fluid pressure = p × area on which p is acting = $p \times (d \times L)$

(bursting force)

Force due to circumferential stress = σ_c × area on which σ_c is acting

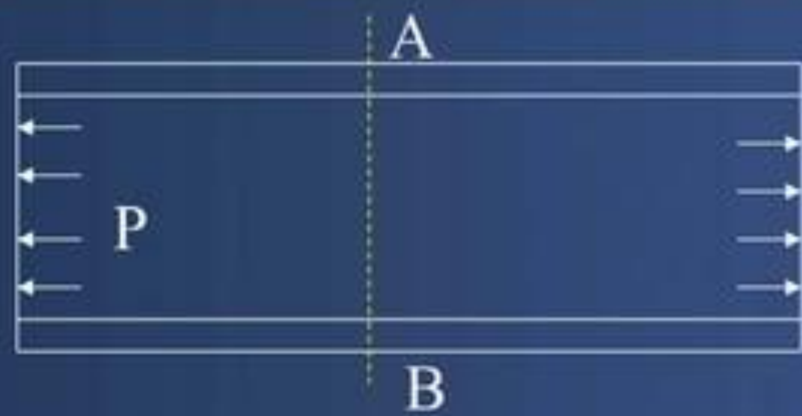
$$\text{(resisting force)} = \sigma_c \times (L \times t + L \times t) = \sigma_c \times 2L \times t$$

Under equilibrium bursting force = resisting force

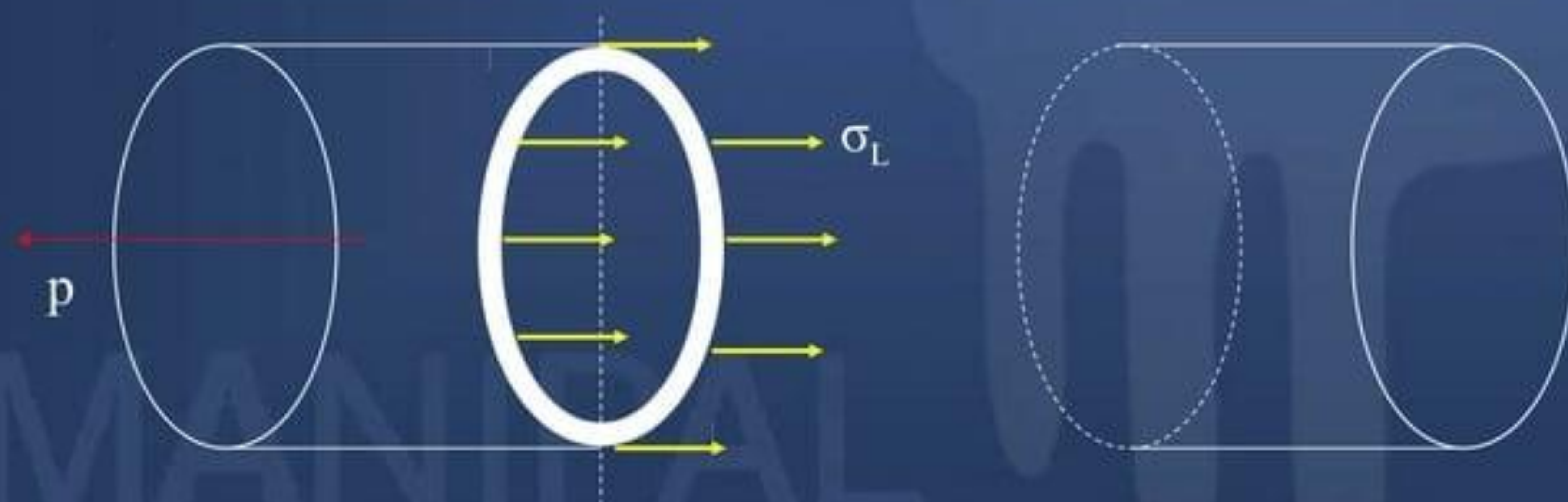
$$p \times (d \times L) = \sigma_c \times 2L \times t$$

$$\therefore \text{Circumferential stress, } \sigma_c = \frac{p \times d}{2 \times t} \text{........................(1)}$$

LONGITUDINAL STRESS (σ_L):

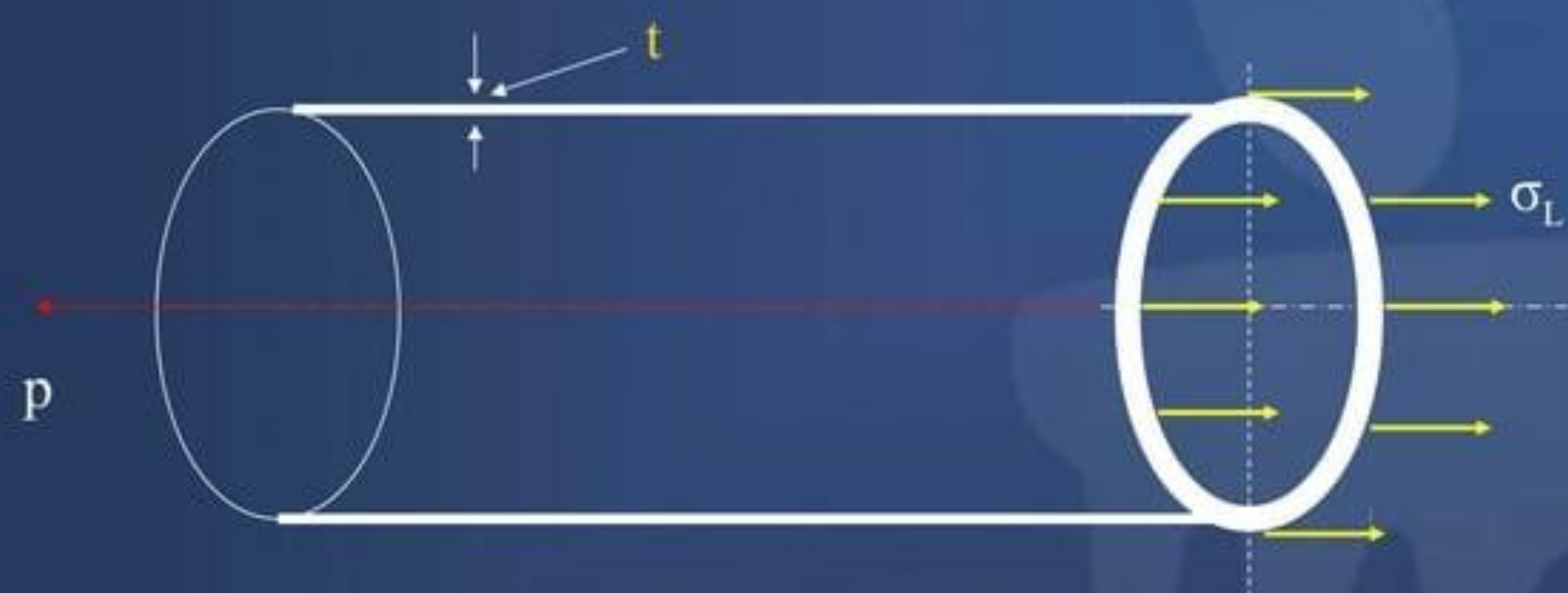


The bursting of the cylinder takes place along the section AB



The force, due to pressure of the fluid, acting at the ends of the thin cylinder, tends to burst the cylinder as shown in figure

EVALUATION OF LONGITUDINAL STRESS (σ_L):



$$\text{Longitudinal bursting force (on the end of cylinder)} = p \times \frac{\pi}{4} \times d^2$$

$$\text{Area of cross section resisting this force} = \pi \times d \times t$$

Let σ_L = Longitudinal stress of the material of the cylinder.

$$\therefore \text{Resisting force} = \sigma_L \times \pi \times d \times t$$

Under equillibrium, bursting force = resisting force

$$\text{i.e., } p \times \frac{\pi}{4} \times d^2 = \sigma_L \times \pi \times d \times t$$

$$\boxed{\therefore \quad \text{Longitudinal stress, } \sigma_L = \frac{p \times d}{4 \times t}} \text{...................}(2)$$

$$\boxed{\text{From eqs (1) \& (2),} \quad \sigma_C = 2 \times \sigma_L}$$

Force due to fluid pressure $= p \times$ area on which p is acting

$$= p \times \frac{\pi}{4} \times d^2$$

Resisting force $= \sigma_L \times$ *area on which* σ_L *is acting*

$$= \sigma_L \times \boxed{\pi \times d} \times t$$

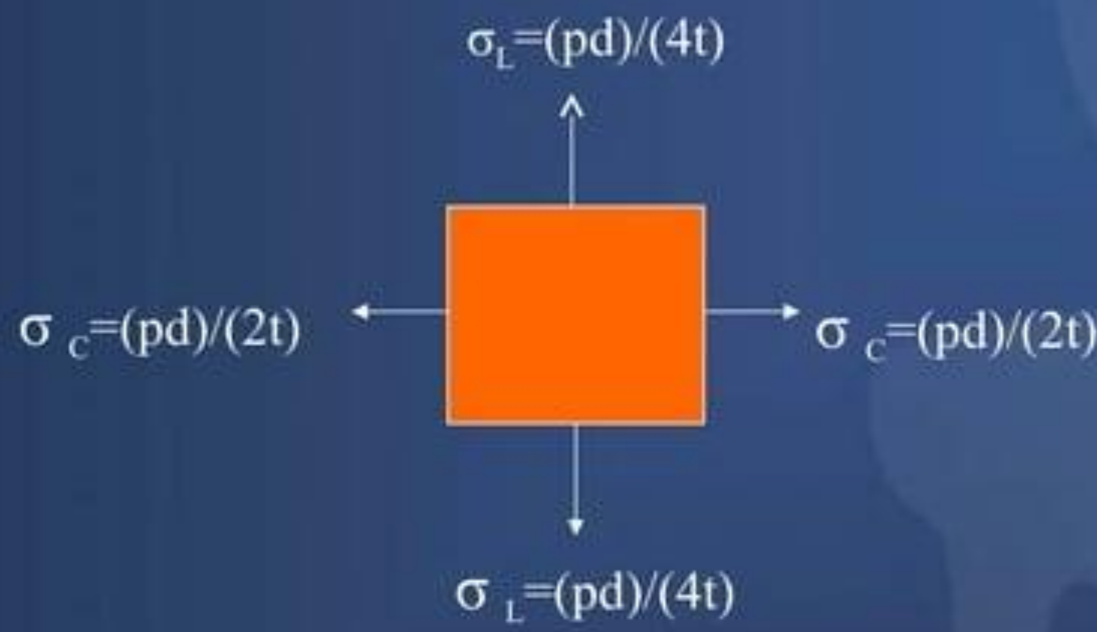
circumference

Under equillibrium, bursting force = resisting force

$$\text{i.e. } p \times \frac{\pi}{4} \times d^2 = \sigma_L \times \pi \times d \times t$$

$$\therefore \quad \text{Longitudinal stress, } \sigma_L = \frac{p \times d}{4 \times t} \text{ ....(2)}$$

EVALUATION OF STRAINS



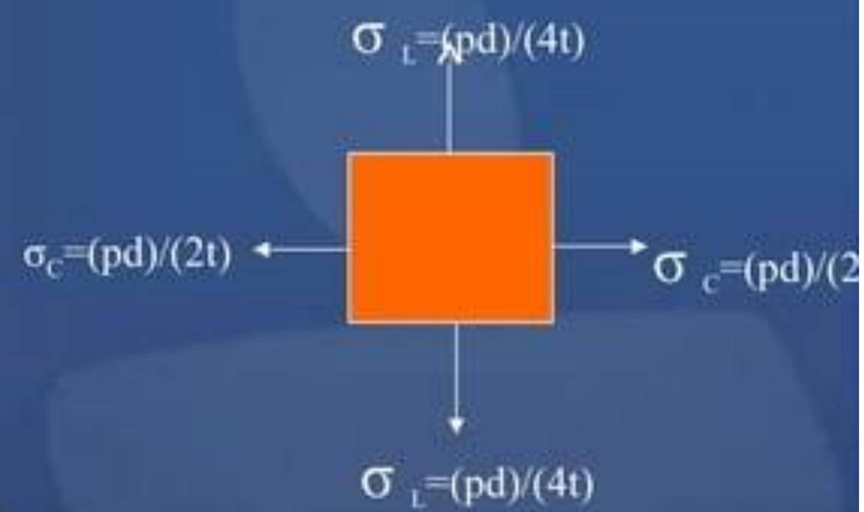
A point on the surface of thin cylinder is subjected to biaxial stress system, (Hoop stress and Longitudinal stress) mutually perpendicular to each other, as shown in the figure. The strains due to these stresses i.e., circumferential and longitudinal are obtained by applying Hooke's law and Poisson's theory for elastic materials.

Circumferential strain, ε_C :

$$\varepsilon_C = \frac{\sigma_C}{E} - \mu \times \frac{\sigma_L}{E}$$

$$= 2 \times \frac{\sigma_L}{E} - \mu \times \frac{\sigma_L}{E}$$

$$= \frac{\sigma_L}{E} \times (2 - \mu)$$



$$\boxed{\text{i.e.,} \quad \varepsilon_C = \frac{\delta d}{d} = \frac{p \times d}{4 \times t \times E} \times (2 - \mu)} \text{..............................} (3)$$

Note: Let δd be the change in diameter. Then

$$\varepsilon_c = \frac{final\ circumference - original\ circumference}{original\ circumference}$$

$$= \left[\frac{\pi(d + \delta d) - \pi d}{\pi d}\right] = \frac{\delta d}{d}$$

Longitudinal strain, ε_L :

$$\varepsilon_L = \frac{\sigma_L}{E} - \mu \times \frac{\sigma_C}{E}$$

$$= \frac{\sigma_L}{E} - \mu \times \frac{(2\times\sigma_L)}{E} = \frac{\sigma_L}{E} \times (1 - 2\times\mu)$$

i.e., $$\varepsilon_L = \frac{\delta l}{L} = \frac{p\times d}{4\times t\times E} \times (1-2\times\mu) \quad \text{.............................} (4)$$

VOLUMETRIC STRAIN, $\frac{\delta v}{V}$

Change in volume = δV = final volume – original volume

original volume = V = area of cylindrical shell × length

$$= \frac{\pi d^2}{4} L$$

final volume = final area of cross section × final length

$$= \frac{\pi}{4}[d+\delta d]^2 \times [L+\delta L]$$

$$= \frac{\pi}{4}\left[d^2+(\delta d)^2+2d\,\delta d\right] \times [L+\delta L]$$

$$= \frac{\pi}{4}\left[d^2 L+(\delta d)^2 L+2Ld\,\delta d+d^2\,\delta L+(\delta d)^2\,\delta L+2d\,\delta d\,\delta L\right]$$

neglecting the smaller quantities such as $(\delta d)^2 L, (\delta d)^2\,\delta L$ *and* $2d\,\delta d\,\delta L$

$$\textit{Final volume} = \frac{\pi}{4}\left[d^2 L+2Ld\,\delta d+d^2\delta L\right]$$

$$\textit{change in volume } \delta V = \frac{\pi}{4}\left[d^2 L+2Ld\,\delta d+d^2\delta L\right] - \frac{\pi}{4}[d]^2 L$$

$$\delta V = \frac{\pi}{4}\left[2Ld\,\delta d+d^2\delta L\right]$$

$$\frac{dv}{V} = \frac{\frac{\pi}{4}\left[2dL\delta d + \delta L d^2\right]}{\frac{\pi}{4} \times d^2 \times L}$$

$$= \frac{\delta L}{L} + 2 \times \frac{\delta d}{d}$$

$$\frac{\mathbf{dV}}{\mathbf{V}} = \varepsilon_L + 2 \times \varepsilon_C$$

$$= \frac{p \times d}{4 \times t \times E}(1 - 2 \times \mu) + 2 \times \frac{p \times d}{4 \times t \times E}(2 - \mu)$$

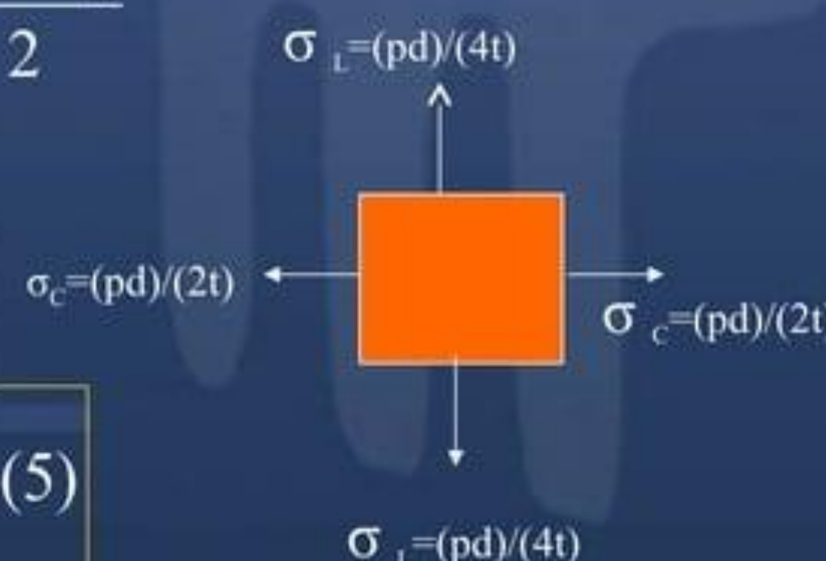
$$\boxed{\text{i.e.,} \quad \frac{dv}{V} = \frac{p \times d}{4 \times t \times E}(5 - 4 \times \mu) \ldots\ldots\ldots\ldots\ldots\ldots (5)}$$

Maximum Shear stress :

There are two principal stresses at any point, viz., Circumferential and longitudinal. Both these stresses are normal and act perpendicular to each other.

$$\therefore \text{ Maximum Shear stress, } \tau_{max} = \frac{\sigma_C - \sigma_L}{2}$$

$$= \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2}$$



$$\boxed{\text{i.e.,} \qquad \tau_{max} = \frac{pd}{8t} \text{.....................(5)}}$$

Maximum Shear stress :

$$\therefore \text{ Maximum Shear stress, } \tau_{max} = \frac{\sigma_C - \sigma_L}{2}$$

$$= \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2}$$

$$\text{i.e.,} \qquad \tau_{max} = \frac{pd}{8t} \text{.....................}(5)$$

ILLUSTRATIVE PROBLEMS

PROBLEM 1:

A thin cylindrical shell is 3m long and 1m in internal diameter. It is subjected to internal pressure of 1.2 MPa. If the thickness of the sheet is 12mm, find the circumferential stress, longitudinal stress, changes in diameter, length and volume . Take E=200 GPa and μ= 0.3.

SOLUTION:

1. Circumferential stress, σ_C:

$$\sigma_C = (p \times d) / (2 \times t)$$
$$= (1.2 \times 1000) / (2 \times 12)$$
$$= \underline{50\ N/mm^2} \underline{= 50\ MPa}\ (Tensile).$$

2. Longitudinal stress, σ_L:

$$\sigma_L = (p \times d) / (4 \times t)$$
$$= \sigma_C / 2 = 50/2$$
$$= \underline{25\ N/mm^2} \underline{= 25\ MPa}\ (Tensile).$$

3. Circumferential strain, ε_c:

$$\varepsilon_c = \frac{(p \times d)}{(4 \times t)} \times \frac{(2-\mu)}{E}$$

$$= \frac{(1.2 \times 1000)}{(4 \times 12)} \times \frac{(2-0.3)}{200 \times 10^3}$$

$$= \underline{2.125 \times 10^{-04}} \text{ (Increase)}$$

Change in diameter, $\delta d = \varepsilon_c \times d$

$= 2.125 \times 10^{-04} \times 1000 = \underline{0.2125}$ mm (Increase).

4. Longitudinal strain, ε_L:

$$\varepsilon_L = \frac{(p \times d)}{(4 \times t)} \times \frac{(1-2 \times \mu)}{E}$$

$$= \frac{(1.2 \times 1000)}{(4 \times 12)} \times \frac{(1-2 \times 0.3)}{200 \times 10^3}$$

$$= \underline{5 \times 10^{-05}} \text{ (Increase)}$$

Change in length $= \varepsilon_L \times L = 5 \times 10^{-05} \times 3000 = \underline{0.15}$ mm (Increase).

Volumetric strain, $\frac{dv}{V}$:

$$\frac{dv}{V} = \frac{(p \times d)}{(4 \times t) \times E} \times (5 - 4 \times \mu)$$

$$= \frac{(1.2 \times 1000)}{(4 \times 12) \times 200 \times 10^{3}} \times (5 - 4 \times 0.3)$$

$$= 4.75 \times 10^{-4} \text{ (Increase)}$$

$\therefore$ Change in volume, $dv = 4.75 \times 10^{-4} \times V$

$$= 4.75 \times 10^{-4} \times \frac{\pi}{4} \times 1000^{2} \times 3000$$

$$= 1.11919 \times 10^{6} \text{ mm}^{3} = 1.11919 \times 10^{-3} \text{ m}^{3}$$

$$= \underline{1.11919 \text{ Litres}}.$$

A copper tube having 45mm internal diameter and 1.5mm wall thickness is closed at its ends by plugs which are at 450mm apart. The tube is subjected to internal pressure of 3 MPa and at the same time pulled in axial direction with a force of 3 kN. Compute: i) the change in length between the plugs ii) the change in internal diameter of the tube. Take E_{CU} = 100 GPa, and μ_{CU} = 0.3.

SOLUTION:

A] Due to Fluid pressure of 3 MPa:

Longitudinal stress, $\sigma_L = (p \times d) / (4 \times t)$

$$= (3 \times 45) / (4 \times 1.5) = 22.50 \text{ N/mm}^2 = 22.50 \text{ MPa.}$$

$$\text{Long. strain, } \varepsilon_L = \frac{(p \times d)}{4 \times t} \times \frac{(1 - 2 \times \mu)}{E}$$

$$= \frac{22.5 \times (1 - 2 \times 0.3)}{100 \times 10^3} = \underline{9 \times 10^{-5}}$$

Change in length, $\delta_L = \varepsilon_L \times L = 9 \times 10^{-5} \times 450 = \underline{+0.0405 \text{ mm}}$ (increase)

$Pd/4t = 22.5$

$$\text{Circumferential strain } \varepsilon_C = \frac{(p \times d)}{(4 \times t)} \times \frac{(2-\mu)}{E}$$

$$= \frac{22.5 \times (2-0.3)}{100 \times 10^3} = \underline{3.825 \times 10^{-4}}$$

Change in diameter, $\delta_d = \varepsilon_c \times d = 3.825 \times 10^{-4} \times 45$

$= +\ \underline{0.0172\text{ mm}}$ (increase)

B] Due to Pull of 3 kN (P=3kN):

Area of cross section of copper tube, $A_c = \pi \times d \times t$

$= \pi \times 45 \times 1.5 = 212.06\text{ mm}^2$

Longitudinal strain, ε_L = direct stress/E = $\sigma/E = P/(A_c \times E)$

$= 3 \times 10^3/(212.06 \times 100 \times 10^3)$

$= \underline{1.415 \times 10^{-4}}$

Change in length, $\delta_L = \varepsilon_L \times L = 1.415 \times 10^{-4} \times 450 = \underline{+0.0637\text{mm}}$ (increase)

Lateral strain, $\varepsilon_{lat} = -\mu \times$ Longitudinal strain $= -\mu \times \varepsilon_L$

$$= -0.3 \times 1.415 \times 10^{-4} = -4.245 \times 10^{-5}$$

Change in diameter, $\delta_d = \varepsilon_{lat} \times d = -4.245 \times 10^{-5} \times 45$

$$= \underline{-1.91 \times 10^{-3}\ \text{mm}}\ \text{(decrease)}$$

C) Changes due to combined effects:

Change in length $= 0.0405 + 0.0637 = \underline{+0.1042\ \text{mm}}$ (increase)

Change in diameter $= 0.01721 - 1.91 \times 10^{-3} = \underline{+0.0153\ \text{mm}}$ (increase)

PROBLEM 3:

A cylindrical boiler is 800mm in diameter and 1m length. It is required to withstand a pressure of 100m of water. If the permissible tensile stress is $20 N/mm^2$, permissible shear stress is $8 N/mm^2$ and permissible change in diameter is 0.2mm, find the minimum thickness of the metal required. Take E = 200GPa, and $\mu = 0.3$.

SOLUTION:

$$\text{Fluid pressure, } p = 100\text{m of water} = 100 \times 9.81 \times 10^3 \ N/m^2$$
$$= 0.981 N/mm^2 .$$

1. Thickness from Hoop Stress consideration: (Hoop stress is critical than long. Stress)

$$\sigma_C = (p \times d)/(2 \times t)$$
$$20 = (0.981 \times 800)/(2 \times t)$$
$$t = 19.62 \text{ mm}$$

2. Thickness from Shear Stress consideration:

$$\tau_{max} = \frac{(p \times d)}{(8 \times t)}$$

$$8 = \frac{(0.981 \times 800)}{(8 \times t)}$$

$$\therefore t = \underline{12.26mm.}$$

3. Thickness from permissible change in diameter consideration (δd=0.2mm):

$$\frac{\delta d}{d} = \frac{(p \times d)}{(4 \times t)} \times \frac{(2 - \mu)}{E}$$

$$\frac{0.2}{800} = \frac{(0.981 \times 800)}{(4 \times t)} \times \frac{(2 - 0.3)}{200 \times 10^3}$$

$$t = \underline{6.67mm}$$

Therefore, required thickness, t = 19.62 mm.

PROBLEM 4:

A cylindrical boiler has 450mm in internal diameter, 12mm thick and 0.9m long. It is initially filled with water at atmospheric pressure. Determine the pressure at which an additional water of 0.187 liters may be pumped into the cylinder by considering water to be incompressible. Take E = 200 GPa, and $\mu = 0.3$.

SOLUTION:

Additional volume of water, $\delta V = 0.187 \text{ liters} = 0.187\times10^{-3}\ m^3$

$= 187\times10^3\ mm^3$

$$V = \frac{\pi}{4}\times450^2\times(0.9\times10^3) = 143.14\times10^6\ mm^3$$

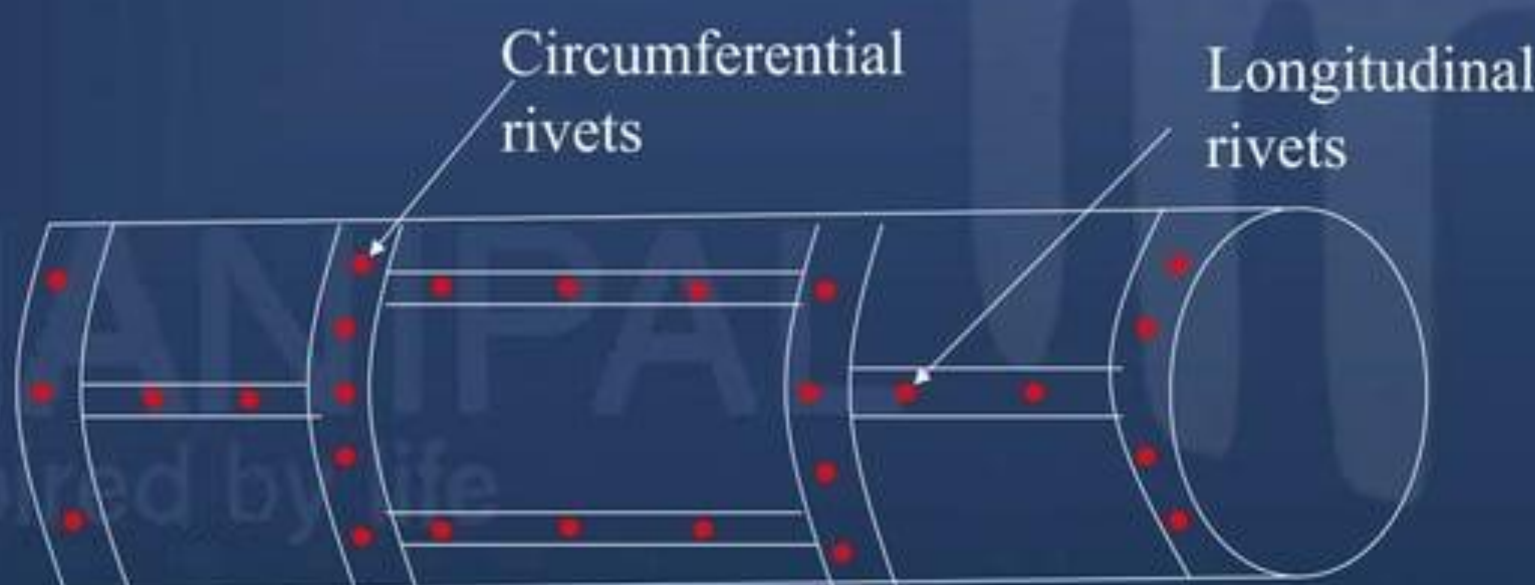
$$\frac{dV}{V} = \frac{p\times d}{4\times t\times E}(5-4\times\mu)$$

$$\frac{187\times10^3}{143.14\times10^6} = \frac{p\times450}{4\times12\times200\times10^3}(5-4\times0.33)$$

Solving, $p = 7.33\ N/mm^2$

JOINT EFFICIENCY

Steel plates of only particular lengths and width are available. Hence whenever larger size cylinders (like boilers) are required, a number of plates are to be connected. This is achieved by using riveting in circumferential and longitudinal directions as shown in figure. Due to the holes for rivets, the net area of cross section decreases and hence the stresses increase.



JOINT EFFICIENCY

The cylindrical shells like boilers are having two types of joints namely Longitudinal and Circumferential joints. Due to the holes for rivets, the net area of cross section decreases and hence the stresses increase. If the efficiencies of these joints are known, the stresses can be calculated as follows.

Let $\eta_L =$ Efficiency of Longitudinal joint

and $\eta_C =$ Efficiency of Circumferential joint.

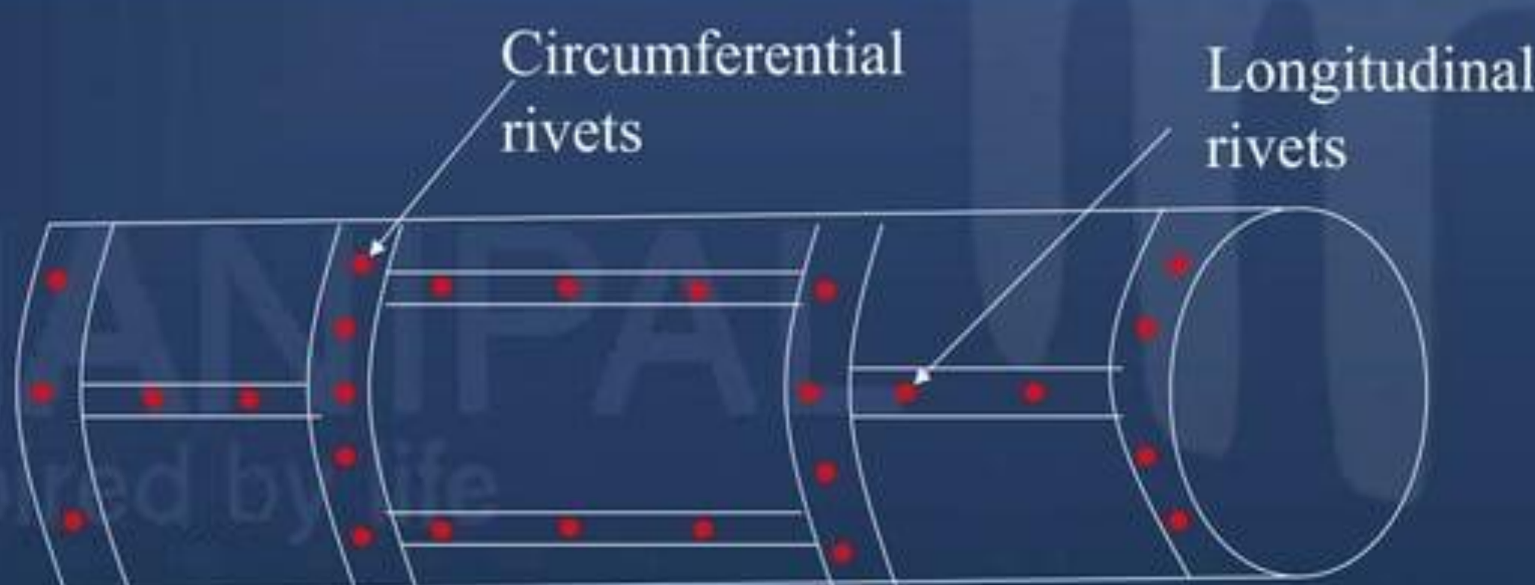
Circumferential stress is given by,

$$\sigma_C = \frac{p \times d}{2 \times t \times \eta_L} \quad \text{.............(1)}$$

Longitudinal stress is given by,

$$\sigma_L = \frac{p \times d}{4 \times t \times \eta_C} \quad \text{.............}(2)$$

Note: In longitudinal joint, the circumferential stress is developed and in circumferential joint, longitudinal stress is developed.



If A is the gross area and A_{eff} is the effective resisting area then,

Efficiency = A_{eff}/A

Bursting force = p L d

Resisting force = $\sigma_c \times A_{eff} = \sigma_c \times \eta_L \times A = \sigma c \times \eta_L \times 2\ t\ L$

Where η_L=Efficiency of Longitudinal joint

Bursting force = Resisting force

$p\ L\ d = \sigma c \times \eta_L \times 2\ t\ L$

$$\sigma_C = \frac{p \times d}{2 \times t \times \eta_L} \quad \text{.............(1)}$$

If η_c=Efficiency of circumferential joint

Efficiency = A_{eff}/A

Bursting force = $(\pi\, d^2/4)p$

Resisting force = $\sigma_L \times A'_{eff} = \sigma_L \times \eta_c \times A' = \sigma_L \times \eta_c \times \pi\, d\, t$

Where η_L=Efficiency of circumferential joint

Bursting force = Resisting force

$$\sigma_L = \frac{p \times d}{4 \times t \times \eta_C} \quad \text{.............(2)}$$

A cylindrical tank of 750mm internal diameter, 12mm thickness and 1.5m length is completely filled with an oil of specific weight 7.85 kN/m^3 at atmospheric pressure. If the efficiency of longitudinal joints is 75% and that of circumferential joints is 45%, find the pressure head of oil in the tank. Also calculate the change in volume. Take permissible tensile stress of tank plate as 120 MPa and E = 200 GPa, and $\mu = 0.3$.

SOLUTION:

Let p = max permissible pressure in the tank.

Then we have, $\sigma_L = (p \times d)/(4 \times t)\ \eta_C$

$$120 = (p \times 750)/(4 \times 12)\ 0.45$$

$$p = 3.456 \text{ MPa.}$$

Also, $\sigma_C = (p \times d)/(2 \times t)\ \eta_L$

$$120 = (p \times 750)/(2 \times 12)\ 0.75$$

$$p = 2.88 \text{ MPa.}$$

Max permissible pressure in the tank, p = 2.88 MPa.

$$\text{Vol. Strain, } \frac{dv}{V} = \frac{(p \times d)}{(4 \times t \times E)} \times (5 - 4 \times \mu)$$

$$= \frac{(2.88 \times 750)}{(4 \times 12 \times 200 \times 10^3)} \times (5 - 4 \times 0.3) = 8.55 \times 10^{-4}$$

$$dv = 8.55 \times 10^{-4} \times V = 8.55 \times 10^{-4} \times \frac{\pi}{4} \times 750^2 \times 1500 = 0.567 \times 10^6 \text{ mm}^3.$$

$$= 0.567 \times 10^{-3} \text{ m}^3 = \underline{0.567} \text{ litres.}$$

A boiler shell is to be made of 15mm thick plate having a limiting tensile stress of 120 N/mm^2. If the efficiencies of the longitudinal and circumferential joints are 70% and 30% respectively determine;

i) The maximum permissible diameter of the shell for an internal pressure of 2 N/mm^2.

(ii) Permissible intensity of internal pressure when the shell diameter is 1.5m.

SOLUTION:

(i) To find the maximum permissible diameter of the shell for an internal pressure of 2 N/mm^2:

a) Let limiting tensile stress = Circumferential stress = σ_c = 120N/mm^2.

$$\text{i. e.,} \quad \sigma_c = \frac{p \times d}{2 \times t \times \eta_L}$$

$$120 = \frac{2 \times d}{2 \times 15 \times 0.7}$$

$$d = 1260 \text{ mm}$$

b) Let limiting tensile stress = Longitudinal stress = σ_L = 120N/mm^2.

$$\text{i. e.,} \quad \sigma_L = \frac{p \times d}{4 \times t \times \eta_C}$$

$$120 = \frac{2 \times d}{4 \times 15 \times 0.3} \qquad . \qquad d = 1080 \text{ mm}$$

The maximum diameter of the cylinder in order to satisfy both the conditions = 1080 mm.

(ii) To find the permissible pressure for an internal diameter of 1.5m: (d=1.5m=1500mm)

a) Let limiting tensile stress = Circumferential stress = σ_c = 120N/mm².

$$\text{i. e.,} \quad \sigma_c = \frac{p \times d}{2 \times t \times \eta_L}$$

$$120 = \frac{p \times 1500}{2 \times 15 \times 0.7}$$

$$p = 1.68 \text{ N/mm}^2.$$

b) Let limiting tensile stress = Longitudinal stress = σ_L = 120N/mm².

$$\text{i. e.,} \quad \sigma_L = \frac{p \times d}{4 \times t \times \eta_C}$$

$$120 = \frac{p \times 1500}{4 \times 15 \times 0.3}$$

$$p = 1.44 \text{ N/mm}^2.$$

The maximum permissible pressure = 1.44 N/mm².

PROBLEMS FOR PRACTICE

PROBLEM 1:

Calculate the circumferential and longitudinal strains for a boiler of 1000mm diameter when it is subjected to an internal pressure of 1MPa. The wall thickness is such that the safe maximum tensile stress in the boiler material is 35 MPa. Take E=200GPa and μ= 0.25.

(Ans: ε_C=0.0001531, ε_L=0.00004375)

PROBLEM 2:

A water main 1m in diameter contains water at a pressure head of 120m. Find the thickness of the metal if the working stress in the pipe metal is 30 MPa. Take unit weight of water = 10 kN/m^3.

(Ans: t=20mm)

PROBLEM 3:

A gravity main 2m in diameter and 15mm in thickness. It is subjected to an internal fluid pressure of 1.5 MPa. Calculate the hoop and longitudinal stresses induced in the pipe material. If a factor of safety 4 was used in the design, what is the ultimate tensile stress in the pipe material?

(Ans: σ_C=100 MPa, σ_L=50 MPa, σ_U=400 MPa)

PROBLEM 4:

At a point in a thin cylinder subjected to internal fluid pressure, the value of hoop strain is 600×10^{-4} (tensile). Compute hoop and longitudinal stresses. How much is the percentage change in the volume of the cylinder? Take E=200GPa and μ= 0.2857.

(Ans: σ_C=140 MPa, σ_L=70 MPa, %age change=0.135%.)

PROBLEM 5:

A cylindrical tank of 750mm internal diameter and 1.5m long is to be filled with an oil of specific weight 7.85 kN/m3 under a pressure head of 365 m. If the longitudinal joint efficiency is 75% and circumferential joint efficiency is 40%, find the thickness of the tank required. Also calculate the error of calculation in the quantity of oil in the tank if the volumetric strain of the tank is neglected. Take permissible tensile stress as 120 MPa, E=200GPa and μ= 0.3 for the tank material. (Ans: t=12 mm, error=0.085%.)

THICK CYLINDERS

INTRODUCTION:

The thickness of the cylinder is large compared to that of thin cylinder.

i. e., in case of thick cylinders, the metal thickness ‘t’ is more than ‘d/20’, where ‘d’ is the internal diameter of the cylinder.

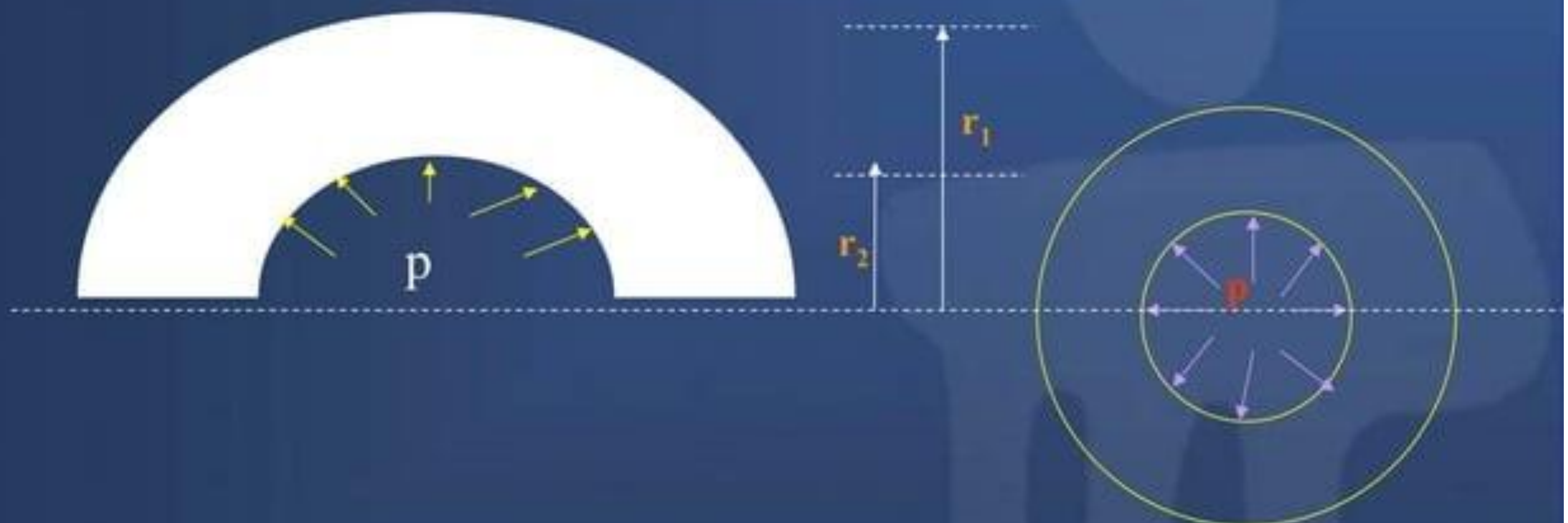
Magnitude of radial stress (p_r) is large and hence it cannot be neglected. The circumferential stress is also not uniform across the cylinder wall. The radial stress is compressive in nature and circumferential and longitudinal stresses are tensile in nature. Radial stress and circumferential stresses are computed by using ‘Lame’s equations’.

LAME'S EQUATIONS (Theory) :

ASSUMPTIONS:

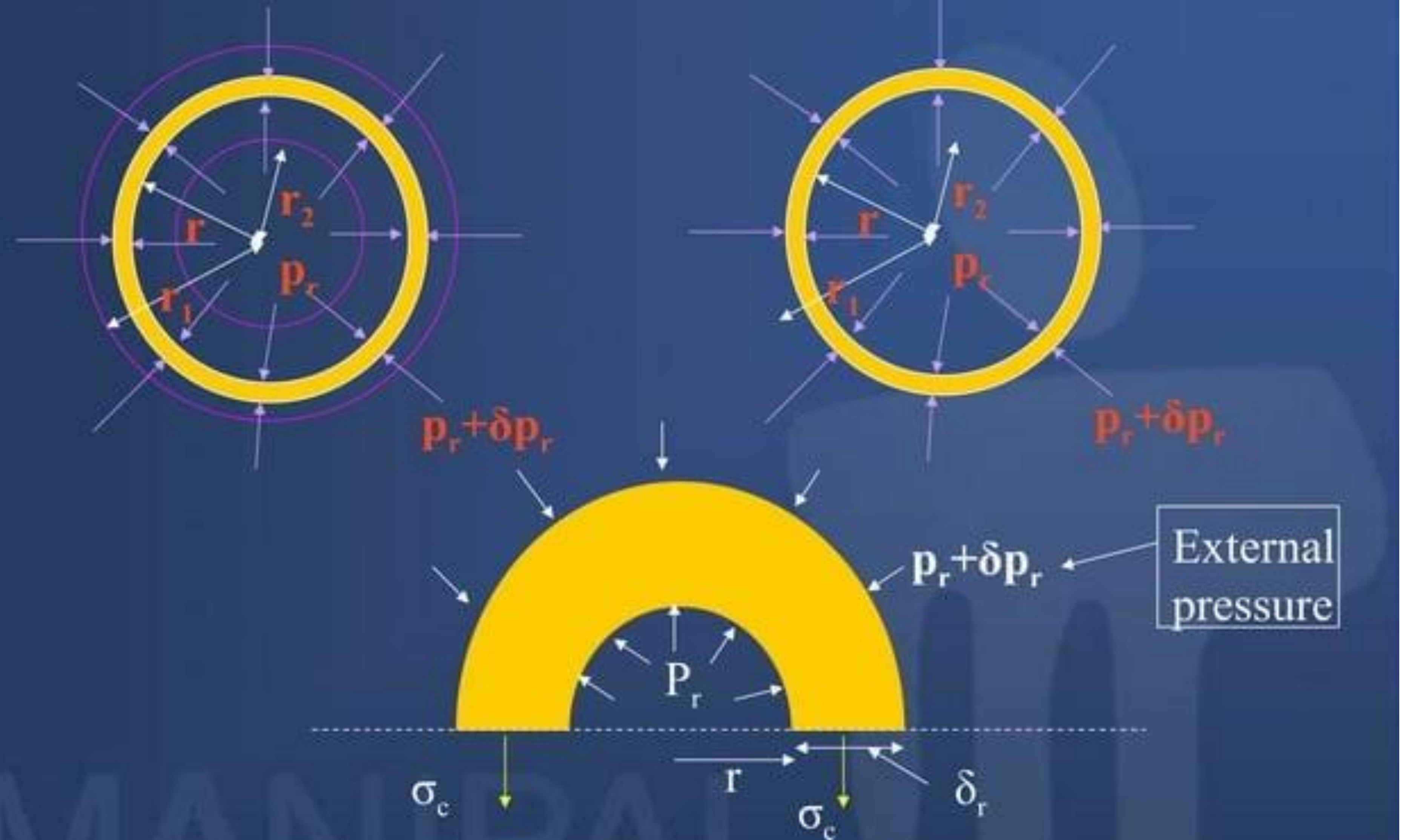
1. Plane sections of the cylinder normal to its axis remain plane and normal even under pressure.
2. Longitudinal stress (σ_L) and longitudinal strain (ε_L) remain constant throughout the thickness of the wall.
3. Since longitudinal stress (σ_L) and longitudinal strain (ε_L) are constant, it follows that the difference in the magnitude of hoop stress and radial stress (p_r) at any point on the cylinder wall is a constant.
4. The material is homogeneous, isotropic and obeys Hooke's law. (The stresses are within proportionality limit).

LAME'S EQUATIONS FOR RADIAL PRESSURE AND CIRCUMFERENTIAL STRESS



Consider a thick cylinder of external radius r_1 and internal radius r_2, containing a fluid under pressure 'p' as shown in the fig.

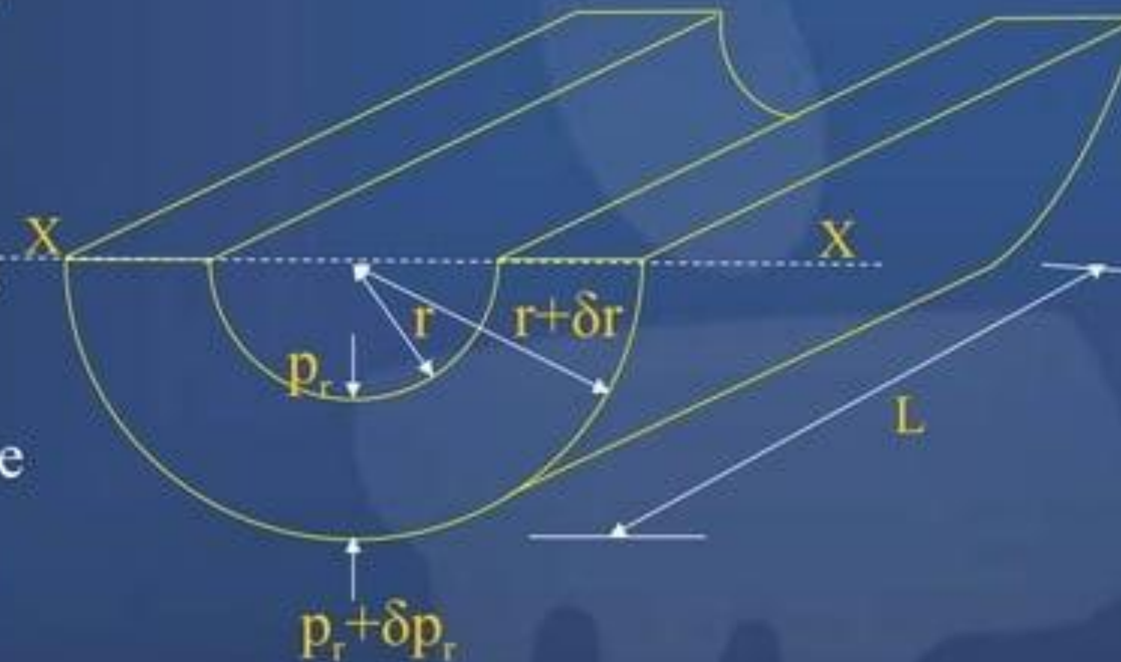
Let 'L' be the length of the cylinder.



Consider an elemental ring of radius 'r' and thickness 'δ_r' as shown in the above figures. Let p_r and $(p_r+ \delta p_r)$ be the intensities of radial pressures at inner and outer faces of the ring.

Consider the longitudinal section XX of the ring as shown in the fig.

The bursting force is evaluated by considering the projected area, '$2 \times r \times L$' for the inner face and '$2 \times (r+\delta_r) \times L$' for the outer face .



The net bursting force, $P = p_r \times 2 \times r \times L - (p_r+\delta p_r) \times 2 \times (r+\delta_r) \times L$

$$=(-p_r \times \delta_r - r \times \delta p_r - \delta p_r \times \delta_r)\, 2L$$

Bursting force is resisted by the hoop tensile force developing at the level of the strip i.e.,

$$F_r = \sigma_c \times 2 \times \delta_r \times L$$

Thus, for equilibrium, $P = F_r$

$$(-p_r \times \delta_r - r \times \delta p_r - \delta p_r \times \delta_r)\, 2L = \sigma_c \times 2 \times \delta_r \times L$$

$$-pr \times \delta r - r \times \delta p_r - \delta p_r \times \delta_r = \sigma_c \times \delta r$$

Neglecting products of small quantities, (i.e., $\delta p_r \times \delta r$)

$$\sigma_c = -p_r - (r \times \delta p_r)/\delta_r \text{ ....(1)}$$

Longitudinal strain is constant. Hence we have,

$$\varepsilon_L = \frac{\sigma_L}{E} - \mu \times \frac{\sigma_C}{E} + \mu \times \frac{p_r}{E} = \text{constant}$$

Since P_r is compressive

$$\varepsilon_L = \frac{\sigma_L}{E} - \frac{\mu}{E}(\sigma_C - p_r) = \text{constant}$$

since σ_L, E and μ are constants $(\sigma_c - P_r)$ should be constant . Let it be equal to 2a. Thus

$\sigma_c - p_r = 2a,$

i.e., $\sigma_c = p_r + 2a,$ ……………….(2)

From (1), $p_r + 2a = - p_r - (r \times \delta p_r) / \delta_r$

i. e., $$2(p_r + a) = -r \times \frac{\delta p_r}{\delta_r}$$

$$-2 \times \frac{\delta_r}{r} = \frac{\delta p_r}{(p_r + a)} \text{ ....(3)}$$

Integrating, $(-2 \times \log_e r) + c = \log_e (p_r + a)$

Where c is constant of integration. Let it be taken as $\log_e b$, where

'b' is another constant.

Thus, $\log_e (p_r + a) = -2 \times \log_e r + \log_e b = - \log_e r^2 + \log_e b = \log_e \frac{b}{r^2}$

i.e., $p_r + a = \frac{b}{r^2}$ or, radial stress, $p_r = \frac{b}{r^2} - a$ ....(4)

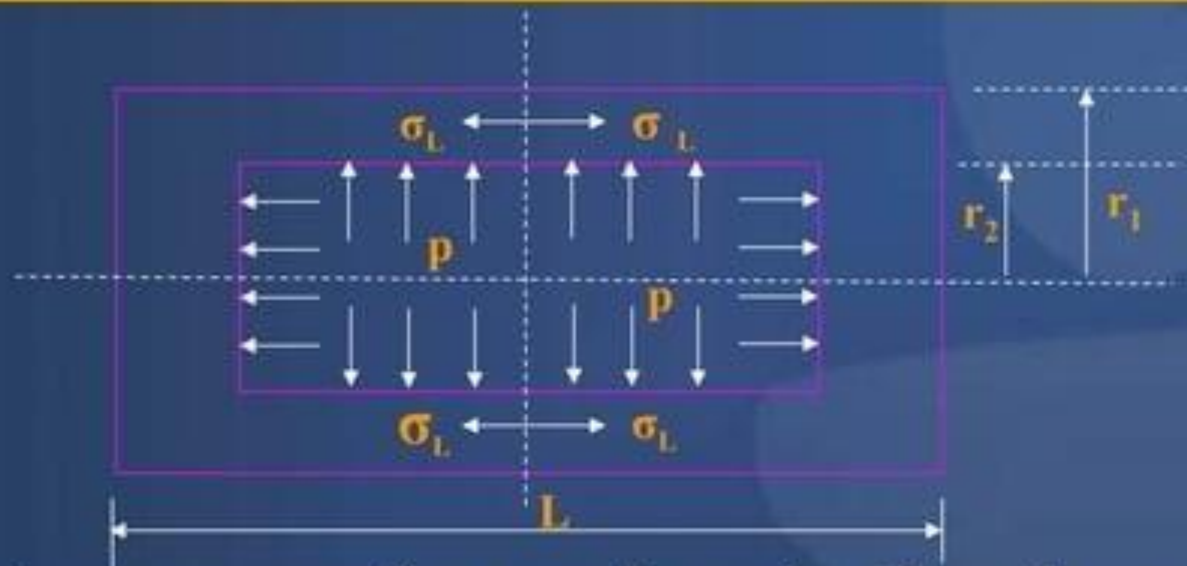
Substituting it in equation 2, we get

Hoop stress, $\sigma_c = p_r + 2a = \frac{b}{r^2} - a + 2a$

i.e., $\sigma_c = \frac{b}{r^2} + a$ ....(5)

The equations (4) & (5) are known as "Lame's Equations" for radial pressure and hoop stress at any specified point on the cylinder wall. Thus, $r_1 \leq r \leq r_2$.

ANALYSIS FOR LONGITUDINAL STRESS



Consider a transverse section near the end wall as shown in the fig.

Bursting force, $P = \pi \times r_2^2 \times p$

Resisting force is due to longitudinal stress 'σ_L'.

i.e., $F_L = \sigma_L \times \pi \times (r_1^2 - r_2^2)$

For equilibrium, $F_L = P$

$\sigma_L \times \pi \times (r_1^2 - r_2^2) = \pi \times r_2^2 \times p$

Therefore, longitudinal stress, $\sigma_L = \dfrac{p \times r_2^2}{(r_1^2 - r_2^2)}$ (Tensile)

NOTE:

1. Variations of Hoop stress and Radial stress are parabolic across the cylinder wall.
2. At the inner edge, the stresses are maximum.
3. The value of 'Permissible or Maximum Hoop Stress' is to be considered on the inner edge.
4. The maximum shear stress (σ_{max}) and Hoop, Longitudinal and radial strains (ε_c, ε_L, ε_r) are calculated as in thin cylinder but separately for inner and outer edges.

ILLUSTRATIVE PROBLEMS

PROBLEM 1:

A thick cylindrical pipe of external diameter 300mm and internal diameter 200mm is subjected to an internal fluid pressure of 20N/mm^2 and external pressure of 5 N/mm^2. Determine the maximum hoop stress developed and draw the variation of hoop stress and radial stress across the thickness. Show at least four points for each case.

SOLUTION:

External diameter = 300mm. External radius, r_1=150mm.

Internal diameter = 200mm. Internal radius, r_2=100mm.

Lame's equations:

For Hoop stress, $$\sigma_c = \frac{b}{r^2} + a \quad ....(1)$$

For radial stress, $$p_r = \frac{b}{r^2} - a \quad ....(2)$$

Boundary conditions:

At r =100mm (on the inner face), radial pressure = 20N/mm²

$$\text{i.e.,} \quad 20 = \frac{b}{100^2} - a \text{ ....(3)}$$

Similarly, at r =150mm (on the outer face), radial pressure = 5N/mm²

$$\text{i.e.,} \quad 5 = \frac{b}{150^2} - a \text{ ....(4)}$$

Solving equations (3) & (4), we get $a = 7$, $b = 2{,}70{,}000$.

Lame's equations are, for Hoop stress, $\sigma_c = \frac{2{,}70{,}000}{r^2} + 7$ ....(5)

For radial stress, $p_r = \frac{2{,}70{,}000}{r^2} - 7$ ....(6)

To draw variations of Hoop stress & Radial stress :

At r =100mm (on the inner face),

$$\text{Hoop stress, } \sigma_c = \frac{2,70,000}{100^2} + 7 = 34 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2,70,000}{100^2} - 7 = 20 \text{ MPa (Comp)}$$

At r =120mm,

$$\text{Hoop stress, } \sigma_c = \frac{2,70,000}{120^2} + 7 = 25.75 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2,70,000}{120^2} - 7 = 11.75 \text{ MPa (Comp)}$$

At r =135mm,

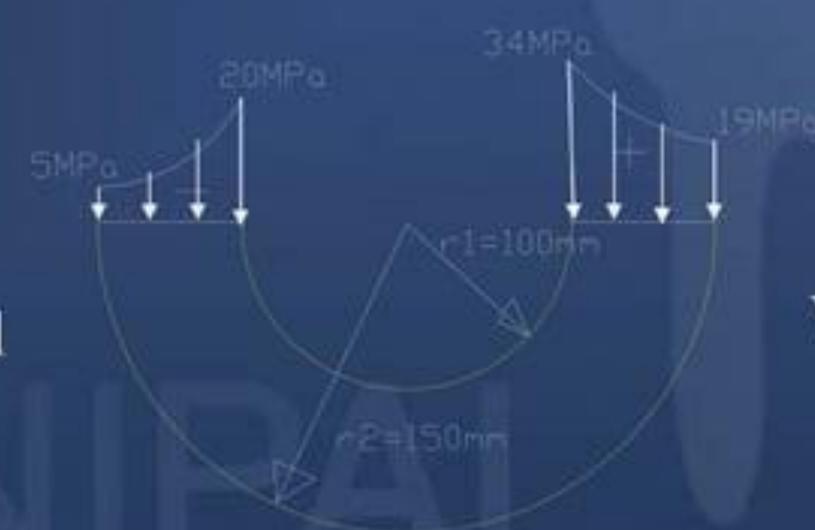
$$\text{Hoop stress, } \sigma_c = \frac{2,70,000}{135^2} + 7 = 21.81 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2,70,000}{135^2} - 7 = 7.81 \text{ MPa (Comp)}$$

At $r = 150\text{mm}$,

$$\text{Hoop stress, } \sigma_c = \frac{2{,}70{,}000}{150^2} + 7 = 19 \text{ MPa (Tensile)}$$

$$\text{Radial stress, } p_r = \frac{2{,}70{,}000}{150^2} - 7 = 5 \text{ MPa (Comp)}$$



Variation of Radial Stress –Comp (Parabolic)

Variation of Hoop Stress-Tensile (Parabolic)

Variation of Hoop stress & Radial stress

PROBLEM 2:

Find the thickness of the metal required for a thick cylindrical shell of internal diameter 160mm to withstand an internal pressure of 8 N/mm^2. The maximum hoop stress in the section is not to exceed 35 N/mm^2.

SOLUTION:

Internal radius, r_2=80mm.

Lame's equations are,

for Hoop Stress, $\sigma_c = \frac{b}{r^2} + a$ ....(1)

for Radial stress, $p_r = \frac{b}{r^2} - a$ ....(2)

Boundary conditions are,

at r = 80mm, radial stress $p_r = 8\ \text{N/mm}^2$,

and Hoop stress, $\sigma_C = 35\ \text{N/mm}^2$. ($\because$ Hoop stress is max on inner face)

i.e.,

$$8 = \frac{b}{80^2} - a \quad \text{.................}(3)$$

$$35 = \frac{b}{80^2} + a \quad \text{.................}(4)$$

Solving equations (3) & (4), we get a = 13.5, b = 1,37,600.

$\therefore$ Lame's equations are,

$$\sigma_c = \frac{1{,}37{,}600}{r^2} + 13.5 \quad \text{............}(5)$$

and

$$p_r = \frac{1{,}37{,}600}{r^2} - 13.5 \quad \text{............}(6)$$

On the outer face, pressure = 0.

i.e., $p_r = 0$ at $r = r_1$.

$$\therefore \quad 0 = \frac{1{,}37{,}600}{r_1^2} - 13.5$$

$$\therefore \quad r_1 = \underline{100.96\text{mm.}}$$

$$\therefore \quad \text{Thickness of the metal} = r_1 - r_2 = \underline{20.96\text{mm.}}$$

PROBLEM 3:

A thick cylindrical pipe of outside diameter 300mm and internal diameter 200mm is subjected to an internal fluid pressure of 14 N/mm^2. Determine the maximum hoop stress developed in the cross section. What is the percentage error if the maximum hoop stress is calculated by the equations for thin cylinder?

SOLUTION:

Internal radius, r_2=100mm.

External radius, r_1=150mm

Lame's equations:

For Hoop stress, $\sigma_c = \frac{b}{r^2} + a$ ....(1)

For radial pressure, $p_r = \frac{b}{r^2} - a$ ....(2)

Boundary conditions:

At x =100mm $\qquad P_r = 14\text{N/mm}^2$

$$\text{i.e.,} \quad 14 = \frac{b}{100^2} - a \quad \text{.................(1)}$$

Similarly, at x =150mm $\qquad P_r = 0$

$$\text{i.e.,} \quad 0 = \frac{b}{150^2} - a \quad \text{.................(2)}$$

Solving, equations (1) & (2), we get a =11.2, b = 2,52,000.

$$\therefore \text{Lame's equation for Hoop stress, } \sigma_r = \frac{22{,}500}{r^2} + 11.2 \quad \text{...........(3)}$$

Max hoop stress on the inner face (where x=100mm):

$$\sigma_{max} = \frac{252000}{100^2} + 11.2 = \underline{36.4\ MPa.}$$

By thin cylinder formula, $\sigma_{max} = \dfrac{p \times d}{2 \times t}$

where $D = 200mm,\ t = 50mm$ and $p = 14MPa.$

$$\therefore \sigma_{max} = \frac{14 \times 200}{2 \times 50} = \underline{28MPa.}$$

$$\text{Percentage error} = \left(\frac{36.4 - 28}{36.4}\right) \times 100 = \underline{23.08\%}.$$

PROBLEM 4:

The principal stresses at the inner edge of a cylindrical shell are 81.88 MPa (T) and 40MPa (C). The internal diameter of the cylinder is 180mm and the length is 1.5m. The longitudinal stress is 21.93 MPa (T). Find,

(i) Max shear stress at the inner edge.
(ii) Change in internal diameter.
(iii) Change in length.
(iv) Change in volume.

Take E=200 GPa and μ=0.3.

SOLUTION:

i) Max shear stress on the inner face :

$$\tau_{max} = \frac{\sigma_C - p_r}{2} = \frac{81.88 - (-40)}{2}$$

$$= 60.94 \text{ MPa}$$

ii) Change in inner diameter :

$$\frac{\delta d}{d}=\frac{\sigma_C}{E}-\frac{\mu}{E}\times p_r-\frac{\mu}{E}\times\sigma_L$$

$$=\frac{81.88}{200\times10^3}-\frac{0.3}{200\times10^3}\times21.93-\frac{0.3}{200\times10^3}\times(-40)$$

$$=4.365\times10^{-4}$$

$$\therefore\quad \delta d=\underline{+0.078\text{mm.}}$$

iii) Change in Length :

$$\frac{\delta l}{L}=\frac{\sigma_L}{E}-\frac{\mu}{E}\times p_r-\frac{\mu}{E}\times\sigma_C$$

$$=\frac{21.93}{200\times10^3}-\frac{0.3}{200\times10^3}\times(-40)-\frac{0.3}{200\times10^3}\times81.88$$

$$=46.83\times10^{-6}$$

$$\therefore\quad \delta l=\underline{+0.070\text{mm.}}$$

iv) Change in volume :

$$\frac{\delta V}{V} = \frac{\delta l}{L} + 2 \times \frac{\delta d}{D}$$

$$= 9.198 \times 10^{-4}$$

$$\therefore \quad \delta V = 9.198 \times 10^{-4} \times \left(\frac{\pi \times 180^2 \times 1500}{4}\right)$$

$$= \underline{35.11 \times 10^3} \text{ mm}^3.$$

PROBLEM 5:

ind the max internal pressure that can be allowed into a thick pipe of uter diameter of 300mm and inner diameter of 200mm so that tensile tress in the metal does not exceed 16 MPa if, (i) there is no external luid pressure, (ii) there is a fluid pressure of 4.2 MPa.

SOLUTION:

External radius, r_1=150mm.

Internal radius, r_2=100mm.

Case (i) – When there is no external fluid pressure:

Boundary conditions:

At r=100mm , $\sigma_c = 16 N/mm^2$

At r=150mm , $P_r = 0$

i.e., $16 = \frac{b}{100^2} + a$ ....(1)

$$0 = \frac{b}{150^2} - a \text{ ....(2)}$$

Solving we get, $a = 4.92$ & $b = 110.77 \times 10^3$

so that $\sigma_c = \frac{110.77 \times 10^3}{r^2} + 4.92$ ....(3)

$$p_r = \frac{110.77 \times 10^3}{r^2} - 4.92 \text{ ....(4)}$$

Fluid pressure on the inner face where $r = 100\text{mm}$,

$$p_r = \frac{110.77 \times 10^3}{100^2} - 4.92 = \underline{6.16 \text{ MPa.}}$$

Case (ii) – When there is an external fluid pressure of 4.2 MPa:

Boundary conditions:

At r=100mm , $\sigma_c = 16\ N/mm^2$

At r=150mm , $p_r = 4.2$ MPa.

i.e., $$16 = \frac{b}{100^2} + a \quad \text{..................(1)}$$

$$4.2 = \frac{b}{150^2} - a \quad \text{..................(2)}$$

Solving we get, a = 2.01 & $b = 139.85 \times 10^3$

so that $$\sigma_r = \frac{139.85 \times 10^3}{r^2} + 2.01 \quad \text{..................(3)}$$

$$p_r = \frac{139.85 \times 10^3}{r^2} - 2.01 \quad \text{..................(4)}$$

Fluid pressure on the inner face where $r = 100\text{mm}$,

$$p_r = \frac{139.85 \times 10^3}{100^2} - 2.01 = \underline{11.975} \text{ MPa.}$$

PROBLEMS FOR PRACTICE

PROBLEM 1:

A pipe of 150mm internal diameter with the metal thickness of 50mm transmits water under a pressure of 6 MPa. Calculate the maximum and minimum intensities of circumferential stresses induced.

(Ans: 12.75 MPa, 6.75 MPa)

PROBLEM 2:

Determine maximum and minimum hoop stresses across the section of a pipe of 400mm internal diameter and 100mm thick when a fluid under a pressure of $8 N/mm^2$ is admitted. Sketch also the radial pressure and hoop stress distributions across the thickness.

(Ans: $\sigma_{max}=20.8 N/mm^2$, $\sigma_{min}=12.8 N/mm^2$)

PROBLEM 3:

A thick cylinder with external diameter 240mm and internal diameter 'D' is subjected to an external pressure of 50 MPa. Determine the diameter 'D' if the maximum hoop stress in the cylinder is not to exceed 200 MPa.

(Ans: 169.7 mm)

PROBLEM 4:

A thick cylinder of 1m inside diameter and 7m long is subjected to an internal fluid pressure of 40 MPa. Determine the thickness of the cylinder if the maximum shear stress in the cylinder is not to exceed 65 MPa. What will be the increase in the volume of the cylinder? E=200 GPa, μ=0.3. (Ans: t=306.2mm, $\delta v = 5.47 \times 10^{-3} m^3$)

PROBLEM 5:

A thick cylinder is subjected to both internal and external pressure. The internal diameter of the cylinder is 150mm and the external diameter is 200mm. If the maximum permissible stress in the cylinder is 20 N/mm^2 and external radial pressure is 4 N/mm^2, determine the intensity of internal radial pressure. (Ans: 10.72 N/mm^2)

WISH YOU ALL GOOD LUCK