(20A01504a) Structural Analysis-II

- **PREPARED BY**

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COURSE OUTCOMES

CO 1	To Analyze rigid frames with and without side sway for end moments, shear forces and support reactions using moment distribution method
CO 2	To Analyze beams and portal frames and draw SFD and BMD using kani's method
CO 3	Construct the bending moment diagram for beams and frames using flexibility method
CO 4	Analyze the beams and frames by system stiffness method.
CO 5	To analyze determinate beams using conjugate beam method

(20A01504a) Structural Analysis-II

UNIT I

MOMENT DISTRIBUTION METHOD FOR FRAMES: Analysis of single bay single storey portal frame including side sway –Substitute frame analysis by two cycle method.

UNIT II

KANI`S METHOD: Analysis of continuous beams with and without settlement of supports -Single Bay single storey portal frames with and without side sway.

UNIT III

FLEXIBILITY METHOD: Flexibility methods- Introduction- Application to continuous beams including support settlements-Analysis of Single Bay single storey portal frames without and with side sway.

UNIT IV

STIFFNESS METHOD: Stiffness methods- Introduction-application to continuous beams including support settlements-Analysis of Single Bay single storey portal frames without and with side sway.

UNIT V

CONJUGATE BEAM METHOD: Real beam and conjugate beam, conjugate beam theorems, Analysis of determinate beams of with uniform and variable cross sections using conjugate beam method.

Textbooks:

1. Analysis of structures by Vazrani&Ratwani – Khanna Publications.
2. Theory of structures by Ramamuratam, jain book depot , New Delhi 9th edition 2015

Reference Books:

1. Strength of materials by R.K Bansal, Lakshmi Publications
2. Strength of materials by S.S Bhavikatti, Vikas Publishing house
3. Structural Analysis: A Unified Approach, by D S Prakash Rao, Universities Press
4. Structural analysis by R.S.Khurmi, S.Chand Publications, New Delhi 2020 edition
5. Basic Structural Analysis by K.U.Muthu*et al.*,I.K.International Publishing House Pvt.Ltd 3rd edition 2017
6. Theory of Structures by Gupta S P, G S Pundit and R Gupta, Vol II, Tata McGrawHillPublications company Ltd.

Moment-Distribution Method

- Classical method.
- Used for Beams and Frames.
- Developed by Hardy Cross in 1924.
- Used by Engineers for analysis of small structures.
- It does not involve the solution of many simultaneous equations.

Moment-Distribution Method

- For beams and frames without sidesway, it does not involve the solution of simultaneous equations.

- For frames with sidesway, number of simultaneous equations usually equals the number of independent joint translations.

- In this method, Moment Equilibrium Equations of joints are solved iteratively by considering the moment equilibrium at one joint at a time, while the remaining joints are considered to be restrained.

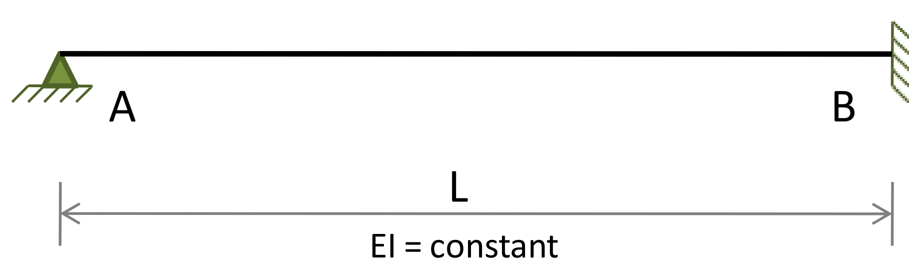
Definitions and Terminology

Sign Convention

- Counterclockwise member end moments are considered positive.
- Clockwise moments on joints are considered positive.

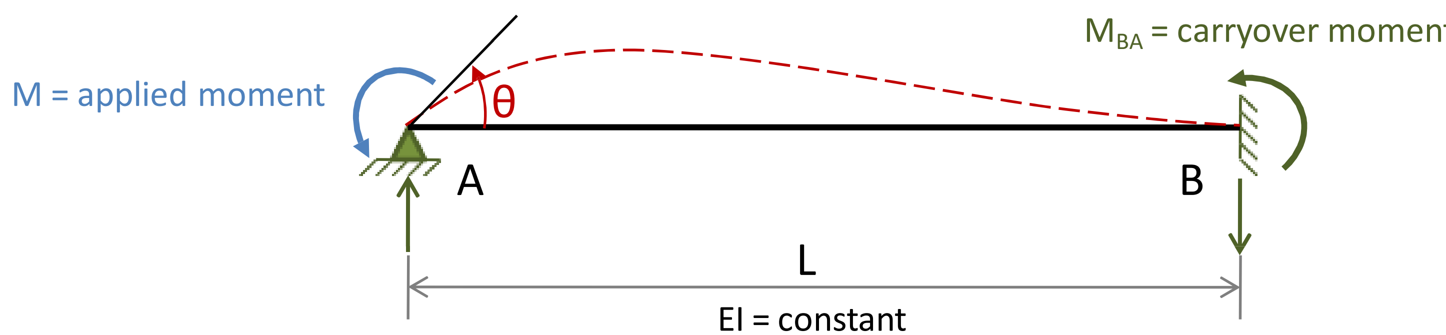
Member Stiffness

- Consider a prismatic beam AB, which is hinged at end A and fixed at end B.



Member Stiffness

If we apply a moment M at the end A, the beam rotates by an angle θ at the hinged end A and develops a moment M_{BA} at the fixed end B, as shown.



The relationship between the applied moment M and the rotation θ can be established using the slope-deflection equation.

Member Stiffness

By substituting $M_{nf} = M$, $\theta_n = \theta$, and $\theta_f = \Psi = FEM_{nf} = 0$ into the slope-deflection equation, we obtain

$$M = \left(\frac{4EI}{L}\right)\theta \quad (1)$$

"The bending stiffness, $\bar{K}$, of a member is defined as the moment that must be applied at an end of the member to cause a unit rotation of that end."

By setting $\theta = 1$ rad in Eq. 1, we obtain the expression for the bending stiffness of the beam of figure to be

$$\bar{K} = \frac{4EI}{L} \quad (2)$$

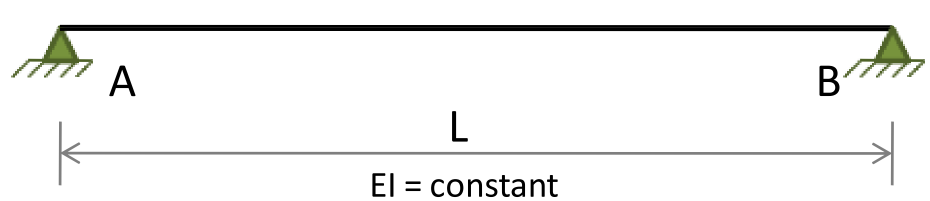
Member Stiffness

when the modulus of elasticity for all the members of a structure is the same (constant), it is usually convenient to work with the relative bending stiffness of members in the analysis.

"The relative bending stiffness, K, of a member is obtained by dividing its bending stiffness, $\overline{K}$ *, by 4E."*

$$K = \frac{\overline{K}}{4E} = \frac{I}{L} \tag{3}$$

- Now suppose that the far end B of the beam is hinged as shown.

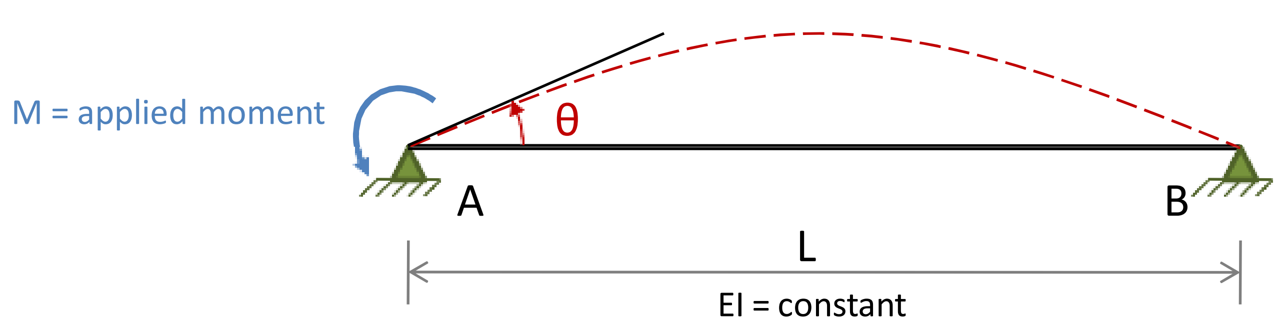


Member Stiffness

The relationship between the applied moment M and the rotation θ of the end A of the beam can now be determined by using the modified slope-deflection equation.

By substituting $M_{rh} = M$, $\theta_r = \theta$, and $\Psi = FEM_{rh} = FEM_{hr} = 0$ into MSDE, we obtain

$$M = \left(\frac{3EI}{L}\right)\theta \qquad (4)$$



Member Stiffness

By setting θ = 1 rad, we obtain the expression for the bending stiffness of the beam of figure to be

$$\bar{K} = \frac{3EI}{L} \tag{5}$$

A comparison of Eq. 2 & Eq. 5 indicates that the stiffness of the beam is reduced by 25% when the fixed support at B is replaced by a hinged support.

The relative bending stiffness of the beam can now be obtained by dividing its bending stiffness by 4E.

$$K = \frac{\bar{K}}{4E} = \frac{3}{4}\left(\frac{I}{L}\right) \tag{6}$$

Member Stiffness

Relationship b/w applied end moment M and the rotation θ

$$M = \begin{cases} \left(\dfrac{4EI}{L}\right)\theta & \text{if far end of member is fixed} \\ \left(\dfrac{3EI}{L}\right)\theta & \text{if far end of member is hinged} \end{cases} \qquad (7)$$

Bending stiffness of a member

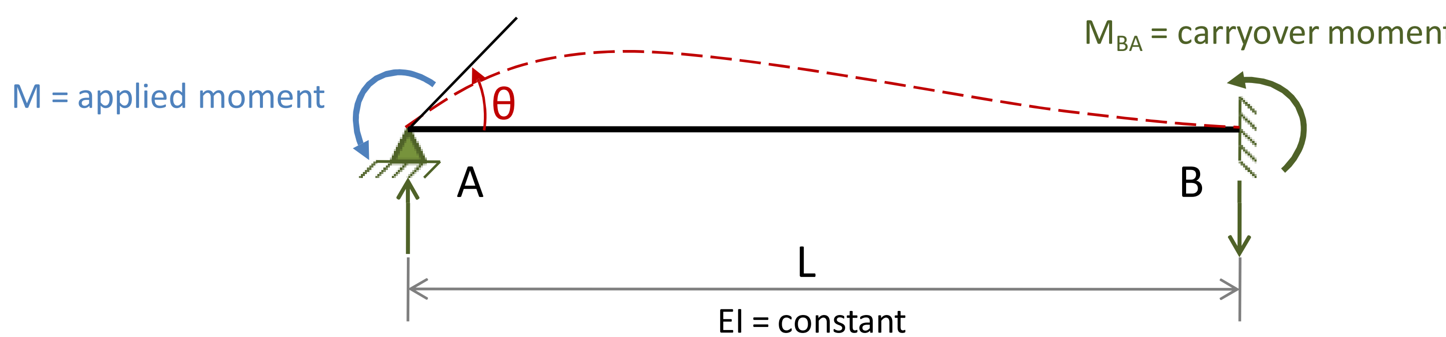
$$\overline{K} = \begin{cases} \dfrac{4EI}{L} & \text{if far end of member is fixed} \\ \dfrac{3EI}{L} & \text{if far end of member is hinged} \end{cases} \qquad (8)$$

Relative bending stiffness of a member

$$K = \begin{cases} \dfrac{I}{L} & \text{if far end of member is fixed} \\ \dfrac{3}{4}\dfrac{I}{L} & \text{if far end of member is hinged} \end{cases} \qquad (9)$$

Carryover Moment

Let us consider again the hinged-fixed beam of Figure.



When a moment M is applied at the hinged end A of the beam, a moment M_{BA} develops at the fixed end B.

The moment M_{BA} is termed the *carryover moment*.

Carryover Moment

To establish the relationship b/w the applied moment M and the carryover moment M_{BA}, we write the slope deflection equation for M_{BA} by substituting $M_{nf} = M_{BA}$, $\theta_f = \theta$, and $\theta_n = \Psi = FEM_{nf} = 0$ into SDE

$$M_{BA} = \left(\frac{2EI}{L}\right)\theta \tag{10}$$

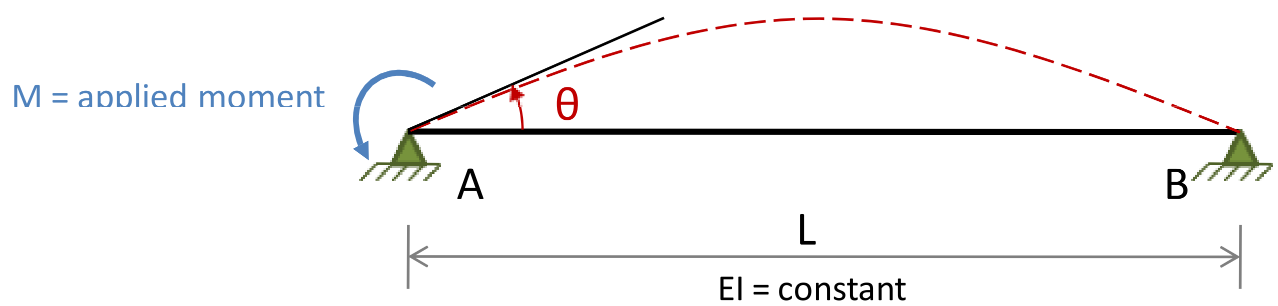
By substituting $\theta = ML/(4EI)$ from Eq. 1 into Eq. 10, we obtain

$$M_{BA} = \frac{M}{2} \tag{11}$$

Eq. 11 indicates, when a moment of magnitude M is applied at the hinged end of the beam, one-half of the applied moment is carried over to the far end, provided that the far end is fixed. The direction of M_{BA} and M is same.

Carryover Moment

When the far end of the beam is hinged as shown, the carryover moment M_{BA} is zero.



$$M_{BA} = \begin{cases} \dfrac{M}{2} & \text{if far end of member is fixed} \\ 0 & \text{if far end of member is hinged} \end{cases} \qquad (12)$$

Carryover Factor (COF)

"The ratio of the carryover moment to the applied moment (M_{BA}/M) is called the carryover factor of the member."

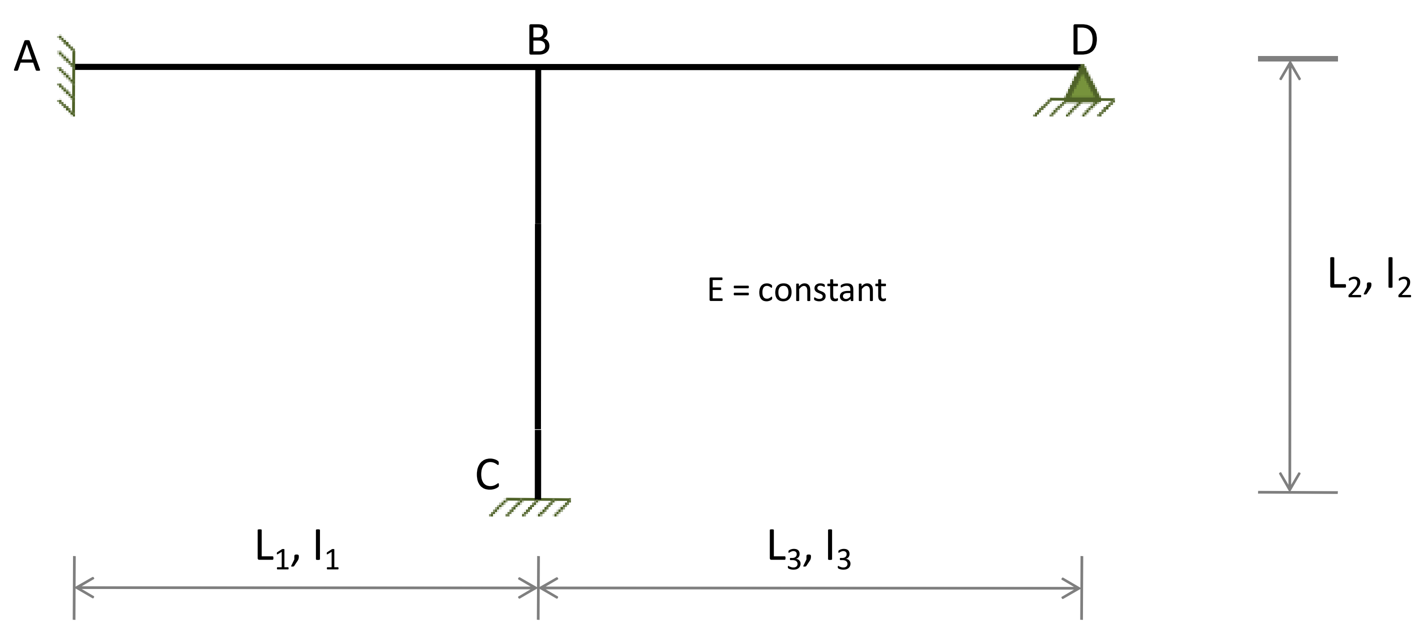
It represents the fraction of the applied moment M that is carried over to the far end of the member. By dividing Eq. 12 by M, we can express the carryover factor (COF) as

$$COF = \begin{cases} \dfrac{1}{2} & \text{if far end of member is fixed} \\ 0 & \text{if far end of member is hinged} \end{cases} \tag{13}$$

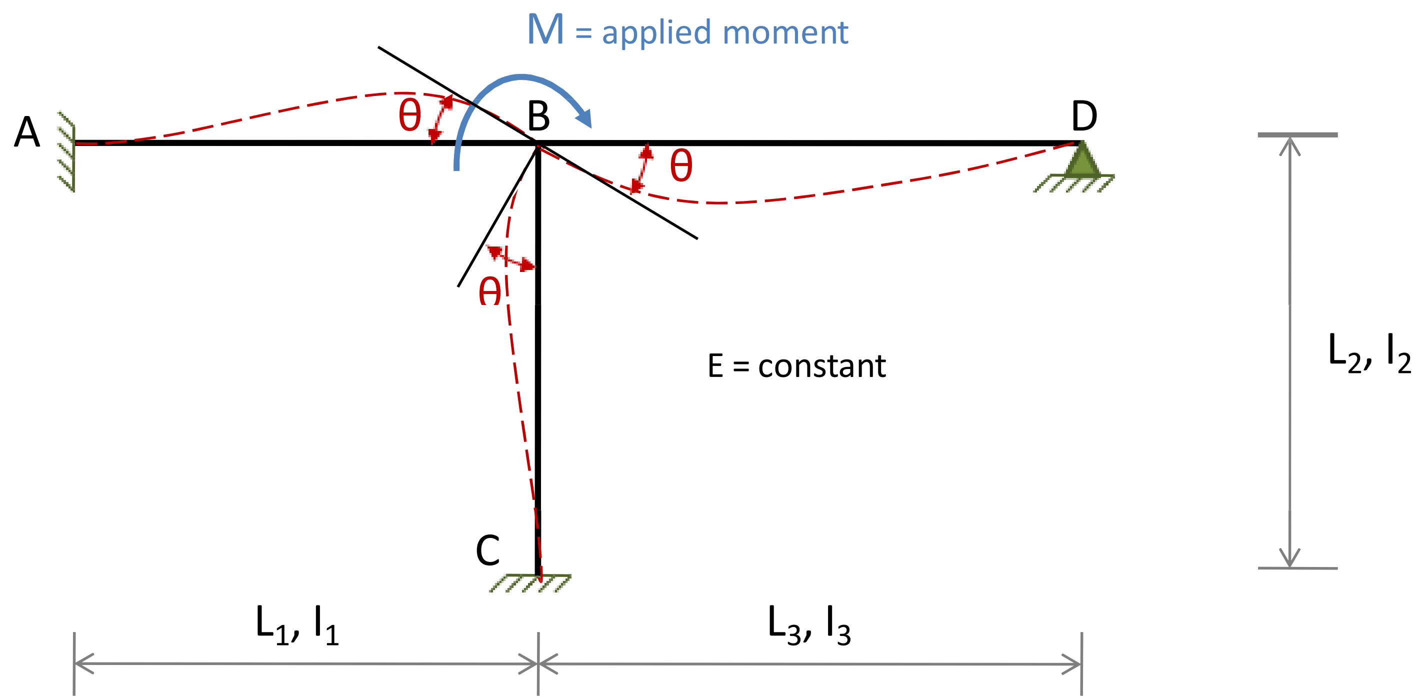
Distribution Factors

When analyzing a structure by the moment-distribution method, an important question that arises is how to distribute a moment applied at a joint among the various members connected to that joint.

Consider the three-member frame shown in figure below.



Suppose that a moment M is applied to the joint B, causing it to rotate by an angle θ as shown in figure below.

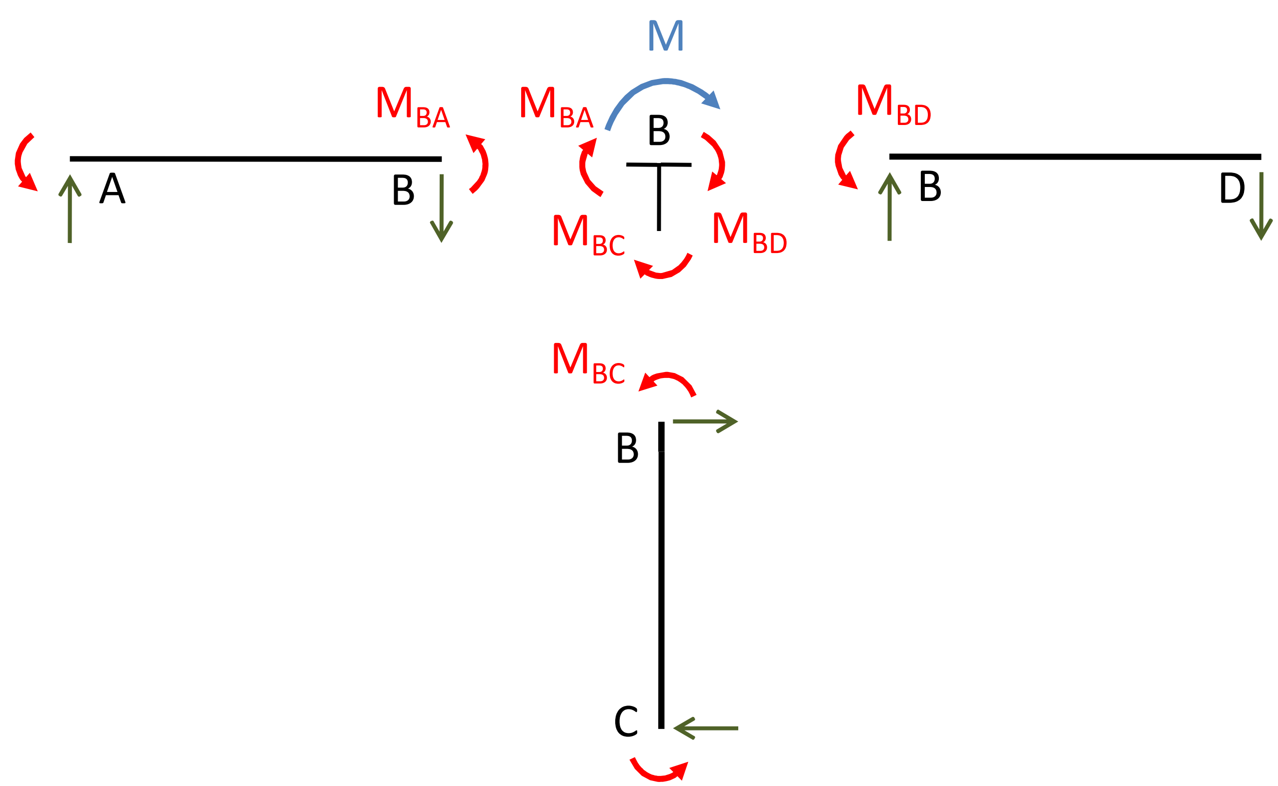


To determine what fraction of applied moment is resisted by each of the three members AB, BC, and BD, we draw free-body diagrams of joint B and of the three members AB, BC, and BD.

By considering the moment equilibrium of the free body of joint B ($\sum M_B = 0$), we write

$$M + M_{BA} + M_{BC} + M_{BD} = 0$$

$$M = -\left(M_{BA} + M_{BC} + M_{BD}\right) \tag{14}$$



Since members AB, BC, and BD are rigidly connected to joint B, the rotations of the ends B of these members are the same as that of the joint.

The moments at the ends B of the members can be expressed in terms of the joint rotation θ by applying Eq. 7.

Noting that the far ends A and C, respectively, of members AB and BC are fixed, whereas the far end D of member BD is hinged, we apply Eq. 7 through Eq. 9 to each member to obtain

$$M_{BA} = \left(\frac{4EI_1}{L_1} \right) \theta = \overline{K}_{BA} \theta = 4EK_{BA} \theta \quad (15)$$

$$M_{BC} = \left(\frac{4EI_2}{L_2} \right) \theta = \overline{K}_{BC} \theta = 4EK_{BC} \theta \quad (16)$$

$$M_{BD} = \left(\frac{3EI_3}{L_3} \right) \theta = \overline{K}_{BD} \theta = 4EK_{BD} \theta \quad (17)$$

Substitution of Eq. 15 through Eq. 17 into the equilibrium equation Eq. 14 yields

$$M = -\left(\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2} + \frac{3EI_3}{L_3}\right)\theta$$
$$= -\left(\bar{K}_{BA} + \bar{K}_{BC} + \bar{K}_{BD}\right)\theta = -\left(\sum \bar{K}_B\right)\theta \qquad (18)$$

in which $\sum \bar{K}_B$ represents the sum of the bending stiffnesses of all the members connected to joint B.

"The rotational stiffness of a joint is defined as the moment required to cause a unit rotation of the joint."

From Eq. 18, we can see that the rotational stiffness of a joint is equal to the sum of the bending stiffnesses of all the members rigidly connected to the joint.

The negative sign in Eq. 18 appears because of the sign convention.

To express member end moments in terms of the applied moment M, we first rewrite Eq. 18 in terms of the relative bending stiffnesses of members as

$$M = -4E\left(K_{BA} + K_{BC} + K_{BD}\right) = -4E \sum K_B$$

$$\theta = -\frac{M}{4E\sum K_B} \qquad (19)$$

By substituting Eq. 19 into Eqs. 15 through 17, we obtain

$$M_{BA} = -\left(\frac{K_{BA}}{\sum K_B}\right)M \qquad (20)$$

$$M_{BC} = -\left(\frac{K_{BC}}{\sum K_B}\right)M \qquad (21)$$

$$M_{BD} = -\left(\frac{K_{BD}}{\sum K_B}\right)M \qquad (22)$$

From Eqs. 20 through 22, we can see that the applied moment M is distributed to the three members in proportion to their relative bending stiffnesses.

"The ratio $K/\sum K_B$ for a member is termed the distribution factor of that member for end B, and it represents the fraction of the applied moment M that is distributed to end B of the member."

Thus Eqs. 20 through 22 can be expressed as

$$M_{BA} = -DF_{BA}M \qquad (23)$$

$$M_{BC} = -DF_{BC}M \qquad (24)$$

$$M_{BD} = -DF_{BD}M \qquad (25)$$

in which $DF_{BA} = K_{BA}/\sum K_B$, $DF_{BC} = K_{BC}/\sum K_B$, and $DF_{BD} = K_{BD}/\sum K_B$, are the distribution factors for ends B of members AB, BC, and BD, respectively.

For example, if joint B of the frame is subjected to a clockwise moment of 150 k-ft ($M = 150$ k-ft) and if $L_1 = L_2 = 20$ ft, $L_3 = 30$ ft, and $I_1 = I_2 = I_3 = I$, so that

$$K_{BA} = K_{BC} = \frac{I}{20} = 0.05I$$

$$K_{BD} = \frac{3}{4}\left(\frac{I}{30}\right) = 0.025I$$

then the distribution factors for the ends B of members AB, BC, and BD are given by

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.05I}{(0.05 + 0.05 + 0.025)I} = 0.4$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.05I}{0.125I} = 0.4$$

$$DF_{BD} = \frac{K_{BD}}{K_{BA} + K_{BC} + K_{BD}} = \frac{0.025I}{0.125I} = 0.2$$

These distribution factors indicate that 40% of the 150 k-ft moment applied to joint B is exerted at end B of member AB, 40% at end B of member BC, and the remaining 20% at end B of member BD.

The moments at ends B of the three members are

$$M_{BA} = -DF_{BA}M = -0.4(150) = -60\ \text{k-ft} \qquad \text{or} \qquad 60\ \text{k-ft}$$

$$M_{BC} = -DF_{BC}M = -0.4(150) = -60\ \text{k-ft} \qquad \text{or} \qquad 60\ \text{k-ft}$$

$$M_{BD} = -DF_{BD}M = -0.2(150) = -30\ \text{k-ft} \qquad \text{or} \qquad 30\ \text{k-ft}$$

Based on the foregoing discussion, we can state that, in general, *"the distribution factor (DF) for an end of a member that is rigidly connected to the adjacent joint equals the ratio of the relative bending stiffness of the member to the sum of the relative bending stiffnesses of all the members framing into the joint"*; that is

$$DF = \frac{K}{\sum K} \tag{26}$$

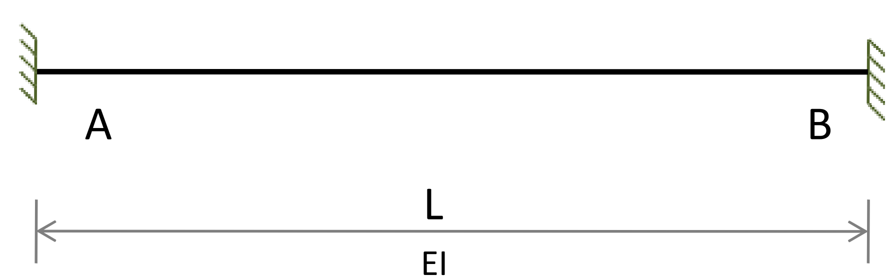
"The moment distributed to (or resisted by) a rigidly connected end of a member equals the distribution factor for that end times the negative of the moment applied to the adjacent joint."

Fixed-End Moments

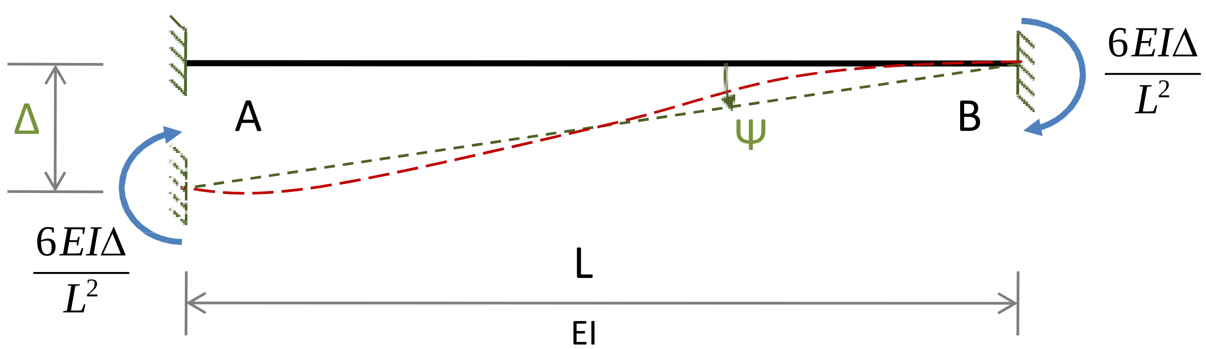
The fixed end moment expressions for some common types of loading conditions as well as for relative displacements of member ends are given inside the back cover of book.

In the MDM, the effects of joint translations due to support settlements and sidesway are also taken into account by means of fixed-end moments.

Consider the fixed beam of Figure.



A small settlement Δ of the left end A of the beam with respect to the right end B causes the beam's chord to rotate counterclockwise by an angle $\Psi = \Delta/L$.

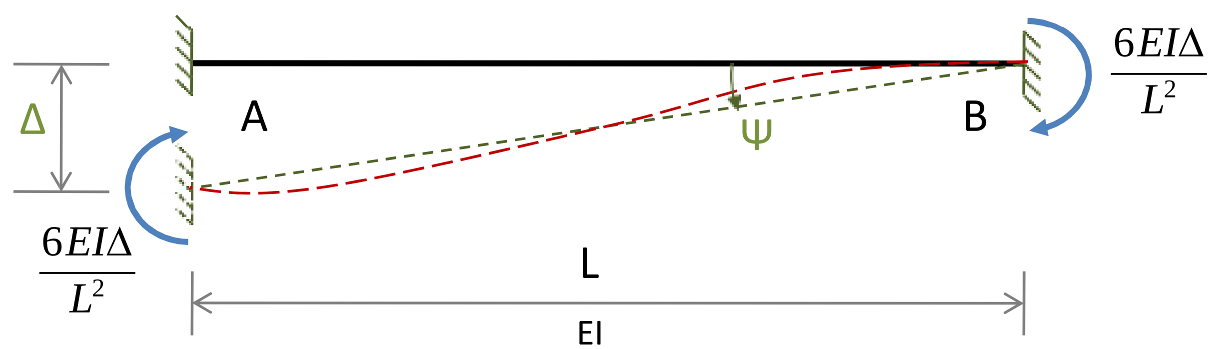


By writing the SDE for the two end moments with $\Psi = \Delta/L$ and by setting θ_A, θ_B, and FEM_{AB} and FEM_{BA} due to external loading, equal to zero, we obtain

$$FEM_{AB} = FEM_{BA} = -\frac{6EI\Delta}{L^2}$$

in which FEM_{AB} and FEM_{BA} denote the FEM due to the relative translation Δ between the two ends of the beam.

Note that the magnitudes as well as the directions of the two FEM are the same.



It can be seen from the figure that when a relative displacement causes a chord rotation in the CCW direction, then the two FEMs act in the CW (-ve) direction to maintain zero slopes at the two ends of the beam.

Conversely, if the chord rotation due to a relative displacement is CW, then both FEM act in CCW (+ve) direction.

Moment-Distribution Method

• MDM	Moment Distribution Method
• MD Table	Moment Distribution Table
• COM	Carryover Moment
• COF	Carryover Factor
• DM	Distributed Moment
• UM	Unbalanced Moment

UNIT 1
PART 2
Kani’s Method

Analysis by Kani's Method:

- Framed structures are rarely symmetric and subjected to side sway, hence Kani's method is best and much simpler than other methods.
- PROCEDURE:
- 1. Rotation stiffness at each end of all members of a structure is determined depending upon the end conditions.
- a. Both ends fixed $K_{ij} = K_{ji} = EI/L$
- b. Near end fixed, far end simply supported $K_{ij} = \frac{3}{4} EI/L$; $K_{ji} = 0$

- 2. Rotational factors are computed for all the members at each joint it is given by Uij= -0.5 (Kij/ ?Kji) {THE SUM OF ROTATIONAL FACTORS AT A JOINT IS -0.5} (Fixed end moments including transitional moments, moment releases and carry over moments are computed for members and entered. The sum of the FEM at a joint is entered in the central square drawn at the joint).

- 3. Iterations can be commenced at any joint however the iterations commence from the left end of the structure generally given by the equation $M?ij = Uij [(Mfi + M??i) + ? M?ji)]$

- 4. Initially the rotational components? Mji (sum of the rotational moments at the far ends of the joint) can be assumed to be zero. Further iterations take into account the rotational moments of the previous joints. 5. Rotational moments are computed at each joint successively till all the joints are processed. This process completes one cycle of iteration

- 6. Steps 4 and 5 are repeated till the difference in the values of rotation moments from successive cycles is neglected.
- 7. Final moments in the members at each joint are computed from the rotational members of the final iterations step. Mij = (Mfij + M??ij) + 2 M?ij + M?jii

- The lateral translation of joints (side sway) is taken into consideration by including column shear in the iterative procedure.
- 8. Displacement factors are calculated for each storey given by $U_{ij} = -1.5 (K_{ij}/?K_{ij})$

- Application Of Analysis Methods For The Portal Frame
- Application of Rotation contribution Method (Kani's Method) for the analysis of portal frame:
- Fixed end moments
- FEMAB = 0
- FEMBA = 0
- FEMBC = -120 kNm
- FEMCB = 120 kNm
- FEMCD = 0
- FEMDC = 0

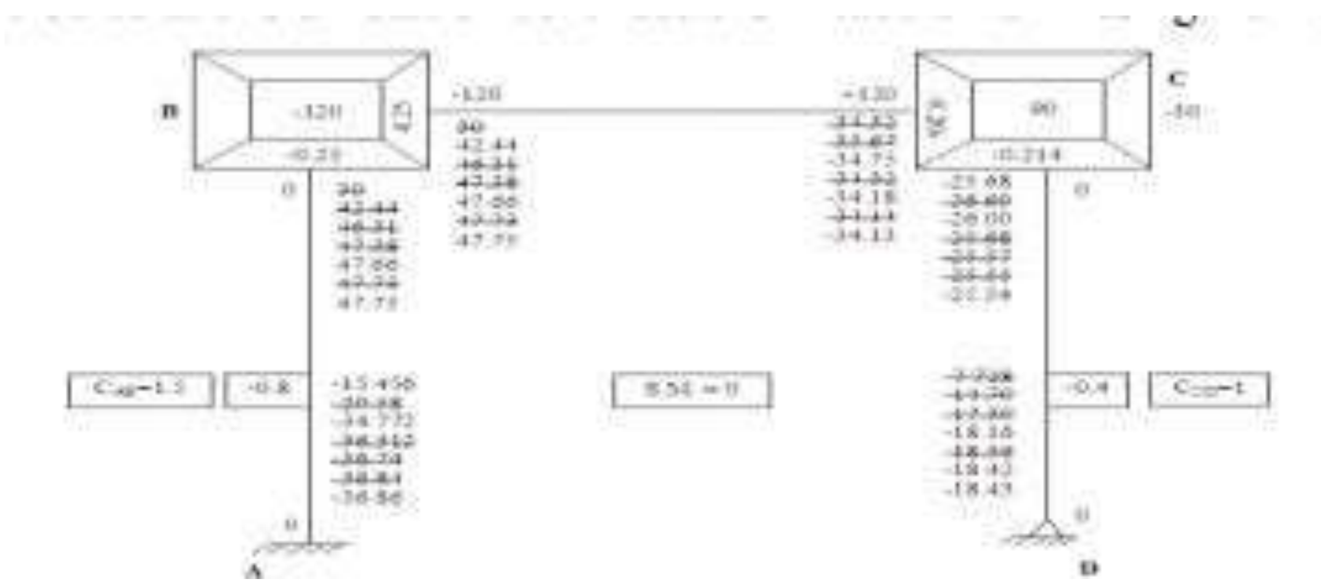
- Stiffness and rotation factor (R.F.)
- Table 1.
- Stiffness and Rotation Factors – Kani’s Method

Stiffness and rotation factor (R.F.)

Table 1. Stiffness and Rotation Factors – Kani's Method

Joint	Member	K	ΣK	RF
B	BA	0.333 I	0.666 I	-0.25
	BC	0.333 I		-0.25
C	CB	0.333 I	0.583 I	-0.286
	CD	0.25 I		-0.214

- 3. Displacement factors (δ)
- Table 2. Calculation of Displacement factors (δ)
- $\Sigma UCD = (-1.2) + (-0.3) = -1.5$
- Checked.
- Hence OK
- Storey Moment (SM) Storey moment = 0 (since lack of nodal loads and lack of loadings on columns, SM=0) Iterations by Kani's Method Figure 2. Calculations of rotation contributions in tabular form using Kani's Method


B
C
A
D

- Final End Moments For columns:
- => F.E.M + 2 (near end contribution) + far end contribution of that particular column + L.D.C. of that column
- For beams: => F.E.M + 2 (near end contribution) + far end contribution of that particular beam or slab.
- MAB = 10.89 kNm
- MBA = 58.64 kNm
- MBC = -58.63 kNm
- MCB = 99.49 kNm
- MCD = -69.51 kNm
- MDC = 0 kNm
- MCE = -30 kNm

Structural Analysis - III

Flexibility Method - 1

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Module I

Matrix analysis of structures

- Definition of flexibility and stiffness influence coefficients – development of flexibility matrices by physical approach & energy principle.

Flexibility method

- Flexibility matrices for truss, beam and frame elements – load transformation matrix-development of total flexibility matrix of the structure –analysis of simple structures – plane truss, continuous beam and plane frame- nodal loads and element loads – lack of fit and temperature effects.

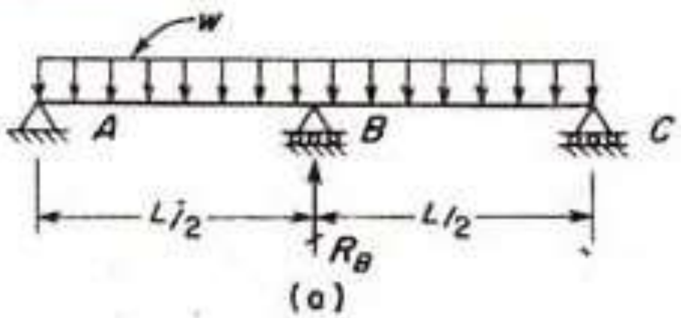
FUNDAMENTALS OF FLEXIBILITY METHOD

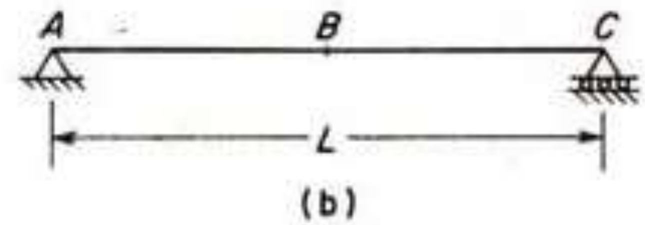
Introduction

- This method is a generalization of the Maxwell-Mohr method(1874)
- Not conducive to computer programming, because the choice of redundants is not unique
- Unknowns are the redundant actions, which are arbitrarily chosen

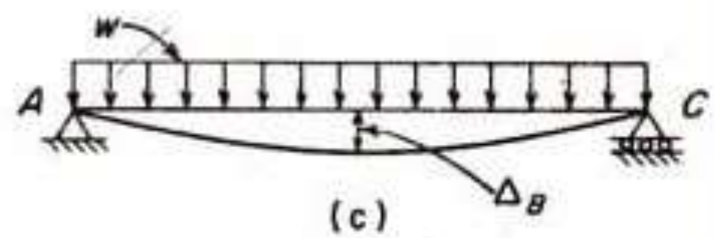
Flexibility method *(Explanation using principle of superposition)*

Example 1: Single redundant - Continuous 2-span beam





Released structure

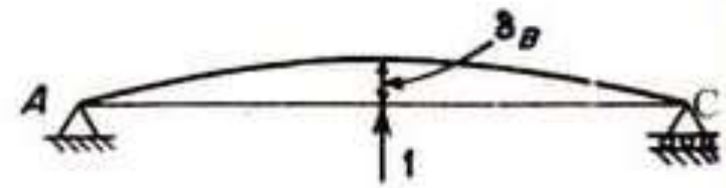


Deflection of released structure due to actual loads

$$\Delta_B = \frac{-5wL^4}{384EI}$$

(Negative, since deflection is downward)

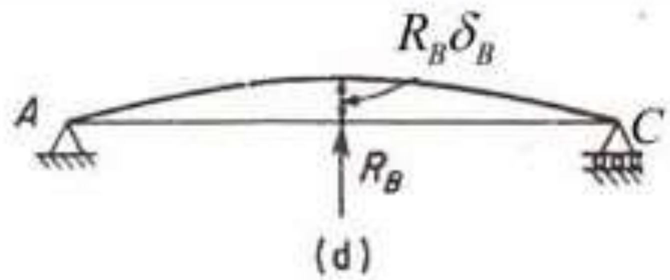
Apply unit load corresponding to R_B



(e)

Displacement due to unit load, $\delta_B = \dfrac{L^3}{48EI}$

Displacement due to R_B is $R_B\delta_B$



Deflection of released structure due to redundant applied as a load

$$\Delta_B + R_B\delta_B = 0$$

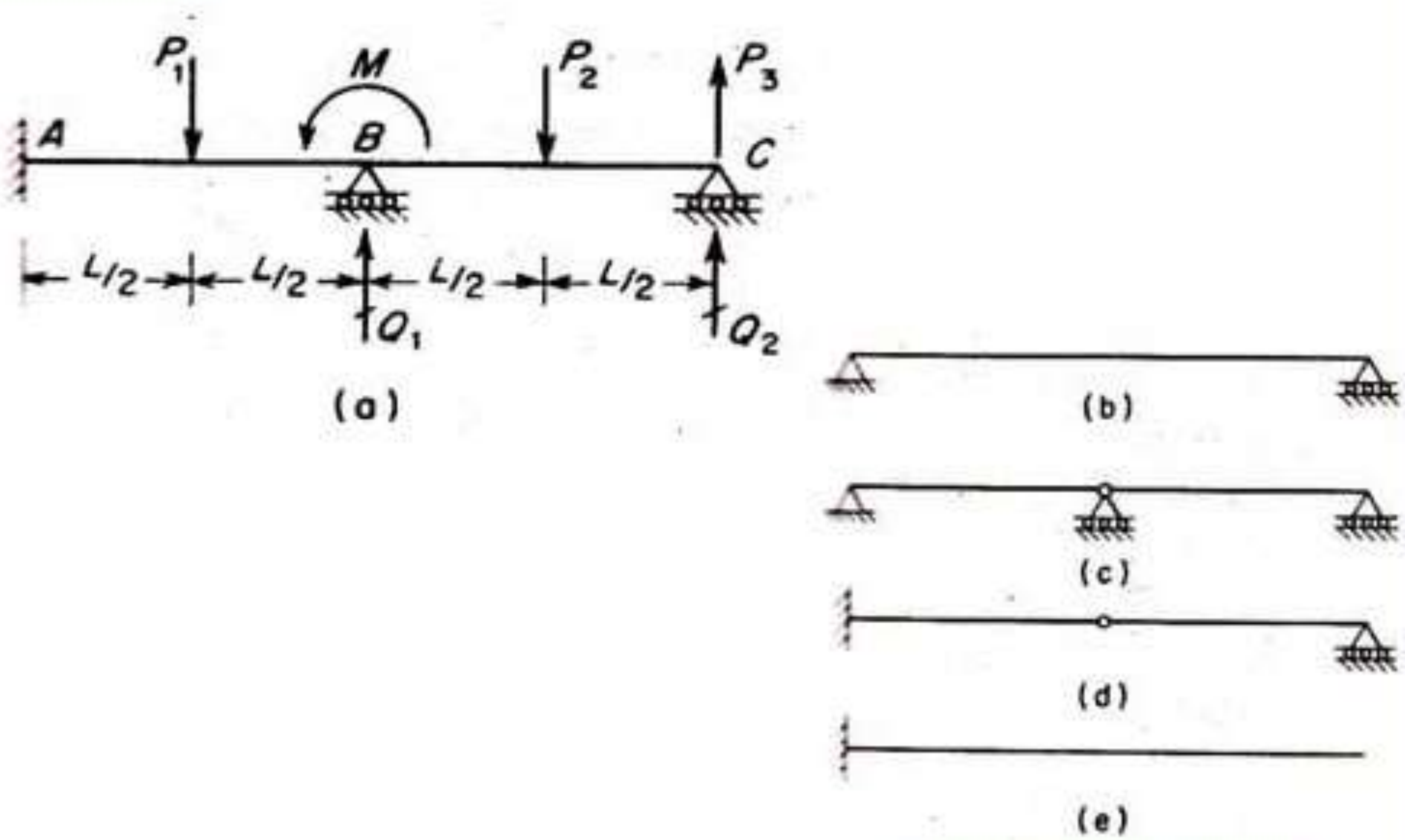
Compatibility condition (or equation of superposition or equation of geometry)

$$\boxed{\therefore R_B = \frac{-\Delta_B}{\delta_B} = \frac{5wL}{8}}$$

δ_B flexibility coefficient

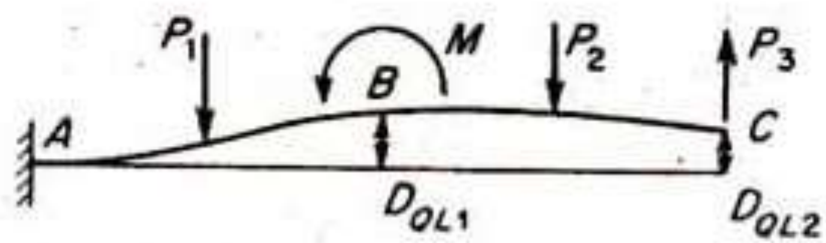
(Displacement due to unit load corresponding to R_B)

Example 2 – More than one redundant



Choice of redundants

Let Q_1, Q_2 be the redundants

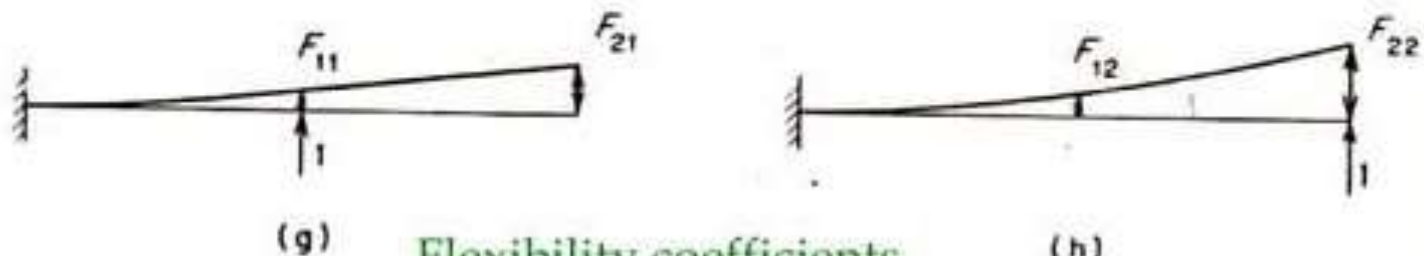


(f)

Deflections corresponding to redundants

D_{QL1} & D_{QL2} Displacements in the released structure corresponding to redundants, due to external loads

- To get flexibility coefficients
  - Apply unit loads corresponding to Q_1 & Q_2



Flexibility coefficients

Net deflection is zero at B and C

$$D_{QL1} + F_{11}Q_1 + F_{12}Q_2 = 0$$

$$D_{QL2} + F_{21}Q_1 + F_{22}Q_2 = 0$$

$$\begin{Bmatrix} D_{QL1} \\ D_{QL2} \end{Bmatrix} + \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- Generally, net deflection need not be zero

$$D_{Q1} = D_{QL1} + F_{11}Q_1 + F_{12}Q_2$$

$$D_{Q2} = D_{QL2} + F_{21}Q_1 + F_{22}Q_2$$

- Where $D_{Q1},\ D_{Q2}$:**support displacements** corresponding to $Q_1,\ Q_2$

$$\{D_Q\} = \{D_{QL}\} + [F]\{Q\}$$

$$\{D_Q\} = \begin{Bmatrix} D_{Q1} \\ D_{Q2} \end{Bmatrix} \qquad \{D_{QL}\} = \begin{Bmatrix} D_{QL1} \\ D_{QL2} \end{Bmatrix} \qquad [F] = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \qquad \{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

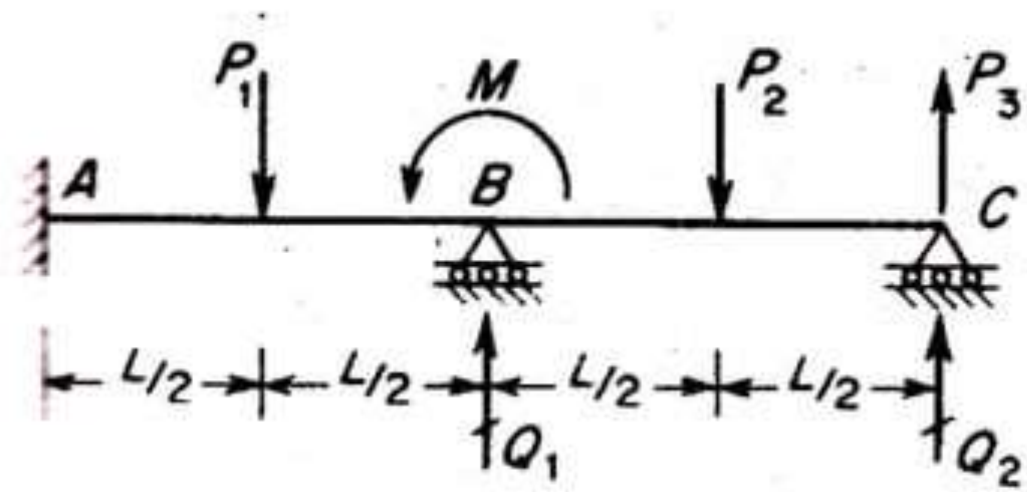
Flexibility coefficient F is sometimes denoted as D_{QQ}

$$\boxed{\{Q\} = [F]^{-1}\left(\{D_Q\} - \{D_{QL}\}\right)}$$

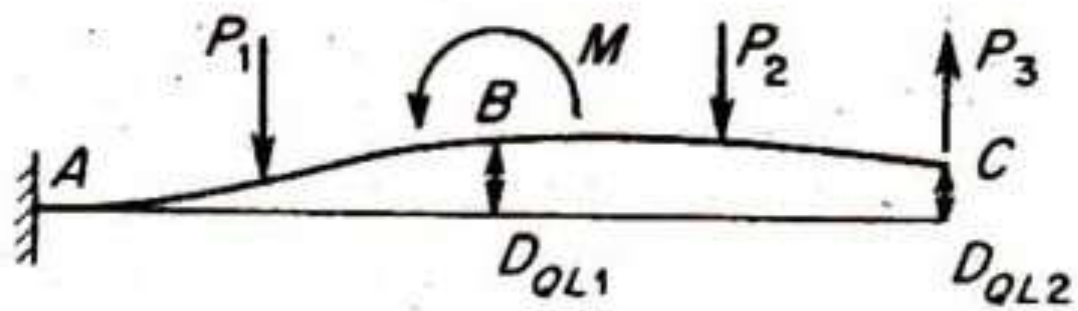
- If there are no support displacements, $\{D_Q\} = \{0\}$

$$\boxed{\therefore \{Q\} = -[F]^{-1}\{D_{QL}\}}$$

- Example: To find out redundants



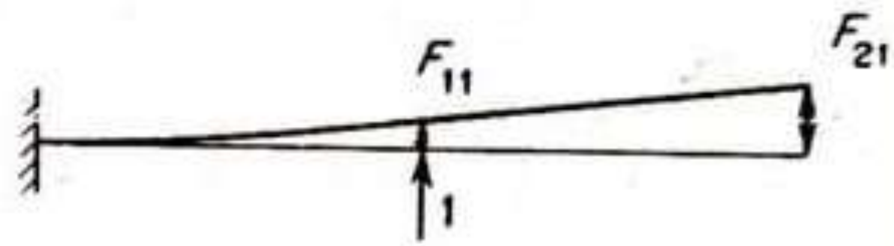
$$Given: \ P_1 = 2P \qquad M = PL \qquad P_2 = P \qquad P_3 = P$$



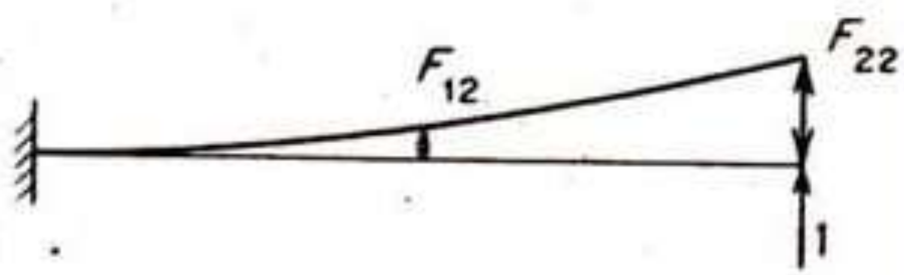
$$D_{QL1} = \frac{13PL^3}{24EI} \qquad D_{QL2} = \frac{97PL^3}{48EI}$$

$$\therefore \left[D_{QL}\right] = \frac{PL^3}{48EI}\begin{bmatrix}26\\97\end{bmatrix}$$

$$F_{11} = \frac{L^3}{3EI}$$



$$F_{21} = \frac{5L^3}{6EI}$$



$$F_{12} = \frac{5L^3}{6EI}$$

$$F_{22} = \frac{8L^3}{3EI}$$

$$\therefore [F] = \frac{L^3}{6EI}\begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix}$$

$$[F]^{-1} = \frac{6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

$$[Q] = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = -[F]^{-1}\left[D_{QL}\right]$$

$$= \frac{-6EI}{7L^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}\frac{PL^3}{48EI}\begin{bmatrix} 26 \\ 97 \end{bmatrix} \quad = \frac{P}{56}\begin{bmatrix} 69 \\ -64 \end{bmatrix}$$

$$i.e., \;\; Q_1 = \frac{69P}{56}, \qquad Q_2 = \frac{-8P}{7}$$

Temperature changes, pre-strains and support displacements not corresponding to redundants

Let:

$\{D_{QT}\}$ Displacements corresponding to redundants due to temperature changes, in the released structure

$\{D_{QP}\}$ Displacements corresponding to redundants due to pre-strains, in the released structure

$\{D_{QR}\}$ Displacements corresponding to redundants due to support displacements not corresponding to redundants, in the released structure

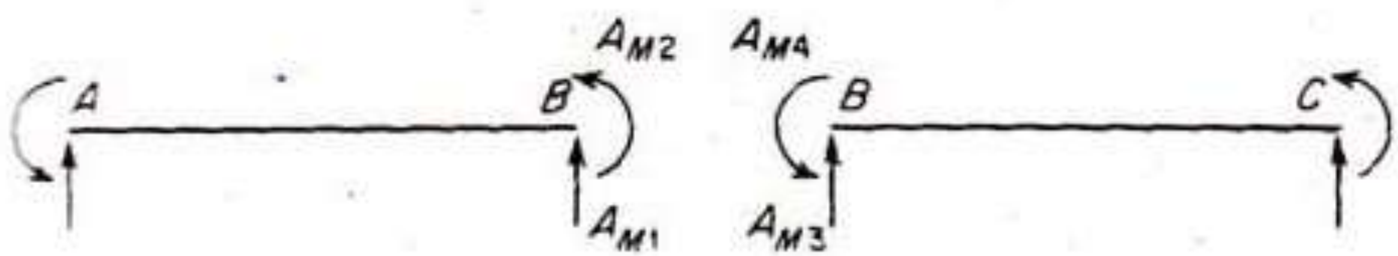
$$\{D_Q\} = \{D_{QL}\} + \{D_{QT}\} + \{D_{QP}\} + \{D_{QR}\} + [F]\{Q\}$$

- Let $\{D_{QC}\} = \{D_{QL}\} + \{D_{QT}\} + \{D_{QP}\} + \{D_{QR}\}$
- Hence, $\{D_Q\} = \{D_{QC}\} + [F]\{Q\}$ and

$$\{Q\} = [F]^{-1}\left(\{D_Q\} - \{D_{QC}\}\right)$$

Member end actions

- Member end actions are the couples and forces that act at the ends of a member when it is considered to be isolated from the remainder of the structure



- In the above case, member end actions are the SFs and BMs at the ends of members *AB* and *BC*

- In the above figure $A_{M1}, A_{M2}, A_{M3}, A_{M4}$ are the member end actions considered (upward forces and anticlockwise moments are +ve).

- The first two are just to the left of *B*, and the last two are just to the right of *B*

$A_{M1} + A_{M3}$ gives the reaction at *B*, and

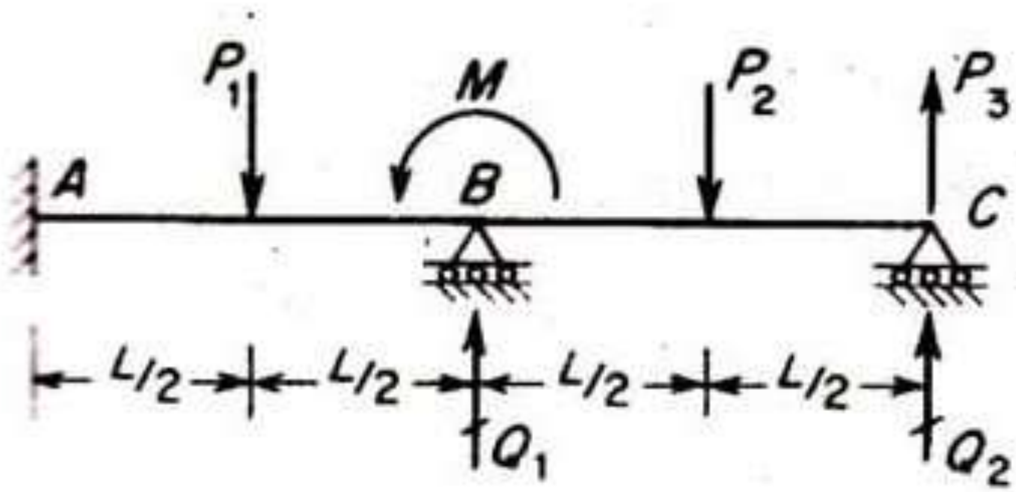
$A_{M2} + A_{M4}$ gives the bending moment at *B*

Joint displacements, member end actions, and support reactions

- Once the redundants are found, all the joint displacements, member end actions, and support reactions can be found subsequently

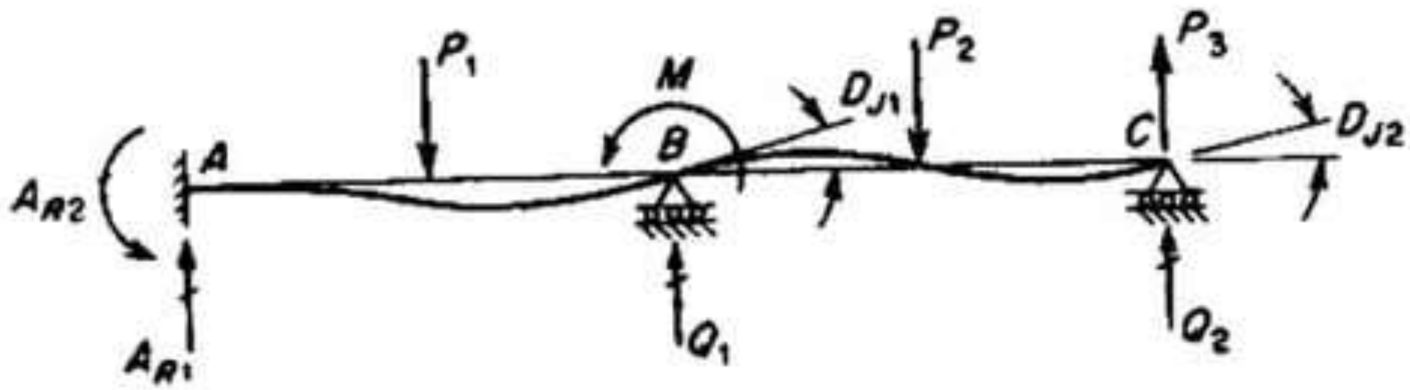
- But it is easier to incorporate such calculations into the basic computations, instead of postponing them as separate calculations

- Example: To find out
  - joint displacements,
  - member end actions and
  - reactions other than redundants



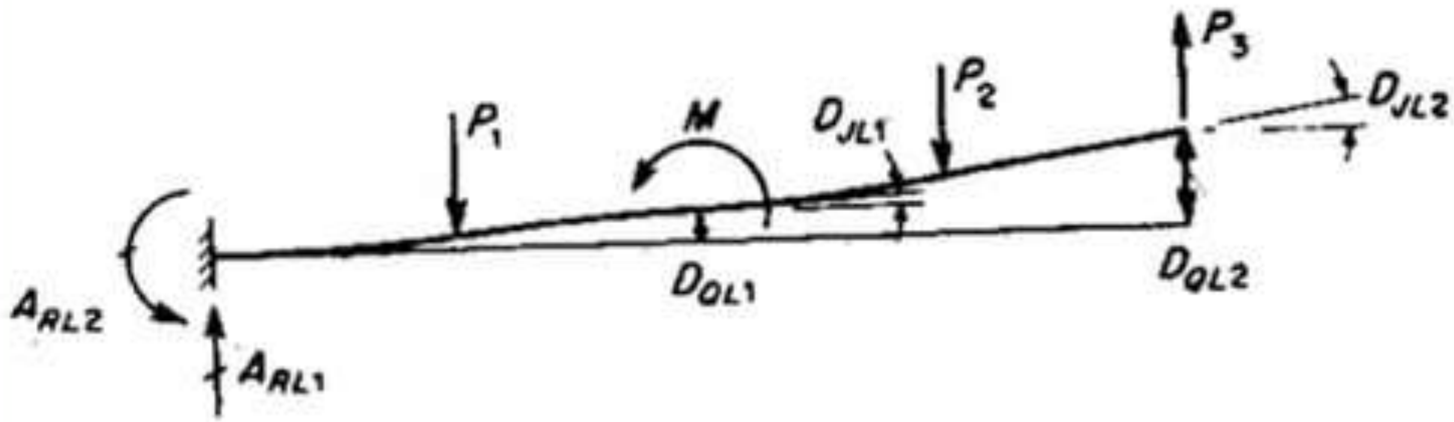
Actual structure

$$Given: \; P_1 = 2P \qquad M = PL \qquad P_2 = P \qquad P_3 = P$$



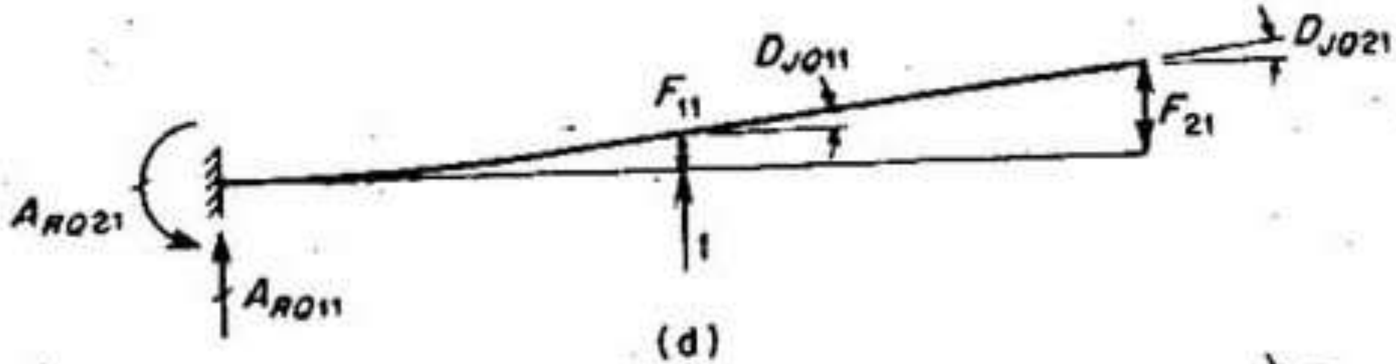
$$\{D_J\} = \begin{Bmatrix} D_{J1} \\ D_{J2} \end{Bmatrix}$$ Joint displacements in the actual structure due to loads

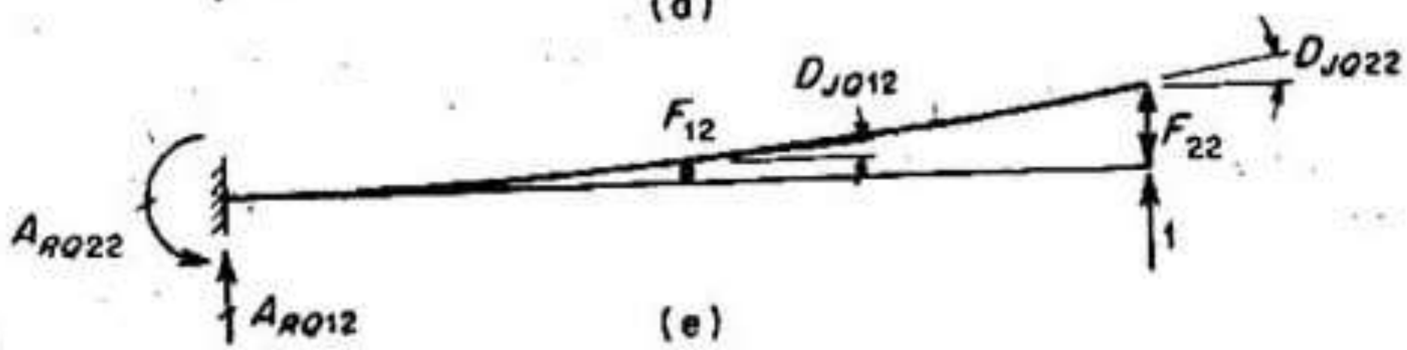
$$\{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$ Redundants



$$\{D_{JL}\} = \begin{Bmatrix} D_{JL1} \\ D_{JL2} \end{Bmatrix}$$ Joint displacements in the released structure due to loads

$$\{A_{RL}\} = \begin{Bmatrix} A_{RL1} \\ A_{RL2} \end{Bmatrix}$$ Reactions in the released structure due to loads





D_{JQij} Joint displacements in the released structure due to unit values of redundants

A_{RQij} Reactions in the released structure due to unit values of redundants

Joint displacements

$$D_{J1} = D_{JL1} + D_{JQ11}Q_1 + D_{JQ12}Q_2$$
$$D_{J2} = D_{JL2} + D_{JQ21}Q_1 + D_{JQ22}Q_2$$

- In matrix form,

$$\boxed{\{D_J\} = \{D_{JL}\} + [D_{JQ}]\{Q\}}$$

$$\textit{where,}\ \{D_J\} = \begin{Bmatrix} D_{J1} \\ D_{J2} \end{Bmatrix},\ \{D_{JL}\} = \begin{Bmatrix} D_{JL1} \\ D_{JL2} \end{Bmatrix},$$

$$[D_{JQ}] = \begin{bmatrix} D_{JQ11} & D_{JQ12} \\ D_{JQ21} & D_{JQ22} \end{bmatrix},\ \{Q\} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

D_{JQij} Joint displacement i in the released structure due to unit value of redundant j

- If there are *j* joint displacements to be obtained, and there are *q* redundants,

$$\underset{j\times 1}{\{D_J\}} = \underset{j\times 1}{\{D_{JL}\}} + \underset{j\times q}{\left[D_{JQ}\right]}\underset{q\times 1}{\{Q\}}$$

Reactions (other than redundants)

$$A_{R1} = A_{RL1} + A_{RQ11}Q_1 + A_{RQ12}Q_2$$

$$A_{R2} = A_{RL2} + A_{RQ21}Q_1 + A_{RQ22}Q_2$$

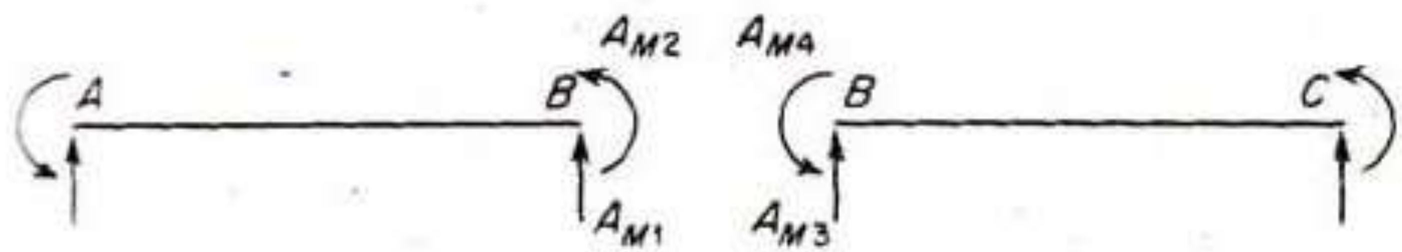
$$\begin{Bmatrix} A_{R1} \\ A_{R2} \end{Bmatrix} = \begin{Bmatrix} A_{RL1} \\ A_{RL2} \end{Bmatrix} + \begin{bmatrix} A_{RQ11} & A_{RQ12} \\ A_{RQ21} & A_{RQ22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

$$\{A_R\} = \{A_{RL}\} + [A_{RQ}]\{Q\}$$

- If there are ***r*** reactions to be obtained (other than redundants) and ***q*** redundants,

$$\underset{r\times 1}{\{A_R\}} = \underset{r\times 1}{\{A_{RL}\}} + \underset{r\times q}{[A_{RQ}]}\underset{q\times 1}{\{Q\}}$$

Member end actions



- Member end actions $\{A_M\} = \{A_{ML}\} + [A_{MQ}]\{Q\}$

$\{A_{ML}\}$ Member end actions in the released structure due to loads

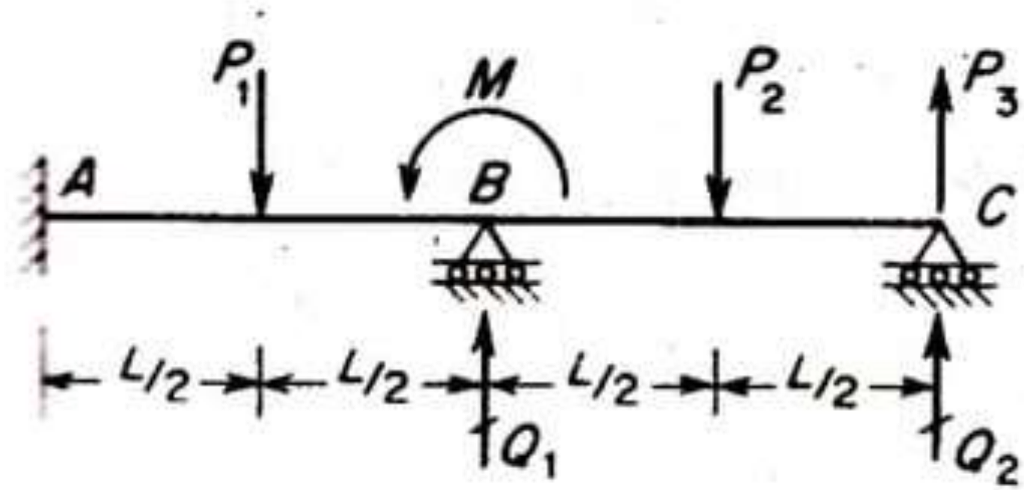
$[A_{MQ}]$ Member end actions in the released structure due to unit values of redundants

If there are m member end actions and q redundants,

$$\underset{m\times 1}{\{A_M\}} = \underset{m\times 1}{\{A_{ML}\}} + \underset{m\times q}{[A_{MQ}]}\underset{q\times 1}{\{Q\}}$$

- In the given example,

$$P_1 = 2P \qquad M = PL \qquad P_2 = P \qquad P_3 = P$$



As found out earlier, $[Q] = \dfrac{P}{56}\begin{bmatrix} 69 \\ -64 \end{bmatrix}$

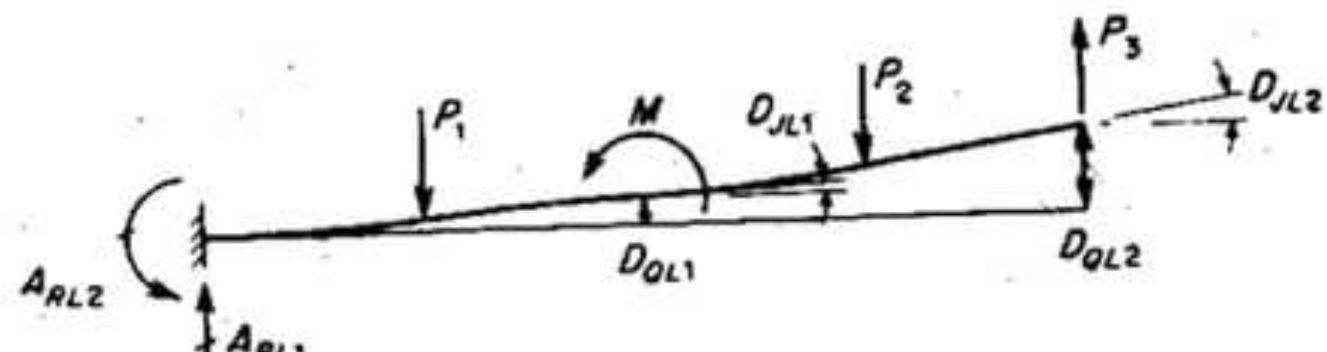
$$\{D_J\} = \{D_{JL}\} + [D_{JQ}]\{Q\}$$

$$\{A_M\} = \{A_{ML}\} + [A_{MQ}]\{Q\}$$

$$\{A_R\} = \{A_{RL}\} + [A_{RQ}]\{Q\}$$

To be found out

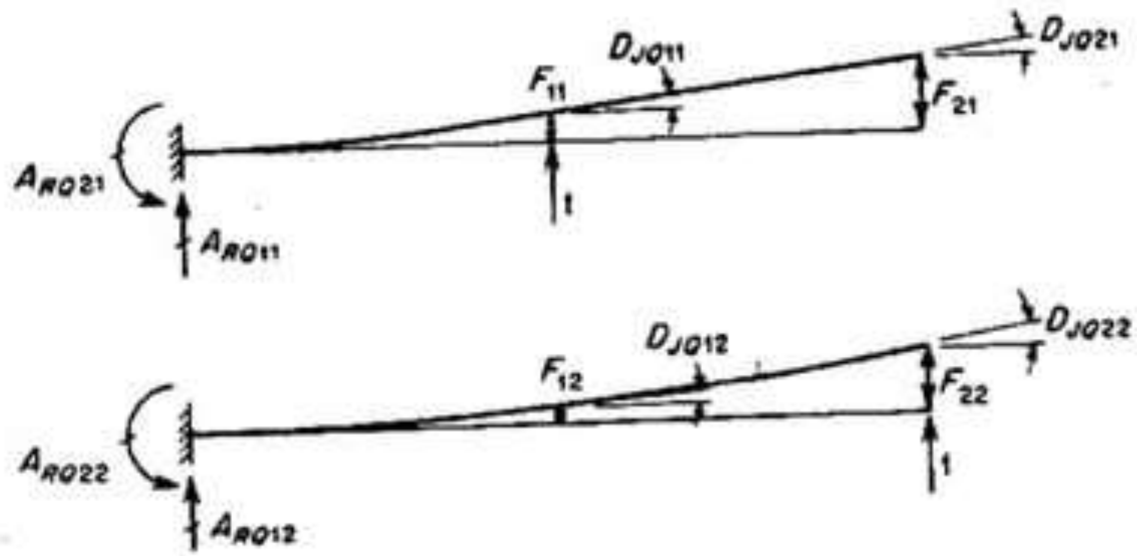
To get $[D_{JL}]$



(c)

$$D_{JL1} = \frac{5PL^2}{4EI} \qquad D_{JL2} = \frac{13PL^2}{8EI} \qquad \therefore [D_{JL}] = \frac{PL^2}{8EI}\begin{bmatrix}10\\13\end{bmatrix}$$

To get $\left[D_{JQ}\right]$



$$\left[D_{JQ}\right]=\begin{bmatrix} D_{JQ11} & D_{JQ12} \\ D_{JQ21} & D_{JQ22} \end{bmatrix}=\begin{bmatrix} \frac{L^2}{2EI} & \frac{3L^2}{2EI} \\ \frac{L^2}{2EI} & \frac{4L^2}{2EI} \end{bmatrix}=\frac{L^2}{2EI}\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

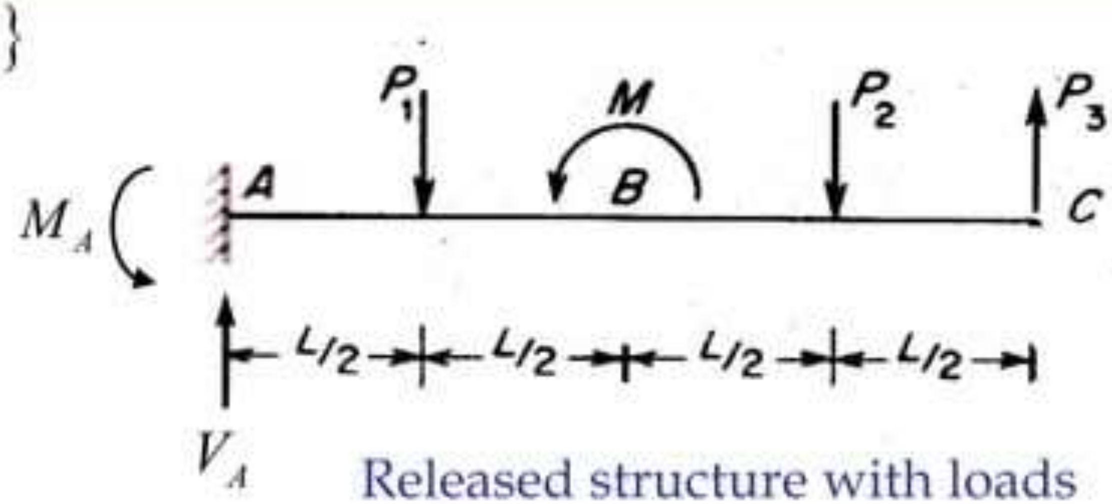
Already we know, $[Q] = \frac{P}{56}\begin{bmatrix} 69 \\ -64 \end{bmatrix}$

Joint displacements

$$\{D_J\} = \{D_{JL}\} + \left[D_{JQ}\right]\{Q\} \quad = \frac{PL^2}{8EI}\begin{bmatrix} 10 \\ 13 \end{bmatrix} + \frac{L^2}{2EI}\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}\frac{P}{56}\begin{bmatrix} 69 \\ -64 \end{bmatrix}$$

$$= \frac{PL^2}{112EI}\begin{bmatrix} 17 \\ -5 \end{bmatrix}$$

To get $\{A_{RL}\}$

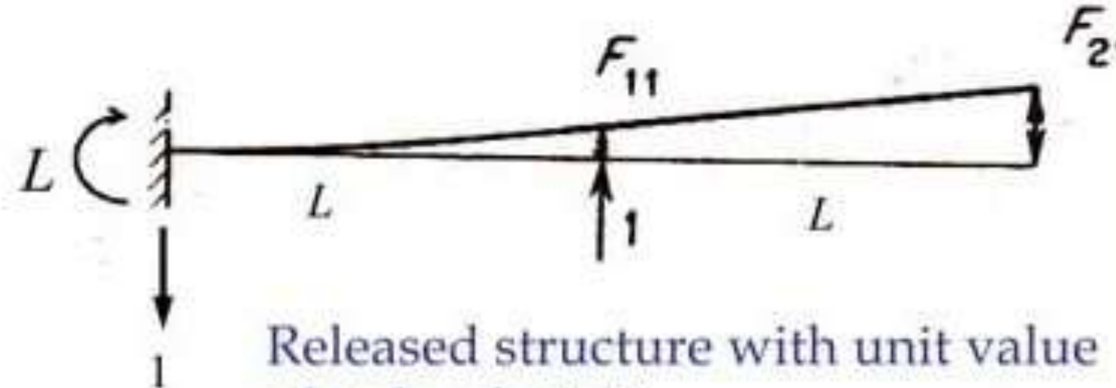


Released structure with loads and reactions

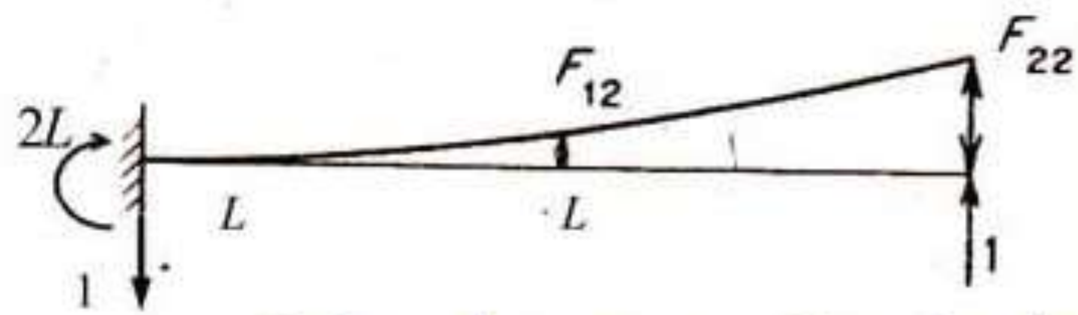
$$V_A = P_1 + P_2 - P_3 = 2P \qquad M_A = \frac{P_1 L}{2} - M + \frac{3P_2 L}{2} - 2P_3 L = \frac{-PL}{2}$$

$$\{A_{RL}\} = \begin{Bmatrix} A_{RL1} \\ A_{RL2} \end{Bmatrix} = \begin{Bmatrix} 2P \\ -\dfrac{PL}{2} \end{Bmatrix}$$

To get $\left[A_{RQ}\right]$



Released structure with unit value of redundant Q_1



Released structure with unit value of redundant Q_2

$$\left[A_{RQ}\right]=\begin{bmatrix}-1 & -1\\ -L & -2L\end{bmatrix}$$

Reactions (other than redundants)

$$\{A_R\} = \{A_{RL}\} + \left[A_{RQ}\right]\{Q\}$$

$$= \begin{Bmatrix} 2P \\ -\dfrac{PL}{2} \end{Bmatrix} + \begin{bmatrix} -1 & -1 \\ -L & -2L \end{bmatrix} \frac{P}{56} \begin{bmatrix} 69 \\ -64 \end{bmatrix}$$

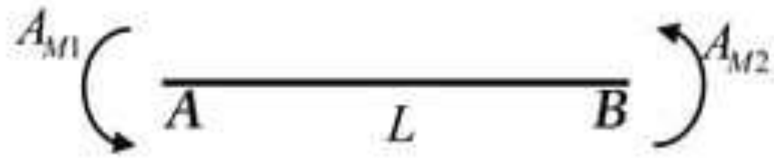
$$= \frac{P}{56} \begin{bmatrix} 107 \\ 31L \end{bmatrix}$$

Choice of member end actions

- Any number among the 4 member end actions can be chosen for analysis
- Usually two among the 4 are chosen
- Any two of the 4 member end actions can be chosen for analysis
- Usually the moments at both ends are chosen

Member end actions

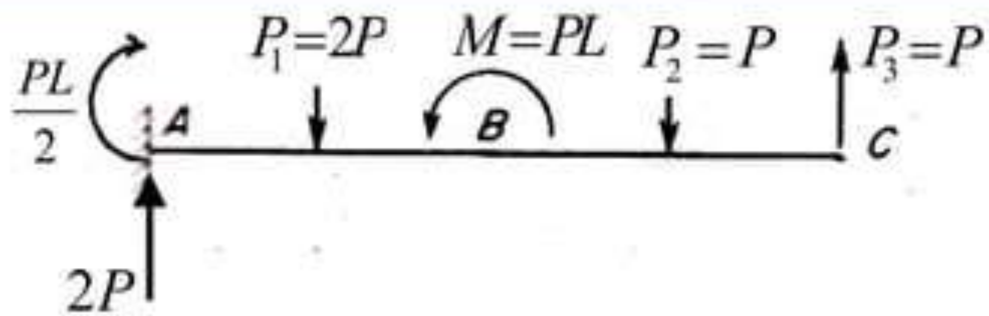
To get $\{A_{ML}\}$



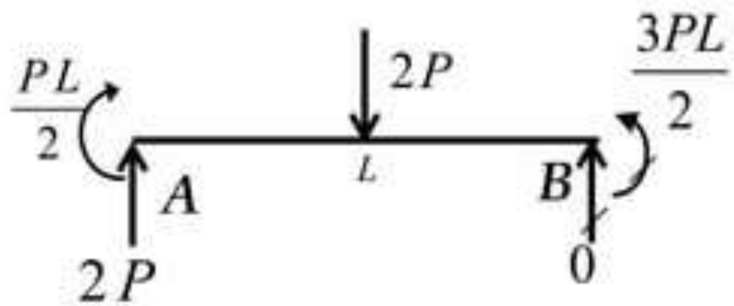
Member end actions considered

$A_{M1}, A_{M2} \cdots$ are the reactive moments at the end of members in the actual structure

$A_{ML1}, A_{ML2} \cdots$ are the reactive moments at the end of members in the released structure

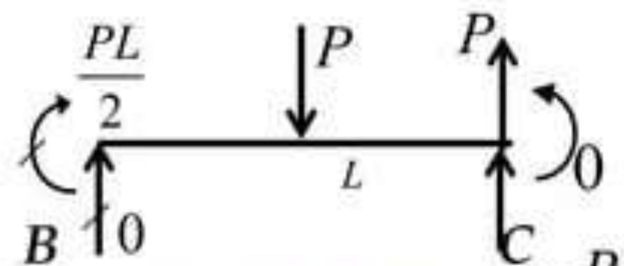


Released structure



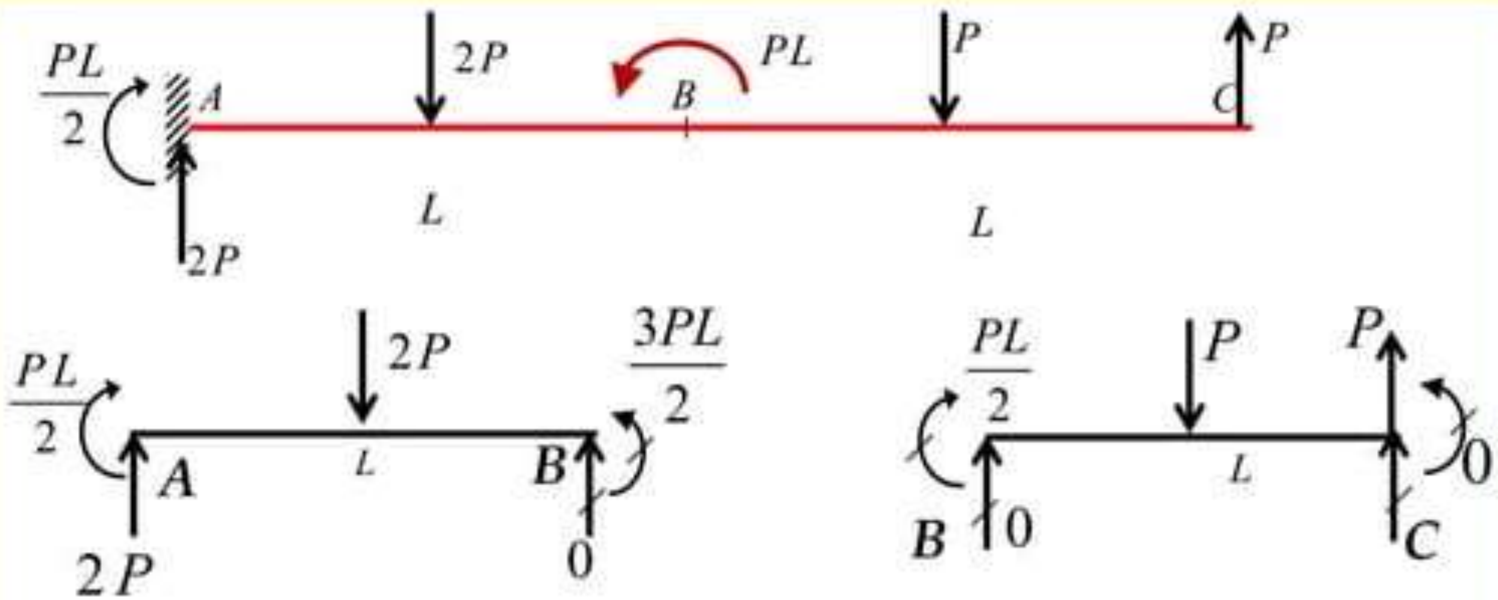
A_{ML1} = reactive moment just to the right of $\boldsymbol{A}$ $= \dfrac{-PL}{2}$

A_{ML2} = reactive moment just to the left of $\boldsymbol{B}$ $= \dfrac{3PL}{2}$



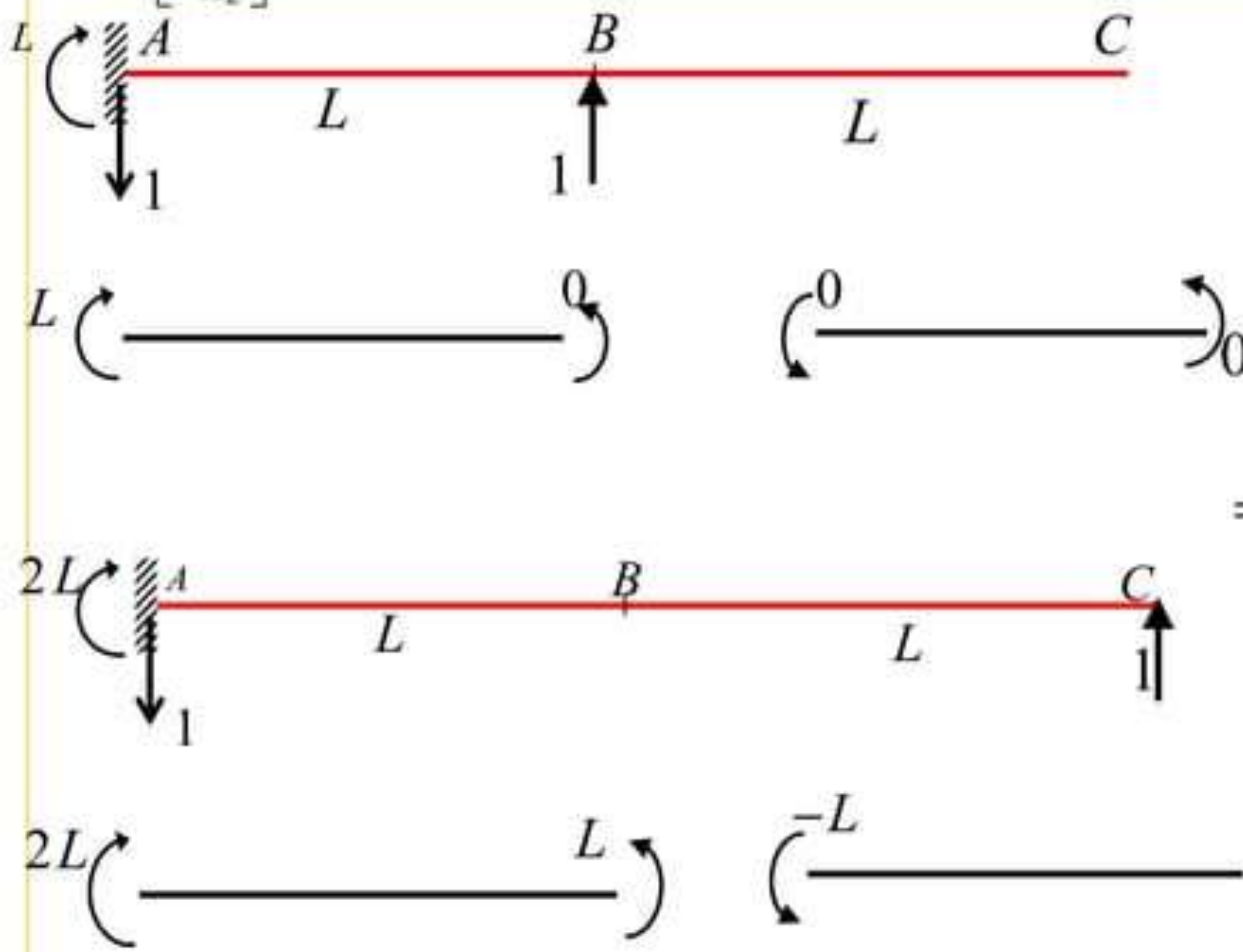
A_{ML3} = reactive moment just to the right of $B = \dfrac{-PL}{2}$

A_{ML4} = reactive moment just to the left of $C \;\; = 0$



$$\{A_{ML}\} = \begin{Bmatrix} A_{ML1} \\ A_{ML2} \\ A_{ML3} \\ A_{ML4} \end{Bmatrix} = \begin{Bmatrix} -PL/2 \\ 3PL/2 \\ -PL/2 \\ 0 \end{Bmatrix}$$

To get $\left[A_{MQ}\right]$



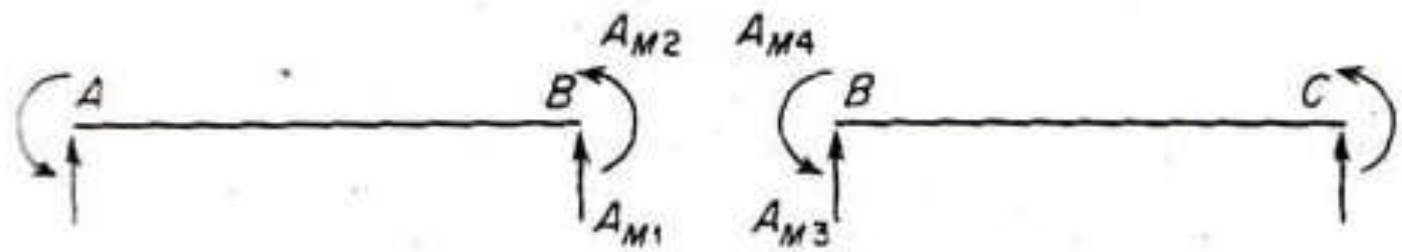
$$\left[A_{MQ}\right] = \begin{array}{c} \begin{array}{cc} Q_1=1 & Q_2=1 \end{array} \\ \begin{bmatrix} -L & -2L \\ 0 & L \\ 0 & -L \\ 0 & 0 \end{bmatrix} \end{array}$$

Hence, member end actions

$$\{A_M\} = \{A_{ML}\} + [A_{MQ}]\{Q\}$$

$$= \begin{Bmatrix} -PL/2 \\ 3PL/2 \\ -PL/2 \\ 0 \end{Bmatrix} + \begin{bmatrix} -L & -2L \\ 0 & L \\ 0 & -L \\ 0 & 0 \end{bmatrix} \frac{P}{56} \begin{bmatrix} 69 \\ -64 \end{bmatrix} = \begin{Bmatrix} 0.554PL \\ 0.357L \\ 0.643L \\ 0 \end{Bmatrix}$$

Member end actions (with a different choice of member end actions for analysis)



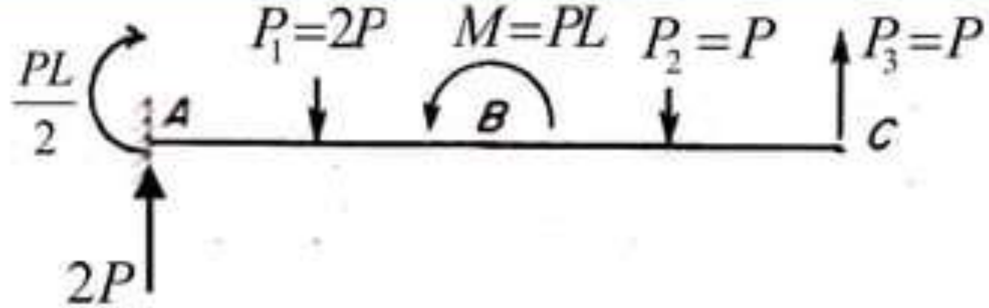
Member end actions considered

To get $\{A_{ML}\}$

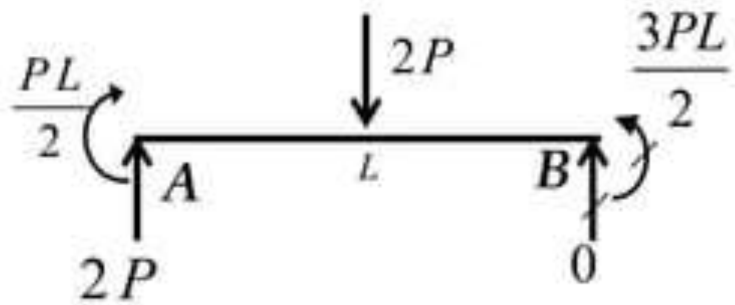
In the released structure,

A_{ML1}, A_{ML2} are SF and BM (equal to reactions) just to the left of B, and

A_{ML3}, A_{ML4} are SF and BM just to the right of B



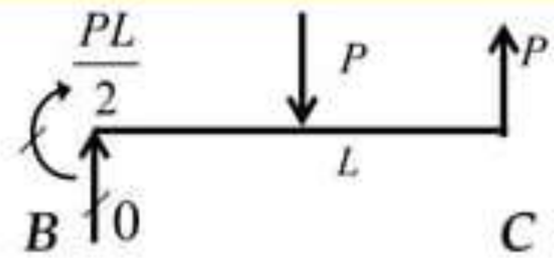
Released structure



A_{ML1} = Shear force just to the left of **B** $= P_3 - P_2 = 0$

A_{ML2} = Bending moment just to the left of **B**

$$= P_3 L - \frac{P_2 L}{2} + M = \frac{3PL}{2}$$



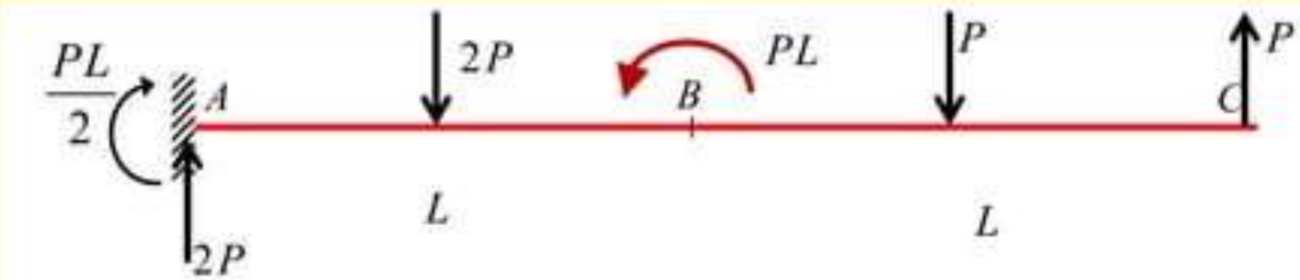
A_{ML3} = Shear force just to the right of B

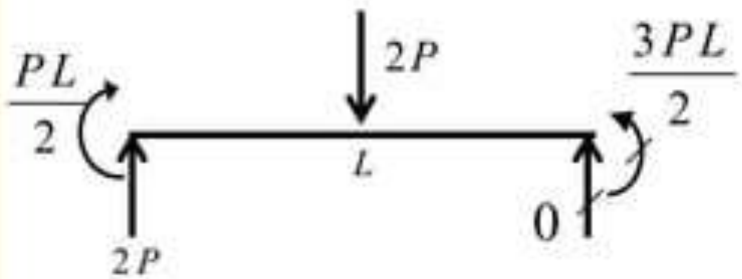
$$= 2P - P_1 = 2P - 2P = 0$$

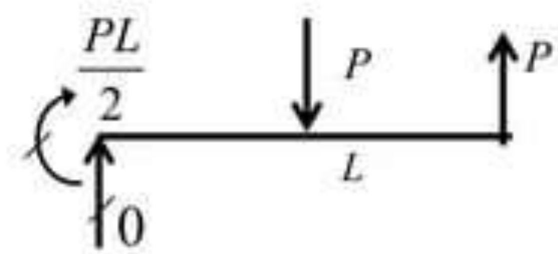
A_{ML4} = Bending moment just to the right of B

$$= \frac{-PL}{2} - 2PL + \frac{P_1L}{2} + M$$

$$= \frac{-PL}{2} - 2PL + \frac{2PL}{2} + PL = \frac{-PL}{2}$$

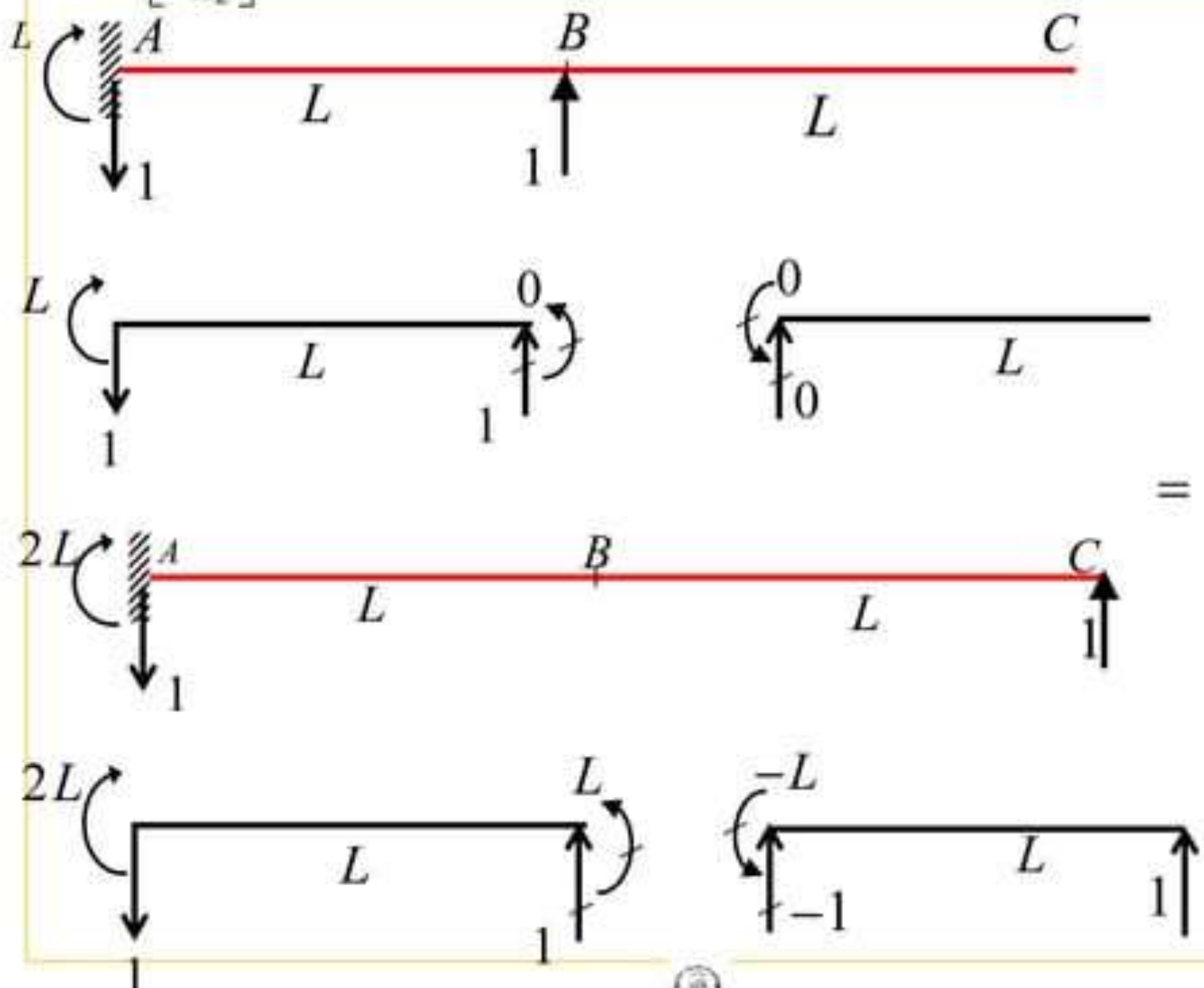






$$\{A_{ML}\} = \begin{Bmatrix} A_{ML1} \\ A_{ML2} \\ A_{ML3} \\ A_{ML4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{3PL}{2} \\ 0 \\ -\frac{PL}{2} \end{Bmatrix}$$

To get $\left[A_{MQ}\right]$



$$\left[A_{MQ}\right] = \begin{array}{cc} Q_1=1 & Q_2=1 \\ \end{array}$$

$$\left[A_{MQ}\right] = \begin{bmatrix} 1 & 1 \\ 0 & L \\ 0 & -1 \\ 0 & -L \end{bmatrix}$$

Hence, member end actions

$$\{A_M\} = \{A_{ML}\} + [A_{MQ}]\{Q\}$$

$$= \begin{Bmatrix} 0 \\ \dfrac{3PL}{2} \\ 0 \\ -\dfrac{PL}{2} \end{Bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & L \\ 0 & -1 \\ 0 & -L \end{bmatrix} \frac{P}{56} \begin{bmatrix} 69 \\ -64 \end{bmatrix} \qquad = \frac{P}{56} \begin{bmatrix} 5 \\ 20L \\ 64 \\ 36L \end{bmatrix}$$

Flexibilities of prismatic members

- Flexibility coefficients of a structure are calculated from the contributions of individual members

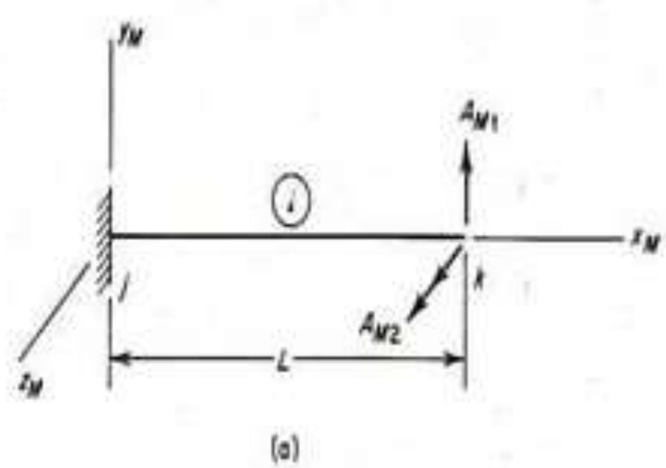
- Hence it is worthwhile to construct member flexibility matrices for various types of actions

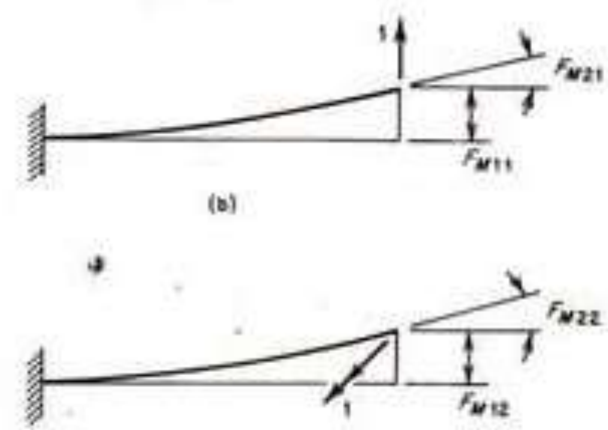
- Member oriented axes (local coordinates) and structure oriented axes (global coordinates)

Member flexibility matrices for prismatic members with one end fixed and the other free

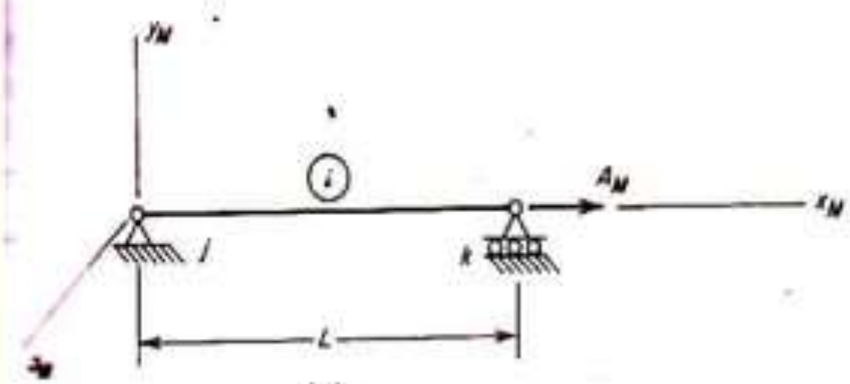
- Beam member

$$[F_{Mi}] = \begin{bmatrix} F_{M11} & F_{M12} \\ F_{M21} & F_{M22} \end{bmatrix} = \begin{bmatrix} \dfrac{L^3}{3EI} & \dfrac{L^2}{2EI} \\ \dfrac{L^2}{2EI} & \dfrac{L}{EI} \end{bmatrix}$$

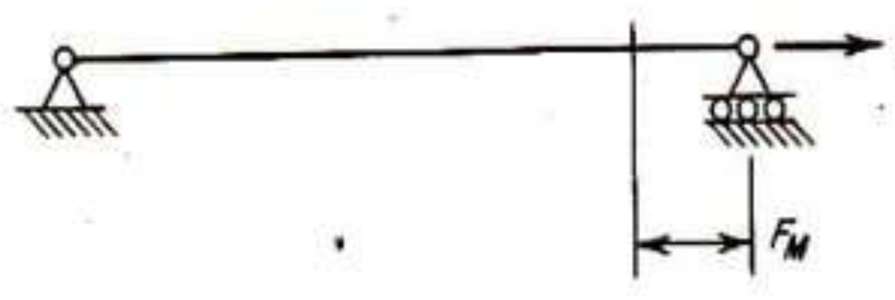




• Truss member $[F_{Mi}] = \dfrac{L}{EA}$

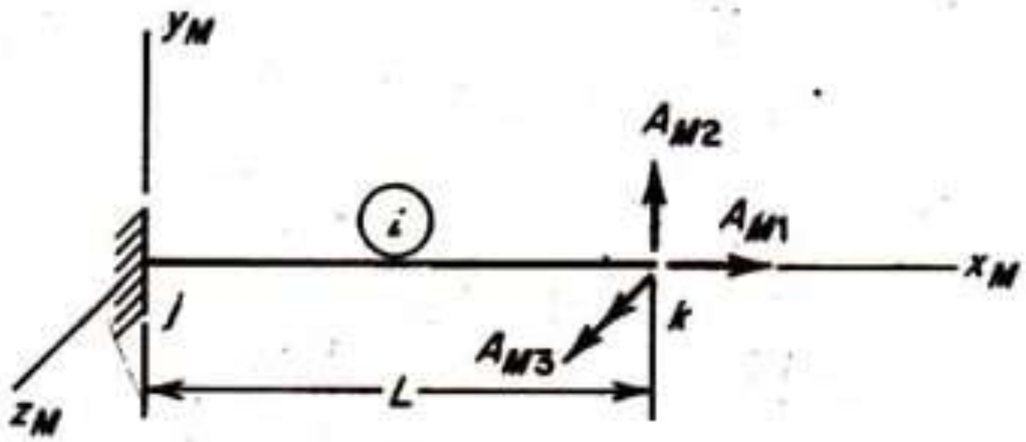


(a)



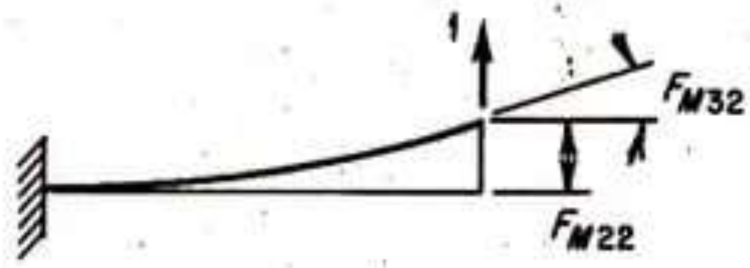
(b)

•Plane frame member



(a)

(b)

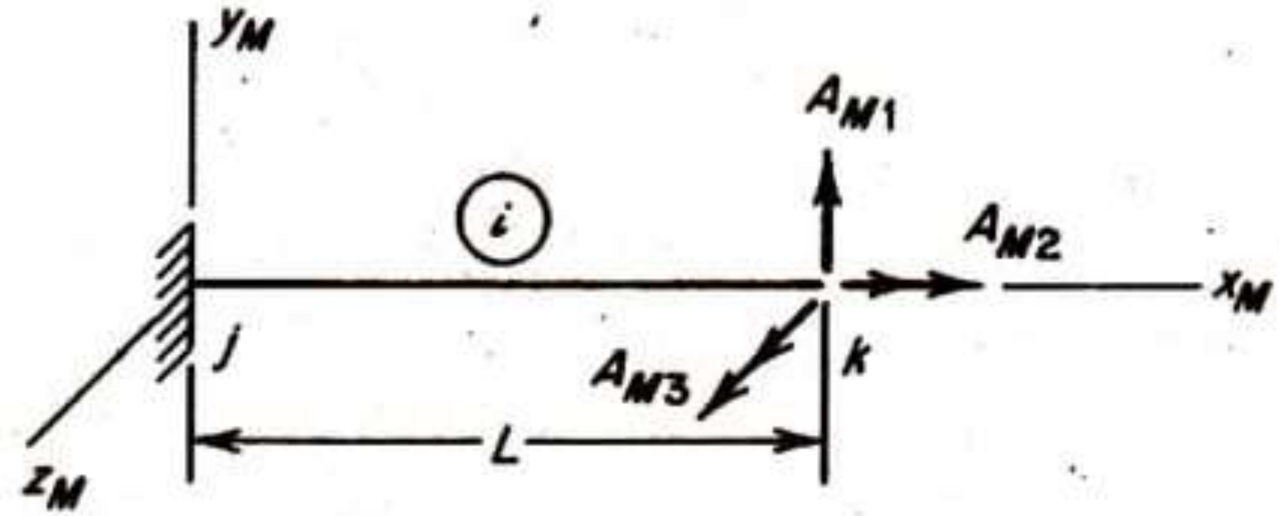


(c)

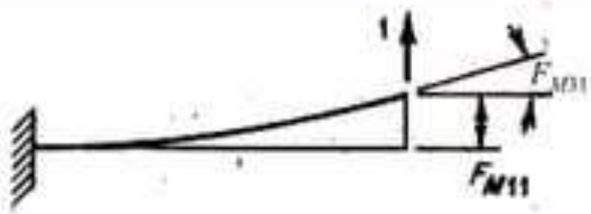
(d)

$$[F_{Mi}] = \begin{bmatrix} F_{M11} & F_{M12} & F_{M13} \\ F_{M21} & F_{M22} & F_{M23} \\ F_{M31} & F_{M32} & F_{M33} \end{bmatrix} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

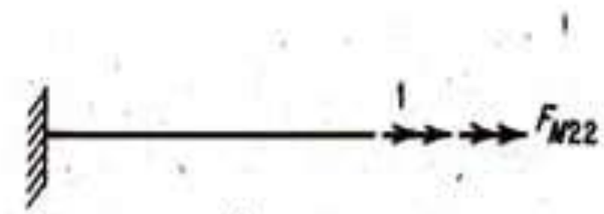
•Grid member



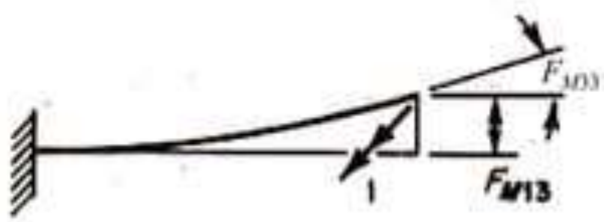
(a)



(b)



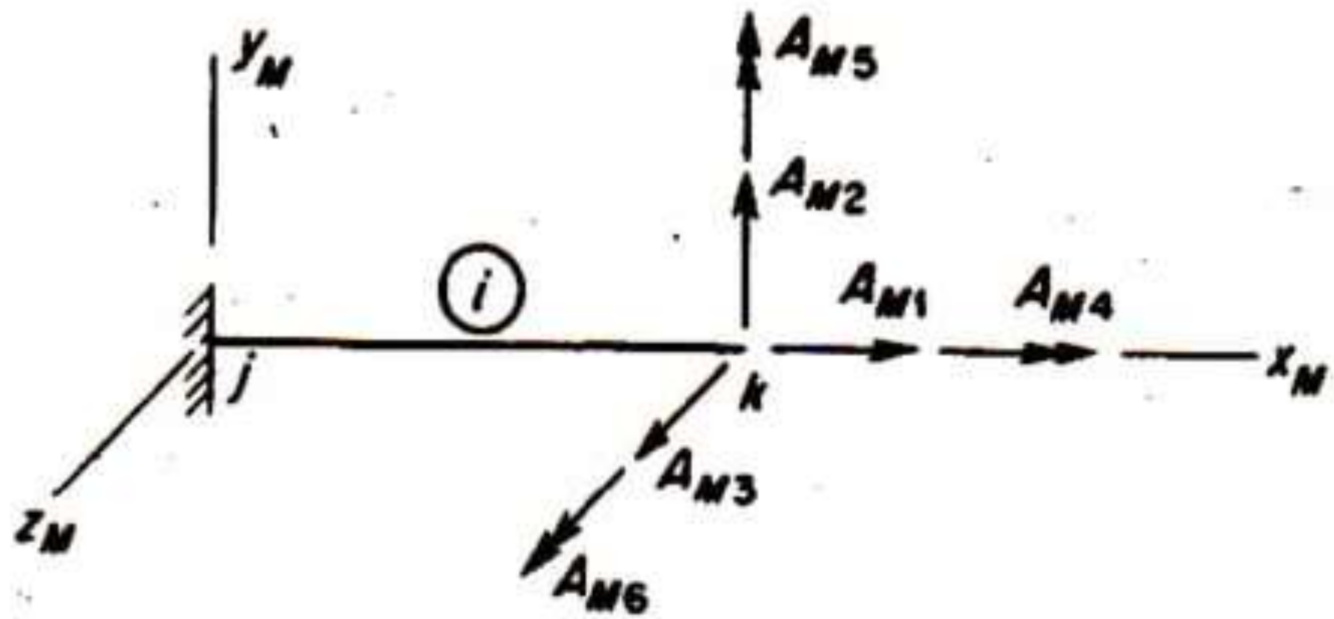
(c)



(d)

$$\left[F_{Mi}\right]=\begin{bmatrix} F_{M11} & F_{M12} & F_{M13} \\ F_{M21} & F_{M22} & F_{M23} \\ F_{M31} & F_{M32} & F_{M33} \end{bmatrix}=\begin{bmatrix} \dfrac{L^3}{3EI} & 0 & \dfrac{L^2}{2EI} \\ 0 & \dfrac{L}{GJ} & 0 \\ \dfrac{L^2}{2EI} & 0 & \dfrac{L}{EI} \end{bmatrix}$$

•Space frame member



(a)

$$[F_{Mi}] = \begin{bmatrix} \frac{L}{EA} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L^3}{3EI_Z} & 0 & 0 & 0 & \frac{L^2}{2EI_Z} \\ 0 & 0 & \frac{L^3}{3EI_Y} & 0 & \frac{-L^2}{2EI_Y} & 0 \\ 0 & 0 & 0 & \frac{L}{GJ} & 0 & 0 \\ 0 & 0 & \frac{-L^2}{2EI_Y} & 0 & \frac{L}{EI_Y} & 0 \\ 0 & \frac{L^2}{2EI_Z} & 0 & 0 & 0 & \frac{L}{EI_Z} \end{bmatrix}$$

Formalization of the Flexibility method

(Explanation using principle of complimentary virtual work)

For each member,

$$\{D_{Mi}\} = [F_{Mi}]\{A_{Mi}\}$$

Here $\{D_{Mi}\}$ contains *relative* displacements of the k end with respect to j end of the i-th member

- If there are *m* members in the structure,

$$\begin{Bmatrix} \{D_{M1}\} \\ \{D_{M2}\} \\ \{D_{M3}\} \\ \vdots \\ \{D_{Mi}\} \\ \vdots \\ \{D_{Mm}\} \end{Bmatrix} = \begin{bmatrix} [F_{M1}] & [0] & [0] & \cdots & [0] & \cdots & [0] \\ [0] & [F_{M2}] & [0] & \cdots & [0] & \cdots & [0] \\ [0] & [0] & [F_{M3}] & \cdots & [0] & \cdots & [0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [0] & [0] & [0] & \cdots & [F_{Mi}] & \cdots & [0] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [0] & [0] & [0] & \cdots & [0] & \cdots & [F_{Mm}] \end{bmatrix} \begin{Bmatrix} \{A_{M1}\} \\ \{A_{M2}\} \\ \{A_{M3}\} \\ \vdots \\ \{A_{Mi}\} \\ \vdots \\ \{A_{Mm}\} \end{Bmatrix}$$

$$\{D_M\} = [F_M]\{A_M\}$$

$$\{D_M\} = [F_M]\{A_M\}$$

$[F_M]$ is the *unassembled* flexibility matrix of the entire structure

- Member end actions in $\{A_M\}$ will be related to the structure actions $\{A_S\}$ applied to the *released structure*.

$\{A_S\}$ consists of joint loads $\{A_J\}$ and redundant actions $\{A_Q\}$

Hence, $\{A_M\} = \boxed{[B_{MS}]}\{A_S\}$

Action transformation matrix (equilibrium matrix)

$$\text{i.e.,}\quad \boxed{\{A_M\} = \begin{bmatrix}[B_{MJ}] & [B_{MQ}]\end{bmatrix}\begin{Bmatrix}\{A_J\} \\ \{A_Q\}\end{Bmatrix}}$$

$\left[B_{MJ}\right]$ relate $\{A_M\}$ to $\{A_J\}$ and

$\left[B_{MQ}\right]$ relate $\{A_M\}$ to $\{A_Q\}$

- Each column in the submatrix $[B_{MJ}]$ consists of member end actions caused by a unit value of a joint load applied to the *released structure*.

- Each column in the submatrix $[B_{MQ}]$ consists of member end actions caused by a unit value of a redundant applied to the *released structure*.

•Suppose an arbitrary set of virtual actions $\{\delta A_S\}$ is applied on the structure.

$$\{\delta A_M\} = [B_{MS}]\{\delta A_S\} = \begin{bmatrix}[B_{MJ}] & [B_{MQ}]\end{bmatrix}\begin{Bmatrix}\{\delta A_J\} \\ \{\delta A_Q\}\end{Bmatrix}$$

External complimentary virtual work produced by the virtual loads $\{\delta A_S\}$ and actual displacements $\{D_S\}$ is

$$\delta W^* = \{\delta A_S\}^{\mathrm{T}}\{D_S\} = \begin{bmatrix}\{\delta A_J\}^{\mathrm{T}} & \{\delta A_Q\}^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}D_J \\ D_Q\end{Bmatrix}$$

Internal complimentary virtual work produced by the virtual member end actions $\{\delta A_M\}$ and actual (relative) end displacements $\{D_M\}$ is

$$\delta U^* = \{\delta A_M\}^{\mathrm{T}} \{D_M\}$$

- Equating the above two (principle of complimentary virtual work),

$$\{\delta A_S\}^{\mathrm{T}}\{D_S\}=\{\delta A_M\}^{\mathrm{T}}\{D_M\}$$

But $\{A_M\}=[B_{MS}]\{A_S\}$ and $\{D_M\}=[F_M]\{A_M\}$

Also, $\{\delta A_M\}=[B_{MS}]\{\delta A_S\}$

Hence, $\{\delta A_S\}^{\mathrm{T}}\{D_S\}=\{\delta A_S\}^{\mathrm{T}}[B_{MS}]^{\mathrm{T}}[F_M][B_{MS}]\{A_S\}$

$$\{D_S\} = [B_{MS}]^T [F_M][B_{MS}]\{A_S\}$$

$$\boxed{\{D_S\} = [F_S]\{A_S\}}$$

Where,

$$\boxed{[F_S] = [B_{MS}]^T [F_M][B_{MS}]}$$

, the *assembled flexibility matrix* for the entire structure.

$[F_S]$ is partitioned into submatrices related to:

joint loads $\{A_J\}$

and redundant actions $\{A_Q\}$

$$\{D_S\} = [F_S]\{A_S\} \Rightarrow \begin{Bmatrix} \{D_J\} \\ \{D_Q\} \end{Bmatrix} = \begin{bmatrix} [F_{JJ}] & [F_{JQ}] \\ [F_{QJ}] & [F_{QQ}] \end{bmatrix} \begin{Bmatrix} \{A_J\} \\ \{A_Q\} \end{Bmatrix}$$

Where,

$$[F_{JJ}] = [B_{MJ}]^{\mathrm{T}}[F_M][B_{MJ}] \qquad [F_{JQ}] = [B_{MJ}]^{\mathrm{T}}[F_M][B_{MQ}]$$

$$[F_{QJ}] = [B_{MQ}]^{\mathrm{T}}[F_M][B_{MJ}] \qquad [F_{QQ}] = [B_{MQ}]^{\mathrm{T}}[F_M][B_{MQ}]$$

$$\{D_J\} = [F_{JJ}]\{A_J\} + [F_{JQ}]\{A_Q\}$$

$$\{D_Q\} = [F_{QJ}]\{A_J\} + [F_{QQ}]\{A_Q\}$$

$$\Rightarrow \boxed{\{A_Q\} = [F_{QQ}]^{-1}\left[\{D_Q\} - [F_{QJ}]\{A_J\}\right]}$$

In the subsequent calculations, the above $\{A_Q\}$ should be used.

However, the final values of redundants are obtained by including actual or equivalent joint loads applied directly to the supports.

Thus, $\{A_Q\}_{FINAL} = -\{A_{QC}\} + \{A_Q\}$

$\{A_{QC}\}$ represents actual and equivalent joint loads applied directly to the supports, corresponding to redundants.

- Once redundants $\{A_Q\}$ are found,

  $\{D_J\}$ can be found out from,

$$\{D_J\} = [F_{JJ}]\{A_J\} + [F_{JQ}]\{A_Q\}$$

- Similarly, support reactions caused by joint loads and redundant can be obtained with an action transformation matrix $[B_{RS}]$

$$\{A_R\} = [B_{RS}]\{A_S\} = \left[[B_{RJ}] \quad [B_{RQ}]\right]\begin{Bmatrix}\{A_J\} \\ \{A_Q\}\end{Bmatrix}$$

- Each column in the submatrix $[B_{RJ}]$ consists of support reactions caused by a unit value of a joint load applied to the *released structure*.

- Each column in the submatrix $[B_{RQ}]$ consists of support reactions caused by a unit value of a redundant applied to the *released structure*.

- If actual or equivalent joint loads are applied directly to the supports,

$$\{A_R\} = -\{A_{RC}\} + [B_{RJ}]\{A_J\} + [B_{RQ}]\{A_Q\}$$

$\{A_{RC}\}$ represents combined joint loads (actual and equivalent) applied directly to the supports.

- As seen earlier, member end actions due to actual loads are obtained by superimposing member end actions due to restraint actions and combined joint loads

$$\boxed{\{A_M\} = \{A_{MF}\} + [B_{MJ}]\{A_J\} + [B_{MQ}]\{A_Q\}}$$

where $\{A_{MF}\}$ represents fixed end actions

Comparison of the procedures explained with *principle of superposition* and *principle of complimentary virtual work*

- For calculating redundants,

Principle of superposition $\{Q\}=[F]^{-1}\left(\{D_Q\}-\{D_{QL}\}\right)$

Principle of complimentary virtual work

$$\{A_Q\}=\left[F_{QQ}\right]^{-1}\left[\{D_Q\}-\left[F_{QJ}\right]\{A_J\}\right]$$

Hence, $[F]=\left[F_{QQ}\right]$

$\{Q\}=\{A_Q\}$ and $\{D_{QL}\}=\left[F_{QJ}\right]\{A_J\}$

- For calculating joint displacements,

Principle of superposition $\{D_J\} = \{D_{JL}\} + [D_{JQ}]\{Q\}$

Principle of complimentary virtual work

$$\{D_J\} = [F_{JJ}]\{A_J\} + [F_{JQ}]\{A_Q\}$$

Hence, $\{D_{JL}\} = [F_{JJ}]\{A_J\}$ and

$$[D_{JQ}] = [F_{JQ}]$$

- For calculating member end actions,

Principle of superposition $\{A_M\} = \{A_{ML}\} + [A_{MQ}]\{Q\}$

Principle of complimentary virtual work

$$\{A_M\} = \{A_{MF}\} + [B_{MJ}]\{A_J\} + [B_{MQ}]\{A_Q\}$$

Hence, $\{A_{ML}\} = \{A_{MF}\} + [B_{MJ}]\{A_J\}$ and

$$[A_{MQ}] = [B_{MQ}]$$

•For calculating support reactions,

Principle of superposition $\{A_R\}=\{A_{RL}\}+\left[A_{RQ}\right]\{Q\}$

Principle of complimentary virtual work

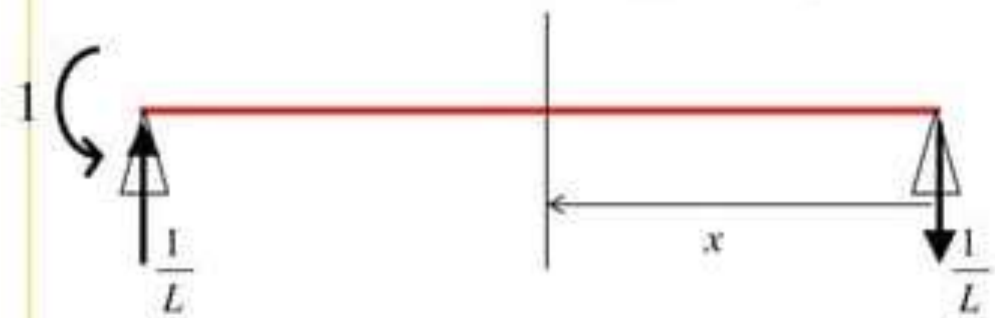
$$\{A_R\}=-\{A_{RC}\}+\left[B_{RJ}\right]\{A_J\}+\left[B_{RQ}\right]\{A_Q\}$$

Hence, $\{A_{RL}\}=-\{A_{RC}\}+\left[B_{RJ}\right]\{A_J\}$ and

$$\left[A_{RQ}\right]=\left[B_{RQ}\right]$$

Member flexibility matrix for a beam member with moments at the ends as member end actions

Required to find out rotations at the ends due to unit moments at each end separately



$$M_x = \frac{-x}{L} = -EI\frac{d^2y}{dx^2}$$

$$EI\frac{dy}{dx} = \frac{x^2}{2L} + C_1$$

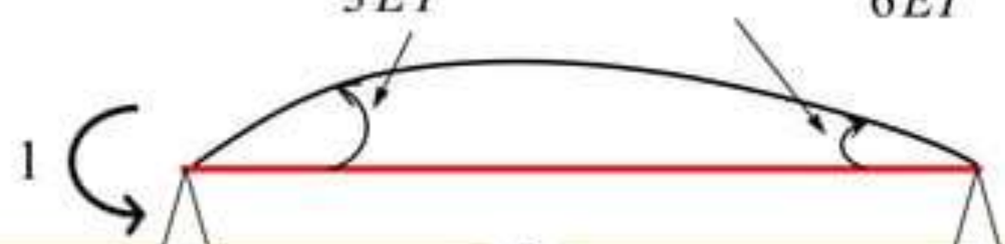
$$EIy = \frac{x^3}{6L} + C_1x + C_2$$

$$\underset{x=0}{y} = 0 \Rightarrow C_2 = 0 \qquad \underset{x=L}{y} = 0 \Rightarrow C_1 = \frac{-L}{6}$$

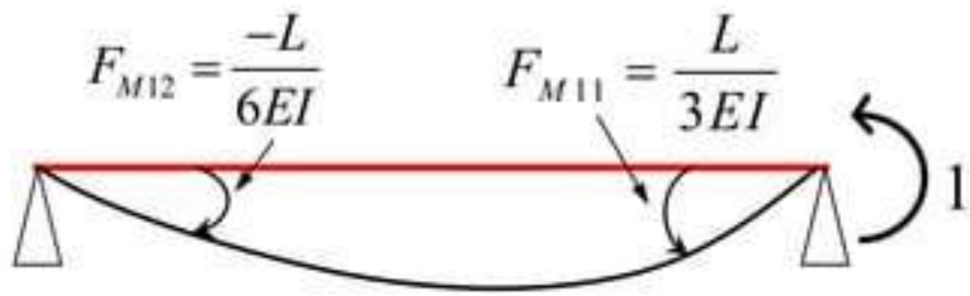
$$\therefore \frac{dy}{dx} = \frac{1}{EI}\left(\frac{x^2}{2L} - \frac{L}{6}\right)$$

$$\underset{x=0}{y'} = \frac{-L}{6EI}, \qquad \underset{x=L}{y'} = \frac{L}{3EI}$$

$$F_{M11} = \frac{L}{3EI} \qquad F_{M21} = \frac{-L}{6EI}$$

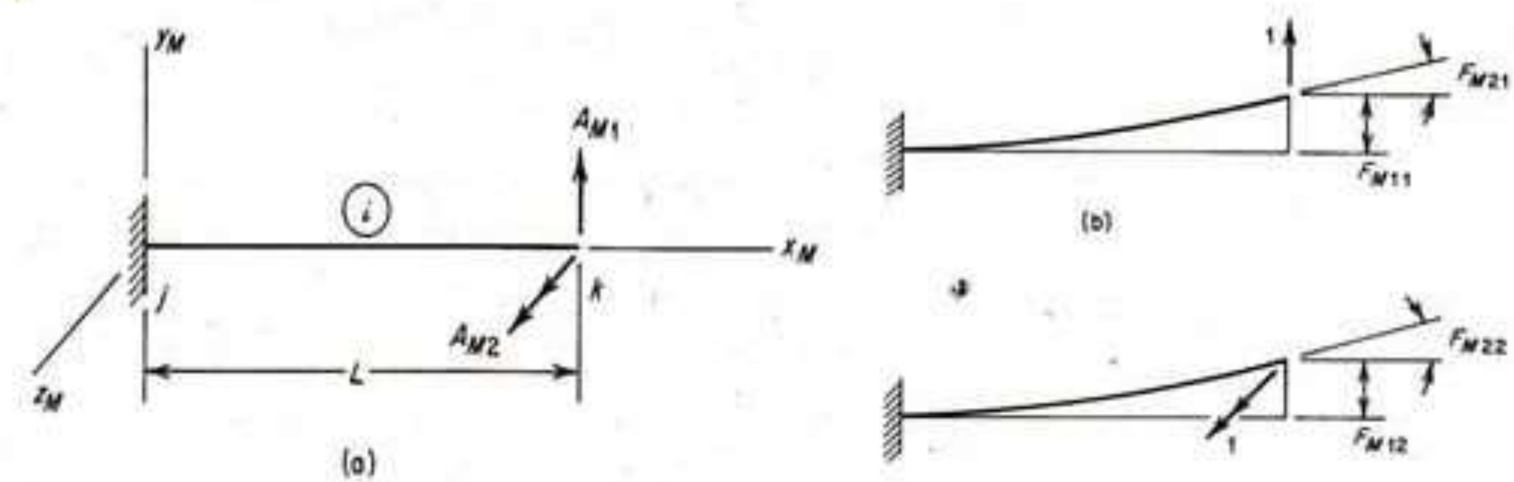


Similarly,



$$\therefore \left[F_{Mi}\right] = \begin{bmatrix} F_{M11} & F_{M12} \\ F_{M21} & F_{M22} \end{bmatrix} = \begin{bmatrix} \frac{L}{3EI} & \frac{-L}{6EI} \\ \frac{-L}{6EI} & \frac{L}{3EI} \end{bmatrix}$$

Member flexibility matrix for a beam member with moment and shear at one end as member end actions



$$[F_{Mi}] = \begin{bmatrix} F_{M11} & F_{M12} \\ F_{M21} & F_{M22} \end{bmatrix} = \begin{bmatrix} \dfrac{L^3}{3EI} & \dfrac{L^2}{2EI} \\ \dfrac{L^2}{2EI} & \dfrac{L}{EI} \end{bmatrix}$$

Summary

Flexibility method

- Flexibility matrices for truss beam and frame elements – load transformation matrix-development of total flexibility matrix of the structure .

James Clerk Maxwell (13 June 1831 – 5 November 1879) was a Scottish theoretical physicist and mathematician. His most important achievement was classical electromagnetic theory. Maxwell also developed the Maxwell–Boltzmann distribution, a statistical means of describing aspects of the kinetic theory of gases. These two discoveries helped usher in the era of modern physics, laying the foundation for such fields as special relativity and quantum mechanics.

Maxwell is also known for creating the first true colour photograph in 1861 and for his foundational work on the rigidity of rod-and-joint frameworks like those in many bridges.

Maxwell is considered by many physicists to be the 19th-century scientist with the greatest influence on 20th-century physics. His contributions to the science are considered by many to be of the same magnitude as those of Isaac Newton and Albert Einstein. Einstein himself described Maxwell's work as the "most profound and the most fruitful that physics has experienced since the time of Newton." Einstein kept a photograph of Maxwell on his study wall, alongside pictures of Michael Faraday and Newton.

Christian Otto Mohr (October 8, 1835 – October 2, 1918) was a German civil engineer, one of the most celebrated of the nineteenth century.
Starting in 1855, his early working life was spent in railroad engineering for the Hanover and Oldenburg state railways, designing some famous bridges and making some of the earliest uses of steel trusses. Even during his early railway years, Mohr had developed an interest in the theories of mechanics and the strength of materials. In 1867, he became professor of mechanics at Stuttgart Polytechnic, and in 1873 at Dresden Polytechnic in 1873. In 1874, Mohr formalised the idea of a statically determinate structure.

In 1882, he famously developed the graphical method for analysing stress known as Mohr's circle and used it to propose an early theory of strength based on shear stress. He also developed the Williot-Mohr diagram for truss displacements and the Maxwell-Mohr method for analysing statically indeterminate structures, it can also be used to determine the displacement of truss nodes and forces acting on each member. The Maxwell-Mohr method is also referred to as the virtual force method for redundant trusses.

Introduction

- The systematic development of slope deflection method in this matrix is called as a stiffness method.
- This method is a powerful tool for analysing indeterminate structures.
- Stiffness method of analysis of structure also called as displacement method.
- In the method of displacement are used as the basic unknowns.

Procedure

1. Determine degree of kinematic indeterminacy of structure.
2. Select unknown displacement
3. Restrain all the joints to set fully restrained structure under given condition.
4. For analysis of the restrain structure to get ADL.
5. Generate stiffness matrix of given structure apply unit positive displacement for members and add all the displacement for members meeting at a joint.

6. Superposition equation.

{AD} = {ADL} + [S] {D}

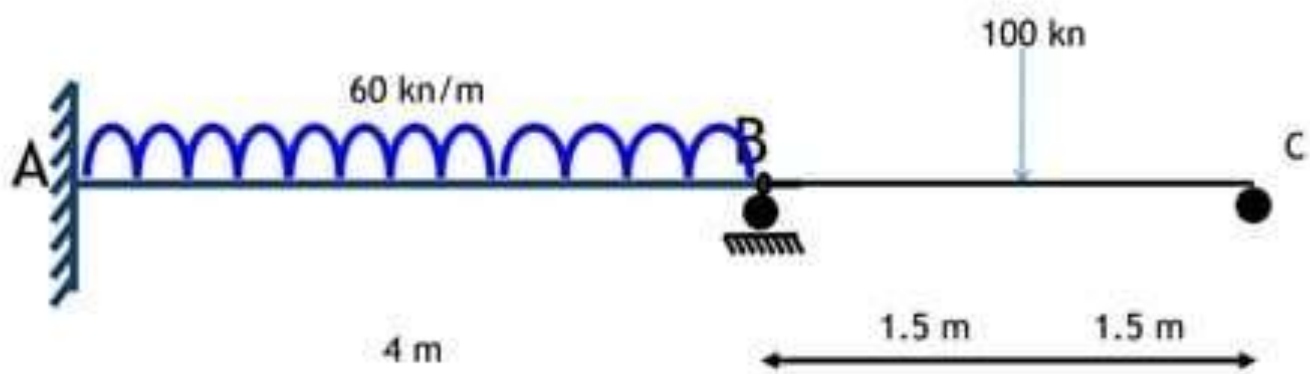
- {AD} = Joint action or forces given in structure
- [ADL]= Force analysis of restrained structure under given loading
- [S] = Stiffness method
- {D} = Unknown

7. **Determine the final moments.**
8. **Calculation for SF and draw SF, BM diagram.**

Properties of stiffness matrix

- Stiffness matrix is a square matrix of order n*n, where n is equal to KI.
- Stiffness matrix is symmetrical matrix. Hence, sij=sji.
- Sii =represents action due to unit positive displacement and while other displacement are 0.
- Sii is the principle diagonal element.
- Stiffness matrix does not exist for unstable structure.
- Stiffness matrix is non-singular matrix [s] is not equal to 0 for stable structure.
- Sii is the action at joint due to unit value of displacement at J joint

Analysis given beam by stiffness method.
Take EI is constant.

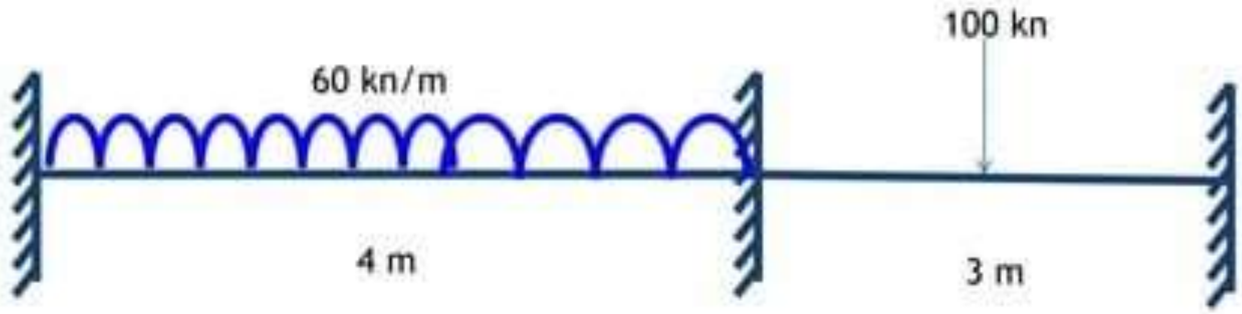


Step 1: Degree of kinematic indeterminancy
Dki = 2

- Step 2: Select unknown displacement

  $D_1 = \theta B$ $D_2 = \theta C$

Step 3: Restrain the Structure to get kinematical determinate structure



Step 4: Free analysis of restrained structure to get ADL

Mab = - Wl^2 / 12 = - 80 kn.m

Mba = + 80 kn.m

Mbc = -W*l / 8 = - 37.5 kn.m

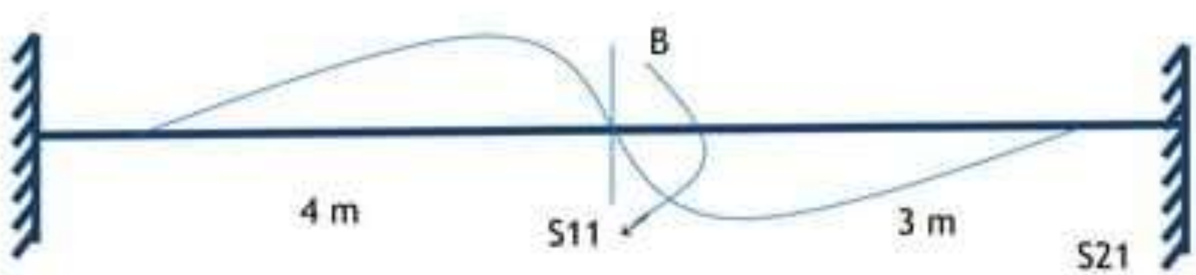
Mcb = +37.5 kn.m

ADL1 = 80-37.5 = 42.5

ADL2 = 37.5

Step 5 : Stiffness matrix

1. Apply unit rotation at joint B



S_{11}= (4EI/L)+(4EI/L)

= (4EI/4)+(4EL/3)

= 2.33EI

S_{21}= 2EI/L

= 2EI/3

= 0.67EI

Apply unit rotation at C

S12= 2EI/L

= 2EI/3

= 0.67EI

S22= 4EI/L

= 4EI/3

= 1.33EI

Superposition equation :

$$\{ AD \} = \{ ADL \} + [S] \{ D \}$$

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 42.5 \\ 37.5 \end{Bmatrix} + \begin{bmatrix} 2.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix} EI \begin{Bmatrix} \theta B \\ \theta C \end{Bmatrix}$$

0= 42.5 + 2.33EIθB+ 0.67EIθC
0= 37.5 + 0.67EIθB+ 1.33EIθC

ΘB= -22.22/EI
Θc= -11.85/EI

- Final End moment :

Mab = Mfab + 2EI / L (2 Θa . Θb - 3Δ/L)

= -80 + 2EI / 4 (-11.85/EI)

= -85.92 kn.m

Mba = 68.16 kn.m

Mbc = 68.15 kn.m

Mcb = 0 kn.m

Span moments :

AB= wl^2/8

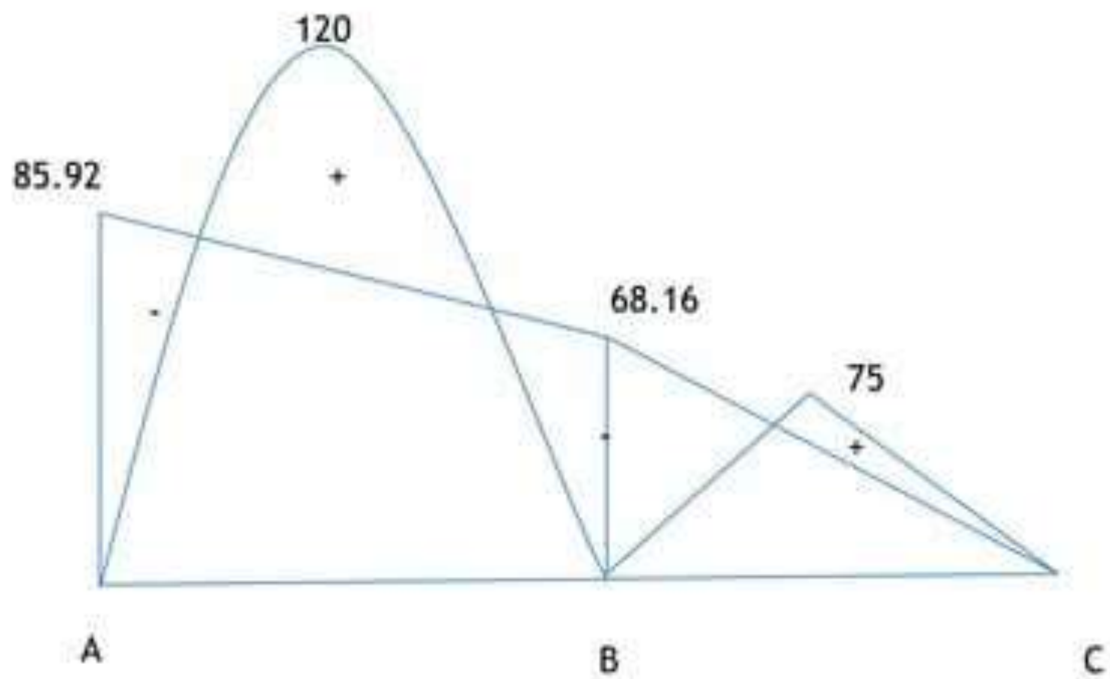
= 60*4^2/8

= 120 kn.m

BC= wl/4

= 100*3/4

= 75 kn.m


120
85.92
+
-
68.16
75
-
+
A
B
C

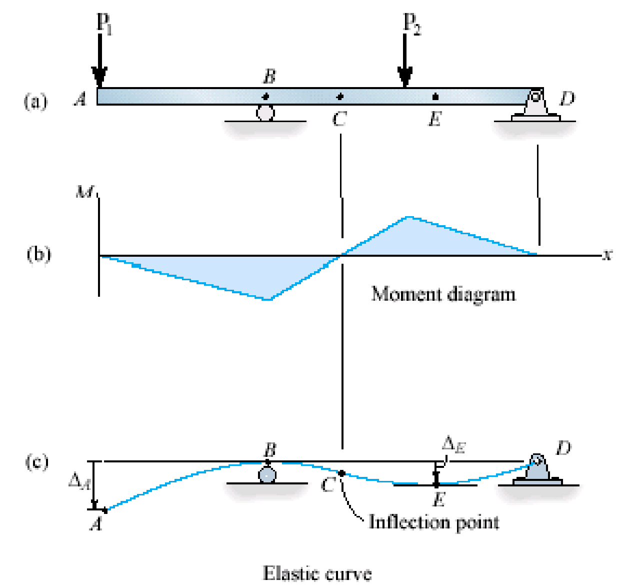
Deflections

Introduction

- Calculation of deflections is an important part of structural analysis
- Excessive beam deflection can be seen as a mode of failure.
  - Extensive glass breakage in tall buildings can be attributed to excessive deflections
  - Large deflections in buildings are unsightly (and unnerving) and can cause cracks in ceilings and walls.
  - Deflections are limited to prevent undesirable vibrations

Beam Deflection

- Bending changes the initially straight longitudinal axis of the beam into a curve that is called the **Deflection Curve** or **Elastic Curve**



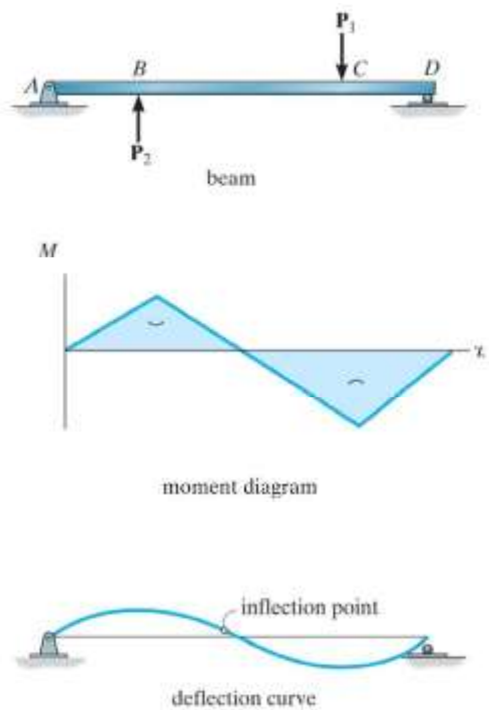
Beam Deflection

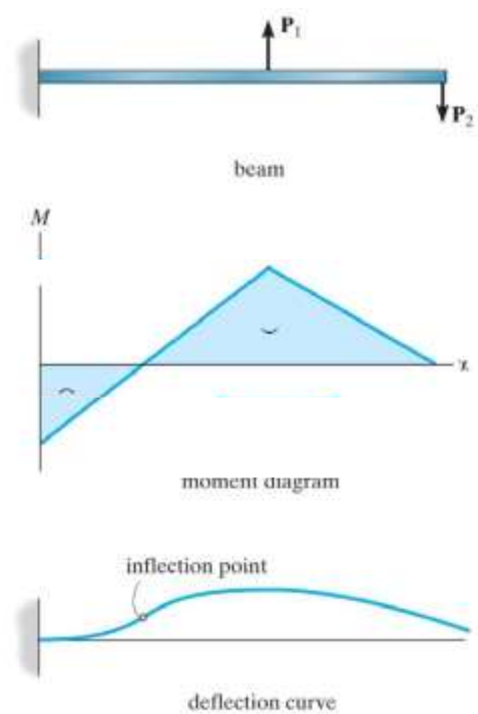
- To determine the deflection curve:
  - Draw shear and moment diagram for the beam
  - Directly under the moment diagram draw a line for the beam and label all supports
  - At the supports displacement is zero
  - Where the moment is negative, the deflection curve is concave downward.
  - Where the moment is positive the deflection curve is concave upward
  - Where the two curve meet is the Inflection Point

positive moment,
concave upward

negative moment,
concave downward

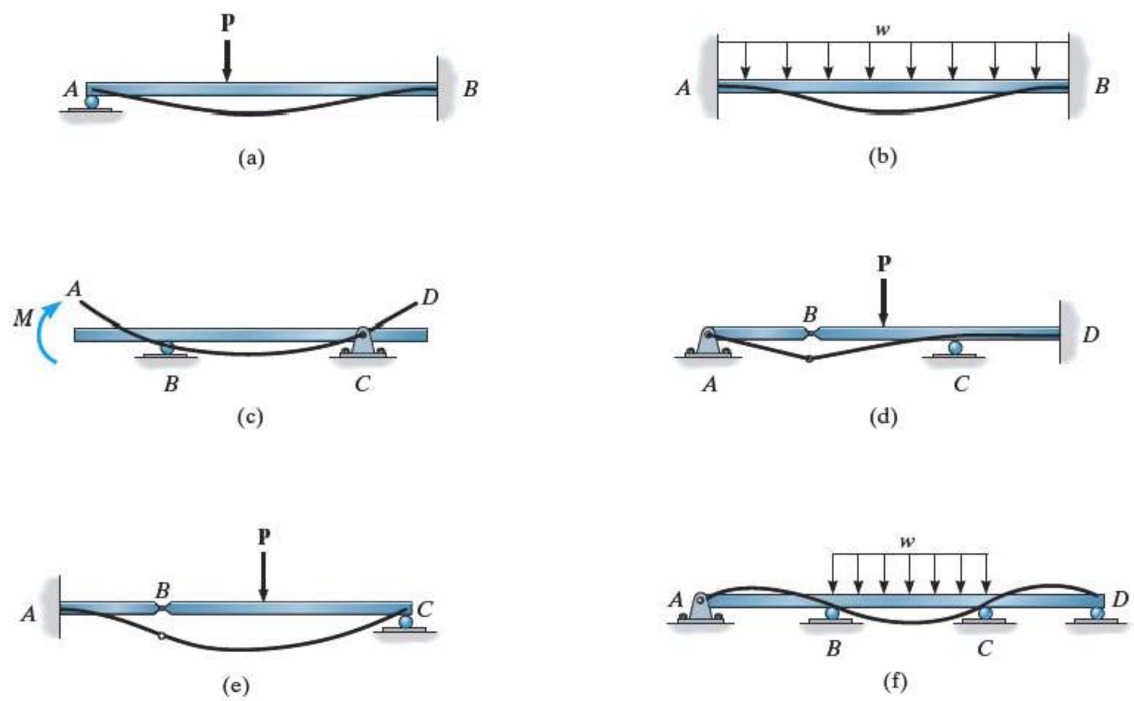
Deflected Shape





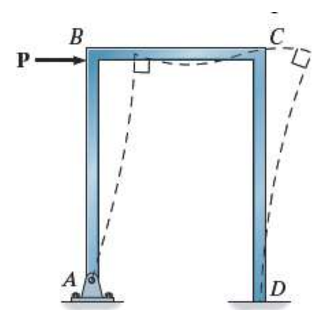
Example 1

Draw the deflected shape for each of the beams shown

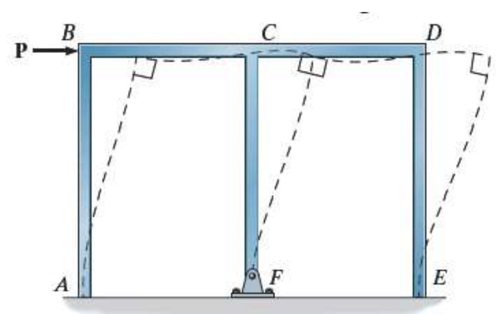


Example 2

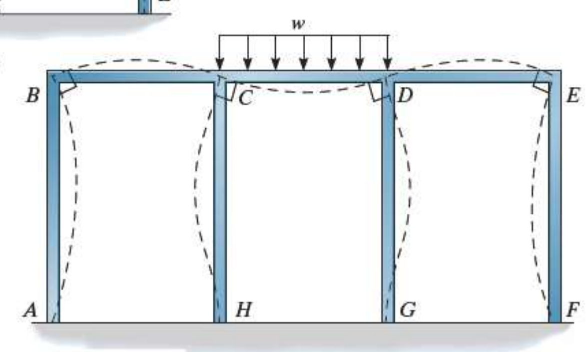
Draw the deflected shape for each of the frames shown



(a)



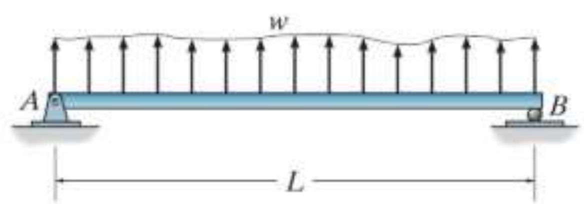
(b)



(c)

Conjugate-Beam Method

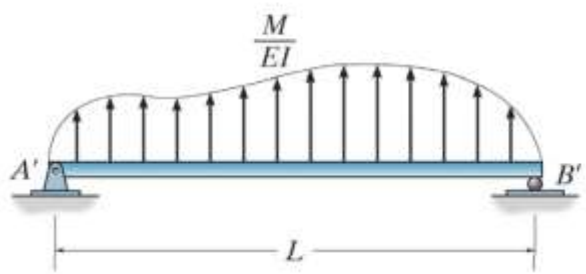
Conjugate-Beam Method



real beam

$$\frac{dV}{dx} = w \qquad \frac{d^2M}{dx^2} = w$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \frac{d^2v}{dx^2} = \frac{M}{EI}$$



conjugate beam

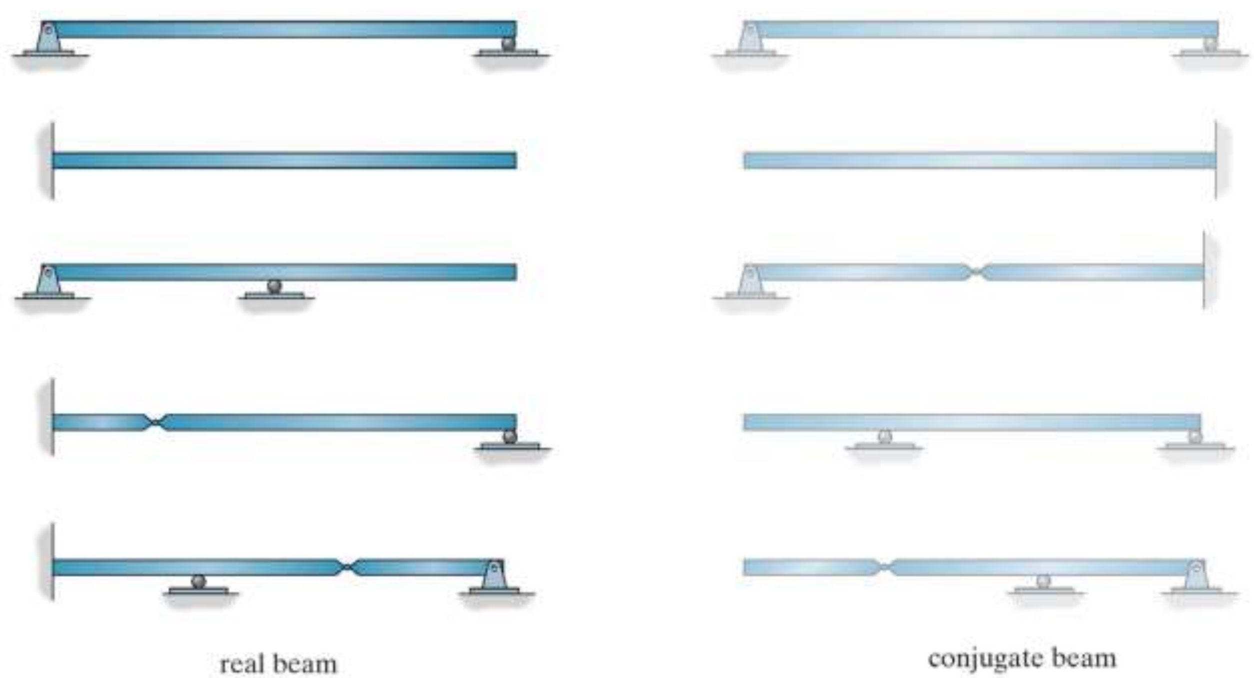
Integrating

$$V = \int w\,dx \qquad M = \int\left[\int w\,dx\right]dx$$

$$\theta = \int\left(\frac{M}{EI}\right)dx \qquad v = \int\left[\int\left(\frac{M}{EI}\right)dx\right]dx$$

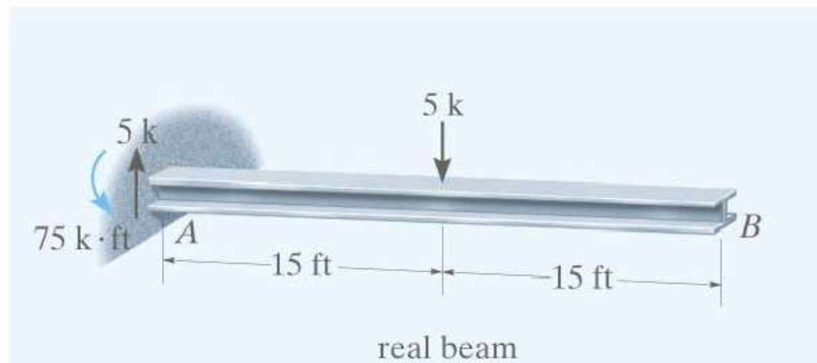
Conjugate-Beam Supports

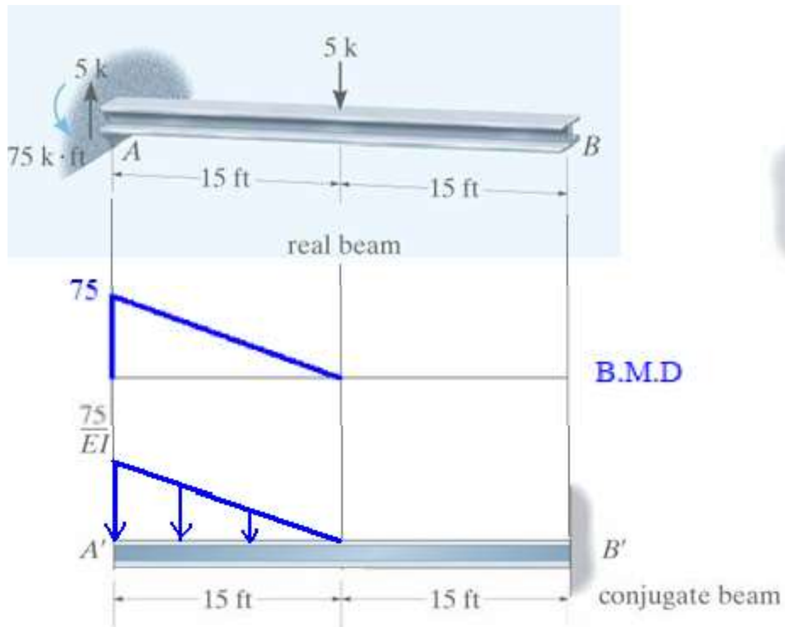
	Real Beam		Conjugate Beam	
1)	θ $\Delta = 0$	pin	V $M = 0$	pin
2)	θ $\Delta = 0$	roller	V $M = 0$	roller
3)	$\theta = 0$ $\Delta = 0$	fixed	$V = 0$ $M = 0$	free
4)	θ Δ	free	V M	fixed
5)	θ $\Delta = 0$	internal pin	V $M = 0$	hinge
6)	θ $\Delta = 0$	internal roller	V $M = 0$	hinge
7)	θ Δ	hinge	V M	internal roller

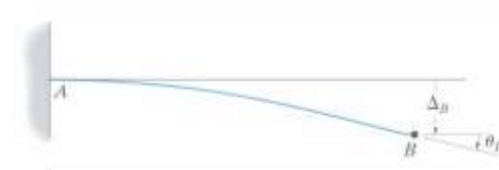

real beam
conjugate beam

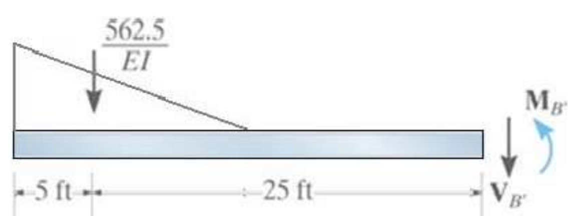
Example 1

Find the Max. deflection *Take* $E=200Gpa$, $I=60(10^6)$







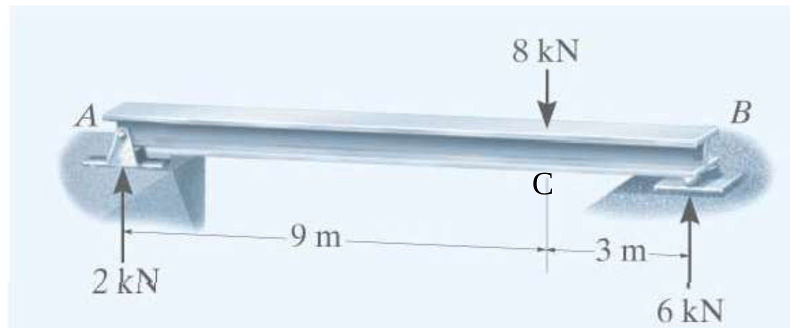


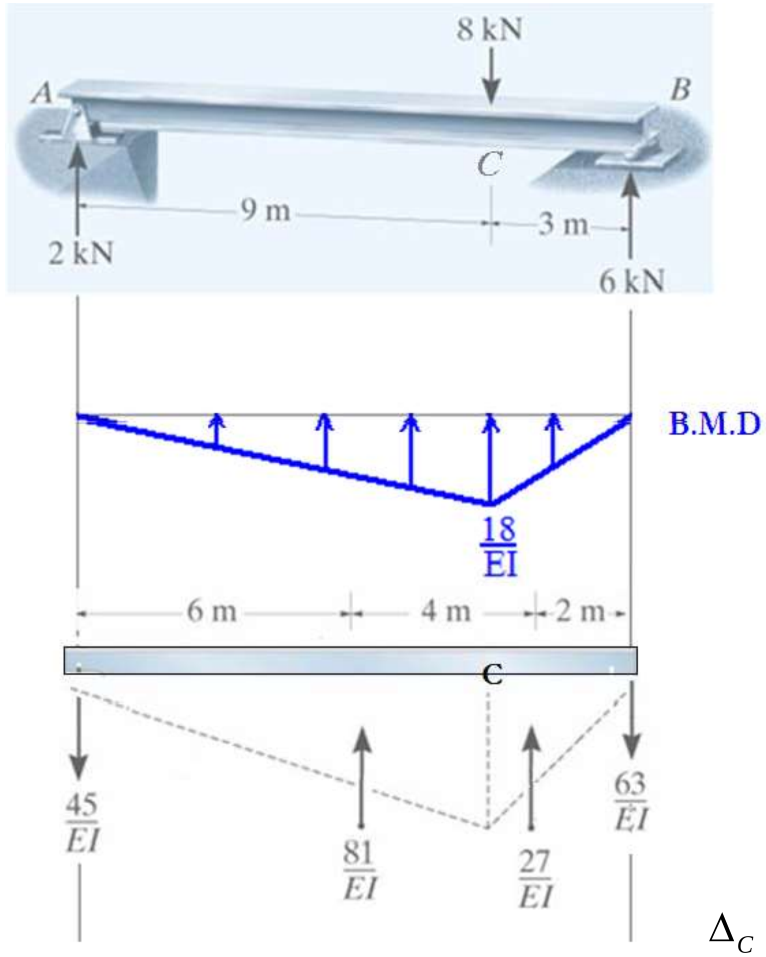
$$\theta_B = V_{B'} = -\frac{562.5}{EI}$$

$$\Delta_B = M_{B'} = \frac{562.5}{EI}(25) = \frac{-14062.5}{EI}$$

Example

Find the deflection at Point C

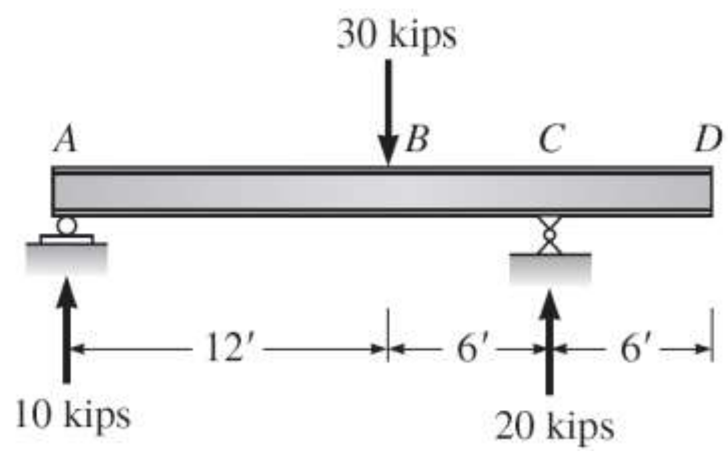


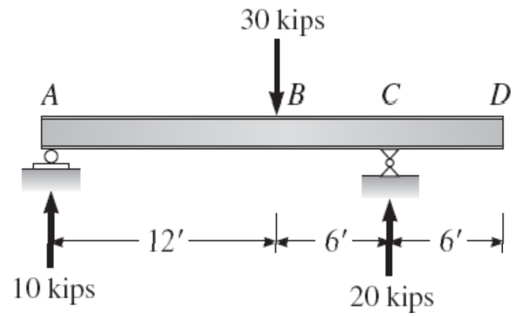


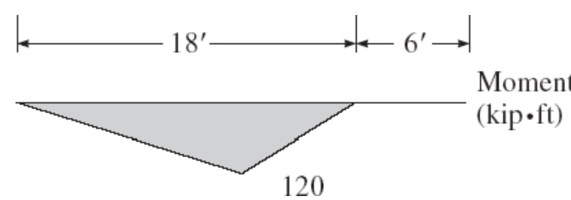
$$\Delta_C = M_{C'} = \frac{27}{EI}(1) - \frac{63}{EI}(3) = \frac{-162}{EI}$$

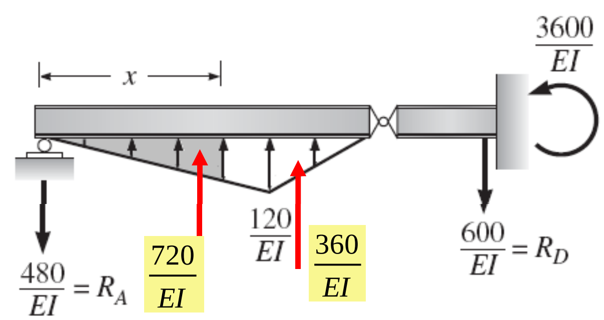
Example

Find the deflection at Point D









$$\circlearrowleft^{+} \quad \Sigma M_{\text{hinge}} = 0$$

$$-18R_A + \frac{720(10)}{EI} + \frac{360(4)}{EI} = 0$$

$$R_A = \frac{480}{EI}$$

$$+\uparrow \quad \Sigma F_y = 0$$

$$\frac{720}{EI} + \frac{360}{EI} - \frac{480}{EI} - R_D = 0$$

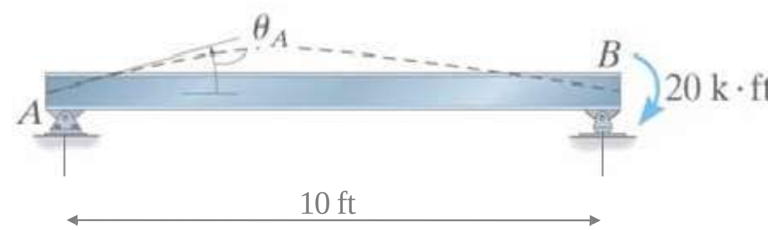
$$R_D = \frac{600}{EI}$$

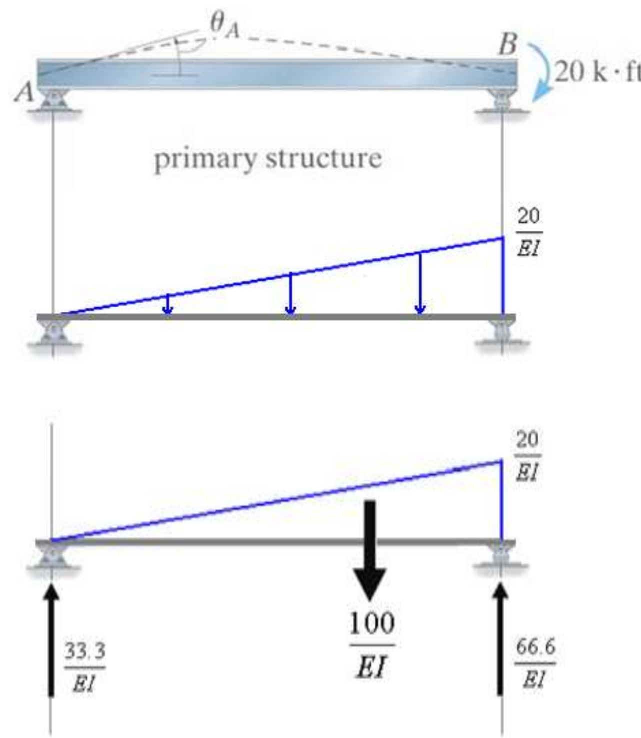
$$M_{D'} = \frac{600}{EI}(6) = \frac{3600}{EI}$$

$$\boxed{\Delta_D = M_{D'} = \frac{3600}{EI}}$$

Example

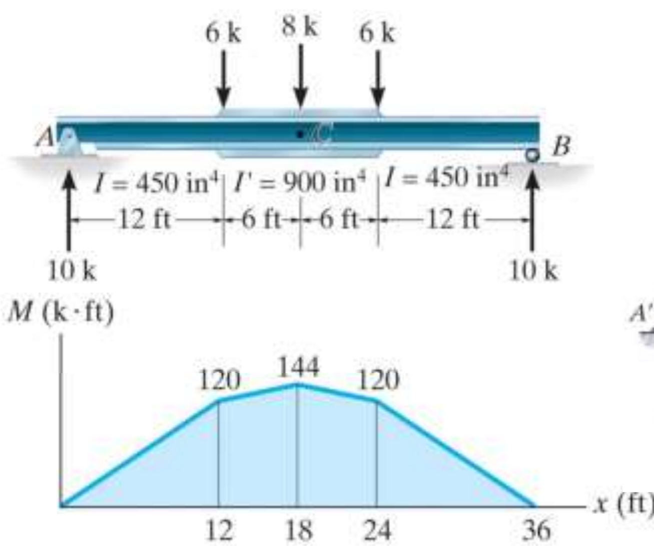
Find the Rotation at A



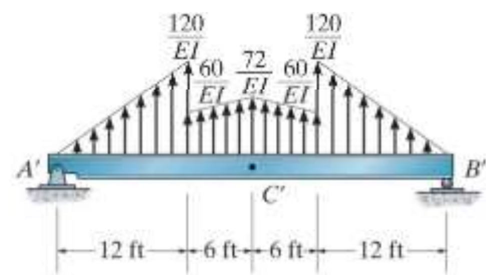


$$_A = \frac{33.3}{EI}$$

Example



moment diagram



conjugate beam

EXAMPLE 8-14 (Continued)

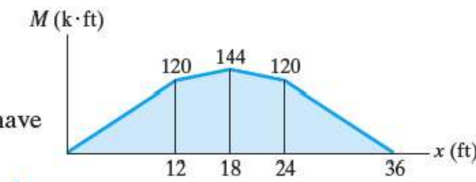
$$\circlearrowleft + \Sigma M_{C'} = 0; \quad \frac{1116}{EI}(18) - \frac{720}{EI}(10) - \frac{360}{EI}(3) - \frac{36}{EI}(2) + M_{C'} = 0$$

$$M_{C'} = -\frac{11\,736 \text{ k}\cdot\text{ft}^3}{EI}$$

Substituting the numerical data for EI and converting units, we have

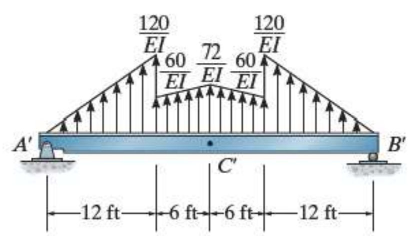
$$\Delta_C = M_{C'} = -\frac{11\,736 \text{ k}\cdot\text{ft}^3(1728 \text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(450 \text{ in}^4)} = -1.55 \text{ in.} \quad \textit{Ans.}$$

The negative sign indicates that the deflection is downward.



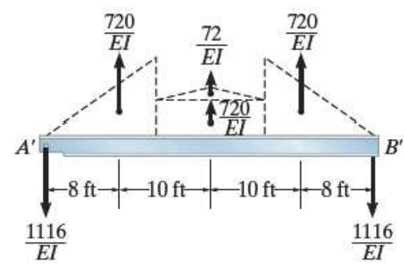
moment diagram

(b)



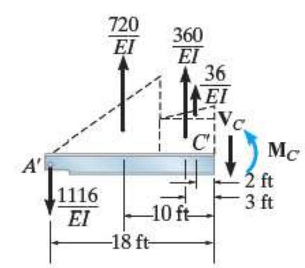
conjugate beam

(c)



external reactions

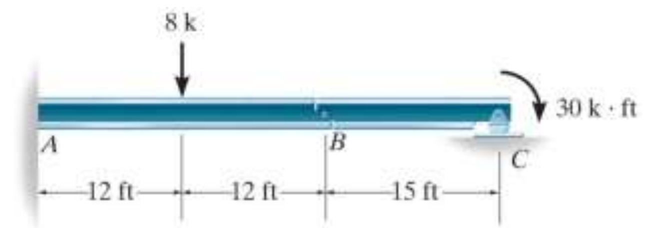
(d)



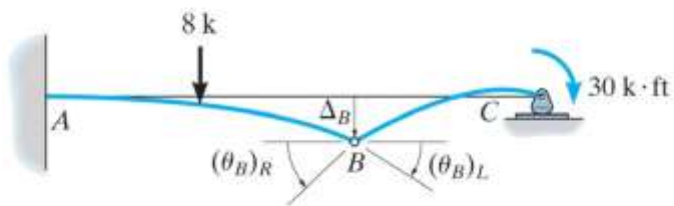
internal reactions

(e)

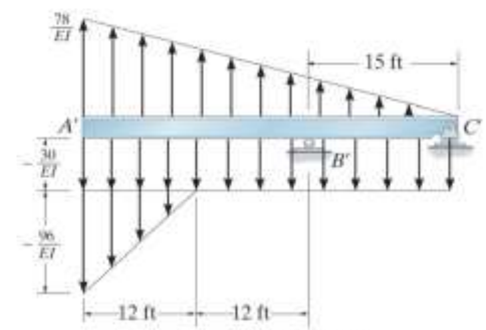
Example



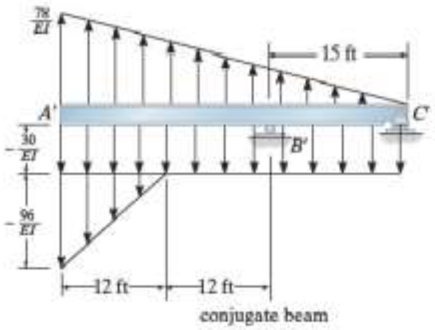
real beam



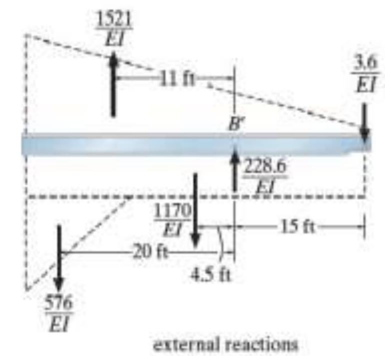
elastic curve



conjugate beam

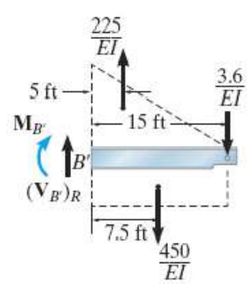


(c)



(d)

Equilibrium. The external reactions at B' and C' are calculated first and the results are indicated in Fig. 8–27*d*. In order to determine $(\theta_B)_R$, the conjugate beam is sectioned just to the *right* of B' and the shear force $(V_{B'})_R$ is computed, Fig. 8–27*e*. Thus,



(e)

$$+\uparrow \Sigma F_y = 0; \qquad (V_{B'})_R + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

$$(\theta_B)_R = (V_{B'})_R = \frac{228.6\ \mathrm{k \cdot ft^2}}{EI}$$

$$= \frac{228.6\ \mathrm{k \cdot ft^2}}{[29(10^3)(144)\ \mathrm{k/ft^2}][30/(12)^4]\ \mathrm{ft^4}}$$

$$= 0.0378\ \mathrm{rad}$$ ***Ans.***

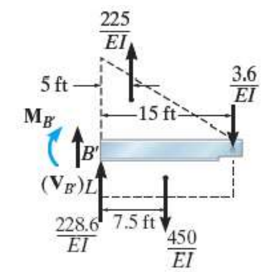
The internal moment at B' yields the displacement of the pin. Thus,

$$\circlearrowleft+\Sigma M_{B'} = 0; \qquad -M_{B'} + \frac{225}{EI}(5) - \frac{450}{EI}(7.5) - \frac{3.6}{EI}(15) = 0$$

$$\Delta_B = M_{B'} = -\frac{2304 \text{ k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-2304 \text{ k}\cdot\text{ft}^3}{[29(10^3)(144)\text{ k/ft}^2][30/(12)^4]\text{ ft}^4}$$

$$= -0.381 \text{ ft} = -4.58 \text{ in.} \qquad \textit{Ans.}$$



(f)

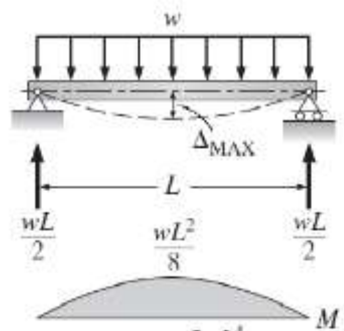
The slope $(\theta_B)_L$ can be found from a section of beam just to the *left* of B', Fig. 8–27*f*. Thus,

$$+\uparrow\Sigma F_y = 0; \qquad (V_{B'})_L + \frac{228.6}{EI} + \frac{225}{EI} - \frac{450}{EI} - \frac{3.6}{EI} = 0$$

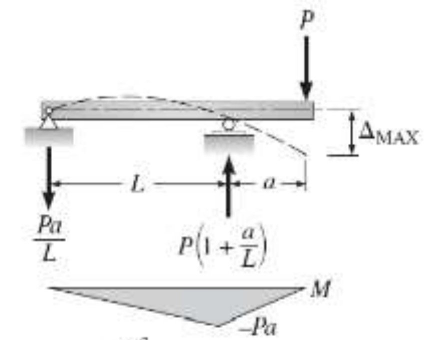
$$(\theta_B)_L = (V_{B'})_L = 0 \qquad \textit{Ans.}$$

Obviously, $\Delta_B = M_{B'}$ for this segment is the *same* as previously calculated, since the moment arms are only slightly different in Figs. 8–27*e* and 8–27*f*.

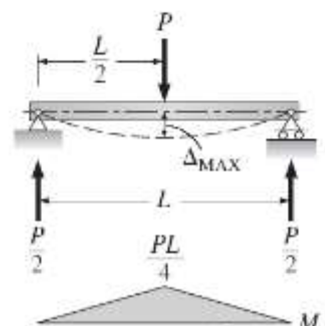
Moment Diagrams and Equations for Maximum Deflection



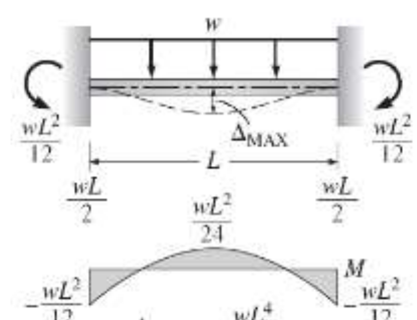
$$\Delta_{MAX} = \frac{5wL^4}{384EI}$$



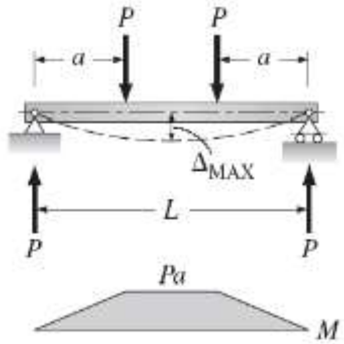
$$\Delta_{MAX} = \frac{Pa^2}{3EI}(L + a)$$



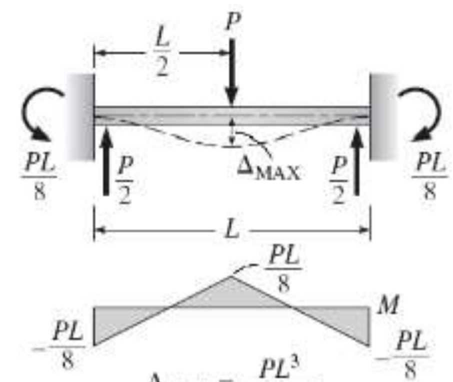
$$\Delta_{MAX} = \frac{PL^3}{48EI}$$



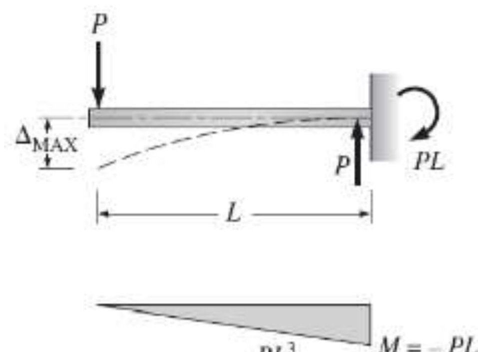
$$\Delta_{MAX} = \frac{wL^4}{384EI}$$



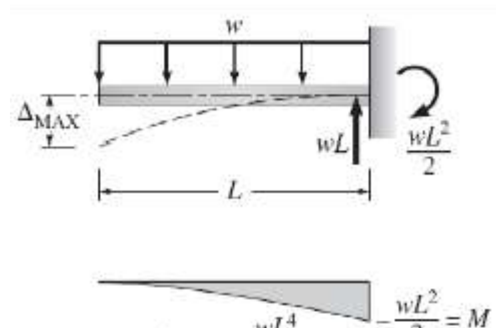
$$\Delta_{MAX} = \frac{Pa}{24EI}(3L^2 - 4a^2)$$



$$\Delta_{MAX} = \frac{PL^3}{192EI}$$



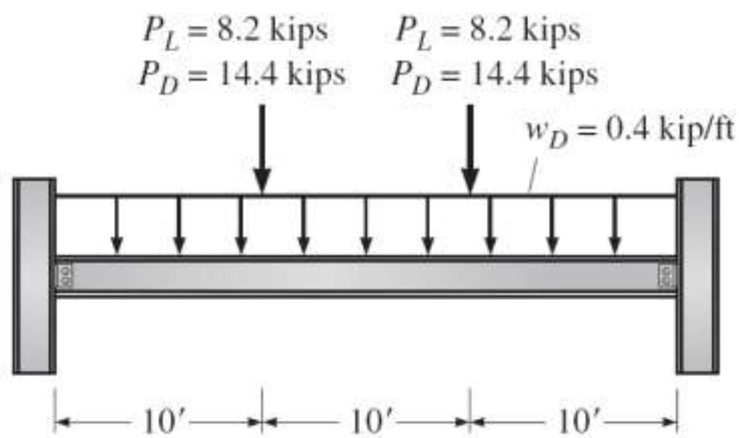
$$\Delta_{MAX} = \frac{PL^3}{3EI}$$



$$\Delta_{MAX} = \frac{wL^4}{8EI}$$

Example 4

Find the Maximum deflection for the following structure based on
The previous diagrams



(*a*) Dead load deflection produced by uniform load is

$$\Delta_{D1} = \frac{5wL^4}{384EI} = \frac{5(0.4)(30)^4(1728)}{384(30{,}000)(758)} = 0.32 \text{ in}$$

Dead load deflection produced by concentrated loads is

$$\Delta_{D2} = \frac{Pa(3L^2 - 4a^2)}{24EI} = \frac{14.4(10)[3(30)^2 - 4(10)^2](1728)}{24(30{,}000)(758)}$$

$$\Delta_{D2} = 1.05 \text{ in}$$

Total dead load deflection, $\Delta_{DT} = \Delta_{D1} + \Delta_{D2} = 0.32 + 1.05 = 1.37$ in

Live load deflection, $\Delta_L = \dfrac{Pa(3L^2 - 4a^2)}{24EI} = \dfrac{8.2(10)[3(30)^2 - 4(10)^2](1728)}{24(30{,}000)(758)}$

$$\Delta_L = 0.6 \text{ in}$$