

# **NUMERICAL METHODS & PROBABILITY THEORY (20A54402)**

## **LECTURE NOTES**

### **II-B.TECH & I-SEM**

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## COURSE MATERIAL

<b>SUBJECT</b>	<b>NUMERICAL METHODS AND PROBABILITY THEORY (20A54402)</b>
<b>UNIT</b>	<b>1</b>
<b>COURSE</b>	<b>B. TECH</b>
<b>SEMESTER</b>	<b>2 - 2</b>
<b>DEPARTMENT</b>	<b>HUMANITIES &amp; SCIENCE</b>
<b>PREPARED BY (Faculty Name/s)</b>	<b>Department of Mathematics</b>

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## 1. COURSE OBJECTIVES

The objectives of this course is to

- This course aims at providing the student with the knowledge on Various numerical methods for solving equations, interpolating the polynomials, evaluation of integral equations and solution of differential equations.
- The theory of Probability and random variables.

## 2. PREREQUISITES

Students should have knowledge on

1. Students should have knowledge on Numerical Techniques
2. Students should have knowledge on Probability Theory

## 3. SYLLABUS

### UNIT I

#### **Solution of Algebraic & Transcendental Equations:**

Introduction-Bisection method-Iterative method-Regula falsi method-Newton Raphson method System of Algebraic equations: Gauss Jordan method-Gauss Siedal method.

## 4. COURSE OUTCOMES

At the end of the course, the student will be able to

- Apply numerical methods to solve algebraic and transcendental equations (L1)
- Derive interpolating polynomials using interpolation formulae (L2)
- Solve differential and integral equations numerically (L3)
- Apply Probability theory to find the chances of happening of events. (L4)
- Understand various probability distributions and calculate their statistical Constants. (L5)

## 5. Co-PO / PSO Mapping

NMPT	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	P10	PO11	PO12	PSO1	PSO2
CO1		3												

## 6. LESSON PLAN

LECTURE	WEEK	TOPICS TO BE COVERED	REFERENCES
1	1	Introduction to Algebraic and Transcendental Equations	TB1/RB2
2		Root of an equation	TB1/RB2
3		Bisection Method	TB1/RB2
4		Problems Bisection method	TB1/RB2
5		Problems on Bisection method	TB1/RB2
6		False Position Method	TB1/RB2
7	2	Problems on FP method	TB1/RB2
8		Problems on FP method	TB1/RB2
9		Newton Raphson method	TB1/RB2
10		Problems on NR method	TB1/RB2
11		Iteration Method	TB1/RB2
12		Problems on Iteration method	TB1/RB2
13	3	Gauss Jordan method	TB1/RB2
14		Problems on GJ method	TB1/RB2
15		Problems on GJ method	TB1/RB2
16		Gauss Siedal Method	TB1/RB2
17		Problems on Gauss Siedal method	TB1/RB2
18		Problems on all methods	TB1/RB2

## 7. ACTIVITY BASED LEARNING

1. To Solve Ordinary Differential Equations by using Numerical Techniques

## 8. LECTURE NOTES

### Solutions of Algebraic and Transcendental equations:

1) **Polynomial function:** A function  $f(x)$  is said to be a polynomial function

if  $f(x)$  is a polynomial in  $x$ .

$$\text{ie, } f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

where  $a_0 \neq 0$ , the co-efficients  $a_0, a_1, \dots, a_n$  are real constants and  $n$  is a non-negative integer.

2) **Algebraic function:** A function which is a sum (or) difference (or) product of

two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

$$\text{Eg: (i) } f(x) = c_1e^x + c_2e^{-x} = 0 \quad \text{(ii) } f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$$

3) **Root of an equation:** A number  $\alpha$  is called a root of an equation  $f(x) = 0$  if

$f(\alpha) = 0$ . We also say that  $\alpha$  is a zero of the function.

Note: The roots of an equation are the abscissae of the points where the graph  $y = f(x)$  cuts the  $x$ -axis.

### Methods to find the roots of $f(x) = 0$

#### Direct method:

We know the solution of the polynomial equations such as linear equation  $ax + b = 0$ , and quadratic equation  $ax^2 + bx + c = 0$ , using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.

## 1.1. Bisection method:

Bisection method is a simple iteration method to solve an equation. This method is also known as Bolzano method of successive bisection. Sometimes it is referred to as half-interval method.

- (i) Suppose we know an equation of the form  $f(x) = 0$  has exactly one real root between two real numbers  $x_0, x_1$ . The number is chosen such that  $f(x_0)$  and  $f(x_1)$  will have opposite sign.
- (ii) Let us bisect the interval  $[x_0, x_1]$  into two half intervals and find the mid point  $x_2 = \frac{x_0 + x_1}{2}$ . If  $f(x_2) = 0$  then  $x_2$  is a root.
- (iii) If  $f(x_1)$  and  $f(x_2)$  have same sign then the root lies between  $x_0$  and  $x_2$ .
- (iv) The interval is taken as  $[x_0, x_2]$ . Otherwise the root lies in the interval  $[x_2, x_1]$ .
- (v) Next calculate  $x_3, x_4, x_5, \dots$ , until two consecutive iterations are equal. Then we stop the process after getting desired accuracy.

This method is known as Bisection Method

### PROBLEMS

1). Find a root of the equation  $x^3 - 5x + 1 = 0$  using the bisection method in 5 – stages

Sol Let  $f(x) = x^3 - 5x + 1$ . We note that  $f(0) > 0$  and  $f(1) < 0$

$\therefore$  One root lies between 0 and 1

Consider  $x_0 = 0$  and  $x_1 = 1$

By Bisection method the next approximation is

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2}(0 + 1) = 0.5$$

$$\Rightarrow f(x_2) = f(0.5) = -1.375 < 0 \text{ and } f(0) > 0$$

We have the root lies between 0 and 0.5

$$\text{Now } x_3 = \frac{0+0.5}{2} = 0.25$$

We find  $f(x_3) = -0.234375 < 0$  and  $f(0) > 0$

Since  $f(0) > 0$ , we conclude that root lies between  $x_0$  and  $x_3$

The third approximation of the root is

$$x_4 = \frac{x_0+x_3}{2} = \frac{1}{2}(0 + 0.25) = 0.125$$

We have  $f(x_4) = 0.37495 > 0$

Since  $f(x_4) > 0$  and  $f(x_3) < 0$ , the root lies between

$$x_4 = 0.125 \text{ and } x_3 = 0.25$$

Considering the 4<sup>th</sup> approximation of the roots

$$x_5 = \frac{x_3+x_4}{2} = \frac{1}{2}(0.125+0.25) = 0.1875$$

$f(x_5) = 0.06910 > 0$ , since  $f(x_5) > 0$  and  $f(x_3) < 0$  the root must lie between

$$x_5 = 0.18758 \text{ and } x_3 = 0.25$$

Here the fifth approximation of the root is

$$\begin{aligned} x_6 &= \frac{1}{2}(x_5 + x_3) \\ &= \frac{1}{2}(0.1875 + 0.25) \\ &= 0.21875 \end{aligned}$$

We are asked to do up to 5 stages

We stop here 0.21875 is taken as an approximate value of the root

and it lies between 0 and 1

2) Find a root of the equation  $x^3 - 4x - 9 = 0$  using bisection method in four stages

Sol Let  $f(x) = x^3 - 4x - 9$

We note that  $f(2) < 0$  and  $f(3) > 0$

∴ One root lies between 2 and 3

Consider  $x_0 = 2$  and  $x_1 = 3$

By Bisection method  $x_2 = \frac{x_0 + x_1}{2} = 2.5$

Calculating  $f(x_2) = f(2.5) = -3.375 < 0$

∴ The root lies between  $x_2$  and  $x_1$

The second approximation is  $x_3 = \frac{1}{2}(x_1 + x_2) = \frac{2.5+3}{2} = 2.75$

Now  $f(x_3) = f(2.75) = 0.7969 > 0$

∴ The root lies between  $x_2$  and  $x_3$

Thus the third approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.625$$

Again  $f(x_4) = f(2.625) = -1.421 < 0$

∴ The root lies between  $x_3$  and  $x_4$

Fourth approximation is  $x_5 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(2.75 + 2.625) = 2.6875$

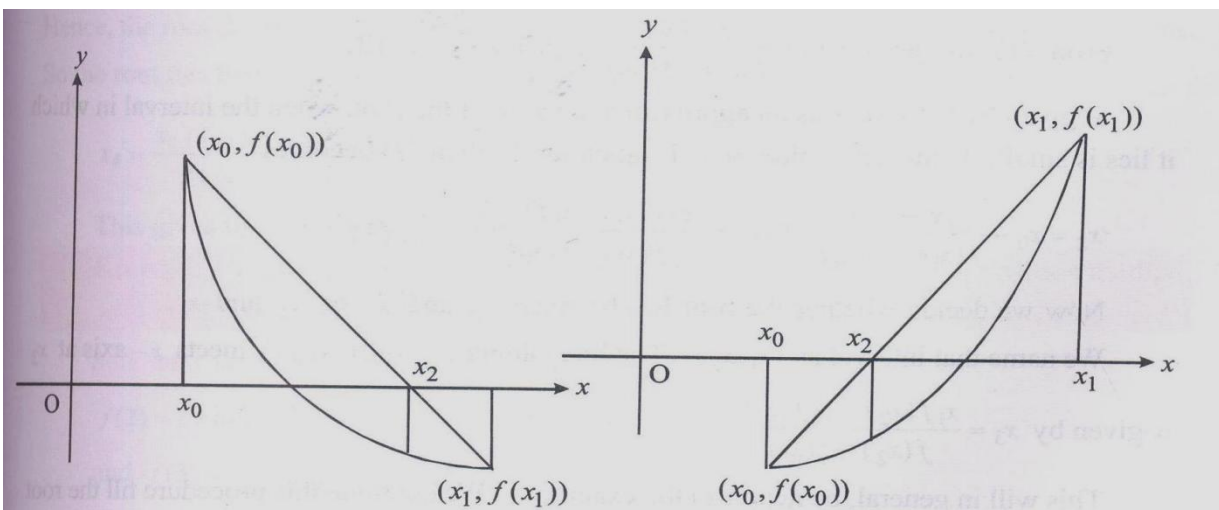
## 1.2. False Position Method ( Regula – Falsi Method)

In the false position method we will find the root of the equation  $f(x)=0$  Consider two initial approximate values  $x_0$  and  $x_1$  near the required root so that  $f(x_0)$  and  $f(x_1)$  have different signs. This implies that a root lies between  $x_0$  and  $x_1$ . The curve  $f(x)$  crosses x- axis only once at the Point  $x_2$  lying between the points  $x_0$  and  $x_1$ . Consider the point  $A=(x_0, f(x_0))$  and  $B=(x_1, f(x_1))$  on the graph and suppose they are connected by a straight line. Suppose this line cuts x-axis at  $x_2$ . We calculate the value of  $f(x_2)$  at the point. If  $f(x_0)$  and  $f(x_2)$  are of opposite signs, then the root lies between  $x_0$  and  $x_2$  and value  $x_1$  is replaced by  $x_2$

Otherwise the root lies between  $x_2$  and  $x_1$  and the value of  $x_0$  is replaced by  $x_2$ . Another line is drawn by connecting the newly obtained pair of values. Again the point here cuts the x-axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points

$x_2, x_3, x_4, \dots$  obtained converge to the expected root of the equation  $y = f(x)$

The below graph shows how to execute Regula Falsi Method



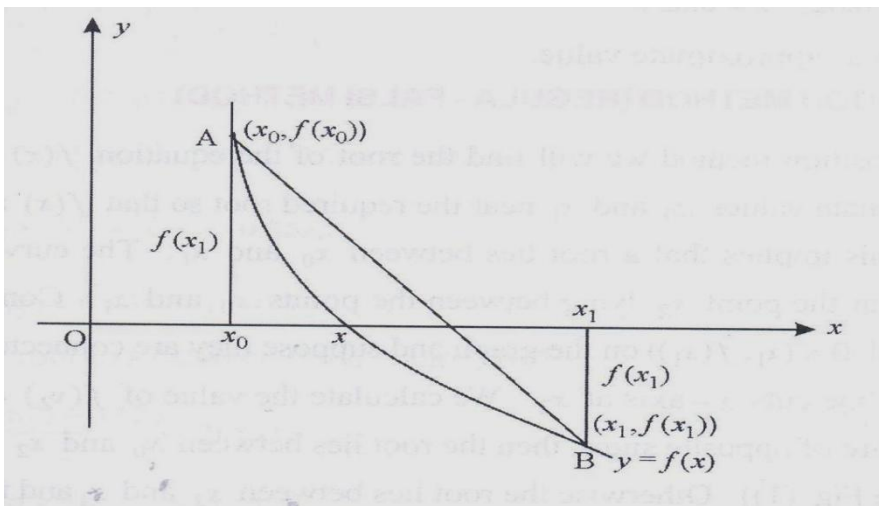
**To Obtain the equation to find the next approximation to the root**

Let  $A = (x_0, f(x_0))$  and  $B = (x_1, f(x_1))$  be the points on the curve  $y = f(x)$ . Then the equation to the chord AB is  $\frac{y-f(x_0)}{x-x_0} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$  -----(1)

At the point C where the line AB crosses the x – axis, where  $f(x) = 0$  ie,  $y = 0$

From (1), we get  $x = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$  -----(2)

$x$  is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of  $x$  is taken as  $x_2$  then (2) becomes



$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \rightarrow (3) \quad \text{-----}(3)$$

Now we decide whether the root lies between

$x_0$  and  $x_2$  (or)  $x_2$  and  $x_1$

We name that interval as  $(x_1, x_2)$ . The line joining  $(x_1, y_1), (x_2, y_2)$  meets x – axis at  $x_3$

is given by  $x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$$(x_0, x_1), (x_1, x_2), (x_2, x_3) \text{ etc}$$

Where  $x_i < x_{i+1}$  and  $f(x_i), f(x_{i+1})$  are of opposite signs.

$$\text{Also } x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

### PROBLEMS:

1. By using Regula - Falsi method, find an approximate root of the equation  $x^4 - x - 10 = 0$  that lies between 1.8 and 2. Carry out three approximations

Sol. Let us take  $f(x) = x^4 - x - 10$  and  $x_0 = 1.8, x_1 = 2$

Then  $f(x_0) = f(1.8) = -1.3 < 0$  and  $f(x_1) = f(2) = 4 > 0$

Since  $f(x_0)$  and  $f(x_1)$  are of opposite signs, the equation  $f(x) = 0$  has a root between  $x_0$  and  $x_1$

The first order approximation of this root is

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3) \\ &= 1.849 \end{aligned}$$

We find that  $f(x_2) = -0.161$  so that  $f(x_2)$  and  $f(x_1)$  are of opposite signs. Hence the root lies between  $x_2$  and  $x_1$  and the second order approximation of the root is

$$\begin{aligned}
 x_3 &= x_2 - \left[ \frac{x_1 - x_2}{f(x_1) - f(x_2)} \right] \cdot f(x_2) \\
 &= 1.8490 - \left[ \frac{2 - 1.849}{0.159} \right] \times (-0.159) \\
 &= 1.8548
 \end{aligned}$$

We find that  $f(x_3) = f(1.8548)$

$$= -0.019$$

So that  $f(x_3)$  and  $f(x_2)$  are of the same sign. Hence, the root does not lie between  $x_2$  and  $x_3$ . But  $f(x_3)$  and  $f(x_1)$  are of opposite signs. So the root lies between  $x_3$  and  $x_1$  and the third order approximate value of the root is  $x_4 = x_3 -$

$$\left[ \frac{x_1 - x_3}{f(x_1) - f(x_3)} \right] f(x_3)$$

$$= 1.8548 - \frac{2 - 1.8548}{4 + 0.019} \times (-0.019)$$

$$= 1.8557$$

This gives the approximate value of  $x$ .

2. Find out the roots of the equation  $x^3 - x - 4 = 0$  using False position method

Sol. Let  $f(x) = x^3 - x - 4 = 0$

Then  $f(0) = -4, f(1) = -4, f(2) = 2$

Since  $f(1)$  and  $f(2)$  have opposite signs the root lies between 1 and 2

By False position method  $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$\begin{aligned}
 x_2 &= \frac{(1 \times 2) - 2(-4)}{2 - (-4)} \\
 &= \frac{2 + 8}{6} = \frac{10}{6} = 1.666
 \end{aligned}$$

$$\begin{aligned}f(1.666) &= (1.666)^3 - 1.666 - 4 \\ &= -1.042\end{aligned}$$

Now, the root lies between 1.666 and 2

$$\begin{aligned}x_3 &= \frac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780 \\ f(1.780) &= (1.780)^3 - 1.780 - 4 \\ &= -0.1402\end{aligned}$$

Now, the root lies between 1.780 and 2

$$\begin{aligned}x_4 &= \frac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794 \\ f(1.794) &= (1.794)^3 - 1.794 - 4 \\ &= -0.0201\end{aligned}$$

Now, the root lies between 1.794 and 2

$$\begin{aligned}x_5 &= \frac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796 \\ f(1.796) &= (1.796)^3 - 1.796 - 4 = -0.0027\end{aligned}$$

Now, the root lies between 1.796 and 2

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796$$

The root is 1.796

### **1.3. Newton- Raphson Method:-**

The Newton- Raphson method is a powerful and elegant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let  $x_0$  be an approximate root of  $f(x)=0$  and let  $x_1 = x_0 + h$  be the correct root which implies that  $f(x_1)=0$ . We use Taylor's theorem and expand  $f(x_1) = f(x_0 + h) = 0$

$$\Rightarrow f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

Substituting this in  $x_1$ , we get

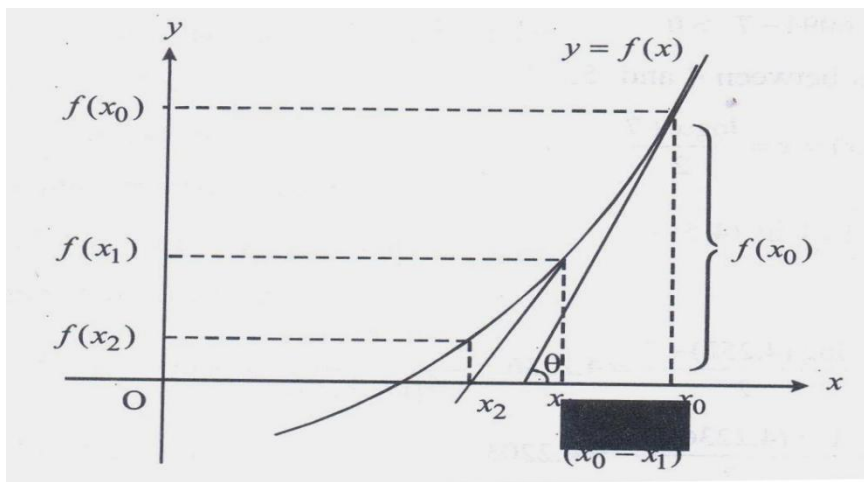
$$x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\therefore x_1$  is a better approximation than  $x_0$

Successive approximations are given by

$$x_2, x_3 \dots \dots \dots x_{n+1} \text{ Where } x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



**PROBLEMS:**

1. Apply Newton – Rapson method to find an approximate root, correct to three decimal places, of the equation  $x^3 - 3x - 5 = 0$ , which lies near  $x = 2$

**Sol:-** Here  $f(x) = x^3 - 3x - 5 = 0$  and  $f'(x) = 3(x^2 - 1)$

$\therefore$  The Newton – Raphson iterative formula

$$x_{i+1} = x_i - \frac{x_i^3 - 3x_i - 5}{3(x_i^2 - 1)} = \frac{2x_i^3 + 5}{3(x_i^2 - 1)}, i = 0, 1, 2, \dots (1)$$

To find the root near  $x = 2$ , we take  $x_0 = 2$  then (1) gives

$$x_1 = \frac{2x_0^3 + 5}{3(x_0^2 - 1)} = \frac{16 + 5}{3(4 - 1)} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3(x_1^2 - 1)} = \frac{2 \times (2.3333)^3 + 5}{3[(2.3333)^2 - 1]} = 2.2806$$

$$x_3 = \frac{2x_2^3 + 5}{3(x_2^2 - 1)} = \frac{2 \times (2.2806)^3 + 5}{3[(2.2806)^2 - 1]} = 2.2790$$

$$x_4 = \frac{2 \times (2.2790)^3 + 5}{3[(2.2790)^2 - 1]} = 2.2790$$

Since  $x_3$  and  $x_4$  are identical up to 3 places of decimal, we take  $x_4 = 2.279$  as the required root, correct to three places of the decimal

## 2. Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

### Sol. a) Square root:-

Let  $f(x) = x^2 - N = 0$ , where  $N$  is the number whose square root is to be found. The solution to  $f(x)$  is then  $x = \sqrt{N}$

$$\text{Here } f'(x) = 2x$$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2} \left[ x_i + \frac{N}{x_i} \right]$$

Using the above iteration formula the square root of any number  $N$  can be found to any desired accuracy. For example, we will find the square root of  $N = 24$ .

Let the initial approximation be  $x_0 = 4.8$

$$x_1 = \frac{1}{2} \left( 4.8 + \frac{24}{4.8} \right) = \frac{1}{2} \left( \frac{23.04 + 24}{4.8} \right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2} \left( 4.9 + \frac{24}{4.9} \right) = \frac{1}{2} \left( \frac{24.01 + 24}{4.9} \right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2} \left( 4.898 + \frac{24}{4.898} \right) = \frac{1}{2} \left( \frac{23.9904 + 24}{4.898} \right) = \frac{47.9904}{9.796} = 4.898$$

Since  $x_2 = x_3$ , therefore the solution to  $f(x) = x^2 - 24 = 0$  is 4.898. That means,

The square root of 24 is 4.898

### **b) Reciprocal:-**

Let  $f(x) = \frac{1}{x} - N = 0$  where  $N$  is the number whose reciprocal is to be found

The solution to  $f(x)$  is then  $x = \frac{1}{N}$ . Also,  $f'(x) = \frac{-1}{x^2}$

To find the solution for  $f(x) = 0$ , apply Newton – Raphson method

$$x_{i+1} = x_i - \frac{\left( \frac{1}{x_i} - N \right)}{-1/x_i^2} = x_i(2 - x_i N)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be  $x_0 = 0.045$

$$\begin{aligned}
\therefore x_1 &= 0.045(2 - 0.045 \times 22) \\
&= 0.045(2 - 0.99) \\
&= 0.0454(1.01) = 0.0454 \\
x_2 &= 0.0454(2 - 0.0454 \times 22) \\
&= 0.0454(2 - 0.9988) \\
&= 0.0454(1.0012) = 0.04545 \\
x_3 &= 0.04545(2 - 0.04545 \times 22) \\
&= 0.04545(1.0001) = 0.04545
\end{aligned}$$

$$\begin{aligned}
x_4 &= 0.04545(2 - 0.04545 \times 22) \\
&= 0.04545(2 - 0.99998) \\
&= 0.04545(1.00002)
\end{aligned}$$

$$= 0.0454509$$

$\therefore$  The reciprocal of 22 is 0.04545

3. Find by Newton's method, the real root of the equation  $xe^x - 2 = 0$  correct to three decimal places.

Sol. Let  $f(x) = xe^x - 2 \rightarrow (1)$

Then  $f(0) = -2$  and  $f(1) = e - 2 = 0.7183$

So root of  $f(x)$  lies between 0 and 1

It is near to 1. So we take  $x_0 = 1$  and  $f'(x) = xe^x + e^x$  and  $f'(1) = e + e = 5.4366$

$\therefore$  By Newton's Rule

First approximation  $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 1 - \frac{0.7183}{5.4366} = 0.8679$$

$$\therefore f(x_1) = 0.0672 \quad f'(x_1) = 4.4491$$

The second approximation  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 0.8679 - \frac{0.0672}{4.4491}$$

$$= 0.8528$$

∴ Required root is 0.853 correct to 3 decimal places.

### 1.4.GAUSS JORDAN METHOD:

#### SYSTEM OF NON HOMOGENEOUS LINEAR EQUATIONS

An equation of the form  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$  where  $x_1, x_2, \dots, x_n$  are unknowns and  $a_1, a_2, \dots, a_n, b$  are constants is called linear equation in  $n$  unknowns .

**Definition:** Consider the system of  $m$  linear equations in  $n$  unknowns  $x_1, x_2, \dots, x_n$  as given below:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

.....

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

The number  $a_{ij}$ 's are known as coefficient and  $b_1, b_2, \dots, b_m$  are constants. An ordered  $n$ -tuple  $(x_1, x_2, \dots, x_n)$  satisfying all the equations simultaneously is called a solution of system.

#### **Non-Homogeneous system:**

If all  $b_i \neq 0$  i.e.at least one  $b_i \neq 0$ .

### Matrix Representation:

The above system of linear non Homogeneous equations can be written in Matrix form as  $AX=B$

$$\begin{bmatrix} a_{11} & a_{12} \dots \dots \dots & a_{1n} \\ a_{21} & a_{22} \dots \dots \dots & a_{2n} \\ a_{m1} & a_{m2} \dots \dots \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}$$

### Augmented Matrix:

It is denoted by  $[A/B]$  or  $[A \ B]$  is obtained by Augmenting A by the column B.

$$\therefore [A \ /B] = \begin{bmatrix} a_{11} & a_{12} \dots \dots \dots & a_{1n} & b_1 \\ a_{21} & a_{22} \dots \dots \dots & a_{2n} & b_2 \\ a_{m1} & a_{m2} \dots \dots \dots & a_{mn} & b_m \end{bmatrix}$$

By reducing  $[A \ /B]$  into its row echelon form the existence and uniqueness of solution

$AX = B$  exists.

### NOTE:

Given a system, we do not know in general whether it has a solution or not .If there is at least one solution , then the system is said to be consistent .If does not have any solution then the system is inconsistent.

**CONSISTENT:** A system is said to be consistent if it has at least one solution

**NOTE:** Here rank is denoted by  $\rho$

**Gauss Jordan Method:** In Gauss Jordan method augmented matrix  $[A/B]$  can be reduced to identity matrix and column matrix by elementary row operations. Finally last column gives solutions of given linear system.

The Augmented matrix  $[A/B]$  can be reduced as follows by elementary row operations

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$$

Then last column is the solution set of given linear system

For Non Homogeneous System, The system  $AX = B$  is consistent i.e it has a solution.

The system is inconsistent i.e. it has no solution.

**NOTE:** Find the rank A and rank  $[A/B]$  by reducing the augmented matrix  $[A/B]$  to Echelon form by elementary row operations. Then the matrix A will be reduced to Echelon form.

This procedure is illustrated through the following examples.

**Example 1:** Find whether the following equations are consistent, if so solve them

By Gauss Jordan method  $x + y + 2z = 4$ ;  $2x - y + 3z = 9$ ;  $3x - y - z = 2$ .

**Solution:** The given equations can be written in the matrix form as 
$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

i.e.  $AX = B$  Use Gauss Jordan method

The Augmented matrix  $[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

Applying  $R_2 \rightarrow R_2 - 2R_1$  and  $R_3 \rightarrow R_3 - 3R_1$

$$[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$$

Applying  $R_3 \rightarrow 3R_3 - 4R_2$

$$[A / B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A / B] = 3

Since the number of non-zero rows of matrix A is 3

Since the number of non-zero rows of matrix [A / B] is 3

∴ Rank of A = Rank of [A B]

i.e.  $\rho(A) = \rho(AB)$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$$

⇒  $R_3 \leftarrow R_3 / (-17)$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

⇒  $R_1 \leftarrow R_1 - 2R_3$  and  $R_2 \rightarrow R_2 + R_3$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Next perform  $R_2 / (-3)$  and  $R_1 \rightarrow R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Then solution set  $X =$

$\therefore x = 1, y = -1, z = 2$  is the solution.

**Example 2:** Using Gauss Jordan method solve linear equations given below

$$x + 2y + 2z = 2; 3x - 2y - z = 5; 2x - 5y + 3z = -4; x + 4y + 6z = 0.$$

**Solution:** The given equations can be written in the matrix form as  $AX = B$

$$\text{i.e. } \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

$$\text{The Augmented matrix } [A/B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$$

Use Gauss Jordan method

Applying  $R_2 \rightarrow R_2 - 3R_1; R_3 \rightarrow R_3 - 2R_1; R_4 \rightarrow R_4 - R_1$

$$[A/B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

Applying  $R_3 \rightarrow 8R_3 - 9R_2; R_4 \rightarrow 4R_4 + R_2$ , we get

$$[A/B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

Applying  $R_3 \rightarrow R_3/55; R_4 \rightarrow R_4/9$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Applying  $R_4 \rightarrow R_4 - R_3$

$$[A/B] \approx \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since Rank of A = 3 & Rank of [A/B] = 3

$\therefore$  Rank of A = Rank of [A/B]

i.e.  $\rho(A) = \rho(AB)$

The given system is consistent, so it has a solution.

$$\text{We have } \begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$\Rightarrow$  Apply  $R_1 - 2R_3, R_2 + 7R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -1 \\ 0 \end{bmatrix}$$

Next  $R_2/(-8)$  and  $R_1 - 2R_2$  then

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$\therefore x = 2, y = 1, z = -1$  is the solution.

### 1.5. Gauss Siedel Method:

**Algorithm:** Consider the linear system of equations as

$$\begin{aligned} \text{(i)} \quad & a_1x + b_1y + c_1z = d_1 \\ & a_2x + b_2y + c_2z = d_2 \\ & a_3x + b_3y + c_3z = d_3 \end{aligned}$$

(ii) If  $a_1, b_2, c_3$  are large as compared with other coefficients, then solve them for  $x, y, z$  respectively.

The system can be written in the below form

$$\begin{aligned} X &= \frac{1}{a_1} (d_1 - b_1y - c_1z) \\ Y &= \frac{1}{b_2} (d_2 - a_2X - c_2z) \\ Z &= \frac{1}{c_3} (d_3 - a_3X - b_3y) \end{aligned}$$

(iii) First iteration: We can calculate first iteration values in the following equations

$$\begin{aligned} X^1 &= \frac{1}{a_1} (d_1 - b_1y^0 + c_1z^0) \\ Y^1 &= \frac{1}{b_2} (d_2 - a_2X^1 - c_2z^0) \\ Z^1 &= \frac{1}{c_3} (d_3 - a_3X^1 - b_3y^1) \end{aligned}$$

(iv) Second iteration: Formulas for second iteration

$$\begin{aligned} X^2 &= \frac{1}{a_1} (d_1 - b_1y^1 + c_1z^1) \\ Y^2 &= \frac{1}{b_2} (d_2 - a_2X^2 - c_2z^1) \\ Z^2 &= \frac{1}{c_3} (d_3 - a_3X^2 - b_3y^2) \end{aligned}$$

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First take initial values zeroes as new approximation for an unknown value found, it is immediately used in next step. We continued these processes up to two successive

iterations are approximately equal. This procedure is called as Gauss Siedal iteration method.

**PROBLEMS:**

1. Solve by Gauss Siedal method  $10x+y+z=12$  ,  $2x+10y+z=13$  ,  $2x+2y+10z=14$

Sol. Given equations are  $10x+y+z=12$ ----(1)

$$2x+10y+z=13$$
----(2)

$$2x+2y+10z=14$$
---(3)

From (1)  $x = \frac{1}{10} (12-y-z)$

From (2)  $y = \frac{1}{10} (13-2x-z)$

From (3)  $z = \frac{1}{10} (14-2x-2y)$

First iteration:

$$X^1 = \frac{1}{10} (12-y^0-z^0) = \frac{1}{10} (12-0-0) = 1.2$$

$$Y^1 = \frac{1}{10} (13-2x^1-z^0) = \frac{1}{10} (13-2(1.2)-0) = 1.06$$

$$Z^1 = \frac{1}{10} (14-2x^1-2y^1) = \frac{1}{10} (14-2(1.2)-2(1.06)) = 0.948$$

Second iteration:

$$X^2 = \frac{1}{10} (12-y^1-z^1) = \frac{1}{10} (12-1.2-1.06) = 0.999$$

$$Y^2 = \frac{1}{10} (13-2x^2-z^1) = \frac{1}{10} (13-2(0.999)-0.948) = 1.005$$

$$Z^2 = \frac{1}{10} (14-2x^2-2y^2) = \frac{1}{10} (14-2(0.999)-2(1.005)) = 0.999$$

Third iteration:

$$X^3 = \frac{1}{10} (12-y^2-z^2) = \frac{1}{10} (12-1.005-0.999) = 1$$

$$Y^3 = \frac{1}{10} (13-2x^3-z^2) = \frac{1}{10} (13-2(1)-0.999) = 1$$

$$Z^3 = \frac{1}{10} (14-2x^3-2y^3) = \frac{1}{10} (14-2(1)-2(1)) = 1$$

Fourth iteration:

$$X^4 = \frac{1}{10} (12-y^3-z^3) = \frac{1}{10} (12-1-1) = 1$$

$$Y^4 = \frac{1}{10} (13-2x^4-z^3) = \frac{1}{10} (13-2(1)-1) = 1$$

$$Z^4 = \frac{1}{10} (14-2x^4-2y^4) = \frac{1}{10} (14-2(1)-2(1)) = 1$$

Since third and fourth iterations are equal then desired set of solutions are

$$X=1, y=1, z=1$$

- Using Gauss Siedal method solve the linear system  $20x+y-2z=17, 3x+20y-z=-18, 2x-3y+20z=25$
- Solve  $6x+y+z=105, 4x+8y+3z=155, 5x+4y-10z=65$  by Gauss Siedal method.

## 9. Practice Quiz

- Newton's iterative formula for finding the Cube root of a number N is  $x_{n+1} = [b \quad ]$

a)  $\frac{1}{3} \left[ 2x_n + \frac{N}{x_n^2} \right]$

b)  $\frac{1}{3} \left[ 2x_n + \frac{N}{x_n^3} \right]$

c)  $\frac{1}{3} \left[ 2x_n - \frac{N}{x_n^2} \right]$

d)  $\frac{1}{3} \left[ 2x_n - \frac{N}{x_n^3} \right]$

- Iteration formula in Newton-Raphson method is [b ]

a)  $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$

b)  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

c)  $x_{n+1} = x_n + \frac{f'(x_n)}{f(x_n)}$

d)  $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$

- Which of the following is an algebraic equation ..... [b ]

a)  $x^2 - \log x - 1.2 = 0$

b)  $x^3 + 2x^2 + x + 1 = 0$

c)  $\cos x = xe^x$

d)  $xe^x - 1 = 0$

4. Which of the following is a transcendental equation...

[a ]

a)  $x^2 - \log x = 1.2$

b)  $x^3 + 2x^2 + x + 1 = 0$

c)  $x^3 - 3x - 5 = 0$

d)  $x^3 - 5x + 1 = 0$

5. Using the false position method, the formula for the approximate root of the equation  $f(x) = 0$  is.....

[a ]

a)  $x = \frac{af(b) - bf(a)}{f(b) - f(a)}$

b)  $x = \frac{bf(b) - af(a)}{f(b) - f(a)}$

c)  $x = \frac{af(b) + bf(a)}{f(b) + f(a)}$

d)  $x = \frac{bf(b) + af(a)}{f(b) + f(a)}$

6. If the root of the equation  $x^3 - 6x + 4 = 0$  lies between 0 & 1, then the first approximation of the required root using Newton-Raphson method is.....

[c ]

a) 0.55555

b) 0.4444

c) 0.77777

d) 0.66666

7. The  $n^{\text{th}}$  order difference of a polynomial of  $n^{\text{th}}$  degree is\_\_\_\_\_

[a ]

a) Constant

- b) 0
- c) Polynomial of first degree
- d) Polynomial of  $(n - 1)^{\text{th}}$  degree

## 10. Assignments

S.No	Question	BL	CO
1	Find and approximate value of the real root of $x^3 - x - 1 = 0$ using Bisection method.	1	1
2	Find real root of the equation $x \log_{10}(x) = 1.2$ using false position method.	1	1
3	Using Newton-Rapshon method find the root of the equation $x + \log_{10}(x) = 3.375$ and correct to four significant figures.	1	1
4	Solve the system $2x - y + 4z = 12; 3x + 2y + z = 10; x + y + z = 6$ by Gauss Jordan method	3	1
5	Express the following system in matrix form and solve by Gauss-seidal method, $4x - y - z = 3; -2x + 6y + z = 9; -x + y + 7z = -6$ .	3	1

## 11. Part A- Question & Answers

S.No	Question& Answers	BL	CO
1	<p><b>Write formula for Regular Falsi method.</b></p> <p><b>Sol.</b> <math>x = \frac{af(b) - bf(a)}{f(b) - f(a)}</math>.</p>	1	1
2	<p><b>Locate the negative root of <math>x^3 - 2x + 5 = 0</math> approximately.</b></p> <p><b>Sol.</b> <math>f(x) = x^3 - 2x + 5</math>  <math>f(-1) = -1 + 2 + 5 = 6 &gt; 0</math>, <math>f(-2) = -8 + 4 + 5 = 1 &gt; 0</math>, <math>f(-3) = -27 + 6 + 5 = -16 &lt; 0</math>.            Root lies between -2 &amp; -3</p>	1	1
3	<p><b>Find <math>f(x_1)</math> an approximate value of the equation <math>x^3 + x - 1 = 0</math>, using the method of regular falsi.</b></p> <p><b>Sol.</b> <math>f(x) = x^3 + x - 1 = 0</math>  <math>f(0.5) = -0.675</math> and <math>f(1) = 1</math>            Hence the root lies between 0.5 and 1  <math>a = 0.5</math> and <math>b = 1</math>  <math>X_0 = \frac{af(b) - bf(a)}{f(b) - f(a)} = 0.64</math>  <math>f(0.64) = -0.0979 &lt; 0</math>  <math>\therefore</math> the root lies between 0.64 and 1</p>	1	1

$$x_1 = \frac{(0.64)(1) - (1)(-0.0979)}{1 - (-0.0979)} = 0.672$$

4	<p><b>Write formula for Newton-Raphson method.</b></p> <p><b>Sol.</b> <math>x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}</math></p>	1	1
5	<p><b>State the order of convergence and convergence condition for NR method?</b></p> <p><b>Sol:</b> The order of convergence is 2</p> <p>Condition of convergence is <math> f(x)f''(x)  &lt;  f'(x) ^2</math></p>	1	1
6	<p><b>Develop an algorithm using Newton-Raphson method, to find the square root of a positive number.</b></p> <p><b>Sol.</b> The square root of the a positive number N is the root of the equation. <math>x^2 - N = 0</math></p> <p><math>f(x) = x^2 - N</math> , <math>f'(x) = 2x</math></p> <p>By Newton-Raphson technique, we have</p> $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - N}{2x_i} = \frac{1}{2} \left( x_i + \frac{N}{x_i} \right)$	1	1
7	<p><b>Write merits and demerits of Newton-Raphson method.</b></p> <p><b>Sol. Merits:</b> 1. In this method convergence is quite fast provided the starting value is close to the desired root.</p> <p>2. It is single point iteration method and it can be used for solving algebraic and transcendental equations. It can also be used when the roots are complex.</p> <p><b>Demerits:</b> 1. It is not applicable to find the approximated values of a repeated root.</p> <p>2. The method should not be used in case where the graph of the function when it crosses the x axis is nearly horizontal.</p>	1	1

## 12. Part B- Questions

S.No	Question	BL	CO
1	Find and approximate value of the real root of $x^3 - x - 1 = 0$ using Bisection method	1	1
2	Find real root of the equation $x \log_{10}(x) = 1.2$ using false position method.	1	1
3	Fit root of equation $x^3 - 4x - 9 = 0$ using Bisection method in four stages.	1	1
4	Find by Iteration method , the real root between 0 and 1 of the equation $x^3 - 6x + 4 = 0$ correct to five decimal places	1	1

5	Find the root of the equation $x \sin x + \cos x = 0$ . Using Newton-Raphson method.	1	1
6	Test the system by Gauss Jordan method $2x + y + 5z = 4$ ; $3x - 2y + 2z = 2$ ; $5x - 8y - 4z = 1$ solve them.	3	1
7	Solve the following system in matrix form and solve by Gauss-seidal method, $4x - y - z = 3$ ; $-2x + 6y + z = 9$ ; $-x + y + 7z = -6$ .	3	1

### 13. Supportive Online Certification Courses:

- An Introductory Method to Numerical Analysis (NPTEL)
- Probability & Statistics (NPTEL)

### 14. Real Time Applications:

- Application of Numerical methods as a way of transforming linear equations and differential equations

### 15. Content beyond the Syllabus:

- Numerical Differentiation and Applications

### 16. Prescribed Text Books and Reference Books:

#### Text Books:

1. B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.
3. Ronald E. Walpole "Probability and Statistics for Engineers and Scientists", PNIE

#### Reference Books:

1. B.V.Ramana, "Higher Engineering Mathematics", Mc Graw Hill publishers.
2. Alan Jeffrey, "Advanced Engineering Mathematics", Elsevier.

## COURSE MATERIAL

<b>SUBJECT</b>	<b>NUMERICAL METHODS AND PROBABILITY STATISTICS (20A54402)</b>
<b>UNIT</b>	<b>2</b>
<b>COURSE</b>	<b>B.TECH</b>
<b>SEMESTER</b>	<b>2 - 2</b>
<b>DEPARTMENT</b>	<b>HUMANITIES &amp; SCIENCE</b>
<b>PREPARED BY (Faculty Name/s)</b>	<b>Department of Mathematics</b>

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## 1. COURSE OBJECTIVES

The objectives of this course is to

- This course aims at providing the student with the knowledge on Various numerical methods for solving equations, interpolating the polynomials, evaluation of integral equations and solution of differential equations.
- The theory of Probability and random variables.

## 2. PREREQUISITES

Students should have knowledge on

1. Students should have knowledge on Numerical Techniques
2. Students should have knowledge on Probability Theory

## 3. SYLLABUS

### UNIT II

#### Interpolation:

Finite differences-Newton's forward and backward interpolation formulae – Lagrange's formulae. Gauss forward and backward formula, Stirling's formula, Bessel's formula.

## 4. COURSE OUTCOMES

At the end of the course, the student will be able to

- Apply numerical methods to solve algebraic and transcendental equations (L1)
- Derive interpolating polynomials using interpolation formulae (L2)
- Solve differential and integral equations numerically (L3)
- Apply Probability theory to find the chances of happening of events. (L4)
- Understand various probability distributions and calculate their statistical Constants. (L5)

## 5. Co-PO / PSO Mapping

NMPT	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	P10	PO11	PO12	PSO1	PSO2
CO1		3												

## 6. LESSON PLAN

LECTURE	WEEK	TOPICS TO BE COVERED	REFERENCES
1	1	Introduction to Finite Differences	TB1/RB2
2		Problems on Forward-Backward Differences	TB1/RB2
3		Newton's Forward formula	TB1/RB2
4		Problems Newton's Forward formula	TB1/RB2
5		Newton's Backward Formula	TB1/RB2
6		Problems on Newton's Backward formula	TB1/RB2
7	2	Lagrange's Interpolation formula	TB1/RB2
8		Problems on Lagrange's Interpolation formula	TB1/RB2
9		Problem's on Lagrange's Interpolation formula	TB1/RB2
10		Gauss's Forward Interpolation formula	TB1/RB2
11		Problems on Gauss's Forward formula	TB1/RB2
12		Problems on on all formulas	TB1/RB2
13	3	Gauss Backward Interpolation formula	TB1/RB2
14		Problems on Gauss Backward formula	TB1/RB2
15		Stirling's Formula	TB1/RB2
16		Problems on Stirling's Formula	TB1/RB2
17		Bessel's Formula	TB1/RB2
18		Problems on Bessel's Formula	TB1/RB2

## 7. ACTIVITY BASED LEARNING

1. To Solve Ordinary Differential Equations by using Numerical Techniques

## 8. LECTURE NOTES

### Interpolation

#### Introduction:-

If we consider the statement  $y = f(x)$   $x_0 \leq x \leq x_n$  we understand that we can find the value of  $y$ , corresponding to every value of  $x$  in the range  $x_0 \leq x \leq x_n$ . If the function  $f(x)$  is single valued and continuous and is known explicitly then the values of  $f(x)$  for certain values of  $x$  like  $x_0, x_1, \dots, x_n$  can be calculated. The problem now is if we are given the set of tabular values

$$\begin{array}{l} x: x_0 \quad x_1 \quad x_2 \dots \dots x_n \\ y: y_0 \quad y_1 \quad y_2 \dots \dots y_n \end{array}$$

Satisfying the relation  $y = f(x)$  and the explicit definition of  $f(x)$  is not known, then it is possible to find a simple function say  $\phi(x)$  such that  $\phi(x)$  and  $f(x)$  agree at the set of tabulated points. This process to finding  $\phi(x)$  is called interpolation. If  $\phi(x)$  is a polynomial then the process is called polynomial interpolation and  $\phi(x)$  is called interpolating polynomial. In our study we are concerned with polynomial interpolation

#### Errors in Polynomial Interpolation:-

Suppose the function  $y(x)$  which is defined at the points  $(x_i, y_i) i = 0, 1, 2, 3 \dots n$  is continuous and differentiable  $(n+1)$  times let  $\phi_n(x)$  be polynomial of degree not exceeding  $n$  such that  $\phi_n(x_i) = y_i, i = 1, 2 \dots n \rightarrow (1)$  be the approximation of  $y(x)$  using this  $\phi_n(x_i)$  for other value of  $x$ , not defined by (1) the error is to be determined

Since  $y(x) - \phi_n(x) = 0$  for  $x = x_0, x_1, \dots, x_n$  we put

$$y(x) - \phi_n(x) = L\pi_{n+1}(x)$$

Where  $\pi_{n+1}(x) = (x-x_0)\dots(x-x_n) \rightarrow (3)$  and  $L$  to be determined such that the equation (2) holds for any intermediate value of  $x$  such as  $x = x^1, x_0 < x^1 < x_n$

Clearly 
$$L = \frac{y(x^1) - \phi_n(x^1)}{\pi_{n+1}(x^1)} \rightarrow (4)$$

We construct a function  $F(x)$  such that  $F(x) = F(x_n) = F(x^1)$ . Then  $F(x)$  vanishes  $(n+2)$  times in the interval  $[x_0, x_n]$ . Then by repeated application of Rolle's theorem.  $F'(x)$  Must be zero  $(n+1)$  times,  $F''(x)$  must be zero  $n$  times..... in the interval  $[x_0, x_n]$ . Also  $F^{(n+1)}(x) = 0$  once in this interval. Suppose this point is  $x = \varepsilon, x_0 < \varepsilon < x_n$  differentiate (5)  $(n+1)$  times with respect to  $x$  and putting  $x = \varepsilon$ , we get

$$y^{(n+1)}(\varepsilon) - L(n+1)! = 0 \text{ Which implies that } L = \frac{y^{(n+1)}(\varepsilon)}{(n+1)!}$$

Comparing (4) and (6), we get

$$y(x^1) - \phi_n(x^1) = \frac{y^{(n+1)}(\varepsilon)}{(n+1)!} \pi_{n+1}(x^1)$$

This can be written as  $y(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!} y^{(n+1)}(\varepsilon)$

This given the required expression  $x_0 < \varepsilon < x_n$  for error

## 2.1. Finite Differences:-

### 1. Introduction:-

In this chapter, we introduce what are called the forward, backward and central differences of a function  $y = f(x)$ . These differences and three standard examples of finite differences and play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics

## 2. Forward Differences:-

Consider a function  $y = f(x)$  of an independent variable  $x$ . Let  $y_0, y_1, y_2, \dots, y_r$  be the values of  $y$  corresponding to the values  $x_0, x_1, x_2, \dots, x_r$  of  $x$  respectively. Then the differences  $y_1 - y_0, y_2 - y_1, \dots$  are called the first forward differences of  $y$ , and we denote them by  $\Delta y_0, \Delta y_1, \dots$  that is

$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, \dots$$

$$\text{In general } \Delta y_r = y_{r+1} - y_r \quad \therefore r = 0, 1, 2, \dots$$

Here, the symbol  $\Delta$  is called the forward difference operator

The first forward differences of the first forward differences are called second forward differences and are denoted by  $\Delta^2 y_0, \Delta^2 y_1, \dots$  that is

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

In general  $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad r = 0, 1, 2, \dots$  similarly, the  $n^{\text{th}}$  forward differences are defined by the formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad r = 0, 1, 2, \dots$$

While using this formula for  $n=1$ , use the notation  $\Delta^0 y_r = y_r$  and we have  $\Delta^n y_r = 0 \quad \forall n = 1, 2, \dots$  and  $r = 0, 1, 2, \dots$  the symbol  $\Delta^n$  is referred as the  $n^{\text{th}}$  forward difference operator.

## 3. Forward Difference Table:-

The forward differences are usually arranged in tabular columns as shown in the following table called a forward difference table

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
$x_0$	$y_0$				
		$\Delta y_0 = y_1 - y_0$			
$x_1$	$y_1$		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
$x_2$	$y_2$		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
$x_3$	$y_3$		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
$x_4$	$y_4$	$= y_4 - y_3$			

Example -finite forward difference table for  $y = x^3$

x	$y = f(x)$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

**4. Backward Differences:-** As mentioned earlier, let  $y_0, y_1, \dots, y_r, \dots$  be the values of a function  $y = f(x)$  corresponding to the values  $x_0, x_1, x_2, \dots, x_r, \dots$  of x respectively. Then,  $\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \dots$  are called the first backward differences

In general  $\nabla y_r = y_r - y_{r-1}, r = 1, 2, 3, \dots \rightarrow (1)$

The symbol  $\nabla$  is called the backward difference operator, like the operator  $\Delta$ , this operator is also a linear operator

Comparing expression (1) above with the expression (1) of section we immediately note that  $\nabla y_r = \nabla y_{r-1}, r = 0, 1, 2, \dots \rightarrow (2)$

The first backward differences of the first backward differences are called second differences and are denoted by  $\nabla^2 y_2, \nabla^2 y_3, \dots, \nabla^2 y_r, \dots$

i.e....  $\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2, \dots$

In general  $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, r = 2, 3, \dots \rightarrow (3)$  similarly, the  $n^{\text{th}}$  backward differences are defined by the formula  $\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, r = n, n+1, \dots \rightarrow (4)$  While using this formula, for  $n = 1$  we employ the notation  $\nabla^0 y_r = y_r$

If  $y = f(x)$  is a constant function, then  $y = c$  is a constant, for all  $x$ , and we get  $\nabla^n y_r = 0 \forall n$  the symbol  $\nabla^n$  is referred to as the  $n^{\text{th}}$  backward difference operator

### 5. Backward Difference Table:-

$x$	$Y$	$\nabla y$	$\nabla^2 y$	$\nabla^3 y$
$x_0$	$y_0$			
		$\nabla y_1 = y_1 - y_0$		
$x_1$	$y_1$		$\nabla^2 y_2$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3$
$x_2$	$y_2$		$\nabla^2 y_3$	
		$\nabla y_3 = y_3 - y_2$		
$x_3$	$y_3$			

## 6. Central Differences:-

With  $y_0, y_1, y_2, \dots, y_r$  as the values of a function  $y = f(x)$  corresponding to the values  $x_1, x_2, \dots, x_r, \dots$  of  $x$ , we define the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2}, \dots$  as follows

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2, \dots$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \rightarrow (1)$$

The symbol  $\delta$  is called the central differences operator. This operator is a linear operator comparing expressions (1) above with expressions earlier used on forward and backward differences we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1, \delta y_{3/2} = \Delta y_1 = \nabla y_2, \dots$$

$$\text{In general } \delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1}, n = 0, 1, 2, \dots \rightarrow (2)$$

The first central differences of the first central differences are called the second central differences and are denoted by  $\delta^2 y_1, \delta^2 y_2, \dots$

$$\text{Thus } \delta^2 y_1 = \delta_{3/2} - \delta_{1/2}, \delta^2 y_2 = \delta_{5/2} - \delta_{3/2}, \dots$$

$$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \rightarrow (3)$$

Higher order central differences are similarly defined. In general the  $n^{\text{th}}$  central differences are given by

$$\text{i) for odd } n: \delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}, r = 1, 2, \dots \rightarrow (4)$$

$$\text{ii) for even } n: \delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}, r = 1, 2, \dots \rightarrow (5)$$

while employing for formula (4) for  $n = 1$ , we use the notation  $\delta^0 y_r = y_r$

If  $y$  is a constant function, that is if  $y = c$  a constant, then

$$\delta^n y_r = 0 \text{ for all } n \geq 1$$

## 7. Central Difference Table

$x_0$	$y_0$	$\delta y$	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
		$\delta y_{1/2}$			
$x_1$	$y_1$		$\delta^2 y_1$		
		$\delta y_{2/2}$		$\delta^3 y_{3/2}$	
$x_2$	$y_2$		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
$x_3$	$y_3$		$\delta^2 y_3$		
		$\delta y_{7/2}$			
$x_4$	$y_4$				

**E**  
**x**

**ample:** Given  $f(-2)=12, f(-1)=16, f(0)=15, f(1)=18, f(2)=20$  from the central difference table and write down the values of  $\delta y_{3/2}, \delta^2 y_0$  and  $\delta^3 y_{7/2}$  by taking  $x_0 = 0$

**Sol.** The central difference table is

$x$	$y = f(x)$	$\delta y$	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		

		2			
2	20				

### 5. Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to  $\Delta$ ,  $\nabla$  and  $\delta$  already defined and establish difference formulae by symbolic methods

**Averaging Operator-Definition:-** The averaging operator  $\mu$  is defined by the equation

$$\mu y_r = \frac{1}{2} [y_{r+1/2} + y_{r-1/2}]$$

**Shift Operator-Definition:-** The shift operator  $E$  is defined by the equation  $Ey_r = y_{r+1}$ .

This shows that the effect of  $E$  is to shift the functional value  $y_r$  to the next higher value  $y_{r+1}$ . A second operation with  $E$  gives  $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

Generalizing  $E^n y^r = y_{r+n}$

### Relationship Between

$\Delta$  and  $E$

We have

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ &= Ey_0 - y_0 = (E - 1)y_0 \\ \Rightarrow \Delta &= E - y \text{ (or) } E = 1 + \Delta \end{aligned}$$

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 \\ &= Ey_0 - y_0 = (E - 1)y_0 \\ \Rightarrow \Delta &= E - y \text{ (or) } E = 1 + \Delta \end{aligned}$$

### Some more relations

$$\begin{aligned} \Delta^3 y_0 &= (E - 1)^3 y_0 = (E^3 - 3E^2 + 3E - 1)y_0 \\ &= y_3 - 3y_2 + 3y_1 - y_0 \end{aligned}$$

### Inverse Shift Operator-Definition

Inverse operator  $E^{-1}$  is defined as  $E^{-1}y_r = y_{r-1}$

In general  $E^{-n}y_n = y_{r-n}$

We can easily establish the following relations

i)  $\nabla \equiv 1 - E^{-1}$

ii)  $\delta \equiv E^{1/2} - E^{-1/2}$

iii)  $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

iv)  $\Delta = \nabla E = E^{1/2}$

v)  $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

**Differential Operator-Definition** The operator D is defined as  $Dy(x) = \frac{\partial}{\partial x}[y(x)]$

**Relation between the Operators D and E**

Using Taylor's series we have,  $y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \dots$

This can be written in symbolic form

$$Ey_x = \left[ 1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \dots \right] y_x = e^{hD} \cdot y_x$$

We obtain in the relation  $E = e^{hD} \rightarrow (3)$

- ❖ If  $f(x)$  is a polynomial of degree n and the values of x are equally spaced then  $\Delta^n f(x)$  is constant

**Proof:**

Let  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$  where  $a_0, a_1, a_2, \dots, a_n$  are constants and  $a_0 \neq 0$ . If h is the step-length, we know the formula for the first forward difference

$$\Delta f(x) = f(x+h) - f(x) = \left[ a_0(x+h)^n + a_1(x+h)^{n-1} + \dots + a_{n-1}(x+h) + a_n \right] - \left[ a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n \right]$$

$$\begin{aligned}
&= a_0 \left[ \left\{ x^n + n.x^{n-1}h + \frac{n(n-1)}{2!} x^{n-2}.h^2 + \dots \right\} - x^n \right] + \\
&a_1 \left[ \left\{ x^{n-1} + (n-1)x^{n-2}.h + \frac{(n-1)(n-2)}{2!} x^{n-3}.h^2 + \dots \right\} - x^{n-1} \right] + \\
&\dots + a_{n-1}h \\
&= a_0nhx^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \dots + b_{n-3}x + b_{n-2}
\end{aligned}$$

Where  $b_2, b_3, \dots, b_{n-2}$  are constants. Here this polynomial is of degree  $(n-1)$ , thus, the first difference of a polynomial of  $n^{\text{th}}$  degree is a polynomial of degree  $(n-1)$

Now

$$\begin{aligned}
\Delta^2 f(x) &= \Delta[\Delta f(x)] \\
&= \Delta[a_0nhx^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \dots + b_{n-1}x + b_{n-2}] \\
&= a_0nh[(x+h)^{n-1} - x^{n-1}] + b_2[(x+h)^{n-2} - x^{n-2}] + \dots + b_{n-1}[(x+h) - x] \\
&= a_0n^{(n-1)}h^2x^{n-2} + c_3x^{n-3} + \dots + c_{n-4}x + c_{n-3}
\end{aligned}$$

Where  $c_3, \dots, c_{n-3}$  are constants. This polynomial is of degree  $(n-2)$

Thus, the second difference of a polynomial of degree  $n$  is a polynomial of degree  $(n-2)$  continuing like this we get  $\Delta^n f(x) = a_0n(n-1)(n-2)\dots 2.1.h^n = a_0h^n (n!)$

$\therefore$  which is constant

### Note:-

1. As  $\Delta^n f(x)$  is a constant, it follows that  $\Delta^{n+1} f(x) = 0, \Delta^{n+2} f(x) = 0, \dots$
2. The converse of above result is also true that is, if  $\Delta^n f(x)$  is tabulated at equal spaced intervals and is a constant, then the function  $f(x)$  is a polynomial of degree  $n$

### Example:-

1. Form the forward difference table and write down the values of  $\Delta f(10)$ ,  $\Delta^2 f(10)$ ,  $\Delta^3 f(15)$  and  $\Delta^4 y(15)$

<b>x</b>	<b>10</b>	<b>15</b>	<b>20</b>	<b>25</b>	<b>30</b>	<b>35</b>
<b>y</b>	<b>19.97</b>	<b>21.51</b>	<b>22.47</b>	<b>23.52</b>	<b>24.65</b>	<b>25.89</b>

**Sol.**

<b>x</b>	<b>Y</b>	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	<b>19.97f(10)</b>					
		<b>1.54<math>\Delta f(10)</math></b>				
15	<b>21.51f(15)</b>		<b>-0.58<math>\Delta^2 f(10)</math></b>			
		0.96		0.67		
20	<b>22.47f(20)</b>		0.09		- 0.68	
		1.05		<b>- 0.01<math>\Delta^3 f(15)</math></b>		0.72
25	<b>23.52f(25)</b>		0.08		<b>0.04<math>\Delta^4 f(15)</math></b>	
		1.13		0.03		
30	<b>24.65f(30)</b>		0.11			
		1.24				
35	<b>25.89f(35)</b>					

**We note that the values of x are equally spaced with step- length h = 5**

**Note:** -  $\therefore x_0 = 10, x_1 = 15 \dots x_5 = 35$  and

$$y_0 = f(x_0) = 19.97$$

$$y_1 = f(x_1) = 21.51$$

-----

-----

$$y_5 = f(x_5) = 25.89$$

$$y_5 = f(x_5) = 25.89$$

## From table

$$\Delta f(10) = \Delta y_0 = 1.54$$

$$\Delta^2 f(10) = \Delta^2 y_0 = -0.58$$

$$\Delta^3 f(15) = \Delta^3 y_1 = -0.01$$

$$\Delta^4 f(15) = \Delta^3 y_1 = 0.04$$

## 2. Evaluate

$$(i) \Delta \cos x$$

$$(ii) \Delta^2 \sin(px+q)$$

$$(iii) \Delta^n e^{ax+b}$$

Sol. Let  $h$  be the interval of differencing

$$(i) \Delta \cos x = \cos(x+h) - \cos x$$

$$= -2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}$$

$$(ii) \Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q) \quad \Delta f(x) = f(x+h) - f(x), \text{for ward formula}$$

$$= 2 \cos\left(px+q + \frac{ph}{2}\right) \sin \frac{ph}{2}$$

$$= 2 \sin \frac{ph}{2} \sin\left(\frac{\pi}{2} + px+q + \frac{ph}{2}\right)$$

$\nabla f(x) = f(x) - f(x-h)$  back ward formula

$$\Delta^2 \sin(px+q) = 2 \sin \frac{ph}{2} \Delta \left[ \sin(px+q) + \frac{1}{2}(\pi + ph) \right]$$

$$= \left[ 2 \sin \frac{ph}{2} \right]^2 \sin \left[ px+q + \frac{1}{2}(\pi + ph) \right]$$

$$\begin{aligned}
(iii) \Delta e^{ax+b} &= e^{a(x+h)+b} - e^{ax+b} \\
&= e^{(ax+b)} (e^{ah}-1) \\
\Delta^2 e^{ax+b} &= \Delta [\Delta (e^{ax+b})] - \Delta [(e^{ah}-1)(e^{ax+b})] \\
&= (e^{ah}-1)^2 \Delta (e^{ax+b}) \\
&= (e^{ah}-1)^2 e^{ax+b}
\end{aligned}$$

Proceeding on, we get  $\Delta^n (e^{ax+b}) = (e^{ah}-1)^n e^{ax+b}$

### 3. Using the method of separation of symbols show that

$$\Delta^n \mu_{x-n} = \mu_{x-n} - n\mu_{x-1} + \frac{n(n-1)}{2} \mu_{x-2} + \dots + (-1)^n \mu_{x-n}$$

Sol. To prove this result, we start with the right hand side. Thus

$$\begin{aligned}
&\mu_{x-n} - n\mu_{x-1} + \frac{n(n-1)}{2} \mu_{x-2} + \dots + (-1)^n \mu_{x-n} \\
&= \mu_{x-n} - nE^{-1} \mu_{x-n} + \frac{n(n-1)}{2} E^{-2} \mu_{x-n} + \dots + (-1)^n E^{-n} \mu_{x-n} \\
&= \left[ 1 - nE^{-1} + \frac{n(n-1)}{2} E^{-2} + \dots + (-1)^n E^{-n} \right] \mu_{x-n} = (1 - E^{-1})^n \mu_{x-n} \\
&= \left( 1 - \frac{1}{E} \right)^n \mu_{x-n} = \frac{(E-1)^n}{E^n} \mu_{x-n} \\
&= \frac{\Delta^n}{E^n} \mu_{x-n} = \Delta^n E^{-n} \mu_{x-n}
\end{aligned}$$

$= \Delta^n \mu_{x-n}$  This is left hand side

### 4. Find the missing term in the following data

X	0	1	2	3	4
Y	1	3	9	-	81

Why this value is not equal to  $3^3$ . Explain

Sol. Consider  $\Delta^4 y_0 = 0$

x	Y	$\Delta$	$\Delta^2$	$\Delta^3$	$\Delta^4$	
0	1	2	4	x-19	124-4x=0	
1	3	6	x-15	105-3x		
2	9	X-9	90-2x			
3	X	81-x				
4	81					

$$\Rightarrow 4y_0 - 4y_3 + 5y_2 - 4y_1 + y_0 = 0$$

$$124 - 4x = 0 \Rightarrow x = 124/4 = 31$$

Substitute given values we get

$$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$$

From the given data we can conclude that the given function is  $y = 3^x$ . To find  $y_3$ , we have to assume that y is a polynomial function, which is not so. Thus we are not getting  $y = 3^3 = 27$

## 2.2. a. Newton's Forward Interpolation Formula:-

Let  $y = f(x)$  be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1}) \rightarrow (1)$$

This polynomial passes through all the points  $[x_i, y_i]$  for  $i = 0$  to  $n$ . therefore, we can obtain the  $y_i$ 's by substituting the corresponding  $x_i$ 's as

$$\text{at } x = x_0, y_0 = b_0$$

$$\text{at } x = x_1, y_1 = b_0 + b_1(x_1 - x_0)$$

$$\text{at } x = x_2, y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \rightarrow (1)$$

Let 'h' be the length of interval such that  $x_i$ 's represent

$$x_0, x_0 + h, x_0 + 2h, x_0 + 3h \dots x_0 + nh$$

This implies  $x_1 - x_0 = h, x_2 - x_0 = 2h, x_3 - x_0 = 3h \dots x_n - x_0 = nh \rightarrow (2)$

From (1) and (2), we get

$$y_0 = b_0$$

$$y_1 = b_0 + b_1 h$$

$$y_2 = b_0 + b_1 2h + b_2 (2h)h$$

$$y_3 = b_0 + b_1 3h + b_2 (3h)(2h) + b_3 (3h)(2h)h$$

.....

.....

$$y_n = b_0 + b_1 (nh) + b_2 (nh)(n-1)h + \dots + b_n (nh) [(n-1)h] [(n-2)h] \rightarrow (3)$$

Solving the above equations for  $b_0, b_1, b_2, \dots, b_n$ , we get  $b_0 = y_0$

$$b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = \frac{y_2 - y_0 - \frac{(y_1 - y_0)}{h} 2h}{2h^2}$$

$$= \frac{y_2 - y_0 - 2y_1 + 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore b_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3!h^3}, b_4 = \frac{\Delta^4 y_0}{4!h^4} \dots b_n = \frac{\Delta^n y_0}{n!h^n}$$

$$\therefore y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1)$$

$$+ \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \dots +$$

$$+ \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1) \dots (x - x_{n-1}) \rightarrow (3)$$

If we use the relationship  $x = x_0 + ph \Rightarrow x - x_0 = ph$ , where  $p = 0, 1, 2, \dots, n$

Then

$$\begin{aligned} x - x_1 &= x - (x_0 + h) = (x - x_0) - h \\ &= ph - h = (p - 1)h \end{aligned}$$

$$\begin{aligned} x - x_2 &= x - (x_1 + h) = (x - x_1) - h \\ &= (p - 1)h - h = (p - 2)h \end{aligned}$$

.....

$$x - x_i = (p - i)h$$

.....

$$x - x_{n-1} = [p - (n - 1)]h$$

Equation (3) becomes for  $p = \frac{x - x_0}{h}$

$$\begin{aligned} y = f(x) = f(x_0 + ph) &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots + \\ &\frac{p(p-1)(p-2)\dots(p-(n-1))}{n!} \Delta^n y_0 \rightarrow (4) \end{aligned}$$

## 2.2. b. Newton's Backward Interpolation Formula:-

If we consider

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \dots + (x - x_i)$$

and impose the condition that  $y$  and  $y_n(x)$  should agree at the tabulated

points  $x_n, x_n - 1, \dots, x_2, x_1, x_0$

We obtain

$$\begin{aligned} y_n(x) &= y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \dots \\ &\frac{p(p+1)\dots[p+(n-1)]}{n!} \nabla^n y_n + \dots \rightarrow (6) \end{aligned}$$

Where  $p = \frac{x - x_n}{h}$

This uses tabular values of the left of  $y_n$ . Thus this formula is useful formula is useful

for interpolation near the end of the table values

### Formula for Error in Polynomial Interpolation:-

If  $y = f(x)$  is the exact curve and  $y = \phi_n(x)$  is the interpolating curve, then the error

in polynomial interpolation is given by

$$\text{Error} = f(x) - \phi_n(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{n+1}(\varepsilon) \rightarrow (7)$$

for any  $x$ , where  $x_0 < x < x_n$  and  $x_0 < \varepsilon < x_n$

The error in Newton's forward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p-1)(p-2)\dots(p-n)}{(n+1)!} \Delta^{n+1} f(\varepsilon)$$

Where  $p = \frac{x-x_0}{h}$

The error in Newton's backward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p+1)(p+2)\dots(p+n)}{(n+1)!} h^{n+1} y^{n+1} f(\varepsilon) \text{ Where } p = \frac{x-x_n}{h}$$

### Examples:-

- Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)-X	50	60	70	80
Temperature ( $Q^\circ c$ )-Y	205	225	248	274

Sol. The difference table is

X	Y	$\Delta$	$\Delta^2$	$\Delta^3$
<b>50=X<sub>0</sub></b>	<b>205=Y<sub>0</sub></b>			
		<b>20=ΔY<sub>0</sub></b>		
60	225		<b>3=Δ<sup>2</sup>Y<sub>0</sub></b>	
		23		<b>0=Δ<sup>3</sup>Y<sub>0</sub></b>
70	248		3	
		26		
80	274			

Let temperature =  $f(x)$ ,  $X=54$

$$x_0 + ph = 54, x_0 = 50, h = 10$$

$$50 + p(10) = 54 \text{ (or) } p = 0.4$$

By Newton's forward interpolation formula

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!} (3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!} (0)$$

$$= 205 + 8 - 0.36$$

$$= 212.64$$

Melting point = 212.64

2. Using Newton's forward interpolation formula, and the given table of values

X	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of  $f(x)$  when  $x=1.4$

Sol.

$x$	$y = f(x)$	$\Delta$	$\Delta^2$	$\Delta^3$	$\Delta^4$
1.1	<b>0.21</b>				
		<b>0.48</b>			
1.3	0.69		<b>0.08</b>		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take  $x_0 = 1.3$  then  $y_0 = 0.69$ ,

$$\Delta y_0 = 0.56, \Delta^2 y_0 = 0.08, \Delta^3 y_0 = 0, L = 0.2, x = 1.3$$

$$x_0 + ph = 1.4 \text{ (or)} 1.3 + p(0.2) = 1.4, p = \frac{1}{2}$$

Using Newton's interpolation formula

$$\begin{aligned} f(1.4) &= 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2} \left( \frac{1}{2} - 1 \right)}{2!} \times 0.08 \\ &= 0.69 + 0.28 - 0.01 = 0.96 \end{aligned}$$

3. The population of a town in the decadal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Population of y	46	66	81	93	101

Sol. Putting  $L = 10, x_0 = 1891, x = 1895$  in the formula  $x = x_0 + ph$  we obtain  $p = 2/5 = 0.4$

X	Y	$\Delta$	$\Delta^2$	$\Delta^3$	$\Delta^4$
1891= $x_0$	46= $y_0$				
		20 $\Delta y_0$			
1901	66		-5 $\Delta^2 y_0$		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$$\begin{aligned}
 y(1895) &= 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{6}(-5) \\
 &\quad + \frac{(0.4-1)0.4(0.4-2)}{6}(2) \\
 &\quad + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24} \\
 &= 54.45 \text{ thousands}
 \end{aligned}$$

**2.3. Gauss's Interpolation Formula:-** We take  $x_0$  as one of the specified of x that lies around the middle of the difference table and denote  $x_0 - rh$  by  $x - r$  and the corresponding value of y by  $y - r$ . Then the middle part of the forward difference table will appear as shown in the next page

X	Y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_{-4}$	$y_{-4}$					
$x_{-3}$	$y_{-3}$	$\Delta y_{-4}$				
$x_{-2}$	$y_{-2}$	$\Delta y_{-3}$	$\Delta^2 y_{-4}$			
$x_{-1}$	$y_{-1}$	$\Delta y_{-2}$	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$		
$x_0$	$y_0$	$\Delta y_{-1}$	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-4}$	
$x_1$	$y_1$	$\Delta y_0$	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-4}$
$x_2$	$y_2$	$\Delta y_1$	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-3}$
$x_3$	$y_3$	$\Delta y_2$	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$
$x_4$	$y_4$	$\Delta y_3$	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^4 y_0$	$\Delta^5 y_{-1}$

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \text{ ----- (1) and}$$

$$\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2}$$

$$\Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2}$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ ----- (2)}$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's forward interpolation formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \text{-----}] \text{ -----} 3$$

Here  $y_p$  is the value of  $y$  at  $x = x_p = x_0 + ph$ ,  $P = (x - x_0)/h$

### 2.3.a. Gauss Forward Interpolation Formula:-

Substituting for  $\Delta^2 y_0, \Delta^3 y_0, \dots$  from (1) in the formula (3), we get

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{-1} + \Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{-1} + \Delta^5 y_{-1} + \dots]$$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1} + \frac{p(p+1)(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + \dots]$$

Substituting  $\Delta^4 y_{-1}$  from (2), this becomes

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + \dots] \dots \dots \dots 4$$

**Note:-** we observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$  and so on. Accordingly the formula

(4) can be written in the notation of central differences as given below

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + \dots] \dots \dots \dots 5$$

### 2.3.b. Gauss's Backward Interpolation formula:-

Let us substitute for  $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots$  from (1) in the formula (3), thus we obtain

$$y_p = [y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{(p-1)p(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{(p-1)(p-2)p(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \dots]$$

$$= [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}(\Delta^2 y_{-1}) + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-1}) + \dots]$$

Substituting for  $\Delta^3 y_{-1}$  and  $\Delta^4 y_{-1}$  from (2) this becomes

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!}(\Delta^2 y_{-1}) + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots]$$

## 2.4. Lagrange's Interpolation Formula:-

Let  $x_0, x_1, x_2, \dots, x_n$  be the  $(n+1)$  values of  $x$  which are not necessarily equally spaced. Let  $y_0, y_1, y_2, \dots, y_n$  be the corresponding values of  $y = f(x)$  let the polynomial of degree  $n$  for the function  $y = f(x)$  passing through the  $(n+1)$  points  $(x_0, f(x_0)), (x_1, f(x_1)) \dots (x_n, f(x_n))$  be in the following form

$$y = f(x) = a_0(x-x_1)(x-x_2)\dots(x-x_n) + a_1(x-x_0)(x-x_2)\dots(x-x_n) + a_2(x-x_0)(x-x_1)\dots(x-x_n) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \rightarrow (1)$$

Where  $a_0, a_1, a_2, \dots, a_n$  are constants

Since the polynomial passes through  $(x_0, f(x_0)), (x_1, f(x_1)) \dots (x_n, f(x_n))$ . The constants can be determined by substituting one of the values of  $x_0, x_1, \dots, x_n$  for  $x$  in the above equation

Putting  $x = x_0$  in (1) we get,  $f(x_0) = a_0(x-x_1)(x-x_2)\dots(x-x_n)$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x-x_1)(x-x_2)\dots(x-x_n)}$$

Putting  $x = x_1$  in (1) we get,  $f(x_1) = a_1(x - x_0)(x_1 - x_2) \dots (x_1 - x_n)$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}$$

Similarly substituting  $x = x_2$  in (1), we get

$$\Rightarrow a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1) \dots (x_2 - x_n)}$$

Continuing in this manner and putting  $x = x_n$  in (1) we

get  $a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})}$

Substituting the values of  $a_0, a_1, a_2, \dots, a_n$ , we get

$$f(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)}$$

$$f(x_1) + \frac{(x - x_0)(x - x_1)(x - x_2) \dots (x - x_n)}{(x_2 - x_0)(x_2 - x_1) \dots (x_2 - x_n)} + \dots + f(x_2) + \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1})} f(x_n)$$

### Examples:-

1. Using Lagrange's formula calculate  $f(3)$  from the following table

x	0	1	2	4	5	6
f(x)	1	14	15	5	6	19

Sol. Given  $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$

$$f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$$

From Lagrange's interpolation formula

$$\begin{aligned}
f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0) \\
&+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1) \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2) \\
&\dots \\
&\dots \\
&+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)
\end{aligned}$$

Here  $x = 3$  then

$$\begin{aligned}
f(3) &= \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1 + \\
&\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14 + \\
&\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 + \\
&\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5 + \\
&\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 + \\
&\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19 \\
&= \frac{12}{240} - \frac{18}{60} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{40} \times 19 \\
&= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95 \\
&= 10
\end{aligned}$$

$$f(3) = 10$$

2. Find  $f(3.5)$  using Lagrange method of 2<sup>nd</sup> and 3<sup>rd</sup> order degree polynomials.

$$x \quad 1 \quad 2 \quad 3 \quad 4$$

$$f(x) \quad 1 \quad 2 \quad 9 \quad 28$$

Sol: By Lagrange's interpolation formula

$$f(x) = \sum_{k=0}^n f(x_k) \frac{(x-x_0)\dots(x-x_{k-1})(x-x_{k+1})(x-x_n)}{(x_k-x_0)\dots(x_k-x_{k-1})\dots(x_k-x_n)}$$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) +$$

$$\therefore f(3.5) = \frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)} (1) + \frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)} (2) +$$

$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)} (9) +$$

$$\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)} (28) + \dots$$

$$= 0.0625 + (-0.625) + 8.4375 + 8.75$$

$$= 16.625$$

$$f(x) = \frac{(x-2)(x-3)(x-4)}{-6} (1) + \frac{(x-1)(x-3)(x-4)}{2} (2) + \frac{(x-1)(x-2)(x-4)}{(-2)} (9) + \frac{(x-1)(x-2)(x-3)}{6} (28)$$

$$= \frac{(x^2-5x+6)(x-4)}{-6} + (x^2-4x+3)(x-4) + \frac{(x^2-3x+2)}{-2} (x-4)(9) + \frac{(x^2-3x+2)}{6} (x-3)(28)$$

$$= \frac{x^3 - 9x^2 + 26x - 24}{-6} + x^3 - 8x^2 + 9x - 12 + \frac{x^3 - 7x^2 + 14x - 8}{-2}(9) + \frac{x^3 - 6x^2 + 11x - 6}{6}(28)$$

$$= \frac{[-x^3 + 9x^2 - 26x + 24 + 6x^3 - 48x^2 + 114x - 72 - 27x^3 + 189x^2 - 378x + 216 + 308x + 28x^3 - 168x^2 - 168]}{6}$$

$$= \frac{6x^3 - 18x^2 + 18x}{6} \Rightarrow f(x) = x^3 - 3x^2 + 3x$$

$$\therefore f(3.5) = (3.5)^3 - 3(3.5)^2 + 3(3.5) = 16.625$$

### Gauss Formula Example:

1. Find  $y(25)$ , given that  $y_{20} = 24, y_{24} = 32, y_{28} = 35, y_{32} = 40$  using Gauss forward difference formula :

Solution: Given

X	20	24	28	32
Y	24	32	35	40

By Gauss Forward difference formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{4!} (\Delta^4 y_{-2}) + \dots] \rightarrow (4)$$

We take  $x_0 = 24$  as origin.

$$X_0 = 24, h = 4, x = 25, p = (x - x_0) / h, p = (25 - 24) / 4 = 0.25$$

Gauss Forward difference table is

X	Y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$
20=x-1	24=y-1			
<b>24=x0</b>	<b>32=y0</b>	$\Delta y_{-1} = 8$		
28=x1	35=y1	$\Delta y_0 = 3$	$\Delta^2 y_{-1} = -5$	
32=x2	40=y2	$\Delta y_1 = 5$	$\Delta^2 y_0 = 2$	$\Delta^3 y_{-1} = 7$

By gauss forward interpolation Formula

$$\text{We get } y(25) = 32 + 0.25(3) + \left(\frac{0.25(0.25-1)}{2}\right)(-5) + \frac{(0.25+1)(0.25)(0.25-1)}{6}(7) = 32 + 0.75$$

$$+ 0.46875 - 0.2734 = 32.945$$

$$Y(25) = 32.945.$$

2. Example:

Use Gauss Backward interpolation formula to find  $f(32)$  given that  $f(25) = 0.2707$ ,  $f(30) = 0.3027$ ,  $f(35) = 0.3386$ ,  $f(40) = 0.3794$ .

Solution: let  $x_0 = 35$  and difference table is

X	Y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$
25=x-2	0.2707=y-2			
30=x-1	0.3027=y-1	0.032		
<b>35=x<sub>0</sub></b>	<b>0.3386=y<sub>0</sub></b>	<b>0.0359</b>	<b>0.0039</b>	
40=x+1	0.3794=y+1	0.0408	0.0049	0.0010

From the table  $y_0 = 0.3386$ ,  $h=5$ ,  $P=(x-x_0)/h=(32-35)/5=-0.6$

$$\Delta y_{-1} = 0.0359, \Delta^2 y_{-1} = 0.0049, \Delta^3 y_{-2} = 0.0010, x_p = 32 \quad p = (x_p - x_0)/h = (32-35)/5 = -0.6$$

By Gauss Backward difference formula

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} (\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!} (\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots]$$

$$f(32) = 0.3386 + (-0.6)(0.0359) + (-0.6)(-0.6+1)(0.0049)/2 + (-0.6)(0.36-1)(0.00010)/6 = 0.3165$$

## 2.5 Stirling's Formula:

Stirling's formula is arithmetic mean of Gauss forward interpolation and Gauss Backward Interpolation formulae

We know that from Gauss forward formula

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)(p-1)p(p-2)}{4!} (\Delta^4 y_{-2}) + \dots] \rightarrow (4)$$

And from Gauss backward formula

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} (\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!} (\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots] \quad \text{----(5)}$$

From 4,5, we found arithmetic mean the Stirling's formula is defined as

$$P = \frac{x-x_0}{h}, \text{ where } h=x_1-x_0$$

$$Y(x) = y_0 + P \left[ \frac{\Delta y_{-1} + \Delta y_0}{2} \right] + \frac{P^2}{2} \Delta^2 y_{-1} + \frac{[P(P^2-1)]}{3!} \left[ \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} \right] + \frac{P^2(P^2-1)}{4!} \Delta^4 y_{-2} + \dots$$

## 2.5 Bessel's Formula:

$Y=f(x)$  is a function with data  $(x_i, y_i)$  with  $P=\frac{x-x_0}{h}$ , where  $h=x_1-x_0$  then Bessel's formula is defined as follows

$$Y(X) = Y_0 + P\Delta Y_0 + \frac{P(P-1)}{2!} \left[ \frac{\Delta^2 Y_0 + \Delta^2 Y_{-1}}{2} \right] + \frac{(P-\frac{1}{2})P(P-1)}{3!} \Delta^3 Y_{-1} + \frac{P(P-1)(P+1)(P-2)}{4!} \left[ \frac{\Delta^4 Y_{-2} + \Delta^4 Y_{-1}}{2} \right] + \dots$$

### Examples:

1. Using Stirling's formula, compute  $f(1.22)$  from the following data

X	1.0	1.1	<b>1.2</b>	1.3	1.4
F(x)	0.841	0.891	0.932	0.963	0.985

Sol. Chose  $X_0=1.2$  is origin and length  $h=0.1$  and  $P=\frac{x-x_0}{h} = \frac{1.22-1.2}{0.1} =0.2$

Next we construct central difference table by using above data and evaluate required value by Stirling's formula

X	Y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2}=1.0$	$y_{-2}=0.841$				
$x_{-1}=1.1$	$y_{-1}=0.891$	$\Delta y_{-2}=0.05$			
$x_0=1.2$	$y_0=0.932$	$\Delta y_{-1}=0.041$	$\Delta^2 y_{-2}=-0.009$		
$x_1=1.3$	$y_1=0.963$	$\Delta y_0=0.031$	$\Delta^2 y_{-1}=-0.01$	$\Delta^3 y_{-2}=-0.001$	
$x_2=1.4$	$y_2=0.985$	$\Delta y_1=0.022$	$\Delta^2 y_0=-0.009$	$\Delta^3 y_{-1}=0.001$	$\Delta^4 y_{-2}=0.02$

Use Stirling's formula

$$Y(x) = y_0 + P \left[ \frac{\Delta y_{-1} + \Delta y_0}{2} \right] + \frac{P^2}{2} \Delta^2 Y_{-1} + [P(P^2 - 1)]/3! \left[ \frac{\Delta^3 Y_{-2} + \Delta^3 Y_{-1}}{2} \right] + \frac{P^2(P^2 - 1)}{4!} \Delta^4 Y_{-2} + \dots$$

$$\text{Then } Y(1.22) = 0.932 + 0.2 \frac{(0.041 + 0.031)}{2} + \frac{0.04}{2!} (-0.01) + \frac{0.2(0.04 - 1)}{4!} (0.002) = 0.93899$$

2. Find  $f(16)$  by Stirling's formula from the following table

x	0	5	10	15	20	25	30
F(x)	0	0.0875	0.1763	0.2679	0.364	0.4663	0.5774

### Examples:

1. Use Bessel's formula to compute  $f(1.95)$  from the following data

X	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F(X)	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Sol. Choose the origin at  $X_0 = 2.0$ , given  $h=0.1$  and  $P = \frac{x-x_0}{h} = \frac{1.95-2.0}{0.1} = -0.5$

Next by using Bessel's formula and central difference table we can evaluate the required solution

X	Y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_{-3}=1.7$	$y_{-3}=2.979$					
$x_{-2}=1.8$	$y_{-2}=3.144$	$\Delta y_{-3}=0.165$				
$x_{-1}=1.9$	$y_{-1}=3.283$	$\Delta y_{-2}=0.139$	$\Delta^2 y_{-3}=-0.026$			
$x_0=2.0$	$y_0=3.391$	$\Delta y_{-1}=0.108$	$\Delta^2 y_{-2}=-0.031$	$\Delta^3 y_{-3}=-0.005$		
$x_1=2.1$	$y_1=3.463$	$\Delta y_0=0.072$	$\Delta^2 y_{-1}=-0.036$	$\Delta^3 y_{-2}=-0.005$	$\Delta^4 y_{-3}=0$	
$x_2=2.2$	$y_2=3.997$	$\Delta y_1=0.53$	$\Delta^2 y_0=0.462$	$\Delta^3 y_{-1}=0.498$	$\Delta^4 y_{-2}=0.503$	$\Delta^5 y_{-3}=0.503$
$x_3=2.3$	$y_3=4.491$	$\Delta y_2=0.494$	$\Delta^2 y_1=-0.04$	$\Delta^3 y_0=-0.502$	$\Delta^4 y_{-1}=-1$	$\Delta^5 y_{-2}=-1.503$

Bessel's formula  $\Delta^6 y_{-3} = -2.006$

$$Y(X) = Y_0 + P\Delta Y_0 + \frac{P(P-1)}{2!} \left[ \frac{\Delta^2 Y_0 + \Delta^2 Y_{-1}}{2} \right] + \frac{(P-\frac{1}{2})P(P-1)}{3!} \Delta^3 Y_{-1} + \frac{P(P-1)(P+1)(P-2)}{4!} \left[ \frac{\Delta^4 Y_{-2} + \Delta^4 Y_{-1}}{2} \right] + \dots$$

$$Y(1.95) = 3.391 + (-0.5)(0.072) + \frac{(-0.5)(-0.5-1)(-0.036+0.462)}{2.2} + \frac{(-0.5-0.5)(-0.5)(-0.5-1)(-0.5-2)(0.503-1)}{24.2}$$

$$Y(1.95) = 3.3629$$

2. Compute Y(25) by using Bessel's formula to the following table

X	20	24	28	32
Y	2854	3162	3544	3992

## 9. Practice Quiz

1. Newton's backward Interpolation formula is .....

[ a ]

a.

$$y_p = [y_0 + p(\Delta y_{-1}) + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} (\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!} (\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots]$$

b.  $y+y_0$

c.  $y_0$

d. None

2. The .....Interpolation formula is used to estimate  $y$ , if the  $x$ -values are unequally spaced. [ c ]

a. Newton formula

b. Gauss formula

c. Lagrange's formula

d. Bessel's formula

3. Averaging Operator formula [ d ]

a.  $\Delta$

b.  $\nabla$

c.  $U$

d.  $\mu y_r = \frac{1}{2} [y_{r+1/2} + y_{r-1/2}]$

4. The relation between  $\Delta$  and  $E$  [ c ]

a.  $\Delta = E$

b.  $\nabla = E$

c.  $\Delta = E - 1$

d.  $\Delta = E + 1$

5. Find the missing term in the following data [ c ]

x	0	1	2	3	4
Y	1	3	9	-	81

a. 10

b. 19

c. 27

d. 0

6. The relation between  $\nabla$  and  $E^{-1}$

[ a ]

a.  $\nabla \equiv 1 - E^{-1}$

b.  $\delta \equiv E^{1/2} - E^{-1/2}$

c.  $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

d.  $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

7. The relation between  $\delta$  and  $E$

[ b ]

a.  $\nabla \equiv 1 - E^{-1}$

b.  $\delta \equiv E^{1/2} - E^{-1/2}$

c.  $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

d.  $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

8. The relation between  $\mu$  and  $E$

[ c ]

a.  $\nabla \equiv 1 - E^{-1}$

b.  $\delta \equiv E^{1/2} - E^{-1/2}$

c.  $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

d.  $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

9. The relation between  $\mu$  and  $\delta$

[ d ]

a.  $\nabla \equiv 1 - E^{-1}$

b.  $\delta \equiv E^{1/2} - E^{-1/2}$

c.  $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

d.  $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

10. The relation between E and D

[ d ]

a.  $\nabla \equiv 1 - E^{-1}$

b.  $\delta \equiv E^{1/2} - E^{-1/2}$

c.  $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

d.  $E = e^{hD}$

### 10. Assignments

S.No	Question	BL	CO												
1	Using Newton's forward interpolation formula, and the given table of values. <table border="1" style="margin-left: 20px;"> <tr> <td>X</td> <td>1.1</td> <td>1.3</td> <td>1.5</td> <td>1.7</td> <td>1.9</td> </tr> <tr> <td>F(X)</td> <td>0.21</td> <td>0.69</td> <td>1.25</td> <td>1.89</td> <td>2.61</td> </tr> </table> Obtain the value of $f(x)$ when $x = 1.4$	X	1.1	1.3	1.5	1.7	1.9	F(X)	0.21	0.69	1.25	1.89	2.61	1	2
X	1.1	1.3	1.5	1.7	1.9										
F(X)	0.21	0.69	1.25	1.89	2.61										
2	Use Gauss back ward interpolation formula to find $f(32)$ given that $f(25) = 0.2707, f(30) = 0.3027, f(35) = 0.3386, f(40) = 0.3794$ .	1	2												
3	Evaluate $f(10)$ given $f(x) = 168,192,336$ at $x = 1,7,15$ respectively. Use Lagrange interpolation.	5	2												
4	Apply Bessel's formula to obtain $f(25)$ given $f(20) = 2854, f(24) = 3162, f(12) = 3544, f(12) = 3992$ .	1	2												
5	Apply Stirling's formula to obtain $f(35)$ given $f(20) = 512, f(30) = 439, f(40) = 346, f(50) = 243$ .	1	2												

## 11. Part A- Question & Answers

S.No	Question& Answers	BL	CO
1	<p><b>Distinguish between interpolation and extrapolation</b>                      Sol. Interpolation: It is the estimation for some such values which lie inside the given Values.                      Extrapolation: It is the estimation for some such values which lie outside the given Values.</p>	4	2
2	<p><b>Write relation between E and Δ.</b>                      Sol. <math>\Delta f(x) = f(x+h) - f(x)</math> Forward Definition  <math>= Ef(x) - f(x)</math> Shift Definition  <math>= (E-1)f(x)</math></p>	1	2
3	<p><b>Prove that</b> <math>(1+\Delta)(1-\nabla) = 1</math>                      Sol. We know that <math>(1+\Delta) = E, (1-\nabla) = E^{-1}</math>  <math>(1+\Delta)(1-\nabla) = EE^{-1} = 1</math></p>	5	2
4	<p><b>Evaluate</b> <math>\Delta \tan^{-1} x</math>                      Sol. <math>\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x</math>  <math>= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right) = \tan^{-1}\left(\frac{h}{1+x^2+hx}\right)</math></p>	5	2
5	<p><b>Evaluate</b> <math>\Delta e^x</math> if <math>h=1</math>.                      Sol. We know that <math>\Delta e^x = e^{x+h} - e^x</math>  <math>= e^{x+1} - e^x</math>  <math>= (e-1)e^x</math></p>	5	2
6	<p><b>Evaluate</b> <math>\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]</math>.                      Sol. Let <math>f(x) = \Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]</math>  <math>f(x)</math> is a polynomial of degree 10 and the coefficient of <math>x^{10}</math> is abcd.  <math>\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = abcd\Delta^{10}x^{10} = abcd10!</math></p>	5	2
7	<p><b>Write Newton-Gregory forward interpolation formula.</b>                      Sol.  <math display="block">y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots</math></p>	1	2
8	<p><b>Write Newton-Gregory backward interpolation formula.</b>                      Sol.  <math display="block">y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots</math></p>	1	2
9	<p><b>State Langrage's interpolation formula.</b>                      Sol. Let <math>x_1, x_2, x_3, \dots, x_n</math> be the values of x which are not equally spaced and <math>y_1, y_2, y_3, \dots, y_n</math> be the Corresponding values of y.                      Thus Lagrange's interpolation formula is</p>	1	2

$$y = \frac{(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)} y_1 + \frac{(x-x_1)(x-x_3)\dots(x-x_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_n)} y_2 + \dots + \frac{(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})} y_n.$$

<b>10</b>	<b>Write Stirling's Formula</b> Sol. $Y(x) = y_0 + P \left[ \frac{\Delta Y_{-1} + \Delta Y_0}{2} \right] + \frac{P^2}{2} \Delta^2 Y_{-1} + \frac{[P(P^2-1)]}{3!} \left[ \frac{\Delta^3 Y_{-2} + \Delta^3 Y_{-1}}{2} \right] + \frac{P^2(P^2-1)}{4!} \Delta^4 Y_{-2} + \dots$	<b>1</b>	<b>2</b>
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## 12. Part B- Questions

S.No	Question	BL	CO												
<b>1</b>	For $x = 0,1,2,4,5$ ; $f(x) = 1,14,15,5,6$ . find $f(3)$ using forward difference table.	<b>1</b>	<b>2</b>												
<b>2</b>	Find the values of $\cos 1.747$ using the values given in the table below <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td>X</td> <td>1.70</td> <td>1.74</td> <td>1.78</td> <td>1.82</td> <td>1.86</td> </tr> <tr> <td>F(X)</td> <td>0.9916</td> <td>0.9857</td> <td>0.9781</td> <td>0.9691</td> <td>0.9584</td> </tr> </table>	X	1.70	1.74	1.78	1.82	1.86	F(X)	0.9916	0.9857	0.9781	0.9691	0.9584	<b>1</b>	<b>2</b>
X	1.70	1.74	1.78	1.82	1.86										
F(X)	0.9916	0.9857	0.9781	0.9691	0.9584										
<b>3</b>	Using Newton's forward interpolation formula, and the given table of values. <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td>X</td> <td>1.1</td> <td>1.3</td> <td>1.5</td> <td>1.7</td> <td>1.9</td> </tr> <tr> <td>F(X)</td> <td>0.21</td> <td>0.69</td> <td>1.25</td> <td>1.89</td> <td>2.61</td> </tr> </table> Obtain the value of $f(x)$ when $x = 1.4$	X	1.1	1.3	1.5	1.7	1.9	F(X)	0.21	0.69	1.25	1.89	2.61	<b>1</b>	<b>2</b>
X	1.1	1.3	1.5	1.7	1.9										
F(X)	0.21	0.69	1.25	1.89	2.61										
<b>4</b>	Use Gauss back ward interpolation formula to find $f(32)$ given that $f(25) = 0.2707$ , $f(30) = 0.3027$ , $f(35) = 0.3386$ , $f(40) = 0.3794$ .	<b>1</b>	<b>2</b>												
<b>5</b>	Use Lagrange interpolation. Evaluate $f(10)$ given $f(x) = 168,192,336$ at $x = 1,7,15$ respectively.	<b>5</b>	<b>2</b>												
<b>6</b>	Find the unique polynomial $p(x)$ of degree 2 or less such that $p(x) = 1$ , $p(3) = 27$ , $p(4) = 64$ using Lagrange's interpolation formula.	<b>1</b>	<b>2</b>												
<b>7</b>	Given $f(2) = 10$ , $f(1) = 8$ , $f(0) = 5$ , $f(-1) = 10$ estimate $f(\frac{1}{2})$ by using Gauss forward formula	<b>6</b>	<b>2</b>												
<b>8</b>	Using Lagrange's Interpolation formula, find $y(10)$ from the following table <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td>X</td> <td>5</td> <td>6</td> <td>9</td> <td>11</td> </tr> <tr> <td>Y</td> <td>12</td> <td>13</td> <td>14</td> <td>16</td> </tr> </table> Fit the second difference of the polynomial $x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing $h = 2$	X	5	6	9	11	Y	12	13	14	16	<b>1</b>	<b>2</b>		
X	5	6	9	11											
Y	12	13	14	16											
<b>9</b>	Apply Bessel's formula to obtain $f(25)$ given $f(20) = 2854$ , $f(24) = 3162$ , $f(12) = 3544$ , $f(12) = 3992$ .	<b>1</b>	<b>2</b>												

<b>10</b>	Apply Stirling's formula to obtain $f(35)$ given $f(20) = 512, f(30) = 439, f(40) = 346, f(50) = 243$ .	<b>1</b>	<b>2</b>
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### 13. Supportive Online Certification Courses:

- An Introductory Method to Numerical Analysis (NPTEL)
- Probability & Statistics (NPTEL)

### 14. Real Time Applications:

- Application of Numerical methods as a way of transforming linear equations and differential equations

### 15. Content beyond the Syllabus:

- Numerical Differentiation and Applications

### 16. Prescribed Text Books and Reference Books:

#### Text Books:

1. B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.
3. Ronald E. Walpole "Probability and Statistics for Engineers and Scientists", PNIE

#### Reference Books:

1. B.V.Ramana, "Higher Engineering Mathematics", Mc Graw Hill publishers.
2. Alan Jeffrey, "Advanced Engineering Mathematics", Elsevier.

## COURSE MATERIAL

<b>SUBJECT</b>	<b>NUMERICAL METHODS AND PROBABILITY THEORY (20A54402)</b>
<b>UNIT</b>	<b>3</b>
<b>COURSE</b>	<b>B.TECH</b>
<b>SEMESTER</b>	<b>2 - 2</b>
<b>DEPARTMENT</b>	<b>HUMANITIES &amp; SCIENCE</b>
<b>PREPARED BY (Faculty Name/s)</b>	<b>Department of Mathematics</b>

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## 1. COURSE OBJECTIVES

The objectives of this course is to

- This course aims at providing the student with the knowledge on Various numerical methods for solving equations, interpolating the polynomials, evaluation of integral equations and solution of differential equations.
- The theory of Probability and random variables.

## 2. PREREQUISITES

Students should have knowledge on

1. Students should have knowledge on Numerical Techniques
2. Students should have knowledge on Probability Theory

## 3. SYLLABUS

### UNIT III

#### **Numerical Integration & Solution of Initial value problems to Ordinary differential equations**

Numerical Integration: Trapezoidal rule – Simpson's 1/3 Rule – Simpson's 3/8 Rule Numerical solution of Ordinary Differential equations: Solution by Taylor's series-Picard's Method of successive Approximations-Modified Euler's Method-Runge-Kutta Methods.

## 4. COURSE OUTCOMES

After the completion of course, students will be able to

1. Apply numerical methods to solve algebraic and transcendental equations
2. Derive interpolating polynomials using interpolation formulae
3. Solve differential and integral equations numerically
4. Apply Probability theory to find the chances of happening of events.
5. Understand various probability distributions and calculate their statistical constants.

## 5. Co-PO / PSO Mapping

NMPT	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	P10	PO11	PO12	PSO1	PSO2
CO1	3	3												
CO2	3	3												
CO3	3	3												
CO4	3	3												
CO5	3	3												

## 6. LESSON PLAN

LECTURE	WEEK	TOPICS TO BE COVERED	REFERENCES
1	1	Introduction to Numerical Integration	TB1/RB2
2		Trapezoidal Rule	TB1/RB2
3		Problems on Trapezoidal Rule	TB1/RB2
4		Simpson's 1/3 Rule	TB1/RB2
5		Problems on Simpson's 1/3 Rule	TB1/RB2
6		Simpson's 3/8 Rule	TB1/RB2
7	2	Problems on Simpson's 3/8 Rule	TB1/RB2
8		Introduction to Numerical solutions of O.D.E.	TB1/RB2
9		Taylor's Series Method	TB1/RB2
10		Problems on Taylor's Series Method	TB1/RB2
11		Picard's Method of Successive Approximation	TB1/RB2
12		Problems on Picard's Method	TB1/RB2
13	3	Euler-Modified Euler's Method	TB1/RB2
14		Problems on Modified Euler's Method	TB1/RB2
15		Runge Kutta Methods	TB1/RB2
16		RK Method of II,III,IV th Orders	TB1/RB2
17		Problems on RK Methods	TB1/RB2
18		Problems on RK Methods	TB1/RB2

## 7. ACTIVITY BASED LEARNING

1. To Solve Ordinary Differential Equations by using Numerical Techniques

## 8. LECTURE NOTES

### 3.1 Numerical Integration

This is the most popular and widely used numerical integration formula. It forms the basis for a number of numerical integration methods.

### 3.2 Trapezoidal Rule

Here the function  $f(x)$  is approximated by a first-order polynomial,  $h$  is the length of interval and  $n$  is number of sub-intervals

$$\text{Then } \int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})]$$

$$\Rightarrow \int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [\text{Sum of first and last ordinates} + 2 \times \text{The remaining ordinates}]$$

First divide the interval  $[x_0, x_n]$  into  $n$  sub-intervals with the length of interval  $h$ , then substitute all values of  $y$  and simply which gives the value of  $\int_{x_0}^{x_n} f(x) dx$

Problems

1. Evaluate  $\int_0^1 x^3 dx$  with five sub-intervals by Trapezoidal rule

Sol: Here  $a = 0, b = 1, n = 5$  and  $h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

The values of  $x$  and  $y$  are tabulated below

$x$	0	0.2	0.4	0.6	0.8	1
$y$	0	0.008	0.064	0.216	0.512	1

Trapezoidal rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$\begin{aligned} \int_0^1 x^3 dx &= \frac{0.2}{2} [(0 + 1) + 2(0.008 + 0.064 + 0.216 + 0.512)] \\ &= 0.26 \end{aligned}$$

2. Solve  $\int_0^{\pi} t \sin t \, dt$  using the Trapezoidal rule

Sol: Divide the interval  $[0, \pi]$  into 6 parts each of width  $h = \frac{\pi}{6}$

The values of  $f(t) = t \sin t$  are given below

t	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\pi$
Y=t sin t	0	0.2618	0.9069	1.5708	1.8138	1.309	0

Trapezoidal rule

$$\int_{x_0}^{x_n} f(x) \, dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$\int_0^{\pi} x^3 \, dx = \frac{\pi}{12} [(0 + 0) + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.309)]$$

$$= 3.07$$

### 3.3 Simpson's 1/3 Rule:

This is another popular method. Here the function  $f(x)$  is approximated by a second order polynomial

Then

$$\int_{x_0}^{x_n} f(x) \, dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2})]$$

First divide the interval  $[x_0, x_n]$  into  $n$  (even) sub-intervals with the length of interval  $h$ , then substitute all values of  $y$  and simply which gives the value of  $\int_{x_0}^{x_n} f(x) \, dx$ .

1. Find the value of  $\int_1^2 \frac{1}{x} \, dx$  by Simpson's 1/3 rule.

Sol: Divide the interval  $[1, 2]$  into 8 parts each of width  $h = 0.125$

The values of  $f(x) = \frac{1}{x}$  are given below

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y	1	0.8888	0.8	0.7272	0.6666	0.6153	0.5714	0.5333	0.5

Then by Simpsons 1/3 rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} [(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$\int_1^2 \frac{1}{x} dx = \frac{0.125}{3} [(1 + 0.5) + 4(0.8888 + 0.7272 + 0.6153 + 0.5333)$$

$$+ 2(0.8 + 0.6666 + 0.5714)]$$

$$= 0.6931$$

2. Evaluate  $\int_0^6 \frac{1}{1+x} dx$  by Simpson's 1/3 rule.

Sol: Divide the interval  $[0, 6]$  into 6 parts each of width  $h = 1$

The values of  $f(x) = \frac{1}{1+x}$  are given below

x	0	1	2	3	4	5	6
$Y=f(x) = \frac{1}{1+x}$	1	0.5	0.3333	0.25	0.2	0.1666	0.1428

Then by Simpsons 1/3 rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$\int_1^2 \frac{1}{x} dx = \frac{1}{3} [(1 + 0.1428) + 4(0.5 + 0.25 + 0.1666) + 2(0.3333 + 0.2)]$$

$$= 1.9659$$

### 3.4 Simpson's 3/8 Rule:

This is another popular method. Here the function  $f(x)$  is approximated by a second order polynomial

Then

$$\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1})$$

$$+ 2(y_3 + y_6 + y_9 + \dots + y_{n-2})]$$

First divide the interval  $[x_0, x_n]$  into  $n$ (even) sub-intervals with the length of interval  $h$ , then substitute all values of  $y$  and simply which gives the value of  $\int_{x_0}^{x_n} f(x) dx$ .

1. Evaluate  $\int_0^6 \frac{1}{1+x^2} dx$  by Simpson's 1/3 rule.

Sol: Divide the interval  $[0, 6]$  into 6 parts each of width  $h = 1$

The values of  $f(x) = \frac{1}{1+x^2}$  are given below

x	0	1	2	3	4	5	6
$Y=f(x) = \frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.0588	0.0385	0.0027

Then by Simpsons 3/8 rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_1^2 \frac{1}{x} dx = \frac{1}{3} [(1 + 0.0027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2(0.1)]$$

$$= 1.3571$$

2. Evaluate  $\int_0^1 \sqrt{1+x^4} dx$  using Simpson's 3/8 rule

Sol: Sol: Divide the interval  $[0, 1]$  into 6 parts each of width  $h = \frac{1}{6}$

The values of  $f(x) = \sqrt{1+x^4}$  are given below

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$Y=f(x) = \sqrt{1+x^4}$	1	1.0004	1.0062	1.0301	1.0943	1.2175	1.4142

Then by Simpsons 3/8 rule

$$\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$\int_1^2 \frac{1}{x} dx = \frac{1}{3} [(1 + 1.4142) + 3(1.0004 + 1.0062 + 1.0943 + 1.2175) + 2(1.0301)]$$

$$= 1.0894$$

### Numerical solutions of ordinary differential equations

1. The important methods of solving ordinary differential equations of first order numerically are as follows
  - 1) Taylor's series method
  - 2) Picard's method
  - 3) Euler's method
  - 4) Modified Euler's method of successive approximations
  - 5) Runge- kutta method

To describe various numerical methods for the solution of ordinary differential eqn's, we consider the general 1<sup>st</sup> order differential eqn

Given O.D.Eqn.  $dy/dx=f(x,y)$ -----(1)

with the initial condition  $y(x_0)=y_0$ ,  $X_1=X_0+h$ ,  $X_2=X_1+h$ , we have to evaluate  $Y_1, Y_2, \dots$  etc

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor and Picard belong to class(i)

The methods of Euler, Runge - kutta method, Adams, Milne etc, belong to class (ii)

### 3.4 TAYLOR'S SERIES METHOD

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \rightarrow (1)$$

With the initial condition  $y(x_0) = y_0 \rightarrow (2)$

$y(x)$  can be expanded about the point  $x_0$  in a Taylor's series in powers of  $(x - x_0)$  as

$$y(x) = y(x_0) + \frac{(x - x_0)}{1} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \dots + \frac{(x - x_0)^n}{n!} y^n(x_0) \rightarrow (3)$$

In equ3,  $y(x_0)$  is known from I.C equ2. The remaining coefficients  $y'(x_0), y''(x_0), \dots, y^n(x_0)$  etc are obtained by successively differentiating equ1 and evaluating at  $x_0$ . Substituting these values in equ3,  $y(x)$  at any point can be calculated from equ3. Provided  $h = x - x_0$  is small.

When  $x_0 = 0$ , then Taylor's series equ3 can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + \dots + \frac{x^n}{n!} y^n(0) + \dots \rightarrow (4)$$

**1. Using Taylor's expansion evaluate the value of  $y' - 2y = 3e^x, y(0) = 0$ , at a)  $x = 0.2$**

b) Compare the numerical solution obtained with exact solution .

Sol: Given equation can be written as  $2y + 3e^x = y', y(0) = 0, x_0 = 0, y_0 = 0$

Differentiating repeatedly w.r.t to 'x' and evaluating at  $x = 0$

$$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$$

$$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$$

$$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$$

$$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$$

$$y^v(x) = 2.y^{iv} + 3e^x, y^v(0) = 2(45) + 3e^0 = 90 + 3 = 93$$

In general,  $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^x$  or  $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^0$

The Taylor's series expansion of  $y(x)$  about  $x_0 = 0$  is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) + \frac{x^5}{5!} y^{(5)}(0) + \dots$$

Substituting the values of  $y(0), y'(0), y''(0), y'''(0), \dots$

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \frac{93}{120}x^5 + \dots$$

$$y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \frac{31}{40}x^5 + \dots \rightarrow \text{equ 1}$$

Now put  $x = 0.1$  in equ 1

$$y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{15}{8}(0.1)^4 + \frac{31}{40}(0.1)^5 = 0.34869$$

Now put  $x = 0.2$  in equ 1

$$y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \frac{31}{40}(0.2)^5 = 0.811244$$

$$y(0.3) = 3(0.3) + \frac{9}{2}(0.3)^2 + \frac{7}{2}(0.3)^3 + \frac{15}{8}(0.3)^4 + \frac{31}{40}(0.3)^5 = 1.41657075$$

Analytical Solution:

The exact solution of the equation  $\frac{dy}{dx} = 2y + 3e^x$  with  $y(0) = 0$  can be found as follows

$$\frac{dy}{dx} - 2y = 3e^x \text{ Which is a linear in } y.$$

$$\text{Here } P = -2, Q = 3e^x$$

$$\text{I.F} = \int_e^{pdx} = \int_e^{-2dx} = e^{-2x}$$

General solution is  $y \cdot e^{-2x} = \int 3e^x \cdot e^{-2x} dx + c = -3e^{-x} + c$ , dividing by  $e^{-2x}$  on b.s.

$$\therefore y = -3e^x + ce^{2x} \text{ where } x = 0, y = 0, 0 = -3 + c \Rightarrow c = 3$$

The particular solution is  $y = 3e^{2x} - 3e^x$  or  $y(x) = 3e^{2x} - 3e^x$

Put  $x = 0.1$  in the above particular solution,

$$y = 3 \cdot e^{0.2} - 3e^{0.1} = 0.34869$$

Similarly put  $x = 0.2$

$$y = 3e^{0.4} - 3e^{0.2} = 0.811265$$

put  $x = 0.3$

$$y = 3e^{0.6} - 3e^{0.3} = 1.416577$$

There is negligible error between numerical solution and analytical solution.

**2. Using Taylor's series method, solve the equation  $\frac{dy}{dx} = x^2 + y^2$  for**

$x = 0.4$  given that  $y = 0$  when  $x = 0$

Sol: Given that  $\frac{dy}{dx} = x^2 + y^2$  and  $y = 0$  when  $x = 0$  i.e.  $y(0) = 0$

Here  $y_0 = 0, x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at  $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$

$$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$$

$$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2$$

$$y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0$$

The Taylor's series for f(x) about  $x_0 = 0$  is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y''''(0) + \dots$$

Substituting the values of  $y(0), y'(0), y''(0), \dots$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + \dots = \frac{x^3}{3} + \text{(Higher order terms are}$$

neglected)

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

**3. Solve  $y' = x - y^2, y(0) = 1$  using Taylor's series method and compute  $y(0.1), y(0.2)$**

Sol: Given that  $y' = x - y^2, y(0) = 1$

Here  $y_0 = 1, x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at  $x=0$

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$

$$y''(x) = 1 - 2y \cdot y', y''(0) = 1 - 2 \cdot y(0) \cdot y'(0) = 1 - 2(-1) = 3$$

$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2 \cdot y(0) \cdot y'(0) - 2 \cdot (y'(0))^2 = -6 - 2 = -8$$

$$y^{(4)}(x) = -2 \cdot y \cdot y'' - 2 \cdot y'' \cdot y' - 4 \cdot y'' \cdot y', y^{(4)}(0) = -2 \cdot y(0) \cdot y''(0) - 6 \cdot y''(0) \cdot y'(0) = 16 + 18 = 34$$

The Taylor's series for  $f(x)$  about  $x_0 = 0$  is

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \dots$$

Substituting the value of  $y(0), y'(0), y''(0), \dots$

$$y(x) = 1 - x + \frac{3}{2} x^2 - \frac{8}{6} x^3 + \frac{34}{24} x^4 + \dots$$

$$y(x) = 1 - x + \frac{3}{2} x^2 - \frac{4}{3} x^3 + \frac{17}{12} x^4 + \dots \rightarrow (1)$$

now put  $x = 0.1$  in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2} (0.1)^2 + \frac{4}{3} (0.1)^3 + \frac{17}{12} (0.1)^4 + \dots$$

$$= 0.91380333 \approx 0.91381$$

Similarly put  $x = 0.2$  in (1)

$$y(0.2) = 1 - 0.2 + \frac{3}{2} (0.2)^2 - \frac{4}{3} (0.2)^3 + \frac{17}{12} (0.2)^4 + \dots$$

$$= 0.8516.$$

**4. Solve  $y' = x^2 - y$ ,  $y(0) = 1$ , using Taylor's series method and compute  $y(0.1)$ ,  $y(0.2)$ ,  $y(0.3)$  and  $y(0.4)$  (correct to 4 decimal places).**

Sol. Given that  $y' = x^2 - y$  and  $y(0) = 1$

Here  $x_0 = 0$ ,  $y_0 = 1$  or  $y = 1$  when  $x = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at  $x = 0$ .

$$Y'(x) = x^2 - y, \quad y'(0) = 0 - 1 = -1$$

$$y''(x) = 2x - y', \quad y''(0) = 2(0) - y'(0) = 0 - (-1) = 1$$

$$y'''(x) = 2 - y'', \quad y'''(0) = 2 - y''(0) = 2 - 1 = 1,$$

$$y^{(4)}(x) = -y''', \quad y^{(4)}(0) = -y'''(0) = -1.$$

The Taylor's series for  $f(x)$  about  $x_0 = 0$  is

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{IV}(0) + \dots$$

substituting the values of  $y(0)$ ,  $y'(0)$ ,  $y''(0)$ ,  $y'''(0)$ ,  $y^{IV}(0)$ , .....

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(-1) + \dots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \dots \quad \rightarrow (1)$$

Now put  $x = 0.1$  in (1),

$$\begin{aligned} y(0.1) &= 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} + \dots \\ &= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416 - 0.905125 \sim 0.9051 \\ &\quad (4 \text{ decimal places}) \end{aligned}$$

Now put  $x = 0.2$  in eq (1),

$$\begin{aligned} y(0.2) &= 1 - 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{64} \\ &= 1 - 0.2 + 0.02 + 0.001333 - 0.000025 \\ &= 1.021333 - 0.200025 \\ &= 0.821308 \sim 0.8213 \text{ (4 decimals)} \end{aligned}$$

Similarly  $y(0.3) = 0.7492$  and  $y(0.4) = 0.6897$  (4 decimal places).

**5. Solve  $\frac{dy}{dx} - 1 = xy$  and  $y(0) = 1$  using Taylor's series method and compute  $y(0.1)$ .**

Sol. Given that  $\frac{dy}{dx} - 1 = xy$  and  $y(0) = 1$

Here  $\frac{dy}{dx} = 1 + xy$  and  $y_0 = 1$ ,  $x_0 = 0$ .

Differentiating repeatedly w.r.t 'x' and evaluating at  $x_0 = 0$

$$y'(x) = 1 + xy, \quad y'(0) = 1 + 0(1) = 1.$$

$$y''(x) = x \cdot y' + y, \quad y''(0) = 0 + 1 = 1$$

$$y'''(x) = x \cdot y'' + y' + y', \quad y'''(0) = 0 \cdot (1) + 2 \cdot (1) = 2$$

$$y^{IV}(x) = xy''' + y'' + 2y'', \quad y^{IV}(0) = 0 + 3(1) = 3.$$

$$y^V(x) = xy^{IV} + y''' + 2y''', \quad y^V(0) = 0 + 2 + 2(3) = 8$$

The Taylor series for  $f(x)$  about  $x_0 = 0$  is

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{IV}(0) + \frac{x^5}{5!} y^V(0) + \dots$$

Substituting the values of  $y(0)$  ,  $y'(0)$  ,  $y''(0)$  , ....

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} (2) + \frac{x^4}{24} (3) + \frac{x^5}{120} (8) + \dots$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \dots \quad \rightarrow(1)$$

Now put  $x = 0.1$  in equ (1),

$$\begin{aligned} y(0.1) &= 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \dots \\ &= 1 + 0.1 + 0.005 + 0.000333 + 0.0000125 + 0.0000006 \\ &= 1.1053461 \end{aligned}$$

### H.W

**6. Given the differential equ  $y' = x^2 + y^2$ ,  $y(0) = 1$ . Obtain  $y(0.25)$ , and  $y(0.5)$  by Taylor's Series method.**

Ans: 1.3333, 1.81667

**7. Solve  $y' = xy^2 + y$ ,  $y(0) = 1$  using Taylor's series method and compute  $y(0.1)$  and  $y(0.2)$ .**

Ans: 1.111, 1.248.

**Note:** We know that the Taylor's expansion of  $y(x)$  about the point  $x_0$  in a power of  $(x - x_0)$  is.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \dots \rightarrow(1)$$

Or

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \dots$$

If we let  $x - x_0 = h$ . (i.e.  $x = x_0 + h = x_1$ ) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \dots$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \dots \quad \rightarrow(2)$$

Similarly expanding  $y(x)$  in a Taylor's series about  $x = x_1$ . We will get.

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \dots \quad \rightarrow(3)$$

Similarly expanding  $y(x)$  in a Taylor's series about  $x = x_2$  We will get.

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \frac{h^4}{4!} y_2^{IV} + \dots \quad \rightarrow(4)$$

In general, Taylor's expansion of  $y(x)$  at a point  $x = x_n$  is

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \frac{h^4}{4!} y_n^{IV} + \dots \quad \rightarrow(5)$$

**8. Solve  $y' = x - y^2$ ,  $y(0) = 1$  using Taylor's series method and evaluate  $y(0.1)$ ,  $y(0.2)$  by step size  $h=0.1$ .**

Sol: Given  $y' = x - y^2 \quad \rightarrow(1)$

and  $y(0) = 1 \quad \rightarrow(2)$

Here  $x_0 = 0$ ,  $y_0 = 1$ .

Differentiating (1) w.r.t 'x', we get.

$$y'' = 1 - 2yy' \quad \rightarrow(3)$$

$$y''' = -2(y \cdot y'' + (y')^2) \quad \rightarrow(4)$$

$$y^{IV} = -2[y \cdot y''' + y' \cdot y'' + 2y' \cdot y''] \quad \rightarrow(5)$$

$$= -2(3y' \cdot y'' + y \cdot y''') \dots\dots$$

Put  $x_0 = 0$ ,  $y_0 = 1$  in (1),(3),(4) and (5),

We get

$$y_0' = 0 - 1 = -1,$$

$$y_0'' = 1 - 2(1)(-1) = 3,$$

$$y_0''' = -2[(-1)^2 + (1)(3)] = -8$$

$$y_0^{IV} = -2[3(-1)(3) + (1)(-8)] = -2(-9 - 8) = 34.$$

Take  $h=0.1$

**Step1:** By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \dots \quad \rightarrow(6)$$

on substituting the values of  $y_0, y_0', y_0'',$  etc in equ (6) we get

$$\begin{aligned} y(0.1) = y_1 &= 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + \dots \\ &= 1 - 0.1 + 0.015 - 0.00133 + 0.00014 + \dots \\ &= 0.91381 \end{aligned}$$

**Step2:** Let us find  $y(0.2)$ , we start with  $(x_1, y_1)$  as the starting value.

Here  $x_1 = x_0 + h = 0 + 0.1 = 0.1$  and  $y_1 = 0.91381$

Put these values of  $x_1$  and  $y_1$  in (1),(3),(4) and (5), we get

$$y_1' = x_1 - y_1^2 = 0.1 - (0.91381)^2 = 0.1 - 0.8350487 = -0.735$$

$$y_1'' = 1 - 2y_1 \cdot y_1' = 1 - 2(0.91381)(-0.735) = 1 + 1.3433 = 2.3433$$

$$y_1''' = -2[(y_1')^2 + y_1 \cdot y_1''] = -2[(-0.735)^2 + (0.91381)(2.3433)] = -5.363112$$

$$\begin{aligned} y_1^{IV} &= -2[3 \cdot y_1' \cdot y_1'' + y_1 \cdot y_1'''] \\ &= -2[3 \cdot (-0.735)(2.3433) + (0.91381)(-5.363112)] \\ &= -2[(-5.16697) - 4.9] = 20.133953 \end{aligned}$$

By Taylor's series expansion,

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \dots$$

$$\therefore y(0.2) = y_2 = 0.91381 + (0.1)(-0.735) + \frac{(0.1)^2}{2}(2.3433) +$$

$$\frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.133953) + \dots$$

$$y(0.2) = 0.91381 - 0.0735 + 0.0117 - 0.00089 + 0.00008 = 0.8512$$

**9. Tabulate  $y(0.1)$ ,  $y(0.2)$  and  $y(0.3)$  using Taylor's series method given that  $y' = y^2 + x$  and  $y(0) = 1$**

Sol: Given  $y^1 = y^2 + x$  →(1)

and  $y(0) = 1$  →(2)

Here  $x_0 = 0, y_0 = 1$ .

Differentiating (1) w.r.t 'x', we get

$$y'' = 2y \cdot y' + 1 \quad \rightarrow(3)$$

$$y''' = 2[y \cdot y'' + (y')^2] \quad \rightarrow(4)$$

$$\begin{aligned} y^{IV} &= 2[y \cdot y''' + y' y'' + 2 y' y''] \\ &= 2[y \cdot y''' + 3 y' y''] \quad \rightarrow(5) \end{aligned}$$

Put  $x_0 = 0, y_0 = 1$  in (1), (3), (4) and (5), we get

$$y_0' = (1)^2 + 0 = 1$$

$$y_0'' = 2(1)(1) + 1 = 3,$$

$$y_0''' = 2((1)(3) + (1)^2) = 8$$

$$\begin{aligned} y_0^{IV} &= 2[(1)(8) + 3(1)(3)] \\ &= 34 \end{aligned}$$

Take  $h = 0.1$ .

**Step1:** By Taylor's series expansion, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \dots \quad \rightarrow(6)$$

on substituting the values of  $y_0, y_0', y_0''$  etc in (6), we get

$$\begin{aligned} y(0.1) = y_1 &= 1 + (0.1)(1) + \frac{(0.1)^2}{2} (3) + \frac{(0.1)^3}{6} (8) + \frac{(0.1)^4}{24} (34) + \dots \\ &= 1 + 0.1 + 0.015 + 0.001333 + 0.000416 \\ y_1 &= 1.116749 \end{aligned}$$

**Step2:** Let us find  $y(0.2)$ , we start with  $(x_1, y_1)$  as the starting values

Here  $x_1 = x_0 + h = 0 + 0.1 = 0.1$  and  $y_1 = 1.116749$

Putting these values in (1), (3), (4) and (5), we get

$$y_1' = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$$

$$y_1'' = 2y_1 y_1' + 1 = 2(1.116749)(1.3471283) + 1 = 4.0088$$

$$y_1''' = 2(y_1 y_1'' + (y_1')^2) = 2((1.116749)(4.0088) + (1.3471283)^2)$$

$$= 12.5831$$

$$y_1^{IV} = 2y_1 y_1''' + 6 y_1' y_1'' = 2(1.116749) (12.5831) + 6(1.3471283) (4.0088) \\ = 60.50653$$

By Taylor's expansion

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \dots$$

$$\therefore y(0.2) = y_2 = 1.116749 + (0.1) (1.3471283) \\ + \frac{(0.1)^2}{2} (4.0088) + \frac{(0.1)^3}{6} (12.5831) + \frac{(0.1)^4}{24} (60.50653)$$

$$y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252 \\ = 1.27385$$

$$y(0.2) = 1.27385$$

**Step3:** Let us find  $y(0.3)$ , we start with  $(x_2, y_2)$  as the starting value.

Here  $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$  and  $y_2 = 1.27385$

Putting these values of  $x_2$  and  $y_2$  in eq (1), (3), (4) and (5),

we get

$$y_2' = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$$

$$y_2'' = 2y_2 y_2' + 1 = 2(1.27385) (1.82269) + 1 = 5.64366$$

$$y_2''' = 2[y_2 y_2'' + (y_2')^2] = 2[(1.27385) (5.64366) + (1.82269)^2] \\ = 14.37835 + 6.64439 = 21.02274$$

$$y_2^{IV} = 2y_2 y_2''' + 6 y_2' y_2'' \\ = 2(1.27385) (21.02274) + 6(1.82269)(5.64366) \\ = 53.559635 + 61.719856 = 115.27949$$

By Taylor's expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \frac{h^4}{4!} y_2^{IV} + \dots$$

$$y(0.3) = y_3 = 1.27385 + (0.1) (1.82269) \\ + \frac{(0.1)^2}{2} (5.64366) + \frac{(0.1)^3}{6} (21.02274) \\ + \frac{(0.1)^4}{24} (115.27949)$$

$$= 1.27385 + 0.182269 + 0.02821$$

$$+ 0.0035037 + 0.00048033 = 1.48831$$

$$y(0.3) = 1.48831$$

**10. Solve  $y' = x^2 - y$ ,  $y(0) = 1$  using Taylor's series method and evaluate**

$y(0.1), y(0.2), y(0.3)$  and  $y(0.4)$  (correct to 4 decimal places)

Sol: Given  $y' = x^2 - y \rightarrow (1)$

and  $y(0) = 1 \rightarrow (2)$

Here  $x_0 = 0, y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$$y'' = 2x - y' \rightarrow (3)$$

$$y''' = 2 - y'' \rightarrow (4)$$

$$y^{IV} = -y''' \rightarrow (5)$$

put  $x_0 = 0, y_0 = 1$  in (1), (3), (4) and (5), we get

$$y_0' = x_0^2 - y_0 = 0 - 1 = -1,$$

$$y_0'' = 2x_0 - y_0' = 2(0) - (-1) = 1$$

$$y_0''' = 2 - y_0'' = 2 - 1 = 1,$$

$$y_0^{IV} = -y_0''' = -1 \quad \text{Take } h = 0.1$$

**Step1:** by Taylor's series expansion

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \dots \rightarrow (6)$$

On substituting the values of  $y_0, y_0', y_0''$  etc in (6), we get

$$y(0.1) = y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \dots$$

$$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416$$

$$= 0.905125 \approx 0.9051 \text{ (4 decimal place).}$$

**Step2:** Let us find  $y(0.2)$  we start with  $(x_1, y_1)$  as the starting values

Here  $x_1 = x_0 + h = 0 + 0.1 = 0.1$  and  $y_1 = 0.905125$ ,

Putting these values of  $x_1$  and  $y_1$  in (1), (3), (4) and (5), we get

From 1,3,4,5 we get

$$y_1' = x_1^2 - y_1 = (0.1)^2 - 0.905125 = -0.895125$$

$$y_1'' = 2x_1 - y_1' = 2(0.1) - (-0.895125) = 1.095125,$$

$$y_1''' = 2 - y_1'' = 2 - 1.095125 = 0.904875,$$

$$y_1^{IV} = -y_1''' = -0.904875,$$

By Taylor's series expansion,

$$y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \dots$$

$$y(0.2) = y_2 = 0.905125 + (0.1)(-0.895125) + \frac{(0.1)^2}{2} (1.095125) \\ + \frac{(0.1)^3}{6} (1.095125) + \frac{(0.1)^4}{24} (-0.904875) + \dots$$

$$y(0.2) = y_2 = 0.905125 - 0.0895125 + 0.00547562 + 0.000150812 \\ = 0.8212351 \simeq 0.8212 \text{ (4 decimal places)}$$

**Step3:** Let us find  $y(0.3)$ , we start with  $(x_2, y_2)$  as the starting value

Here  $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$  and  $y_2 = 0.8212351$

Putting these values of  $x_2$  and  $y_2$  in (1), (3), (4), and (5) we get

$$y_2' = x_2^2 - y_2 = (0.2)^2 - 0.8212351 = 0.04 - 0.8212351 = -0.7812351$$

$$y_2'' = 2x_2 - y_2' = 2(0.2) - (-0.7812351) = 1.1812351,$$

$$y_2''' = 2 - y_2'' = 2 - 1.1812351 = 0.818765,$$

$$y_2^{IV} = -y_2''' = -0.818765,$$

By Taylor's series expansion,

$$y(x_3) = y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \frac{h^4}{4!} y_2^{IV} + \dots$$

$$y(0.3) = y_3 = 0.8212351 + (0.1)(-0.7812351) + \frac{(0.1)^2}{2} (1.1812351) \\ + \frac{(0.1)^3}{6} (0.818765) + \frac{(0.1)^4}{24} (-0.818765) + \dots$$

$$y(0.3) = y_3 = 0.8212351 - 0.07812351 + 0.005906 + 0.000136 - \\ = 0.749150 \simeq 0.7492 \text{ (4 decimal places)}$$

**Step4:** Let us find  $y(0.4)$ , we start with  $(x_3, y_3)$  as the starting value

Here  $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$  and  $y_3 = 0.749150$

Putting these values of  $x_3$  and  $y_3$  in (1),(3),(4), and (5) we get

$$y_3^I = x_3^2 - y_3 = (0.3)^2 - 0.749150 = -0.65915,$$

$$y_3^{II} = 2x_3 - y_3^I = 2(0.3) + (0.65915) = 1.25915,$$

$$y_3^{III} = 2 - y_3^{II} = 2 - 1.25915 = 0.74085,$$

$$y_3^{IV} = -y_3^{III} = -0.74085,$$

By Taylor's series expansion,

$$y(x_4) = y_4 = y_3 + \frac{h}{1!} y_3^I + \frac{h^2}{2!} y_3^{II} + \frac{h^3}{3!} y_3^{III} + \frac{h^4}{4!} y_3^{IV} + \dots$$

$$y(0.4) = y_4 = 0.749150 + (0.1)(-0.65915) + \frac{(0.1)^2}{2} (1.25915) + \frac{(0.1)^3}{6} (0.74085) + \frac{(0.1)^4}{24} (-0.74085) + \dots$$

$$y(0.4) = y_4 = 0.749150 - 0.065915 + 0.0062926 + 0.000123475 - 0.0000030 \\ = 0.6896514 \approx 0.6896 \text{ (4 decimal places)}$$

11. Solve  $y^I = x^2 - y$ ,  $y(0) = 1$  using T.S.M and evaluate  $y(0.1), y(0.2), y(0.3)$  and  $y(0.4)$  (correct to 4 decimal place ) Ans : 0.9051, 0.8212, 0.7492, 0.6896

12. Given the differentiating equation  $y^I = x^I + y^2$ ,  $y(0) = 1$ . Obtain  $y(0.25)$  and  $y(0.5)$  by T.S.M.

Ans: 1.3333, 1.81667

13. Solve  $y^I = xy^2 + y$ ,  $y(0) = 1$  using Taylor's series method and evaluate  $y(0.1)$  and  $y(0.2)$

Ans: 1.111, 1.248.

### 3.6 Picard's Method

Consider the differential equation  $\frac{dy}{dx} = f(x, y)$

Given that  $y = y_0$  for  $x = x_0$

Then  $y^{(n)} = y_0 + \int_{x_0}^{x_n} f(x, y^{(n-1)}) dx, \quad n = 1, 2, 3, \dots$

Problems

1. Find the value of  $y$  for  $x=0.4$  by Picard's method, given that  $\frac{dy}{dx} = x^2 + y^2$ ,  $y(0) = 0$ .

Sol: Given  $\frac{dy}{dx} = x^2 + y^2$ ,  $y(0) = 0$

By Picard's method  $y^{(n)} = y_0 + \int_{x_0}^x f(x, y^{(n-1)}) dx$ ,  $n = 1, 2, 3, \dots$

For the first approximation, replace  $y_0$  by 0

$$y^{(1)} = 0 + \int_0^x (x^2 + 0) dx = \frac{x^3}{3}$$

Second approximation is  $y^{(2)} = \int_0^x (x^2 + \left(\frac{x^3}{3}\right)^2) dx = \int_0^x (x^2 + \frac{x^6}{9}) dx = \frac{x^3}{3} + \frac{x^7}{63}$

Calculation of  $y^{(3)}$  is tedious and hence approximate value is  $y^{(2)}$

For  $x=0.4$ ,  $y = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63} = 0.02133 + 0.00026 = 0.0214$

2. Solve Find the value of  $y$  at  $x=0.1$  by Picard's method, given that

$$\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$$

Sol: Given  $\frac{dy}{dx} = \frac{y-x}{y+x}$ ,  $y(0) = 1$

By Picard's method  $y^{(n)} = y_0 + \int_{x_0}^x f(x, y^{(n-1)}) dx = y_0 + \int_0^x \frac{y-x}{y+x} dx$

For the first approximation, replace  $y_0$  by 1

$$y^{(1)} = 1 + \int_0^x \frac{1-x}{1+x} dx = 1 + \int_0^x -1 + \frac{2}{1+x} dx$$

$$y^{(1)} = 1 + [-x + 2\log(1+x)]_0^x$$

$$y^{(1)} = 1 - x + 2\log(1+x)$$

Second approximation is  $y^{(2)} = 1 + \int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x} dx$

Which is very difficult to integrate

Hence we use the first approximation itself as the value of  $y$

$$\therefore y(x) = y^{(1)} = 1 - x + 2\log(1+x)$$

Put  $x=0.1$ , we get

$$y(0.1) = 1 - 0.1 + 2\log(1 + 0.1) = 1.0906$$

### 3.6 EULER'S METHOD

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation  $\frac{dy}{dx} = f(x,y) \rightarrow (1)$

With  $y(x_0) = y_0 \rightarrow (2)$

Consider the first two terms of the Taylor's expansion of  $y(x)$  at  $x = x_0$

$$y(x) = y(x_0) + (x - x_0) y'(x_0) \rightarrow (3)$$

from equation (1)  $y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$

Substituting in equation (3)

$$\therefore y(x) = y(x_0) + (x - x_0) f(x_0, y_0)$$

At  $x = x_1$ ,  $y(x_1) = y(x_0) + (x_1 - x_0) f(x_0, y_0)$

$$\therefore y_1 = y_0 + h f(x_0, y_0) \quad \text{where } h = x_1 - x_0$$

Similarly at  $x = x_2$ ,  $y_2 = y_1 + h f(x_1, y_1)$ ,

Proceeding as above,  $y_{n+1} = y_n + h f(x_n, y_n)$

This is known as Euler's Method

1. **Using Euler's method solve for  $x = 2$  from  $\frac{dy}{dx} = 3x^2 + 1, y(1) = 2$ , by**

**taking step size**

(I)  $h = 0.5$  and (II)  $h=0.25$

Sol: Here  $\frac{dy}{dx} = f(x,y) = 3x^2 + 1, x_0 = 1, y_0 = 2$

Euler's algorithm is  $y_{n+1} = y_n + h f(x_n, y_n), n = 0, 1, 2, 3, \dots \rightarrow (1)$

(I)  $h = 0.5 \quad \therefore x_1 = x_0 + h = 1 + 0.5 = 1.5$

Taking  $n = 0$  in (1), we have  $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$$y_1 = y_0 + h f(x_0, y_0)$$

i.e.  $y_1 = y(1.5) = 2 + (0.5) f(1, 2) = 2 + (0.5) (3 + 1) = 2 + (0.5)(4) = 4$

Here  $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$$\therefore y(1.5) = 4 = y_1$$

Taking  $n = 1$  in (1), we have

$$y_2 = y_1 + h f(x_1, y_1)$$

$$\text{i.e. } y(x_2) = y_2 = 4 + (0.5) f(1.5, 4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$$

$$\text{Here } x_2 = x_1 + h = 1.5 + 0.5 = 2$$

$$\therefore y(2) = 7.875$$

$$\text{(II) } \quad h = 0.25 \quad x_0 = 1, y_0 = 2 \quad \therefore x_1 = 1.25, x_2 = 1.50, x_3 = 1.75, x_4 = 2$$

Taking  $n = 0$  in (1), we have

$$y_1 = y_0 + h f(x_0, y_0)$$

$$\text{i.e. } y(x_1) = y_1 = 2 + (0.25) f(1, 2) = 2 + (0.25) (3 + 1) = 3 = y(1.25)$$

$$y(x_2) = y_2 = y_1 + h f(x_1, y_1)$$

$$\text{i.e. } y(x_2) = y_2 = 3 + (0.25) f(1.25, 3)$$

$$= 3 + (0.25)[3(1.25)^2 + 1]$$

$$= 4.42188$$

$$\text{Here } x_2 = x_1 + h = 1.25 + 0.25 = 1.5$$

$$\therefore y_2 = y(1.5) = 4.42188$$

Taking  $n = 2$  in (1), we have

$$\text{i.e. } y(x_3) = y_3 = y_2 + h f(x_2, y_2)$$

$$= 4.42188 + (0.25) f(1.5, 2)$$

$$= 4.42188 + (0.25) [3(1.5)^2 + 1]$$

$$= 6.35938$$

$$\text{Here } x_3 = x_2 + h = 1.5 + 0.25 = 1.75$$

$$\therefore y(1.75) = 6.35938 = y_3$$

Taking  $n = 4$  in (1), we have

$$y(x_4) = y_4 = y_3 + h f(x_3, y_3)$$

$$\text{i.e. } y(x_4) = y_4 = 6.35938 + (0.25) f(1.75, 2)$$

$$= 6.35938 + (0.25)[3(1.75)^2 + 1]$$

$$y(x_4) = 8.90626 = y(2)$$

Note that the difference in values of  $y(2)$  in both cases (i.e. when  $h = 0.5$  and when  $h = 0.25$ ). The accuracy is improved significantly when  $h$  is reduced to  $0.25$  (Example significantly of the eqn is  $y = x^3 + x$  and with this  $y(2) = y_2 = 10$ )

**2. Solve by Euler's method,  $y' = x + y$ ,  $y(0) = 1$  and find  $y(0.3)$  taking step size  $h = 0.1$ . compare the result obtained by this method with the result obtained by analytical solution**

Sol: Given D.E. is  $y' = f(x,y) = x + y$ ,  $y(0) = 1$ ,  $h=0.1$ ,  $x_0=0$ ,  $x_1=0.1$ ,  $x_2=0.2$ ,  $x_3=0.3$ ,  $y_0=1$

From Euler's method

$$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y(x_2) = y_2 = y_1 + h f(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

$$y(x_3) = y_3 = y_2 + h f(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

$$y_1 = 1.1 = y(0.1),$$

$$y_2 = y(0.2) = 1.22$$

$$y_3 = y(0.3) = 1.362$$

Analytical method (linear d.e. method)

$$y' = x + y$$

$$dy/dx - y = x$$

$$P = -1, Q = x$$

$$I.F. = e^{\int P dx} = e^{\int -dx} = e^{-x}$$

$$\text{Sol. } Y(I.F.) = \int(Q(i.f.)) dx + c$$

$$Y(e^{-x}) = \int(xe^{-x}) dx + c = e^{-x}(-x-1) + c$$

Divide by  $e^{-x}$  on b.s.

Solution  $y = -x - 1 + ce^{\lambda x}$

Put  $x=0, y=1$  then

$$1 = -0 - 1 + c$$

$$c = 2$$

General solution  $y = -x - 1 + 2e^{\lambda x}$

Particular solution is  $y(x) = 2e^x - (x + 1)$

Hence analytical values  $y(0.1) = 1.11034$ ,  $y(0.2) = 1.3428$ ,  $y(0.3) = 1.5997$

We shall tabulate the result as follows

X	0	$X_1=0.1$	$X_2=0.2$	$X_3=0.3$
Euler $y(\text{numerical})$	1	1.1	1.22	1.362
Linear $y(\text{analytical})$	1	1.11034	1.3428	1.3997

The value of  $y$  deviate from the exact value as  $x$  increases. This indicate that the method is not accurate

**3. Solve by Euler's method  $y' + y = 0$  given  $y(0) = 1$  and find  $y(0.04)$  taking step size**

$$h = 0.01 \quad \text{Ans: } 0.9606$$

**4. Using Euler's method, solve  $y$  at  $x = 0.1$  from  $y' = x + y + xy$ ,  $y(0) = 1$  taking step size  $h = 0.025$ .**

**5. Given that  $\frac{dy}{dx} = xy$ ,  $y(0) = 1$  determine  $y(0.1)$ , using Euler's method.  $h = 0.1$**

Sol: The given differentiating equation is  $\frac{dy}{dx} = xy$ ,  $y(0) = 1$ ,  $a = 0, b = 0.1$

Here  $f(x,y) = xy$ ,  $x_0 = 0$  and  $y_0 = 1$

Since  $h$  is not given much better accuracy is obtained by breaking up the interval  $(0,0.1)$  into five steps.

$$\text{i.e. } h = \frac{b-a}{5} = \frac{0.1}{5} = 0.02$$

Euler's algorithm is  $y_{n+1} = y_n + h f(x_n, y_n)$   $\rightarrow (1)$

$\therefore$  From (1) for  $n = 0$ , we have

$$\begin{aligned} y_1 &= y_0 + h f(x_0, y_0) \\ &= 1 + (0.02) f(0, 1) \\ &= 1 + (0.02) (0) \\ &= 1 \end{aligned}$$

Next we have  $x_1 = x_0 + h = 0 + 0.02 = 0.02$

$\therefore$  From (1), for  $n = 1$ , we have

$$\begin{aligned} y_2 &= y_1 + h f(x_1, y_1) \\ &= 1 + (0.02) f(0.02, 1) \\ &= 1 + (0.02) (0.02) \\ &= 1.0004 \end{aligned}$$

Next we have  $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

$\therefore$  From (1), for  $n = 2$ , we have

$$\begin{aligned} y_3 &= y_2 + h f(x_2, y_2) \\ &= 1.0004 + (0.02) (0.04) (1.0004) \\ &= 1.0012 \end{aligned}$$

Next we have  $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

$\therefore$  From (1), for  $n = 3$ , we have

$$\begin{aligned} y_4 &= y_3 + h f(x_3, y_3) \\ &= 1.0012 + (0.02) (0.06) (1.00012) \\ &= 1.0024. \end{aligned}$$

Next we have  $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

$\therefore$  From (1), for  $n = 4$ , we have

$$\begin{aligned}
 y_5 &= y_4 + h f(x_4, y_4) \\
 &= 1.0024 + (0.02) (0.08) (1.00024) \\
 &= 1.0040.
 \end{aligned}$$

Next we have  $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When  $x = x_5$ ,  $y \simeq y_5$

$$\therefore y = 1.0040 \text{ when } x = 0.1$$

**6. Solve by Euler's method  $y' = \frac{2y}{x}$  given  $y(1) = 2$  and find  $y(2)$ .**

**7. Given that  $\frac{dy}{dx} = 3x^2 + y$ ,  $y(0) = 4$ . Find  $y(0.25)$  and  $y(0.5)$  using Euler's method**

Sol: given  $\frac{dy}{dx} = 3x^2 + y$  and  $y(0) = 4$ .

Here  $f(x, y) = 3x^2 + y$ ,  $x_0 = 0$ ,  $y_0 = 4$

Consider  $h = 0.25$

Euler's algorithm is  $y_{n+1} = y_n + h f(x_n, y_n) \rightarrow (1)$

$\therefore$  From (1), for  $n = 0$ , we have

$$\begin{aligned}
 y_1 &= y_0 + h f(x_0, y_0) \\
 &= 2 + (0.25)[0 + 4] \\
 &= 2 + 1 \\
 &= 3
 \end{aligned}$$

Next we have  $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When  $x = x_1$ ,  $y_1 \simeq y$

$$\therefore y_1 = 3 \text{ when } x_1 = 0.25$$

$\therefore$  From (1), for  $n = 1$ , we have

$$\begin{aligned}
 y_2 &= y_1 + h f(x_1, y_1) \\
 &= 3 + (0.25)[3 \cdot (0.25)^2 + 3] \\
 &= 3.7968
 \end{aligned}$$

Next we have  $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

When  $x = x_2$ ,  $y \simeq y_2$

$$\therefore y = 3.7968 \text{ when } x = 0.5.$$

8. Solve first order differential equation  $\frac{dy}{dx} = \frac{y-x}{y+x}$ ,  $y(0) = 1$  and estimate  $y(0.1)$  using Euler's method (5 steps),  $h=0.02$

Ans: 1.0928

9. Use Euler's method to find approximate value of solution of  $\frac{dy}{dx} = y-x + 5$  at  $x = 2-1$  and  $2-2$  with initial condition  $y(0.2) = 1$

### 3.7 Modified Euler's method

It is given by  $y_{k+1}^{(i)} = y_k + h/2f\left[(x_k, y_k) + f(x_{k+1}, 1)_{k+1}^{(i-1)}\right]$ ,  $i = 1, 2, \dots, k_i = 0, 1, \dots$

#### Working rule :

##### i) Modified Euler's method

$$y_{k+1}^{(i)} = y_k + h/2f\left[(x_k, y_k) + f(x_{k+1}, 1)_{k+1}^{(i-1)}\right], i = 1, 2, \dots, k_i = 0, 1, \dots$$

ii) When  $i = 1$   $y_{k+1}^{(0)}$  can be calculated from Euler's method

iii)  $K=0, 1, \dots$  gives number of iteration.  $i = 1, 2, \dots$

gives number of times, a particular iteration  $k$  is repeated

Suppose consider  $dy/dx=f(x, y)$  ----- (1) with  $y(x_0) = y_0$ ----- (2)

To find  $y(x_1) = y_1$  at  $x=x_1=x_0+h$

Now take  $k=0$  in modified Euler's method

$$\text{We get } y_1^{(i)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(i-1)})\right] \dots \dots \dots (3)$$

Taking  $i=1, 2, 3, \dots, k+1$  in eqn (3), we get

$$y_1^{(0)} = y_0 + h/2\left[f(x_0, y_0)\right] \text{ (By Euler's method)}$$

$$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

-----

$$y_1^{(k+1)} = y_0 + h/2\left[f(x_0, y_0) + f(x_1, y_1^{(k)})\right]$$

If two successive values of  $y_1^{(k)}, y_1^{(k+1)}$  are sufficiently close to one another, we will take the common value as  $y_2 = y(x_2) = y(x_1 + h)$

We use the above procedure again

1) using modified Euler's method find the approximate value of  $x$  when  $x = 0.3$  given that  $dy/dx = x + y$  and  $y(0) = 1$

sol: Given  $dy/dx = x + y$  and  $y(0) = 1$

Here  $f(x, y) = x + y, x_0 = 0$ , and  $y_0 = 1$

Take  $h = 0.1$  which is sufficiently small

Here  $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2 \left[ f(x_k + y_k) + f(x_{k+1}, y_{k+1}^{(i-1)}) \right] \rightarrow (1)$$

**Step1:** To find  $y_1 = y(x_1) = y(0.1)$

Taking  $k = 0$  in eqn(1)

$$y_{k+1}^{(i)} = y_0 + h/2 \left[ f(x_0 + y_0) + f(x_1, y_1^{(i-1)}) \right] \rightarrow (2)$$

when  $i = 1$  in eqn (2)

$$y_1^{(i)} = y_0 + h/2 \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

First apply Euler's method to calculate  $y_1^{(0)} = y_1$

$$\begin{aligned} \therefore y_1^{(0)} &= y_0 + h f(x_0, y_0) \\ &= 1 + (0.1)f(0,1) \\ &= 1 + (0.1) \\ &= 1.10 \end{aligned}$$

now  $[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10]$

$$\begin{aligned} \therefore y_1^{(1)} &= y_0 + 0.1/2 \left[ f(x_0, y_0) + f(x_1, y_1^{(0)}) \right] \\ &= 1 + 0.1/2 [f(0,1) + f(0.1, 1.10)] \\ &= 1 + 0.1/2 [(0+1) + (0.1+1.10)] \\ &= 1.11 \end{aligned}$$

When  $i=2$  in eqn (2)

$$y_1^{(2)} = y_0 + h/2 \left[ f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$$

$$\begin{aligned}
&= 1+0.1/2[f(0.1)+f(0.1,1.11)] \\
&= 1 + 0.1/2[(0+1)+(0.1+1.11)] \\
&= 1.1105
\end{aligned}$$

$$\begin{aligned}
y_1^{(3)} &= y_0 + h/2 \left[ f(x_0, y_0) + f(x_1, y_1^{(2)}) \right] \\
&= 1+0.1/2[f(0,1)+f(0.1, 1.1105)] \\
&= 1+0.1/2[(0+1)+(0.1+1.1105)] \\
&= 1.1105
\end{aligned}$$

Since  $y_1^{(2)} = y_1^{(3)}$

$$\therefore y_1 = 1.1105$$

**Step:2** To find  $y_2 = y(x_2) = y(0.2)$

Taking  $k = 1$  in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2 \left[ f(x_1, y_1) + f(x_2, y_2^{(i-1)}) \right] \rightarrow (3) \quad i = 1,2,3,4,\dots$$

For  $i = 1$

$$y_2^{(1)} = y_1 + h/2 \left[ f(x_1, y_1) + f(x_2, y_2^{(0)}) \right]$$

$y_2^{(0)}$  is to be calculate from Euler's method

$$\begin{aligned}
y_2^{(0)} &= y_1 + h f(x_1, y_1) \\
&= 1.1105 + (0.1) f(0.1, 1.1105) \\
&= 1.1105+(0.1)[0.1+1.1105] \\
&= 1.2316
\end{aligned}$$

$$\begin{aligned}
\therefore y_2^{(1)} &= 1.1105+0.1/2 \left[ f(0.1,1.1105) + f(0.2,1.2316) \right] \\
&= 1.1105 + 0.1/2[0.1+1.1105+0.2+1.2316] \\
&= 1.2426
\end{aligned}$$

$$\begin{aligned}
y_2^{(2)} &= y_1 + h/2 \left[ f(x_1, y_1) + f(x_2, y_2^{(1)}) \right] \\
&= 1.1105 + 0.1/2[f(0.1, 1.1105) , f(0.2, 1.2426)] \\
&= 1.1105 + 0.1/2[1.2105 + 1.4426]
\end{aligned}$$

$$= 1.1105 + 0.1(1.3266)$$

$$= 1.2432$$

$$y_2^{(3)} = y_1 + h/2 \left[ f(x_1, y_1) + f(x_2, y_2^{(2)}) \right]$$

$$= 1.1105 + 0.1/2 [f(0.1, 1.1105) + f(0.2, 1.2432)]$$

$$= 1.1105 + 0.1/2 [1.2105 + 1.4432]$$

$$= 1.1105 + 0.1(1.3268)$$

$$= 1.2432$$

Since  $y_2^{(3)} = y_2^{(3)}$

Hence  $y_2 = 1.2432$

### Step:3

To find  $y_3 = y(x_3) = y(0.3)$

Taking  $k=2$  in eqn (1) we get

$$y_3^{(i)} = y_2 + h/2 \left[ f(x_2, y_2) + f(x_3, y_3^{(i-1)}) \right] \rightarrow (4)$$

For  $i=1$ ,

$$y_3^{(1)} = y_2 + h/2 \left[ f(x_2, y_2) + f(x_3, y_3^{(0)}) \right]$$

$y_3^{(0)}$  is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h f(x_2, y_2)$$

$$= 1.2432 + (0.1) f(0.2, 1.2432)$$

$$= 1.2432 + (0.1)(1.4432)$$

$$= 1.3875$$

$$\therefore y_3^{(1)} = 1.2432 + 0.1/2 [f(0.2, 1.2432) + f(0.3, 1.3875)]$$

$$= 1.2432 + 0.1/2 [1.4432 + 1.6875]$$

$$= 1.2432 + 0.1(1.5654)$$

$$= 1.3997$$

$$y_3^{(2)} = y_2 + h/2 \left[ f(x_2, y_2) + f(x_3, y_3^{(1)}) \right]$$

$$= 1.2432 + 0.1/2 [1.4432 + (0.3 + 1.3997)]$$

$$= 1.2432 + (0.1)(1.575)$$

$$= 1.4003$$

$$y_3^{(3)} = y_2 + h/2 \left[ f(x_2, y_2) + f(x_3, y_3^{(2)}) \right]$$

$$= 1.2432 + 0.1/2 [f(0.2, 1.2432) + f(0.3, 1.4003)]$$

$$= 1.2432 + 0.1(1.5718)$$

$$= 1.4004$$

$$y_3^{(4)} = y_2 + h/2 \left[ f(x_2, y_2) + f(x_3, y_3^{(3)}) \right]$$

$$= 1.2432 + 0.1/2 [1.4432 + 1.7004]$$

$$= 1.2432 + (0.1)(1.5718)$$

$$= 1.4004$$

Since  $y_3^{(3)} = y_3^{(4)}$

Hence  $y_3 = 1.4004 \therefore$  The value of  $y$  at  $x = 0.3$  is  $1.4004$

**2 . Find the solution of  $\frac{dy}{dx} = x-y$  ,  $y(0)=1$  at  $x=0.1$  ,  $0.2$  ,  $0.3$  ,  $0.4$  and  $0.5$**

**. Using modified Euler's method**

Sol . Given  $\frac{dy}{dx} = x-y$  and  $y(0) = 1$

Here  $f(x,y) = x-y$  ,  $x_0 = 0$  and  $y_0 = 1$

Consider  $h = 0.1$  so that

$x = 0.1$  ,  $x_2 = 0.2$  ,  $x_3 = 0.3$  ,  $x_4 = 0.4$  and  $x_5 = 0.5$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2 f(x_k, y_k) + f_{x_{k+1}, y_{k+1}}^{(i-1)} \rightarrow (1)$$

Where  $k = 0, 1, 2, 3, \dots$

$i = 1, 2, 3, \dots$

x	$f(x_k, y_k) = x_k - y_k$	$\frac{1}{2} [f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(i-1)})]$	$y_{k+1}^{(i)} = y_k + h/2 [f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(i-1)})]$
K = 0			
0. 0	0-1=-1	-	$1+(0.1)(-1)=0.9 = y_1^{(0)}$

0.1(i=1)	0-1=-1	$\frac{1}{2}(-1-0.8) = -0.9$	$1+(0.1)(-0.9)=0.91$
0.1(i=2)	0-1=-1	$\frac{1}{2}(-1-0.81) = -0.905$	$1+(0.1)(-0.905)=0.9095$
0.1(i=3)	0-1=-1	$\frac{1}{2}(-1-0.8095) = -0.90475$	$1+(0.1)(-0.90475)=0.9095$
K=1			
0.1	$0.1-0.9095 = -0.8095$	-	$0.9095+(0.1)(-0.8095)=0.82855$
0.2(i=1)	-0.8095	$\frac{1}{2}(-0.8095-0.62855)$	$0.9095+(0.1)(-0.719025)=0.8376$
0.2(i=2)	-0.8095	$\frac{1}{2}(-0.8095-0.6376)$	$0.9095+(0.1)(-0.72355)=0.8371$
0.2(i=3)	-0.8095	$\frac{1}{2}(-0.8095-0.6371)$	$0.9095+(0.1)(-0.7233)=0.8372$
0.2(i=4)	-0.8095	$\frac{1}{2}(-0.8095-0.6372)$	$0.9095+(0.1)(-0.72355)=0.8371$
K=2			
0.2	$0.2-0.8371 = -0.6371$	-	$0.8371+(0.1)(-0.6371)=0.7734$

0.3(i=1)	= -0.6371	$\frac{1}{2}(-0.6371-0.4734)$	$0.8371+(0.1)(-0.555)=0.7816$
0.3(i=2)	= -0.6371	$\frac{1}{2}(-0.6371-0.4816)$	$0.8371-0.056=0.7811$
0.3(i=3)	= -0.6371	$\frac{1}{2}(-0.6371-0.4811)$	$0.8371-0.05591=0.7812$
0.3(i=4)	= -0.6371	$\frac{1}{2}(-0.6371-0.4812)$	$0.8371-0.055915=0.7812$
K =3			
0.3(i=1)	0.3-0.7812	-	$0.7812+(0.1)(-0.4812) = 0.7331$
0.4(i=1)	-0.4812	$\frac{1}{2}(-0.4812-0.4311)$	$0.7812-0.0457 = 0.7355$
0.4(i=2)	-0.4812	$\frac{1}{2}(-0.4812-0.4355)$	$0.7812-0.0458 = 0.7354$
0.4(i=3)	-0.4812	$\frac{1}{2}(-0.4812-0.4354)$	$0.7812-0.0458 = 0.7354$
K=4			
0.4	-0.3354	-	$0.7354-0.03354 = 0.70186$
0.5	-0.3354	$\frac{1}{2}(-0.3354-0.301816)$	$0.7354-0.03186 = 0.7035$
0.5	-0.3354	$\frac{1}{2}(-0.3354-0.30354)$	$0.7354-0.0319 = 0.7035$

3. Find  $y(0.1)$  and  $y(0.2)$  using modified Euler's formula given that  $dy/dx=x^2-y, y(0)=1$

[consider  $h=0.1, y_1=0.90523, y_2=0.8214$ ]

4. Given  $dy/dx = -xy^2, y(0) = 2$  compute  $y(0.2)$  in steps of 0.1

Using modified Euler's method

$$[h=0.1, y_1=1.9804, y_2=1.9238]$$

5. Given  $y' = x + \sin y$ ,  $y(0)=1$  compute  $y(0.2)$  and  $y(0.4)$  with  $h=0.2$  using modified Euler's

method

$$[y_1=1.2046, y_2=1.4644]$$

### **3.8 Runge – Kutta Methods**

#### **I. Second order R-K Formula**

$$y_{i+1} = y_i + \frac{1}{2} (K_1 + K_2),$$

Where  $K_1 = h (x_i, y_i)$

$$K_2 = h (x_i + h, y_i + k_1)$$

For  $i = 0, 1, 2, \dots$

#### **II. Third order R-K Formula**

$$y_{i+1} = y_i + \frac{1}{6} (K_1 + 4K_2 + K_3),$$

Where  $K_1 = h (x_i, y_i)$

$$K_2 = h (x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$

$$K_3 = h (x_i + h, y_i + 2k_2 - k_1)$$

For  $i = 0, 1, 2, \dots$

#### **III. Fourth order R-K Formula**

$$y_{i+1} = y_i + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4),$$

Where  $K_1 = h (x_i, y_i)$

$$K_2 = h (x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$

$$K_3 = h (x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$$

$$K_4 = h (x_i + h, y_i + k_3)$$

For  $i = 0, 1, 2, \dots$

1. Using Runge-Kutta method of second order, find  $y(2.5)$  from  $\frac{dy}{dx} = \frac{x+y}{x}$ ,  $y(2)=2$ ,  $h = 0.25$ .

Sol: Given  $\frac{dy}{dx} = \frac{x+y}{x}$ ,  $y(2) = 2$ .

Here  $f(x, y) = \frac{x+y}{x}$ ,  $x_0 = 2$ ,  $y_0 = 2$  and  $h = 0.25$

$\therefore x_1 = x_0 + h = 2 + 0.25 = 2.25$ ,  $x_2 = x_1 + h = 2.25 + 0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + h/2(k_1 + k_2), k_1 = hf(x_i, y_i), k_2 = hf(x_i + h, y_i + k_1), i = 0, 1, \dots \rightarrow (1)$$

### Step -1:-

To find  $y(x_1)$  i.e  $y(2.25)$  by second order R - K method taking  $i=0$  in eqn(i)

We have  $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where  $k_1 = hf(x_0, y_0)$ ,  $k_2 = hf(x_0 + h, y_0 + k_1)$

$f(x_0, y_0) = f(2, 2) = 2 + 2/2 = 2$

$k_1 = hf(x_0, y_0) = 0.25(2) = 0.5$

$k_2 = hf(x_0 + h, y_0 + k_1) = (0.25)f(2.25, 2.5)$

$= (0.25)(2.25 + 2.5/2.25) = 0.528$

$\therefore y_1 = y(2.25) = 2 + 1/2(0.5 + 0.528)$

$= 2.514$

### Step2:

To find  $y(x_2)$  i.e.,  $y(2.5)$

$i=1$  in (1)

$x_1 = 2.25, y_1 = 2.514$ , and  $h = 0.25$

$$y_2 = y_1 + 1/2(k_1 + k_2)$$

$$\text{where } k_1 = h f(x_1, y_1) = (0.25)f(2.25, 2.514)$$

$$= (0.25)[2.25 + 2.514/2.25] = 0.5293$$

$$k_2 = h f(x_0 + h, y_0 + k_1) = (0.1)f(0.1, 1 - 0.1) = (0.1)(-0.9) = -0.09$$

$$= (0.25)[2.5 + 2.514 + 0.5293/2.5]$$

$$= 0.55433$$

$$y_2 = y(2.5) = 2.514 + 1/2(0.5293 + 0.55433)$$

$$= 3.0558$$

$$\therefore y = 3.0558 \text{ when } x = 2.5$$

Obtain the values of  $y$  at  $x=0.1, 0.2$  using R-K method of

(i) second order (ii) third order (iii) fourth order for the diff eqn  $y' + y = 0, y(0) = 1$

Sol: Given  $dy/dx = -y, y(0) = 1$

$$f(x, y) = -y, x_0 = 0, y_0 = 1$$

Here  $f(x, y) = -y, x_0 = 0, y_0 = 1$  take  $h = 0.1$

$$\therefore x_1 = x_0 + h = 0.1,$$

$$x_2 = x_1 + h = 0.2$$

### **Second order:**

**step1:** To find  $y(x_1)$  i.e  $y(0.1)$  or  $y_1$

by second-order R-K method, we have

$$y_1 = y_0 + 1/2(k_1 + k_2)$$

$$\text{where } k_1 = hf(x_0, y_0) = (0.1)f(0, 1) = (0.1)(-1) = -0.1$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = (0.1)f(0.1, 1 - 0.1) = (0.1)(-0.9) = -0.09$$

$$y_1 = y(0.1) = 1 + 1/2(-0.1 - 0.09) = 1 - 0.095 = 0.905$$

$\therefore y = 0.905$  when  $x = 0.1$

### Step2:

To find  $y_2$  i.e  $y(x_2)$  i.e  $y(0.2)$

Here  $x_1 = 0.1$ ,  $y_1 = 0.905$  and  $h = 0.1$

By second-order R-K method, we have

$$y_2 = y(x_2) = y_1 + 1/2(k_1 + k_2)$$

Where  $k_1 = h f(x_1, y_1) = (0.1)f(0.1, 0.905) = (0.1)(-0.905) = -0.0905$

$$\begin{aligned} k_2 &= h f(x_1 + h, y_1 + k_1) = (0.1) f(0.2, 0.905 - 0.0905) \\ &= (0.1) f(0.2, 0.8145) = (0.1)(-0.8145) \\ &= -0.08145 \end{aligned}$$

$$\begin{aligned} y_2 &= y(0.2) = 0.905 + 1/2(-0.0905 - 0.08145) \\ &= 0.905 - 0.085975 = 0.819025 \end{aligned}$$

### Third order

#### Step1:

To find  $y_1$  i.e  $y(x_1) = y(0.1)$

By Third order Runge kutta method

$$y_1 = y_0 + 1/6(k_1 + 4k_2 + k_3)$$

where  $k_1 = h f(x_0, y_0) = (0.1) f(0.1) = (0.1)(-1) = -0.1$

$$\begin{aligned} k_2 &= h f(x_0 + h/2, y_0 + k_1/2) = (0.1) f(0.1/2, 1 - 0.1/2) = (0.1) f(0.05, 0.95) \\ &= (0.1)(-0.95) = -0.095 \end{aligned}$$

and  $k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1)$

$$(0.1) f(0.1, 1 + 2(-0.095) + 0.1) = -0.905$$

Hence  $y_1 = 1 + 1/6(-0.1 + 4(-0.095) - 0.09) = 1 + 1/6(-0.57) = 0.905$

$y_1 = 0.905$  i.e  $y(0.1) = 0.905$

## Step2:

To find  $y_2$ , i.e.  $y(x_2) = y(0.2)$

Here  $x_1=0.1, y_1=0.905$  and  $h = 0.1$

Again by 2<sup>nd</sup> order R-K method

$$y_2 = y_1 + 1/6(k_1 + 4k_2 + k_3)$$

$$\text{Where } k_1 = h f(x_1, y_1) = (0.1)f(0.1, 0.905) = -0.0905$$

$$k_2 = h f(x_1 + h/2, y_1 + k_1/2) = (0.1)f(0.1 + 0.05, 0.905 - 0.04525) = -(0.1)f(0.15, 0.85975) = (0.1)(-0.85975)$$

$$\text{and } k_3 = h f(x_1 + h, y_1 + 2k_2 - k_1) = (0.1)f(0.2, 0.905 + 2(-0.085975) + 0.0905) = -0.082355$$

$$\text{hence } y_2 = 0.905 + 1/6(-0.0905 + 4(-0.085975) - 0.082355) = 0.818874$$

$$\therefore y = 0.905 \text{ when } x = 0.1$$

And  $y = 0.818874$  when  $x = 0.2$

## fourth order:

### step1:

$x_0=0, y_0=1, h=0.1$  To find  $y_1$  i.e.  $y(x_1) = y(0.1)$

By 4<sup>th</sup> order R-K method, we have

$$y_1 = y_0 + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\text{Where } k_1 = h f(x_0, y_0) = (0.1)f(0, 1) = -0.1$$

$$k_2 = h f(x_0 + h/2, y_0 + k_1/2) = -0.095$$

$$\text{and } k_3 = h f(x_0 + h/2, y_0 + k_2/2) = (0.1)f(0.05, 1 - 0.0475/2)$$

$$= (0.1)f(0.05, 0.9525)$$

$$= -0.09525$$

$$\text{and } k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= (0.1)f(0.1, 1 - 0.09525) = (0.1)f(0.1, 0.90475)$$

$$= -0.090475$$

$$\text{Hence } y_1 = 1 + 1/6(-0.1) + 2(-0.095) + 2(0.09525) - 0.090475$$

$$= 1 + 1/6(-0.570975) + 1 - 0.951625 = 0.9048375$$

**Step2:**

To find  $y_2$ , i.e.,  $y(x_2) = y(0.2)$ ,  $y_1 = 0.9048375$ , i.e.,  $y(0.1) = 0.9048375$

Here  $x_1 = 0.1$ ,  $y_1 = 0.9048375$  and  $h = 0.1$

Again by 4<sup>th</sup> order R-K method, we have

$$y_2 = y_1 + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

Where  $k_1 = h f(x_1, y_1) = (0.1)f(0.1, 0.9048375) = -0.09048375$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = (0.1)f(0.1 + 0.1/2, 0.9048375 - 0.09048375/2) = -0.08595956$$

and  $k_3 = hf(x_1 + h/2, y_1 + k_2/2) = (0.1)f(0.15, 0.8618577) = -0.08618577$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)f(0.2, 0.86517)$$

$$= -0.08186517$$

$$\text{Hence } y_2 = 0.9048375 + 1/6(-0.09048375 - 2(0.08595956) - 2(0.08618577) - 0.08186517)$$

$$= 0.9048375 - 0.0861065$$

$$= 0.818731$$

$y = 0.9048375$  when  $x = 0.1$  and  $y = 0.818731$

**3. Apply the 4<sup>th</sup> order R-K method to find an approximate value of y when  $x=1.2$  in steps of 0.1, given that  $y^1 = x^2 + y^2, y(1) = 1.5$**

sol. Given  $y^1 = x^2 + y^2$ , and  $y(1) = 1.5$

Here  $f(x, y) = x^2 + y^2$ ,  $y_0 = 1.5$  and  $x_0 = 1, h = 0.1$

So that  $x_1 = 1.1$  and  $x_2 = 1.2$

**Step1:**

To find  $y_1$  i.e.,  $y(x_1)$

by 4<sup>th</sup> order R-K method we have

$$y_1 = y_0 + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_0, y_0) = (0.1)f(1, 1.5) = (0.1) [1^2 + (1.5)^2] = 0.325$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2) = (0.1)f(1 + 0.05, 1.5 + 0.325) = 0.3866$$

$$\text{and } k_3 = hf(x_0 + h/2, y_0 + k_2/2) = (0.1)f(1.05, 1.5 + 0.3866/2) = (0.1)[(1.05)^2 + (1.6933)^2]$$

$$= 0.39698$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(1.1, 1.89698)$$

$$= 0.48085$$

Hence

$$y_1 = 1.5 + \frac{1}{6} [0.325 + 2(0.3866) + 2(0.39698) + 0.48085] \\ = 1.8955$$

### Step2:

To find  $y_2$ , i.e.,  $y(x_2) = y(1.2)$

Here  $x_1 = 0.1, y_1 = 1.8955$  and  $h = 0.1$

by 4<sup>th</sup> order R-K method we have

$$y_2 = y_1 + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_1, y_1) = (0.1)f(0.1, 1.8955) = (0.1) [1^2 + (1.8955)^2] = 0.48029$$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = (0.1)f(1.1 + 0.1, 1.8937 + 0.4796) = 0.58834$$

$$\text{and } k_3 = hf(x_1 + h/2, y_1 + k_2/2) = (0.1)f(1.15, 1.8937 + 0.58743) \\ = (0.1)[(1.05)^2 + (1.6933)^2]$$

$$= 0.611715$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)f(1.2, 1.8937 + 0.610728)$$

$$= 0.77261$$

$$\text{Hence } y_2 = 1.8937 + 1/6(0.4796 + 2(0.58834) + 2(0.611715) + 0.7726) = 2.5043$$

$\therefore y = 2.5043$  where  $x = 0.2$

**4. Using R-K method, find  $y(0.2)$  for the eqn  $dy/dx=y-x, y(0)=1$ , take  $h=0.2$**

Ans: 1.15607

**5. Given that  $y' = y - x, y(0) = 2$  find  $y(0.2)$  using R-K method take  $h = 0.1$**

Ans: 2.4214

**6. Apply the 4<sup>th</sup> order R-K method to find  $y(0.2)$  and  $y(0.4)$  for one**

**equation  $10 \frac{dy}{dx} = x^2 + y^2, y(0) = 1$  take  $h = 0.1$       Ans. 1.0207, 1.038**

**7. using R-K method, estimate  $y(0.2)$  and  $y(0.4)$  for the eqn  $dy/dx = y^2 - x^2 / y^2 + x^2, y(0) = 1, h = 0.2$**

Ans: 1.19598, 1.3751

**8. use R-K method, to approximate  $y$  when  $x = 0.2$  given that  $y' = x + y, y(0) = 1$**

Sol: Here  $f(x, y) = x + y, y_0 = 1, x_0 = 0$

Since  $h$  is not given for better approximation of  $y$

Take  $h = 0.1$

$\therefore x_1 = 0.1, x_2 = 0.2$

Step 1

To find  $y_1$  i.e  $y(x_1) = y(0.1)$

By R-K method, we have

$y_1 = y_0 + 1/6 (k_1 + 2k_2 + 2k_3 + k_4)$

Where  $k_1 = hf(x_0, y_0) = (0.1)f(0, 1) = (0.1)(1) = 0.1$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2) = (0.1)f(0.05, 1.05) = 0.11$

$$\text{and } k_3 = hf((x_0+h/2, y_0+k_2/2)) = (0.1)f(0.05, 1+0.11/2) = (0.1)[(0.05) + (4.0.11/2)]$$

$$= 0.1105$$

$$k_4 = h f(x_0+h, y_0+k_3) = (0.1)f(0.1, 1.1105) = (0.1)[0.1+1.1105]$$

$$= 0.12105$$

$$\text{Hence } \therefore y_1 = y(0.1) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.240 + 0.12105)$$

$$y = 1.11034$$

### **Step2:**

To find  $y_2$  i.e  $y(x_2) = y(0.2)$

Here  $x_1=0.1$ ,  $y_1=1.11034$  and  $h=0.1$

Again By R-K method, we have

$$y_2 = y_1 + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_1, y_1) = (0.1)f(0.1, 1.11034) = (0.1)[1.21034] = 0.121034$$

$$k_2 = h f(x_1+h/2, y_1+k_1/2) = (0.1)f(0.1+0.1/2, 1.11034+0.121034/2)$$

$$= 0.1320857$$

$$\text{and } k_3 = h f((x_1+h/2, y_1+k_2/2)) = (0.1)f(0.15, 1.11034+0.1320857/2)$$

$$= 0.1326382$$

$$k_4 = h f(x_1+h, y_1+k_3) = (0.1)f(0.2, 1.11034+0.1326382)$$

$$(0.1)(0.2+1.2429783) = 0.1442978$$

$$\text{Hence } y_2 = 1.11034 + 1/6(0.121034 + 0.2641714 + 0.2652764 + 0.1442978)$$

$$= 1.11034 + 0.1324631 = 1.242803$$

$$\therefore y = 1.242803 \text{ when } x=0.2$$

**9. Using Runge-kutta method of order 4, compute  $y(1.1)$  for the eqn  $y' = 3x + y^2, y(1) = 1.2$   $h = 0.05$**

Ans:1.7278

**10. Using Runge-kutta method of order 4, compute  $y(2.5)$  for the eqn  $dy/dx = x+y/x$ ,  $y(2)=2$  [hint  $h = 0.25$ (2 steps)]**

Ans:3.058

### 9. Practice Quiz

1. The trapezoidal rule is  $\int_{x_0}^{x_n} y dx =$  [a]

a)  $\frac{h}{2} [y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$

b)  $h[y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$

c)  $\frac{h}{2} [y_0 + (y_1 + y_2 + \dots + y_{n-1}) + y_n]$

d)  $h[y_0 + 3(y_1 + y_2 + \dots + y_{n-1}) + y_n]$

2. To apply Simpson's 1/3rd and 3/8th rules both the interval must be divided into minimum of \_\_\_\_\_ intervals. [c]

a) 10

b) 12

**c) 6**

d) 4

3. Simpson's 1/3 rule for  $n = 2$  is [a]

a)  $y = \frac{h}{3} [y_0 + 4y_1 + y_2]$

b)  $y = \frac{3h}{8} [y_0 + 4y_1 + y_2]$

c)  $y = \frac{4}{3} [y_0 + 2y_1 + y_2]$

d)  $y = \frac{3h}{8} [y_0 + 2y_1 + y_2]$

4. To evaluate  $\int_0^4 \frac{1}{1+x^2} dx$  for  $h = 1$ , we take  $n =$  [ c ]

a) 2

b) 3

**c) 4**

d) 5

5. In the case of Simpson's  $\frac{1}{3}$  rule, the number of sub-intervals must be [ b ]

a) Small

**b) Even**

c) Odd

d) Larger

6. In Trapezoidal rule, if the interval of  $\int_2^9 f(x) dx$  is divided into 7 equal sub-intervals, then  $h$  is ..... [ c ]

a) 2

b)  $\frac{1}{2}$

**c) 1**

d)  $\frac{3}{2}$

7. The Taylor's series for  $f(x) = \log(1 + x)$  is [ b ]

a)  $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$

**b)  $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$**

c)  $x + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$

d)  $x - x^2 + x^3 - x^4 + \dots$

8. If  $\frac{dy}{dx} = x + y$ ,  $y = 1$  at  $x = 1$ , using picard method the first

approximation is [ b ]

a)  $1 + x$

**b)  $1 + x + \frac{x^2}{2}$**

c)  $1 - x - x^2$

d) none.

9. R. K method 2nd order formula for  $y_1 =$  [ a ]

**a)  $y = y_0 + \frac{1}{2}(K_1 + K_2)$**

b)  $y_1 = \frac{K_1 + K_2}{2}$

c)  $y_1 = y_0 - \frac{1}{2}(K_1 + K_2)$

d) none

10. The first order Range-Kutta method is equal to\_\_\_\_\_ [a]

**a) Taylors series**

b) Eulers method

c) Picards

d) none

## 10. Assignments

S.No	Question	BL	CO
1	Solve $\int_0^{\pi} t \sin t dt$ using the Trapezoidal rule	3	3
2	Evaluate $\int_0^1 \sqrt{1+x^4} dx$ using Simpson's 3/8 rule	5	3
3	Solve $y' = x^2 - y$ , $y(0) = 1$ using Taylor's series method and evaluate $y(0.1), y(0.2), y(0.3)$ and $y(0.4)$ (correct to 4 decimal places)	3	3
4	Find the solution of $\frac{dy}{dx} = x-y$ , $y(0)=1$ at $x = 0.1, 0.2, 0.3, 0.4$ and $0.5$ . Using modified Euler's method	1	3
5	Apply the 4 <sup>th</sup> order R-K method to find an approximate value of $y$ when $x=1.2$ in steps of $0.1$ , given that $y' = x^2+y^2, y(1)=1.5$	3	3

## 11. Part A- Question & Answers

S.N	Question & Answers	B	C
o		L	O
1	State Trapezoidal rule $\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})]$ $\int_{x_0}^{x_n} f(x) dx = \frac{h}{2} [\text{Sum of first and last ordinates} + 2 \times \text{The remaining ordinates}]$	1	3
2	State Simpson's 1/3 rule $\int_{x_0}^{x_n} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2})]$	1	3
3	State Simpson's 3/8 rule $\int_{x_0}^{x_n} f(x) dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots + y_{n-2})]$	1	3
4	<b>State Taylor's series formula</b> $y(x) = y(0) + x.y'(0) + \frac{x^2}{2!} y''(0) + \dots + \frac{x^n}{n!} y^n(0) + \dots$	1	3
5	Explain modified Euler's method	5	3

### Modified Euler's method

$$y_{k+1}^{(i)} = y_k + h/2 f \left[ (x_k, y_k) + f(x_{k+1}, y_{k+1}^{(i-1)}) \right], i = 1, 2, \dots, k_i = 0, 1, \dots$$

ii) When  $i = 1$   $y_{k+1}^0$  can be calculated from Euler's method

iii)  $K=0, 1, \dots$  gives number of iteration.  $i = 1, 2, \dots$

gives number of times, a particular iteration  $k$  is repeated

Suppose consider  $dy/dx = f(x, y)$  ----- (1) with  $y(x_0) = y_0$  ----- (2)

To find  $y(x_1) = y_1$  at  $x = x_1 = x_0 + h$

Now take  $k=0$  in modified Euler's method

$$\text{We get } y_1^{(1)} = y_0 + h/2 \left[ f(x_0, y_0) + f(x_1, y_1^{(i-1)}) \right]$$

6	<p>State Euler's formula It is the simplest one-step method and it is less accurate. Hence it has a limited application.</p> <p>Consider the differential equation <math>\frac{dy}{dx} = f(x, y) \rightarrow (1)</math></p> <p>With <math>y(x_0) = y_0 \rightarrow (2)</math></p> $y_{n+1} = y_n + h f(x_n, y_n)$ <p>This is known as Euler's Method</p>	1	3
7	<p>State R-K method 4<sup>th</sup> order formula</p> $y_{i+1} = y_i + 1/6 (K_1 + 2K_2 + 2K_3 + K_4),$ <p>Where <math>K_1 = h f(x_i, y_i)</math></p> $K_2 = h f(x_i + h/2, y_i + K_1/2)$ $K_3 = h f(x_i + h/2, y_i + K_2/2)$ $K_4 = h f(x_i + h, y_i + K_3)$	1	3
8	<p>Explain Picard's method consider the differential equation <math>\frac{dy}{dx} = f(x, y)</math></p>	5	3

	Given that $y = y_0$ for $x = x_0$ Then $y^{(n)} = y_0 + \int_{x_0}^{x_n} f(x, y^{(n-1)}) dx, \quad n = 1, 2, 3, \dots$		
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**Solve the equation  $\frac{dy}{dx} = x^2 + y^2$  for  $x = 0.4$  given that  $y = 0$  when  $x = 0$  by Taylor's series method**

Sol: Given that  $\frac{dy}{dx} = x^2 + y^2$  and  $y = 0$  when  $x = 0$  i.e.  $y(0) = 0$

Here  $y_0 = 0, x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at  $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$

$$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$$

$$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y''(0) + 2.y'(0)^2 = 2 \quad 3 \quad 3$$

$$y''''(x) = 2.y.y''' + 2.y''.y' + 4.y''.y', y''''(0) = 0$$

The Taylor's series for  $f(x)$  about  $x_0 = 0$  is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y''''(0) + \dots$$

Substituting the values of  $y(0), y'(0), y''(0), \dots$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + \dots = \frac{x^3}{3} + \text{(Higher order terms are neglected)}$$

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

<b>10</b>	Solve Find the value of $y$ at $x=0.1$ by Picard's method, given that $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$ Sol: Given $\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 1$ By Picard's method $y^{(n)} = y_0 + \int_{x_0}^{x_n} f(x, y^{(n-1)}) dx = y_0 +$	<b>3</b>	<b>3</b>
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	$\int_0^x \frac{y-x}{y+x} dx$ <p>For the first approximation, replace <math>y_0</math> by 1</p> $y^{(1)} = 1 + \int_0^x \frac{1-x}{1+x} dx = 1 + \int_0^x -1 + \frac{2}{1+x} dx$ $y^{(1)} = 1 + [-x + 2\log(1+x)]_0^x$ $y^{(1)} = 1 - x + 2\log(1+x)$ <p>Second approximation is <math>y^{(2)} = 1 +</math></p> $\int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x} dx$ <p>Which is very difficult to integrate</p> <p>Hence we use the first approximation itself as the value of <math>y</math></p> $\therefore y(x) = y^{(1)} = 1 - x + 2\log(1+x)$ <p>Put <math>x=0.1</math>, we get</p> $y(0.1) = 1 - 0.1 + 2\log(1 + 0.1) = 1.0906$		
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## 12. Part B- Questions

S.No	Question	BL	CO
1	Solve $\int_0^\pi t \sin t dt$ using the Trapezoidal rule	3	3
2	Evaluate $\int_0^6 \frac{1}{1+x} dx$ by Simpson's 1/3 rule	5	3
3	Evaluate $\int_0^1 \sqrt{1+x^4} dx$ using Simpson's 3/8 rule	5	3
4	Solve $y' = x - y^2, y(0) = 1$ using Taylor's series method and compute $y(0.1), y(0.2)$	3	3
5	Determine $y(0.1)$ , using Euler's method. Given that $\frac{dy}{dx} = xy, y(0) = 1, h = 0.1$	5	3
6	Find the value of $y$ for $x=0.4$ by Picard's	1	3

	method, given that $\frac{dy}{dx} = x^2 + y^2$ , $y(0) = 0$ .		
7	Find the solution of $\frac{dy}{dx} = x-y$ , $y(0)=1$ at $x=0.1$ , $0.2, 0.3, 0.4$ and $0.5$ . Using modified Euler's method	1	3
8	Solve first order differential equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ , $y(0) = 1$ and estimate $y(0.1)$ using Euler's method (5 steps). $h=0.02$	3	3
9	Apply the 4 <sup>th</sup> order R-K method to find an approximate value of $y$ when $x=1.2$ in steps of $0.1$ , given that $y' = x^2+y^2$ , $y(1)=1.5$	4	3
10	<b>Apply the 4<sup>th</sup> order R-K method to find (0.2) &amp; (0.4) for <math>10 \frac{dy}{dx} = x^2 + y^2</math>, <math>y(0)=1</math> take <math>h = 0.1</math></b>	4	3

### 13. Supportive Online Certification Courses:

- An Introductory Method to Numerical Analysis (NPTEL)
- Probability & Statistics (NPTEL)

### 14. Real Time Applications:

- Application of Numerical methods as a way of transforming linear equations and differential equations

### 15. Content beyond the Syllabus:

- Numerical Differentiation and Applications

## **16. Prescribed Text Books and Reference Books:**

### **Text Books:**

1. B. S. Grewal, Higher Engineering Mathematics, 44/e, Khanna Publishers, 2017.
2. Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.
3. Ronald E. Walpole "Probability and Statistics for Engineers and Scientists", PNIE

### **Reference Books:**

1. B.V.Ramana, "Higher Engineering Mathematics", Mc Graw Hill publishers.
2. Alan Jeffrey, "Advanced Engineering Mathematics", Elsevier.

## COURSE MATERIAL

<b>SUBJECT</b>	<b>NUMERICAL METHODS AND PROBABILITY THEORY (20A54402)</b>
<b>UNIT</b>	<b>IV</b>
<b>COURSE</b>	<b>II B. TECH</b>
<b>SEMESTER</b>	<b>2 - 2</b>
<b>DEPARTMENT</b>	<b>HUMANITIES &amp; SCIENCE</b>
<b>PREPARED BY (Faculty Name/s)</b>	<b>Department of Mathematics</b>

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## 1. COURSE OBJECTIVES

This course aims at providing the student with the knowledge on

- Various numerical methods for solving equations, interpolating the polynomials, evaluation of integral equations and solution of differential equations.
- The theory of Probability and random variables.

## 2. PREREQUISITES

Students should have knowledge on

- Basic Mathematics & Calculus
- Permutations and combinations

## 3. SYLLABUS

### UNIT IV: Probability theory

Probability, probability axioms, addition law and multiplicative law of probability, conditional probability, Baye's theorem, random variables (discrete and continuous), probability density functions, properties, mathematical expectation.

## 4. COURSE OUTCOMES

After the completion of course, students will be able to

1. Apply numerical methods to solve algebraic and transcendental equations
2. Derive interpolating polynomials using interpolation formulae
3. Solve differential and integral equations numerically
4. Apply Probability theory to find the chances of happening of events.
5. Understand various probability distributions and calculate their statistical constants.

## 5. Co-PO / PSO Mapping

NMPT	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	P10	PO11	PO12	PSO1	PSO2
CO1	3	3												
CO2	3	3												
CO3	3	3												
CO4	3	3												
CO5	3	3												

## 6. LESSON PLAN

Lecture No.	Weeks	Topics to be covered	References
1	1	Introduction of Probability and its basic terms	T1, R1
2		Axioms of probability	T1, R1
3		Problems on Probability	T1, R1
4		Addition law & Problems	T1, R1
5	2	Multiplication Law	T1, R1
6		Conditional Probability and problems	T1, R1
7		Bayes Theorem and problems	T1, R1
8		Problems on Bayes theorem	T1, R1
9	3	Introduction of Random Variables	T1, R1
10		Discrete Random Variables and Probability density function	T1, R1
11		Mathematical Expectation and variance	T1, R1
12		Problems on Discrete Random Variables	T1, R1
13	4	Properties of Random variables	T1, R1
14		Continuous Random Variables and problems	T1, R1

## 7. ACTIVITY BASED LEARNING

1. Probability game on Cards
2. Probability game on dice

## 8. LECTURE NOTES

### 4.1 INTRODUCTION

Suppose a coin is tossed. The toss may result in the occurrence of 'Head' or in the occurrence of 'Tail'. Here, the chances of head and tail are equal. In other words, the probability of occurrence of head is  $\frac{1}{2}$  and the probability of occurrence of tail is  $\frac{1}{2}$ . Thus, Probability is a numerical measure which indicates the chance of occurrence.

#### **RANDOM EXPERIMENT:**

There are two types of experiments. They are

- (i) Deterministic experiment and
- (ii) Random experiment.

A deterministic experiment, when repeated under the same conditions, results in the same outcome. It has a unique outcome.

Random experiment is an experiment which may not result in the same outcome when repeated under the same conditions. It is an experiment which does not have a unique outcome.

For example:

1. The experiment of 'Toss of a coin' is a random experiment. It is so because when a coin is tossed the result may be 'Head' or it may be 'Tail'.
2. The experiment of 'Drawing a card randomly from a pack of playing cards' is a random experiment. Here, the result of the draw may be any one of the 52 cards.

#### **SAMPLE SPACE**

The set of all possible outcomes of a random experiment is the Sample space. The sample space is denoted by  $S$ . The outcomes of the random experiment (elements of the sample space) are called sample points or outcomes or cases.

A sample space with finite number of outcomes is a finite sample space. A sample space with infinite number of outcomes is an infinite sample space.

Ex1. While throwing a die, the sample space is

$S = \{1, 2, 3, 4, 5, 6\}$ . This is a finite sample space.

Ex2. While tossing two coins simultaneously, the sample space is

$S = \{HH, HT, TH, TT\}$ . This is a finite sample space.

Ex3. Consider the toss of a coin successively until a head is obtained. Let the number of tosses be noted. Here, the sample space is

$S = \{1, 2, 3, 4, \dots\}$  This is an infinite sample space

### **EVENT:**

Even is a subnet of the sample space. Events are denoted by A, B, C etc.

- An event which does not contains any outcome is a null event (impossible event). It is denoted by  $\Phi$ .
- An event which has only one outcome is an ELEMENTARY EVENT OR SIMPLEEVENT.
- An event which has more than outcome is a compound event.
- An event which contains all the outcomes is equal to the sample and it is called sure event or certain event.

Ex.1. While throwing a die,  $A = \{2, 4, 6\}$  is an event. It is the event that the throw results in an even number. Here, A is a compound event.

Ex.2. While tossing two coins,  $A = \{TT\}$  is an event. It is the event that the toss results in two tails. Here, A is a simple event.

The outcomes which belong to an event are said to be favourable to that event. The event happens whenever the experiment results in a favourable outcome. Otherwise, the event does not happen

While throwing a die, the event  $A = \{2, 4, 6\}$  has three favourable outcomes, namely, 2, 4 and 6. Where the throw results in 2, 4 or 6, event A occurs.

### **COMPLEMENT OF AN EVENT:**

Let A be an event. Then, Complement of A is the event of non-occurrence of A. It is the event constituted by the outcomes which are not favourable to A. The complement of A is denoted by  $A'$  or  $\bar{A}$  or  $A^c$ .

While throwing a die, If  $A = \{2,4,6\}$ , its complement is  $A' = \{1,3,5\}$ . Here, A is the event that throw result in an even number.  $A'$  is the event that throw does not result in an even number. That is,  $A'$  is the event that throw result in an odd number.

### **EQUALLY LIKELY EVENTS (Equiprobable events)**

Two or more events are equally likely if they have equal chance of occurrence. That is, equally likely events are such that none of them has greater chance of occurrence than the others.

Ex. 1. While tossing a fair coin, the outcomes 'Head' and 'Tail' are equally likely.

Ex.2. While throwing a fair die, the events  $A = \{2,4,6\}$ ,  $B = \{1,3, 5\}$  &  $C = \{1,2, 3\}$  are equally likely.

A sample space is called an equiprobable space if the outcomes are equally likely. For instance, the sample space  $S = \{1, 2, 3,4, 5, 6\}$  of throw of a fair die is equiprobable space because the six outcomes are equally likely.

### **MUTUALLY EXCLUSIVE EVENTS (Disjoint events)**

Two or more events are mutually exclusive if only one of them can occur at a time. That is, the occurrence of any of these events totally excludes the occurrence of the other events. Mutually exclusive events cannot occur together.

Ex. 1. While tossing a coin, the outcomes 'Head' and 'Tail' are mutually exclusive because when the coin is tossed once, the result cannot be Head as well as Tail.

Ex.2. While throwing a die, the events  $A = \{2, 4, 6\}$ ,  $B = \{3,5\}$  and  $C = \{1\}$  are mutually exclusive.

If A is an event, A and  $A'$  are mutually exclusive. It should be noted that intersection of mutually exclusive events is a null event.

### **EXHAUSTIVE EVENTS (Exhaustive set of events)**

A set of events is exhaustive if one or the other of the events in the set occurs whenever the experiment is conducted.

That is, the set of events exhausts all the outcomes of the experiment

The union of exhaustive events is equal to the sample space.

Ex.1. While throwing a die, the six outcomes together are exhaustive. But here, if any one of these outcomes is left out, the remaining five outcomes are not exhaustive.

Ex.2. While throwing a die, events  $A = \{2, 4, 6\}$ ,  $B = \{3, 6\}$  and  $C = \{1, 5, 6\}$  together are exhaustive.

### **CLASSICAL DEFINITION:**

Let a random experiment have 'n' equally likely, mutually exclusive and exhaustive outcomes. Let 'm' of these outcomes be favorable to an event A. Then, probability of A is —

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}} = \frac{m}{n}$$

### **Limitations of classical definition:**

This definition is applicable only when

- (i) The outcomes are equally likely, mutually exclusive and exhaustive.
- (ii) The number of outcomes n is finite.

## **4.2 THE AXIOMATIC APPROACH**

Consider a random experiment with sample space S. Associated with this random experiment, many events can be defined. Let for every event A, a real number  $P(A)$  be assigned. Then,  $P(A)$  is the probability of event A, if the following axioms are satisfied.

Axiom 1:  $P(A) \geq 0$

Axiom 2:  $P(S) = 1$ , S being the sure event or Sample space.

Axiom 3: For two mutually exclusive events A and B,

$$P(A \cup B) = P(A) + P(B)$$

Note that the third axiom can be generalised for any number of mutually exclusive events.

RESULT 1:

$P(A)$  is a value between 0 and 1. That is,  $0 < P(A) < 1$ .

Proof:

Let a random experiment have 'n' equally likely, mutually exclusive and exhaustive outcomes. Let 'm' of these outcomes be favourable to event A.

$$\text{Then } P(A) = \frac{m}{n}$$

Here, the least possible value of m is 0. Also, the highest possible value of m is n.

And so,  $0 \leq m \leq n$ .

$$\frac{0}{n} \leq \frac{m}{n} \leq \frac{n}{n}$$

$$\Rightarrow 0 \leq p(A) \leq 1$$

Thus,  $P(A)$  is a value between 0 and 1.

RESULT 2:

$P(A') = 1 - P(A)$ . That is,  $P(A) = 1 - P(A')$ .

Proof:

In a random experiment with n equally likely, mutually exclusive and exhaustive outcomes, if m outcomes are favourable to event A, the remaining (n-m) outcomes are favourable to the complementary event A'. Therefore,

$$P(A') = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - P(A)$$

Thus,  $P(A') = 1 - P(A)$ . That is,  $P(A) = 1 - P(A')$ .

1. Find the probability that a throw of an unbiased die results in

- (i) an One      (ii) an even number      (iii) a multiple of 3.

Sol: The sample space is  $S = \{1, 2, 3, 4, 5, 6\}$ . There is  $n = 6$  equally likely, mutually exclusive and exhaustive outcomes. Let events A, B and C be—

A: throw results in an ace (number 1)

B: throw results in an even number

C: throw results in a multiple of 3

(i) Event A has one favourable outcome.

$$\therefore P[\text{ace}] = P(A) = \frac{m}{n} = \frac{1}{6}$$

(ii) Event B has 3 favourable outcomes, namely, 2, 4 and 6.

$$\therefore P[\text{even number}] = P(B) = m/n = 3/6 = \frac{1}{2}$$

(iii) Event C has 2 favourable outcomes, namely, 3 and 6

$$P[\text{multiple of 3}] = P(C) = m/n = 2/6 = \frac{1}{3}$$

2. A bag contains 3 white, 4 red and 2 green balls. One ball is selected at random from the bag. Find the probability that the selected ball is

(i) white            (ii) non-white            (iii) white or green.

Sol: The bag totally has 9 balls. Since the ball drawn can be any one of them, there are 9 equally likely, mutually exclusive and exhaustive outcomes. Let events A, B and C be

A: selected ball is white

B: selected ball is non-white

C: selected ball is white or green

(i) There are 3 white balls in the bag. Therefore, out of the 9 outcomes, 3 are favourable to event A.

$$\therefore P[\text{white ball}] = P(A) = 3/9 = \frac{1}{3}$$

(ii) Event B is the complement of event A. Therefore,

$$\therefore P(\text{non-white ball}) = P(B) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$$

(iii) There are 3 white and 2 green balls in the bag. Therefore, out of 9 outcomes, 5 are either white or green.

$$\therefore P[\text{white or green ball}] = P(C) = \frac{5}{9}$$

3. One card is drawn from a well-shuffled pack of playing cards. Find the probability that the card drawn (i) is a Heart (ii) is a King (iii) belongs to red suit (iv) is a King or a Queen (v) is a King or a Heart.

Sol:

A pack of playing cards has 52 cards. There are four suits, namely, Spade, Club, Heart and Diamond (Dice). In each suit, there are thirteen denominations - Ace (1), 2, 3, 10, Jack (Knave), Queen and King.

A card selected at random may be any one of the 52 cards. Therefore, there are 52 equally likely, mutually exclusive and exhaustive outcomes. Let events A, B, C, D and E be —

A: selected card is a Heart

B: selected card is a King

C: selected card belongs to a red suit.

D: selected card is a King or a Queen

E: selected card is a King or a Heart

(i) There are 13 Hearts in a pack. Therefore, 13 outcomes are favourable to event A.

$$\therefore P[\text{Heart}] = P(A) = 13/52 = 1/4$$

(ii) There are 4 Kings in a pack. Therefore, 4 outcomes are favourable to event B.

$$\therefore P[\text{King}] = P(B) = 4/52 = 1/13$$

(iii) There are 13 Hearts and 13 Diamonds in a pack. Therefore, 26 outcomes are favorable to event C.

$$\therefore P[\text{Red card}] = P(C) = 26/52 = 1/2$$

(iv) There are 4 Kings and 4 Queens in a pack. Therefore, 8 outcomes are favourable to event D.

$$\therefore P[\text{King or Queen}] = P(D) = 8/52 = 2/13$$

(v) There are 4 Kings and 13 Hearts in a pack. Among these, one card is Heart-King.

Therefore,  $(4+13-1) = 16$  outcomes are favourable to event E.

$$\therefore P[\text{King or Heart}] = P(E) = 16/52 = 4/13$$

4. A bag contains 8 tickets which are marked with the numbers 1, 2, 3, .. 8. Find the probability that a ticket drawn at random from the bag is marked with (i) an even number (ii) a multiple of 3.

Sol: The selection can be any one of the eight numbers. Therefore, there are 8 equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be—

A: selected number is even.

B: selected number is a multiple of 3.

(i) Four of the selections, namely, 2, 4, 6 and 8 are favourable to event A.

$$\therefore P[\text{even number}] = P(A) = 4/8 = 1/2$$

(ii) Two of the selections, namely, 3 and 6 are favourable to event B.  
 $\therefore P[\text{multiple of 3}] = P(B) = 2/8 = 1/4.$

5. A fair coin is tossed twice. Find the probability that the tosses result in  
 (i) two heads      (ii) at least one head.

Sol: The sample space is  $S = \{HH, HT, TH, TT\}$ , There are four equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be—

A: the tosses result in 2 heads

B: the tosses result in at least one head.

(i) One outcome, HH is favourable to event A.

$$\therefore P[\text{two heads}] = P(A) = 1/4$$

(ii) 3 outcomes HH, HT and TH are favourable to event B.

$$\therefore P[\text{at least one head}] = P(B) = 3/4$$

6. Two fair dice are rolled. Find the probability that (i) both the dice show number 6 (ii) the sum of numbers obtained is 7 or 10 (iii) the sum of the numbers obtained is less than 11 (iv) the sum is divisible by 3.

Sol: The sample space is  $S = \{(1, 1), (1, 2), (1, 3), \dots, (1, 6)$   
 $(2, 1), (2, 2), (2, 3) \dots (2, 6)$   
 $\dots\dots\dots$   
 $(6, 1), (6, 2), (6, 3) \dots (6, 6)\}$

There are  $6 \times 6 = 36$  equally likely, mutually exclusive and exhaustive outcomes.

Let events A, B, C and D be —

A: both the dice show number 6

B: sum of the numbers obtained is 7 or 10

C: sum of the numbers obtained is less than 11.

D: sum of the numbers obtained is divisible by 3.

(i) One outcome, namely, (6, 6) is favourable to event A.

$$\therefore P[6 \text{ on both the dice}] = P(A) = 1/36$$

(ii) Nine outcomes, namely, (6,1), (5,2), (4, 3), (3,4), (2,5), (1,6), (6, 4), (5, 5) and (4,6) are favourable to event B.

$$\therefore P [\text{sum is 7 or 10}] = 9/36 = 1/4$$

(iii) The complement of event C is— C': sum is 11 or 12.

Event C' has three favourable outcomes, namely, (6,5), (5, 6) and (6, 6).

$$\begin{aligned} P [\text{sum is less than 11}] &= 1 - P [\text{sum is 11 or 12}] \\ &= 1 - 3/36 = 1 - 1/12 = 11/12 \end{aligned}$$

The sum is divisible by 3 if it is 3, 6, 9 or 12. Therefore, the outcomes favourable to event D are (2, 1), (1, 2), (5,1), (4,2), (3,3), (2, 4), (1,5), (6, 3), (5, 4), (4, 5), (3, 6) and (6, 6). Thus, 12 outcomes are favourable.

$$P [\text{sum is divisible by 3}] = 12/36 = 1/3.$$

**7.** A box has 5 white, 4 red and 3 green balls. Two balls are drawn at random from the box. Find the probability that they are (i) of the same colour (ii) of different colours.

Sol: The box totally has 12 balls. A random draw of two balls has  $12C_2$  equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be—

A: the balls drawn are of the same colour

B: the balls drawn are of different colours.

(i). Events happens when the drawn balls are both white or both red or both green.

Out of  $12C_2$  selections,  $5C_2$  selections are both white,  $4C_2$  selections are both red

And  $3C_2$  selections are both green. Thus,  $5C_2 + 4C_2 + 3C_2$  outcomes are

Favourable to event A.

$$\begin{aligned} P [\text{balls of same colour}] &= \frac{5C_2 + 4C_2 + 3C_2}{12C_2} \\ &= \frac{10+6+3}{66} = \frac{19}{66} = 0.2879 \end{aligned}$$

(ii). Event B is the complement of event A. Therefore,

$$\begin{aligned} P [\text{balls of different colours}] &= 1 - P[\text{same colour}] \\ &= 1 - P(A) \\ &= 1 - 19/66 \\ &= 47/66 \end{aligned}$$

8. Two cards are drawn at random from a pack of cards. Find the probability that (i) both are Spades (ii) both are Kings (iii) one is Spade and the other is a Heart (v) the cards belong to the same suit (v) the cards belong to different suits.

Sol:

A random draw of 2 cards from a pack of 52 cards has  $52C_2$  equally likely, mutually exclusive and exhaustive outcomes. Let events A, B, C, D and E be—

A: both the cards drawn are Spades

B: both the cards drawn are Kings.

C: the cards drawn are one Spade and one Heart.

D: the cards belong to the same suit.

E: the cards belong to different suits.

(i) Since there are 13 Spades in a pack, event A has  $13C_2$  favourable outcomes.

$$\text{Therefore, } P[\text{both spades}] = \frac{13C_2}{52C_2} = \frac{13 \times 6}{26 \times 51} = \frac{1}{17}$$

(ii) Since there are 4 Kings in a pack, event B has  $4C_2$  favourable outcomes.

$$\text{Therefore, } P[\text{both Kings}] = \frac{4C_2}{52C_2} = \frac{2 \times 6}{26 \times 51} = \frac{1}{221}$$

(iii) Here, one card should be a Spade and the other should be a Heart.

From 13 Spades, one Spade can be had in  $13C_1$  ways. From 13 Hearts, one Heart can be had in  $13C_1$  ways. Thus,  $13C_1 \times 13C_1$  outcomes are favourable to event C.

$$\begin{aligned} \text{Therefore, } P[\text{a Spade and a Heart}] &= \frac{13C_1 \times 13C_1}{52C_2} \\ &= \frac{13 \times 13}{26 \times 51} = \frac{13}{102} \end{aligned}$$

(iv) Here, the cards should be 2 Spades or 2 Clubs or 2 Hearts or 2 Diamonds. There are 13 cards of each suit. In each case, a selection of two cards can be made in  $13C_2$  ways. Thus, totally the number of favourable cases is  $13C_2 + 13C_2 + 13C_2 + 13C_2$ .

$$\begin{aligned} P[\text{cards of same suit}] &= \frac{13C_2 + 13C_2 + 13C_2 + 13C_2}{52C_2} \\ &= \frac{4 \times 78}{26 \times 51} = \frac{4}{17} \end{aligned}$$

(v) Events E is the complement of event D. Therefore,

$$\begin{aligned} P[\text{cards of different suits}] &= 1 - P[\text{cards of same suit}] \\ &= 1 - 4/17 = 13/17 \end{aligned}$$

9. A bag has 9 tickets marked with numbers 1, 2, 3,.....9. Two tickets are drawn at random from the bag. Find the probability that both the numbers drawn are  
(i) even (ii) odd.

Sol:

There are  ${}^9C_2$  equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be

A: both the selected numbers are even.

B: both the selected numbers are odd.

(i) Out of 9 numbers, 4 numbers, namely, 2,4,6 and 8 are even. Therefore,  ${}^4C_2$  selections will have two even numbers. Therefore,

$$P[\text{both even}] = P(A) = \frac{{}^4C_2}{{}^9C_2} = \frac{6}{36} = \frac{1}{6}$$

(ii) Out of 9 numbers, 5 numbers, namely, 1,3,5,7 and 9 are odd. Therefore,  ${}^5C_2$  selections will have two odd numbers. Therefore,

$$P[\text{both odd}] = P(B) = \frac{{}^5C_2}{{}^9C_2} = \frac{10}{36} = \frac{5}{18}$$

10. A bag contains 3 red, 4 green and 3 yellow marbles. Three marbles are randomly drawn from the bag. What is the probability that they are of (i) the same colour (ii) different colours (one of each colour)?

Sol: There are  ${}^{10}C_3$  equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be

A: Selected marble are of the same colour.

B: Selected marbles are of different colours

(i) The marbles drawn should be 3 red or 4 green or 3 yellow.

Therefore,  ${}^3C_3 + {}^4C_3 + {}^3C_3$  outcomes are favourable to events A, Therefore,

$$P[\text{marbles of the same colour}] = \frac{{}^3C_3 + {}^4C_3 + {}^3C_3}{{}^{10}C_3} = \frac{1+4+1}{120} = \frac{1}{20}$$

(ii) The marbles should be one of each colour. Therefore,  ${}^3C_1 \times {}^3C_1 \times {}^3C_1$  outcomes are favourable. Therefore,

$$P[\text{marbles of different colours}] = \frac{{}^3C_1 + {}^4C_1 + {}^3C_1}{{}^{10}C_3} = \frac{3}{10}$$

### 4.3 ADDITION THEOREM ON PROBABILITY

**Statement:** For any two events A and B,  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:

For events A and B,  $B = (A \cap B) \cup (A^1 \cap B)$

Here,  $A \cap B$  and  $A^1 \cap B$  are mutually exclusive. Therefore, by axiom 3,

$$P(B) = P(A \cap B) + P(A^1 \cap B)$$

$$P(A^1 \cap B) = P(B) - P(A \cap B)$$

Also,  $A \cup B = A \cup (A^1 \cap B)$

Here,  $A \cap B$  and  $A^1 \cap B$  are mutually exclusive therefore,

$$P(A \cup B) = P(A) + P(A^1 \cap B)$$

By above results

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

1. Three unbiased dice are thrown once. Find the probability that all the three dice show the number 6.

**Sol:**

When 3 dice are thrown, there are  $6 \times 6 \times 6 = 216$  equally, mutually exclusive and exhaustive outcomes. of these 216 outcomes, 1 outcome, namely, (6, 6, 6) is favourable. Therefore, probability of all the three dice showing the number 6 is  $P[\text{all the three result in the number 6}] = 1/216$

2. A fair coin is tossed five times. Find the probability of obtaining

(i) head in all the tosses, (ii) head in at least one of the tosses.

**Sol:** There are 32 equally likely, mutually exclusive and exhaustive outcomes.

(i) Out of them, one outcome is HHHHH and another outcome is TTTTT. Therefore,

$$P[\text{head in all tosses}] = 1/32$$

(ii)  $P[\text{at least one head}] = 1 - P[\text{tail in all tosses}]$

$$= 1 - 1/32 = 31/32$$

**Note:** Whenever probability of the event "at least one" has to be found, it is easier to find it by using the probability of the complementary event as follows.

$$P[\text{at least one}] = P[\text{none}]$$

3. In a college, there are five lecturers. Among them, three are doctorates. If a committee consisting three lecturers is formed, what is the probability that at least two of them are doctorates?

**Sol:**

From the five lecturers, three lecturers can be selected in  ${}^5C_3$  ways. Thus, there are  ${}^5C_3$  equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be —

A: Two of the selected lecturers are doctorates.

B: All the three selected lecturers are doctorates.

Then, events have  ${}^3C_2 \times {}^2C_1$  favourable outcomes. And, event B has  ${}^3C_3$  favourable outcomes. Here, events A and B are mutually exclusive.

$\therefore P$  [at least two doctorates] =  $P$  [two or three doctorates] =  $P(A \cup B)$

$$= P(A) + P(B) = \frac{{}^3C_2 \times {}^2C_1}{{}^5C_3} + \frac{{}^3C_3}{{}^5C_3}$$

$$= \frac{3 \times 2}{10} + \frac{1}{10} = \frac{7}{10} = 0.7$$

4. What is the probability that there will be 53 Sundays in a randomly selected  
(i) leap year (ii) non-leap year?

**Sol:**

(i) A leap year has 366 days, out of them,  $7 \times 52 = 364$  days make 52 complete weeks. The remaining two days may occur in any of the following pattern ---

(Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday) and (Saturday, Sunday).

Out of these 7 cases which are equally likely, mutually exclusive and exhaustive, 2 cases namely (Sunday, Monday) and (Saturday, Sunday) have Sunday. Therefore,

$$P \text{ [leap year has 53 Sundays]} = 2/7$$

(ii) A non-leap year has 365 days. Out of them, 364 days make 52 complete weeks. The remaining one day may be Sunday, Monday, --- Saturday. Out of these 7 possibilities, only one is Sunday. Therefore,

$$P \text{ [non-leap year has 53 Sundays]} = 1/7$$

5. If from a pack of cards, a single card is drawn. What is the probability that it is either a spade or a king?

**Sol:**  $P(A) = P(\text{a spade card}) = 12/52 = 1/4$

$P(B) = P(\text{a king card}) = 4/52$

$P(\text{either a spade or a king card}) = P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= (12/52) + (4/52) - (4/52)$$

$$= 12/52$$

6. A person is known to hit the target in 3 out of 4 shots, whereas another person is known to hit the target in 2 out of 3 shots. Find the probability of the targets being hit at all when they both persons try.

**Sol:** The prob. that the first person hit the target =  $P(A) = 3/4$

The prob. that the second person hit the target =  $P(B) = 2/3$

The two events are not mutually exclusive, since both persons hit the same target.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 3/4 + 2/3 - (6/12)$$

$$= 11/12$$

## 4.4 CONDITIONAL PROBABILITY

Let A and B be two events. Then, conditional probability of B given A is the probability of happening of B when it is known that A has already happened. On the other hand, the probability of happening of B when nothing is known about happening of A is called unconditional probability of B.

The conditional probability of B given A is denoted by  $P(B|A)$ . The unconditional probability is  $P(B)$ .

Let  $P(A) > 0$ . Then, conditional probability of event B given A is defined as-----

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

If  $P(A) = 0$ , the conditional probability  $P(B|A)$  is not defined.

If A and B are independent events, occurrence of B will be independent of occurrence of A. Therefore, the conditional and unconditional probabilities are equal. That is,  $P(B|A) = P(B)$ .

$$P(B) = \frac{P(A \cap B)}{P(A)} \quad \text{That is, } P(A \cap B) = P(A) \cdot P(B)$$

### INDEPENDENT & DEPENDENT EVENTS:

Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other. Events which are not independent are called dependent events.

If two events are independent, the occurrence or non-occurrence of one does not depend on the occurrence or non-occurrence of the other.

**Note:** Two events A and B are independent if and only if  $P(A \cap B) = P(A) \cdot P(B)$

## 4.5 MULTIPLICATION LAW OF PROBABILITY

If A & B are two independent events, then

$$\begin{aligned} P(A \cap B) &= P(\text{Both A \& B will happen}) \\ &= P(A) \times P(B) \end{aligned}$$

1. A bag contains 8 white and 10 black balls. Two balls are drawn in succession. What is the prob. that first is white and second is black?

Sol: Total no. of balls =  $8 + 10 = 18$

$$P(\text{drawing one white ball from 8 balls}) = 8/18$$

$$P(\text{drawing one black ball from 10 balls}) = 10/18$$

$$\begin{aligned} P(\text{drawing first white \& second black}) &= 10 \cdot 8 / (18 \cdot 18) \\ &= 80/324 \end{aligned}$$

2. Two persons A & B appear in an interview for 2 vacancies for the same post. The probability of A's selection is  $1/7$  and that of B's selection is  $1/8$ . What is the probability that, i) both of them will be selected?  
ii) none of them will be selected.

Sol:  $P(\text{A selected}) = 1/7$

$$P(\text{B selected}) = 1/8$$

$$P(\text{A will not be selected}) = 1 - (1/7) = 6/7$$

$$P(\text{B will not be selected}) = 1 - (1/8) = 7/8$$

$$\begin{aligned} P(\text{Both of them will be selected}) &= P(\text{A}) \times P(\text{B}) \\ &= 1 / (7 \cdot 8) \\ &= 1/56 \end{aligned}$$

3. A card is drawn at random from a pack of cards.

- (i) What is the probability that it is a heart?  
(ii) If it is known that the card drawn is red, what is the probability that it is a heart?

Sol: There are 52 equally likely, mutually exclusive and exhaustive outcomes. Let events A and B be —

A: card drawn is red.

B: card drawn is heart.

There are 26 red cards and 13 hearts in a pack of cards. Therefore, event A has 26 favourable outcomes and event B has 13 favourable outcomes. Event  $A \cap B$  has 13 favourable outcomes because when any of the 13 hearts is drawn  $A \cap B$  happens.

Therefore,  $P(A) = 26/52$ .  $P(B) = 13/52$  and  $P(A \cap B) = 13/52$

(i) The unconditional probability of drawing a heart is ---

$$P(B) = 13/52 = 1/4$$

(ii) The conditional probability of drawing a heart given that it is red card is-----

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{13/52}{26/52} = 1/2$$

4. A fair coin is tossed thrice. What is the probability that all the three tosses result in heads?

**Sol:**

Let events A, B, and C be-----

A: the first toss results in head

B: the second toss results in head.

C: the third toss results in head.

Then,  $P(A) = P(B) = P(C) = 1/2$

Since A, B, and C are results of three different tosses, they are independent.

Therefore, probability that all the three tosses result in head is ---

$P[3 \text{ heads}] = P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

5. Two fair dice are rolled. If the sum of the numbers obtained is 4, find the probability that the numbers obtained on both the dice are even-

**Sol:** Let events A and B be —

A: the sum of the numbers is 4

B: the numbers on both the dice are even

Here, we have to find -----

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Event A has 3 favourable outcomes, namely, (1,3), (2,2) and (3,1)

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$\therefore P[\text{Sum 4}] = P(A) = 3/36$

Event (AnB) has 1 favourable outcome, namely, (2,2).

$$\therefore P [\text{Sum 4 and number even}] = P(\text{AnB}) = 1/36$$

$$\text{Thus, } P [\text{Number even given Sum 4}] = \frac{1/36}{3/36} = 1/3$$

6. A box has 1 red and 3 white balls. Balls are drawn one after one from the box. Find the probability that the two balls drawn would be red if

a. the ball drawn first is returned to the box before the second draw is made. (Draw with replacement).

b. the ball drawn first is not returned before the second draw is made. (Draw without replacement).

**Sol:** Let A: the first ball drawn is red

B: the second ball drawn is red.

Draw with replacement:

Here,  $P(A) = 1/4$  Also, since the first ball- is returned before the second draw is made,

$$P(B | A) = 1/4$$

$$\begin{aligned} \therefore P [\text{Two balls are red}] &= P(\text{AnB}) \\ &= P(A). P (B | A) \\ &= 1/4 * 1/4 = 1/16 \end{aligned}$$

Draw without replacement:

Here, Since the first ball drawn is not returned before the second draw is made,

$$P(B | A) = 0/4$$

$$\begin{aligned} \therefore P [\text{Two balls are red}] &= P(\text{AnB}) \\ &= P(A). P(B | A) \\ &= 1/4 * 0/4 = 0 \end{aligned}$$

7. The probability that a contractor will get a plumbing contract is  $2/3$  and probability that he will not get an electrical contract is  $5/9$ . If the probability of getting at least one of these contracts is  $4/5$ , what is the probability that he will get both?

**Sol:** Let A: contractor gets plumbing contract

B: contractor gets electrical contract

Then,  $P(A) = 2/3$ ,  $P(B^c) = 5/9$  and  $P(A \cup B) = 4/5$

Therefore,  $P(B) = 1 - P(B^c) = 4/9$

By addition theorem we have,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

That is,  $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Therefore,

$$\begin{aligned} P[\text{he gets both plumbing and electrical contract}] &= P(A \cap B) \\ &= P(A) + P(B) - P(A \cup B) \\ &= 2/3 + 4/9 - 4/5 = 14/45 \end{aligned}$$

- 8.** A can solve 90 percent of the problems given in a book and B can solve 70 percent. What is the probability that at least one of them will solve a problem selected at random?

**Sol:** event A: student A solve the problem

event B: student B solve the problem.

$P(\text{at least one solves the problem}) = 1 - P(\text{none solve the problem})$

$$\begin{aligned} &= 1 - P(\bar{A} \cap \bar{B}) \\ &= 1 - P(\bar{A}) \cdot P(\bar{B}) \\ &= 1 - (0.10)(0.30) \\ &= 0.97 \end{aligned}$$

The probability that a trainee will remain with a company 0.6, The probability that an employee earns more ten Rs.10,000 per year 0.5. The probability an employee is trainee who remained with the company or who earn more then Rs.10,000 per year is 0.7. What is the probability earn more than Rs.10,000 per year given that he is a trainee who stayed with the company?

**Sol:** Event A: A trainee will remain with the company

Event B: A trainee earns more than Rs. 10,000.

Given  $P(A) = 0.6$   $P(B) = 0.5$   $P(A \cup B) = 0.7$

We need to find

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A) + P(B) - P(A \cup B)}{P(A)} = \frac{0.4}{0.6} = 0.67$$

9. Suppose that one of the three men, a politician a bureaucrat and an educationist will be appointed as VC of the university. The probabilities of their appointment are respectively 0.3,0.25, and 0.45. The probability that these people will promote research activities if there are appointed is 0.4,0.7 and 0.8 respectively. What is the probability that research will be promoted by the new VC

**Sol:**

event A: Politician appointed as VC

event B: bureaucrat appointed as VC

event C: Educationist appointed as VC

event D: promotion of research activities

$$\begin{aligned} &= P(A \cap D) + P(B \cap D) + P(C \cap D). \\ &= P(D|A).P(A) + P(D|B).P(B) + P(D|C).P(C) \\ &= (0.3)(0.4) + (0.25)(0.7) + (0.45)(0.8) = 0.655 \end{aligned}$$

10. A box contains 4 green and 6 white balls another box contains 7 green and 8 white balls. Two balls are transferred from box 1 to box 2 and then a ball is drawn from box 2. What is the probability that it is white?

**Sol:**

event A: transferred balls are green

event B: transferred balls are white

event C: Among transferred balls one green & 1 white

event D: selection of a white ball from box 2.

$$\begin{aligned} &= P(A \cap D) + P(B \cap D) + P(C \cap D) \\ &= P(D|A).P(A) + P(D|B).P(B) + P(D|C).P(C) \\ &= \frac{4C_2}{10C_2} \cdot \frac{8}{17} + \frac{6C_2}{10C_2} \cdot \frac{10}{17} + \frac{4C_1 \times 6C_1}{10C_2} \times \frac{9}{17} \\ &= 0.5412 \end{aligned}$$

11. Probabilities of Husband's and wife's selection to a post are  $1/5$  and  $1/7$  respectively, what is the probability that.

- (i) Both of them will be selected.
- (ii) Exactly one of them will be selected
- (iii) None of them will be selected

**Sol:**

event A: selection of Husband  $P(A) = \frac{1}{5}$

event B: selection of Wife  $P(B) = \frac{1}{7}$

(i)  $P(\text{both of them will be selected}) = P(A \cap B) = P(A) \cdot P(B)$   
 $= \frac{1}{5} \times \frac{1}{7} = \frac{1}{35}$

(ii)  $P(\text{exactly one of them will be selected}) = P(\bar{A} \cap B) + P(\bar{B} \cap A)$   
 $= P(\bar{A}) \cdot P(B) + P(\bar{B}) \cdot P(A)$   
 $= \frac{4}{5} \times \frac{1}{7} + \frac{6}{7} \times \frac{1}{5} = \frac{10}{35}$

(iii)  $P(\text{none of them will be selected}) = P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$   
 $= \frac{4}{5} \times \frac{6}{7} = \frac{24}{35}$

#### 4.6 BAYE'S THEOREM

**Statement:**

If  $E_1, E_2, E_3, \dots, E_n$  are 'n' mutually exclusive events with  $P(E_i) \neq 0$  for each i in the sample space S and for any arbitrary event A which is a subset of  $\cup_{i=1}^n E_i$  with

$$P(A) > 0 \text{ then } P\left(\frac{E_i}{A}\right) = \frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^n P(E_i)P\left(\frac{A}{E_i}\right)} \quad \text{for } i=1 \text{ to } n.$$

Proof:

Given  $E_1, E_2, E_3, \dots, E_n$  are n mutually exclusive

i. e.,  $E_i \cap E_j = \phi$  for  $i \neq j$

Also  $E_1, E_2, E_3, \dots, E_n$  are exhaustive events

i. e.,  $E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n = S$

$$\begin{aligned} A &= A \cap S \\ &= A \cap (E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) \\ &= (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \dots \cup (A \cap E_n) \end{aligned}$$

$$P(A) = P((A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \dots \cup (A \cap E_n))$$

Since  $(A \cap E_1), (A \cap E_2), (A \cap E_3), \dots, (A \cap E_n)$  are  $n$  mutually exclusive even

By Axiom-III

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \dots + P(A \cap E_n)$$

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)}$$

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{P(A)}$$

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \dots + P(A \cap E_n)}$$

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum P(A \cap E_i)}$$

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^n P(E_i)P(A/E_i)} \quad (\text{By the definition of Conditional Probability})$$

#### PROBLEMS:

- In a certain college, 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the students. (a) What is the probability that mathematics is being studied? (b) If a student is selected at random and is found to be studying mathematics, Find the probability that the student is a girl? © a boy?

Sol: Given  $P(\text{Boy}) = P(B) = \frac{40}{100} = \frac{2}{5}$

$$P(\text{Girl}) = P(G) = \frac{60}{100} = \frac{3}{5}$$

Probability that mathematics is studied given that the student is a boy

$$P(M/B) = \frac{25}{100} = \frac{1}{4}$$

Probability that mathematics is studied given that the student is a girl

$$P(M/G) = \frac{10}{100} = \frac{1}{10}$$

Probability that the student studied Mathematics  $P(M) = P(G)P(M/G) + P(B)P(M/B)$

$$= \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{1}{4} = \frac{5}{25}$$

By Bayes theorem, probability of mathematics student is a girl

$$P(G/M) = \frac{P(G)P(M/G)}{P(G)P(M/G)+P(B)P(M/B)} = \frac{\frac{3}{5} \times \frac{1}{10}}{\frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{1}{4}} = \frac{3}{10}$$

By Bayes theorem, probability of mathematics student is a boy

$$P(B/M) = \frac{P(B)P(M/B)}{P(G)P(M/G)+P(B)P(M/B)} = \frac{\frac{2}{5} \times \frac{1}{4}}{\frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{1}{4}} = \frac{1}{2}$$

2. The chance that doctor A will diagnose a disease x correctly is 60%. The chance that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease x died. What is the probability that his disease was diagnosed correctly?

Sol: Let  $E_1$  be the event that "disease x is diagnosed correctly by doctor A" and

$E_2$  be the event that "a patient of doctor A who disease x died"

$$\text{Then } P(E_1) = \frac{60}{100} = 0.6, \quad P(E_2/E_1) = \frac{40}{100} = 0.4$$

$$\text{Which implies that } P(\bar{E}_1) = 1 - 0.6 = 0.4 \text{ and } P(E_2/\bar{E}_1) = \frac{70}{100} = 0.7$$

By Bayes theorem

$$\begin{aligned} P(E_1/E_2) &= \frac{P(E_1)P(E_2/E_1)}{P(E_1)P(E_2/E_1) + P(\bar{E}_1)P(E_2/\bar{E}_1)} \\ &= \frac{0.6 \times 0.4}{0.6 \times 0.4 + 0.4 \times 0.7} = \frac{6}{13} \end{aligned}$$

3. Of the three men, the chances that a politician, a business man or an academician will be appointed as a vice-chancellor (V.C) of a university are 0.5, 0.3, 0.2 respectively. Probability that research is promoted by these persons if they are appointed as VC are 0.3, 0.7, 0.8 respectively. Determine the probability that research is promoted. If research is promoted, what is the probability that VC is an academician?

Sol: Let A, B, C be the events that a politician, businessmen or an academician will be appointed as VC of the three men.

Then  $P(A)=0.5$ ,  $P(B)=0.3$ ,  $P(C)=0.2$

The probabilities that research is promoted if they are appointed as VC s are

$$P(R/A) = 0.3, P(R/B) = 0.7, P(R/C) = 0.8$$

The Probability that the research is promoted

$$\begin{aligned} &= P(A)P(R/A) + P(B)P(R/B) + P(C)P(R/C) \\ &= (0.5)(0.3) + (0.3)(0.7) + (0.2)(0.8) \\ &= 0.52 \end{aligned}$$

The probability that research is promoted when the VC is an academician

$$\begin{aligned} P(C/R) &= \frac{P(C)P(R/C)}{P(C)P(R/C)+P(B)P(R/B)+P(A)P(R/A)} \\ &= \frac{0.16}{0.15+0.21+0.16} = \frac{4}{13} = 0.3077 \end{aligned}$$

4. A businessman goes to hotels X, Y, Z, 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbings. What is the probability that business man's room having faulty plumbing is assigned to hotel Z?

**Sol:** Let the probabilities of business man going to hotels X, Y, Z be receptively  $P(X)$ ,  $P(Y)$ ,  $P(Z)$ .

$$\text{Then } P(X) = \frac{20}{100} = 0.2, P(Y) = \frac{50}{100} = 0.5, P(Z) = \frac{30}{100} = 0.3$$

Let E be the event that the hotel room has faulty plumbing. Then the probabilities that hotels X, Y, Z have faculty plumbing are

$$P(E/X) = \frac{5}{100} = 0.05, P(E/Y) = \frac{4}{100} = 0.04, P(E/Z) = \frac{8}{100} = 0.08,$$

The probability that the business man's room having faulty plumbing is assigned to hotel z

$$\begin{aligned} P(Z/E) &= \frac{P(Z)P(E/Z)}{P(Z)P(E/Z)+P(Y)P(E/Y)+P(X)P(E/X)} \\ &= \frac{0.3 \times 0.08}{0.3 \times 0.08 + 0.5 \times 0.04 + 0.2 \times 0.05} = \frac{4}{9} \end{aligned}$$

5. There are two boxes, in box I, 11 cards are there numbered 1 to 11 and in box II, 5 cards are there numbered 1 to 5. A box is chosen and a card is drawn. If the card shows an even number, then another card is drawn from the same box. If card shows an odd number another card is drawn from the other box. Find the probability that they are from box I?

**Sol:**

Number of cards in box I = 11

Number of cards with even number = 5

Number of cards with odd number = 6

Number of cards in box II = 5

Number of cards with even number = 2

Number of cards with odd number = 3

The probability of choosing any one box =  $\frac{1}{2}$

Let E be the event that both the cards are even.

For this box is chosen and a card is picked, if the first card is even then the second card is also picked from the same box and the card is also even.

Let  $E_1$  be the event that both the cards are from box I

$$P(E_1) = \frac{1}{2} \left( \frac{5}{11} \right) \left( \frac{4}{10} \right) = \frac{1}{11}$$

Let  $E_2$  be the event that both the cards are from box II. Then

$$P(E_2) = \frac{1}{2} \left( \frac{2}{5} \right) \left( \frac{1}{4} \right) = \frac{1}{20}$$

$$\therefore P(E) = P(E_1) + P(E_2) = \frac{1}{11} + \frac{1}{20} = \frac{31}{220}$$

Let E be the event that both the cards are odd

Then a box is chosen, first card is odd and second card is picked from another box and that is also odd

Let  $E_1$  be the event that the first card odd from box I and second card is odd from box II.

$$P(E_1) = \frac{1}{2} \left( \frac{6}{11} \right) \left( \frac{3}{5} \right) = \frac{9}{55}$$

Let  $E_2$  be the event that first card is odd from box II and second card is odd from box I

$$\text{Then } P(E_2) = \frac{1}{2} \left( \frac{3}{5} \right) \left( \frac{6}{11} \right) = \frac{9}{55}$$

$$\therefore P(E) = P(E_1) + P(E_2) = \frac{9}{55} + \frac{9}{55} = \frac{18}{55}$$

The probability that both cards from box I are even =  $\frac{1}{2} \left( \frac{5}{11} \right) \left( \frac{4}{10} \right) = \frac{1}{11}$

The probability that both cards from box II are even =  $\frac{1}{2} \left( \frac{2}{5} \right) \left( \frac{1}{4} \right) = \frac{1}{20}$

The Probability that if both cards are even then they are from box I

$$\frac{\frac{1}{2} \left( \frac{5}{11} \right) \left( \frac{4}{10} \right)}{\frac{1}{2} \left( \frac{5}{11} \right) \left( \frac{4}{10} \right) + \frac{1}{2} \left( \frac{2}{5} \right) \left( \frac{1}{4} \right)} = \frac{\frac{1}{11}}{\frac{1}{11} + \frac{1}{20}} = \frac{20}{31}$$

**6.** Two bolts are drawn from a box containing 4 good and 6 bad bolts. Find the probability that the second bolt is good if the first one is found to be bad.

Sol: Since in the problem it is given that "probability that the second bolt is good, if the first one is bad", we have to understand that the two bolts are drawn in succession (Without replacement).

Probability that the first bolt is found to be bad =  $\frac{6}{10}$

Probability that the bolt drawn now, is found to be good =  $\frac{4}{9}$

P (first is bad and second is good) =  $\frac{6}{10} \times \frac{4}{9} = \frac{4}{15}$

7. In a factory, machine A produce 40% of the output and machine B produces 60%. On the average, 9 items in 1000 produced by A are defective and 1 item in 250 produced by B is defective. An item drawn at random from a day's output is defective. What is the probability that it was produced by A or B?

Sol:

Output produced by A=40%

$$\therefore P(A) = 0.4$$

Output produced by B=60%

$$\therefore P(B) = 0.6$$

$$P(D/A) = \text{Probability that items produced by A are defective} = \frac{9}{1000} = 0.009$$

Similarly

$$P(D/B) = \text{Probability that items produced by B are defective} = \frac{1}{250} = 0.004$$

$P(A/D)$  = Probability that the bolt is produced by A given that it is defective

$$\begin{aligned} &= \frac{P(A) \times P(D/A)}{P(A) \times P(D/A) + P(B) \times P(D/B)} \\ &= \frac{0.4 \times 0.009}{0.4 \times 0.009 + 0.6 \times 0.004} = \frac{0.0036}{0.006} = 0.6 \end{aligned}$$

$P(B/D)$  = Probability that the bolt is produced by B given that it is defective

$$\begin{aligned} &= \frac{P(B) \times P(D/B)}{P(A) \times P(D/A) + P(B) \times P(D/B)} \\ &= \frac{0.6 \times 0.004}{0.4 \times 0.009 + 0.6 \times 0.004} = \frac{0.0024}{0.006} = 0.4 \end{aligned}$$

$$\text{Probability that it was produced by A or B} = P(A/D) + P(B/D) = 0.6 + 0.4 = 1$$

## 4.7 RANDOM VARIABLE

Suppose two fair coins are tossed. Here, the sample space is  $S = \{TT, TH, HT, HH\}$ . Suppose each of the four sample points in this sample space, a number is assigned as follows.

Sample point	TT	TH	HT	HH
Number	0	1	1	2

Here, the assigned numbers indicate the number of heads obtained in each case. Let 'the number of heads' be denoted by  $X$ . Then,  $X$  is a function on the sample space. It takes the values 0, 1 and 2 with probabilities —

$$P[X=0] = P[\text{no head}] = \frac{1}{4}$$

$$P[X=1] = P[\text{one head}] = \frac{1}{2}$$

$$P[X=2] = P[\text{two head}] = \frac{1}{4}$$

Here,  $X$  is called Random variable or Variate.

Random variable is a function which assigns a real number to every sample point in the sample space. The set of such real values is the range of the random variable.

There are two types of random variable, namely, Discrete random variable and Continuous random variable.

A Variable  $X$  which takes values  $x_1, x_2, \dots, x_n$  with probabilities  $p_1, p_2, \dots, p_n$  is a Discrete random variable. Here, the value  $x_1, x_2, \dots, x_n$  from the range of the random variable.

A random variable whose range is uncountable infinite is a Continuous random variable.

Ex1. Let  $X$  denote the number of heads obtained while tossing two fair coins. Then,  $X$  is a random variable which takes the values 0, 1 and 2 with respective probabilities  $\frac{1}{4}$ ,  $\frac{1}{2}$  and  $\frac{1}{4}$ . Here,  $X$  is a discrete random variable.

Ex. 2. Let  $X$  denote the number obtained while throwing a fair die. Then,  $X$  is a discrete random variable taking values 1, 2, 3, 4, 5 and 6 with probability  $\frac{1}{6}$  each.

Ex. 3. Let  $X$  denote the weight of apples. Then,  $X$  is a continuous random variable.

Generally, random variables are denoted by  $X, Y, Z$ , etc. If  $X$  is a random variable, the values taken by  $X$  are denoted by  $x$  (small letter).

### PROBABILITY DISTRIBUTION

A systematic presentation of the values taken by a random variable and the corresponding probabilities is called probability distribution of the random variable.

### PROBABILITY MASS FUNCTION

Let  $X$  be a discrete random variable. And let  $p(x)$  be a function such that  $p(x) = P[X=x]$ . Then,  $p(x)$  is the probability mass function of  $X$  if it satisfies following conditions.

$$(i) p(x) \geq 0 \text{ for all } x$$

$$(ii) \sum p(x) = 1$$

A similar function is defined for a continuous random variable  $X$ . It is called **probability density function** (p.d.f.). It is denoted by  $f(x)$ .

### MATHEMATICAL EXPECTATION:

Let  $X$  be a discrete random variable with probability mass function  $p(x)$ . Then, mathematical expectation of  $X$  is  $E(X) = \sum x \cdot p(x)$

### Mathematical expectation of a function $h(x)$ of $X$ :

Let  $X$  be a discrete random variable with probability mass function  $p(x)$ . Then, mathematical expectation of any function  $h(X)$  of  $X$  is  $E[h(X)] = \sum h(x) \cdot p(x)$

1. Two fair coins are tossed once. Find the mathematical expectation of the number of heads obtained.

Sol: Let X denote the number of heads obtained. Then, X is a random variable which takes the values 0, 1 and 2 with respective probabilities  $\frac{1}{4}$ ,  $\frac{1}{2}$  and  $\frac{1}{4}$  and That is,

x	0	1	2
p(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The mathematical expectation of the number of head is

$$E(X) = \sum x.p(x) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$$

**Note:**

1. For a random variable X, the Arithmetic Mean is E(X).
2. For a random variable X, the Variance is

$$\begin{aligned} \text{Var}(X) &= E[X-E(X)]^2 \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

The Standard Deviation is the square – root of the variance.

2. A bag has 3 white and 4 red balls. Two balls are randomly drawn from the bag. Find the expected number of white balls in the draw.

Sol: Let X denote the number of white balls obtained in the draw. Then, X is a random variable which takes the values 0, 1 and 2 with respective probabilities –

$$P(x=0) = P[\text{both red}] = \frac{{}^4C_2}{{}^7C_2} = \frac{2}{7}$$

$$P(x=1) = P[\text{one white \& one red}] = \frac{{}^3C_2 \times {}^4C_1}{{}^7C_2} = \frac{4}{7}$$

$$P(x=2) = P[\text{both white}] = \frac{{}^3C_2}{{}^7C_2} = \frac{1}{7}$$

The probability distribution of X is –

x	0	1	2
p(x)	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$

$$E(X) = \sum x.p(x) = 0 \times \frac{2}{7} + 1 \times \frac{4}{7} + 2 \times \frac{1}{7} = \frac{6}{7}$$

3. A Random Variable 'X' has the following values of 'X'

x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	K <sup>2</sup>	2K <sup>2</sup>	K+7K <sup>2</sup>

- (i) Find 'K'
- (ii) Find mean
- (iii) Find Variance

Sol:

(i) If X is a random variable, then  $\sum P(x_i) = 1$

$$\Rightarrow P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) + P(x = 5) + P(x = 6) + P(x = 7) = 1$$

$$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow 10k^2 + 9k - 1 = 0$$

$$\Rightarrow k = \frac{1}{10}, -1$$

$$\therefore k = \frac{1}{10} \text{ (Since probability is positive)}$$

(ii)  $E(X) = \sum x \cdot p(x)$

$$= 0 \times 0 + 1 \times k + 2 \times 2k + 3 \times 2k + 4 \times 3k + 5 \times k^2 + 6 \times 2k^2 + 7 \times (7k^2 + k)$$

$$= 30k + 66k^2$$

$$= \frac{30}{10} + \frac{66}{100} = 3.66$$

(iii) Variance =  $E(X^2) - [E(X)]^2$

$$= \sum x^2 \cdot p(x) - (3.66)^2$$

$$= 0 \times 0 + 1 \times k + 4 \times 2k + 9 \times 2k + 16 \times 3k + 25 \times k^2 + 36 \times 2k^2 + 49 \times (7k^2 + k) - (3.66)^2$$

$$= 124k + 440k^2 - (3.66)^2 = 3.404$$

4. Two dice are thrown, let X assign to each of (a, b) in S the maximum of its numbers i.e.,  $X(a, b) = \max(a, b)$ . Find the probability distribution and also find the mean and variance of the distribution.

Sol: If we throw two dice maximum number could be 1, 2, 3, 4, 5, 6,

$$\therefore X = \{1, 2, 3, 4, 5, 6\}$$

If we throw two dice then the sample has 36 events

$$P(x = 1) = \frac{1}{36}, P(x = 2) = \frac{3}{36}, P(x = 3) = \frac{5}{36}, P(x = 4) = \frac{7}{36},$$

$$P(x = 5) = \frac{9}{36}, P(x = 6) = \frac{11}{36}$$

The probability distribution of X is –

x	1	2	3	4	5	6
p(x)	1/36	3/36	5/36	7/36	9/36	11/36

$$\text{Mean} = E(X) = \sum x \cdot p(x) = 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36} = \frac{161}{36} = 4.47$$

$$\text{Variance} = E(X^2) - [E(X)]^2 = \sum x^2 \cdot p(x) - (4.47)^2 = 1.99$$

5. The probability density f(x) of a continuous random variable is given by

$f(x) = ce^{-|x|}, -\infty < x < \infty$ . Show that  $c = \frac{1}{2}$  and also find the mean and variance of the distribution.

Sol: We know that  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} ce^{-|x|} dx = 1$$

$$\Rightarrow 2 \int_0^{\infty} ce^{-|x|} dx = 1 \quad \text{since the function is Even.}$$

$$\Rightarrow 2 \int_0^{\infty} ce^{-x} dx = 1$$

$$\Rightarrow 2c \left[ \frac{e^{-x}}{-1} \right]_0^{\infty} = 1$$

$$\Rightarrow -2c[e^{-\infty} - e^{-0}] = 1$$

$$\Rightarrow -2c[0 - 1] = 1$$

$$\Rightarrow -2c[0 - 1] = 1$$

$$\Rightarrow 2c = 1$$

$$\therefore c = \frac{1}{2}$$

$$\text{Mean} = \int_{-\infty}^{\infty} xf(x) dx$$

$$= \int_{-\infty}^{\infty} x ce^{-|x|} dx$$

$$= c \int_{-\infty}^{\infty} x e^{-|x|} dx = 0 \quad \text{since } x e^{-|x|} \text{ is odd function}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x) dx - 0$$

$$= \int_{-\infty}^{\infty} x^2 ce^{-|x|} dx$$

$$\begin{aligned}
&= 2 \int_0^{\infty} x^2 c e^{-x} dx && \text{Since } x^2 e^{-|x|} \text{ is even function} \\
&= 2 \cdot \frac{1}{2} \int_0^{\infty} x^2 e^{-x} dx \\
&= [-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}]_0^{\infty} \\
&= [0 - (-2)] = 2
\end{aligned}$$

6. A continuous random variable has the probability density function

$$f(x) = \begin{cases} kx e^{-\lambda x}, & \text{for } x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{Determine (i) } k \quad \text{(ii) Mean} \quad \text{(iii) variance}$$

**Sol:** (i) We know that  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\begin{aligned}
\Rightarrow \int_0^{\infty} kx e^{-\lambda x} dx &= 1 \\
\Rightarrow k \left[ x \frac{e^{-\lambda x}}{-\lambda} - 1 \cdot \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty} &= 1 \\
\Rightarrow k \left[ 0 - \left( -\frac{1}{\lambda^2} \right) \right] &= 1 \\
\Rightarrow k &= \lambda^2
\end{aligned}$$

(ii) Mean =  $\int_{-\infty}^{\infty} x f(x) dx$

$$\begin{aligned}
&= \int_0^{\infty} x k x e^{-\lambda x} dx \\
&= \int_0^{\infty} k x^2 e^{-\lambda x} dx \\
&= k \left[ x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x \cdot \frac{e^{-\lambda x}}{\lambda^2} - 2 \frac{e^{-\lambda x}}{\lambda^3} \right]_0^{\infty} \\
&= k \left[ 0 - \left( -\frac{2}{\lambda^3} \right) \right] \\
&= \lambda^2 \cdot \frac{2}{\lambda^3} = \frac{2}{\lambda}
\end{aligned}$$

(iii) Variance =  $E(X^2) - [E(X)]^2$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} x^2 f(x) dx - \left( \frac{2}{\lambda} \right)^2 \\
&= \int_0^{\infty} k x^3 e^{-\lambda x} dx - \frac{4}{\lambda^2} \\
&= k \left[ x^3 \frac{e^{-\lambda x}}{-\lambda} - 3x^2 \cdot \frac{e^{-\lambda x}}{\lambda^2} - 6x \cdot \frac{e^{-\lambda x}}{\lambda^3} - 6 \cdot \frac{e^{-\lambda x}}{\lambda^4} \right]_0^{\infty} - \frac{4}{\lambda^2} \\
&= k \left[ 0 - \left( -\frac{6}{\lambda^4} \right) \right] - \frac{4}{\lambda^2} = \lambda^2 \cdot \frac{6}{\lambda^4} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}
\end{aligned}$$

## 9. Practice Quiz

1. The chance that a leaf year contains 53 Fridays [ B ]
- A)  $\frac{1}{7}$
- B)  $\frac{2}{7}$
- C)  $\frac{1}{365}$
- D)  $\frac{2}{365}$
2. If a card is drawn from a well shuffled packet 53 cards, then the probability that it is spade (or) queen is [ B ]
- A)  $\frac{17}{52}$
- B)  $\frac{4}{13}$
- C)  $\frac{13}{52} + \frac{4}{51}$
- D) None
3. If a coin is tossed twice the probability of getting at least one head is [ C ]
- A)  $\frac{1}{2}$
- B)  $\frac{1}{4}$
- C)  $\frac{3}{4}$
- D) None
4. A bag contains 3 red balls, 4 white balls and 7 black balls. The probability of drawing a red or a black ball is [ B ]
- A)  $\frac{2}{7}$
- B)  $\frac{5}{7}$
- C)  $\frac{3}{7}$
- D)  $\frac{4}{7}$
5.  $P(\bar{A} \cap \bar{B}) = \dots\dots\dots$  Where A, B are independents event [ C ]
- A)  $P(\bar{A}) + P(\bar{B})$
- B)  $P(\bar{A}) - P(\bar{B})$
- C)  $P(\bar{A}) \cdot P(\bar{B})$
- D) None

6. The probability of drawing a king from pack of cards is [ A ]
- A)  $\frac{1}{13}$
- B)  $\frac{1}{26}$
- C)  $\frac{1}{52}$
- D) None
7. A, B, C are three mutually exclusive events then  $P(A \cup B \cup C) =$  [ A ]
- A)  $P(A) + P(B) + P(C)$
- B) 0
- C)  $P(A)P(B)P(C)$
- D) None
8.  $P(A \cap B) = \frac{1}{6}, P(A) = \frac{1}{2}$  then  $P\left(\frac{B}{A}\right) =$  [ A ]
- A)  $\frac{1}{3}$
- B)  $\frac{2}{3}$
- C) 1
- D) None
9. The total no. of sample events when two dice are thrown simultaneously [ C ]
- A) 20
- B) 25
- C) 36
- D) None
10.  $P(A) = P(B)$  and  $P(A) = 2P(C)$  then  $P(A) =$  [ B ]
- A)  $\frac{1}{5}$
- B)  $\frac{2}{5}$
- C)  $\frac{3}{5}$
- D) 1
11. Tossing of a coin is an example for ..... variable [ A ]
- A) Discrete
- B) Continuous
- C) both A & B

D) None

12. If  $K$  is constant then  $var(k) = \dots\dots\dots$  [ A ]

A) 0

B) 1

C)  $k$

D) None

13.  $V(x) = 2$  then  $V(2x + 3) =$  [ B ]

A) 4

B) 8

C) 16

D) 12

14. Which of the following is an example for a continuous random variable [ C ]

A) The no. of heads when die rolled

B) No. of road accidents at a specific place in given period of time

C) Life of an electric bulb

D) None

15. The maximum value of probability is [ B ]

A) 0

B) 1

C) -1 to 1

D) None

## 10. Assignments

S.No	Question	BL	CO
1	A box has 5 white, 4 red and 3 green balls. Two balls are drawn at random from the box. Find the probability that they are (i) of the same colour (ii) of different colours	L1	CO4
2	State and prove Bayes theorem.	L1	CO4

3	In a toy manufacturing company three machines namely A, B, C manufacture 20%, 30%, 50% of toys respectively. A quality control officer examined the machines and found that A, B, C produces 2%, 3%, 4% defectives of the total output. A toy is selected at a random and is found to be defective, what are the probabilities that this toy came from machines A, B, C, respectively.	L3	CO4																		
4	A Random Variable 'X' has the following values of 'X' <table border="1" style="margin: 10px auto;"> <tr> <td>x</td> <td>0</td> <td>1</td> <td>2</td> <td>3</td> <td>4</td> <td>5</td> <td>6</td> <td>7</td> </tr> <tr> <td>P(x)</td> <td>0</td> <td>k</td> <td>2k</td> <td>2k</td> <td>3k</td> <td>K<sup>2</sup></td> <td>2K<sup>2</sup></td> <td>K+7K<sup>2</sup></td> </tr> </table> (i) Find 'K' (ii) Find mean (iii) Find Variance	x	0	1	2	3	4	5	6	7	P(x)	0	k	2k	2k	3k	K <sup>2</sup>	2K <sup>2</sup>	K+7K <sup>2</sup>	L3	CO4
x	0	1	2	3	4	5	6	7													
P(x)	0	k	2k	2k	3k	K <sup>2</sup>	2K <sup>2</sup>	K+7K <sup>2</sup>													
5	The probability density f(x) of a continuous random variable is given by $f(x) = ce^{- x }, -\infty < x < \infty$ . Show that $c = \frac{1}{2}$ and also find the mean and variance of the distribution.	L3	CO4																		

### 11. Part A- Question & Answers

S.No	Question& Answers	BL	CO
1	What is meant by mutually exclusive event? Sol: Two or more events are mutually exclusive if only one of them can occur at a time. That is, the occurrence of any of these events totally excludes the occurrence of the other events. Mutually exclusive events cannot occur together.	L1	4
2	If A, B and C are any three events then $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$ Sol: Let D = B ∪ C	L1	4

	$P(A \cup B \cup C) = P(A \cup D)$ $= P(A) + P(D) - P(A \cap D)$ $= P(A) + P(B \cup C) - P[A \cap (B \cup C)]$ $= P(A) + P(B) + P(C) - P(B \cap C) - P[(A \cap B) \cup (A \cap C)]$ $= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P[(A \cap B) \cap (A \cap C)]$ $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$		
3	<p>State Baye's theorem.</p> <p>Sol: If <math>E_1, E_2, E_3, \dots, E_n</math> are 'n' mutually exclusive events with <math>P(E_i) \neq 0</math> for each i in the sample space S and for any arbitrary event A which is a subset of <math>U_{i=1}^n E_i</math> with <math>P(A) &gt; 0</math> then</p> $P\left(\frac{E_i}{A}\right) = \frac{P(E_i)P\left(\frac{A}{E_i}\right)}{\sum_{i=1}^n P(E_i)P\left(\frac{A}{E_i}\right)} \quad \text{for } i=1 \text{ to } n.$	L1	4
4	<p>If <math>P(A) = 0.37, P(B) = 0.42, P(A \cap B) = 0.09</math> then find <math>P(A \cup B)</math></p> <p>Sol: <math>P(A) = 0.37, P(B) = 0.42, P(A \cap B) = 0.09</math></p> $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = 0.37 + 0.42 - 0.09 = 0.7$	L1	4
5	<p>Define the Axioms of probability.</p> <p>Sol: Axiom 1: <math>P(A) \geq 0</math></p> <p>Axiom 2: <math>P(S) = 1, S</math> being the sure event.</p> <p>Axiom 3: For two mutually exclusive events A and B,</p> $P(A \cup B) = P(A) + P(B)$	L1	4
6	<p>Define the conditional probability.</p> <p>Sol: conditional probability of event B given A is defined as-----</p> $P(B/A) = \frac{P(A \cap B)}{P(A)}$	L1	4
7	<p>What is the probability of drawing either a king or a queen in a single draw from a well shuffled pack of 52 cards?</p> <p>Sol: Total number of cards = 52</p> <p>Number of king cards = 4</p> <p>Probability of drawing a king card = <math>4/52</math></p> <p>Number of queen cards = 4</p> <p>Probability of drawing a queen card = <math>4/52</math></p> <p>Both the events of drawing a king and a queen are mutually exclusive</p> $\Rightarrow P(A \cup B) = P(A) + P(B)$	L1	4

	Therefore, probability of drawing either a king or a queen = $\frac{4}{52} + \frac{4}{52} = \frac{2}{13}$										
8	Define probability distribution. Sol: A systematic presentation of the values taken by a random variable and the corresponding probabilities is called probability distribution of the random variable.	L1	4								
9	Define probability mass function. Sol: Let X be a discrete random variable. And let p(x) be a function such that $p(x) = P[X=x]$ . Then, p(x) is the probability mass function of X if it satisfies following conditions.  (i) $p(x) \geq 0$ for all x (ii) $\sum p(x) = 1$	L1	4								
10	Two fair coins are tossed once. Find the mathematical expectation of the number of heads obtained. Sol: Let X denote the number of heads obtained. Then, X is a random variable which takes the values 0, 1 and 2 with respective probabilities $\frac{1}{4}$ , $\frac{1}{2}$ and $\frac{1}{4}$ and That is, <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td>x</td> <td>0</td> <td>1</td> <td>2</td> </tr> <tr> <td>p(x)</td> <td><math>\frac{1}{4}</math></td> <td><math>\frac{1}{2}</math></td> <td><math>\frac{1}{4}</math></td> </tr> </table> The mathematical expectation of the number of head is $E(X) = \sum x.p(x) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$	x	0	1	2	p(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	L1	4
x	0	1	2								
p(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$								

## 12. Part B- Questions

S.No	Question	BL	CO
1	Box A contains 5 red and 3 white marbles and box B contains 2 red and 6 white marbles. If a marble is drawn from each box, what is the probability that they are both of same colour.	L2	4

2	<p>Of three men, the chances that a politician, business man or an academician will be appointed as vice chancellor of a university are 0.5,0.3,0.2 respectively. The probability that the research is promoted by these persons if they are appointed as V.C are 0.3,0.7 and 0.8.</p> <p>a) Find the probability that the research is promoted?  b) if research is promoted then find the probability that V.C is an academician?</p>	L3	<b>4</b>
3	A businessman goes to the hotels X, Y, Z are 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbing. What is the probability that business man's room having faulty plumbing is assigned to hotel Z?	L3	<b>4</b>
4	State and prove Addition theorem and Bayes theorem	L1	<b>4</b>
5	A can hit a target 3 times in 5 shots, B hits target 2 time in 5 shots, C hits target 3 times in 4 shots. Find the probability of the target being hit when all of them try.	L3	<b>4</b>
6	Two dice are thrown, let X assign to each of (a, b) in S the maximum of its numbers i.e., $X(a, b) = \max(a, b)$ . Find the probability distribution and also find the mean and variance of the distribution.	L3	<b>4</b>
7	<p>A continuous random variable has the probability density function <math>f(x) = \begin{cases} kxe^{-\lambda x}, &amp; \text{for } x \geq 0, \lambda &gt; 0 \\ 0, &amp; \text{otherwise} \end{cases}</math></p> <p>Determine (i) k (ii) Mean (iii) variance</p>	L3	<b>4</b>

### 13. Supportive Online Certification Courses

1. Introduction to Probability and Statistics by Prof. G. Srinivasan  
IIT Madras – 4 weeks

### 14. Real Time Applications

S. No	Application	CO
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## 15. Contents Beyond the Syllabus

1. Probability theory with applications to engineering problems such as the reliability of circuits and systems.

## 16. Prescribed Text Books & Reference Books

### Text Book

1. B. S. Grewal, Higher Engineering Mathematics, Khanna Publishers.
2. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons.

### References:

1. B. V. Ramana, Higher Engineering Mathematics, McGraw Hill Education
2. Alan Jeffrey, "Advanced Engineering Mathematics", Elsevier Publishers

## COURSE MATERIAL

<b>SUBJECT</b>	<b>NUMERICAL METHODS AND PROBABILITY THEORY (20A54402)</b>
<b>UNIT</b>	<b>V</b>
<b>COURSE</b>	<b>II B. TECH</b>
<b>SEMESTER</b>	<b>2 - 2</b>
<b>DEPARTMENT</b>	<b>HUMANITIES &amp; SCIENCE</b>
<b>PREPARED BY (Faculty Name/s)</b>	<b>Department of Mathematics</b>

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## 1. COURSE OBJECTIVES

This course aims at providing the student with the knowledge on

- Various numerical methods for solving equations, interpolating the polynomials, evaluation of integral equations and solution of differential equations.
- The theory of Probability and random variables.

## 2. PREREQUISITES

Students should have knowledge on

- Basic Mathematics & Calculus
- Permutations and combinations

## 3. SYLLABUS

**UNIT V:** Random Variables & Distributions:

Probability distribution - Binomial, Poisson approximation to the binomial distribution and normal distribution-their properties-Uniform distribution-exponential distribution

## 4. COURSE OUTCOMES

After the completion of course, students will be able to

1. Apply numerical methods to solve algebraic and transcendental equations
2. Derive interpolating polynomials using interpolation formulae
3. Solve differential and integral equations numerically
4. Apply Probability theory to find the chances of happening of events.
5. Understand various probability distributions and calculate their statistical constants.

## 5. Co-PO / PSO Mapping

NMPT	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	P10	PO11	PO12	PSO1	PSO2
CO1	3	3												
CO2	3	3												
CO3	3	3												
CO4	3	3												
CO5	3	3												

## 6. LESSON PLAN

Lecture No.	Weeks	Topics to be covered	References
1	1	Introduction of Binomial Distribution	T1, R1
2		Problems on binomial Distribution	T1, R1
3		Problems on binomial Distribution	T1, R1
4		Introduction of Poisson Distribution	T1, R1
5	2	Problems on Poisson distribution	T1, R1
6		Problems on Poisson distribution	T1, R1
7		Normal distribution and properties	T1, R1
8		Problems on Normal distribution	T1, R1
9	3	Problems on Normal distribution	T1, R1
10		Problems on Normal distribution	T1, R1

## 7. ACTIVITY BASED LEARNING

1. Probability game on Cards
2. Probability game on dice

## 8. LECTURE NOTES

### 5.1 INTRODUCTION

In day-to-day life, we come across many random variables such as -----

1. Number of male children in a family having three children.
2. Number of passengers getting into a bus at the bus stand.
3. I.Q. of children
4. Number of stones thrown successively at a mango on the tree until the mango is hit
5. Marks scored by a candidate in the P.U.E. examination.

For a quick analysis of distributions of such random variables, we consider their theoretical equivalents. These equivalent distributions are originated according to certain theoretical assumptions and restrictions. Such theoretically designed distributions are called theoretical distributions.

There are many types (families) of theoretical distributions. Some of them

- (i) Binomial distribution
- (ii) Poisson distribution
- (iii) Normal distribution
- (iv) Uniform distribution
- (v) Exponential distribution.

#### **BERNOULLI EXPERIMENT:**

Suppose a random experiment has two outcomes, namely, "Success" and "Failure". Let probability of Success be  $p$  and let probability of Failure be  $q=(1-p)$ . Such an experiment is called Bernoulli experiment or Bernoulli trial.

Let a Bernoulli experiment be conducted (repeated)  $n$  times. Let the variable  $X_i$  ( $i=1,2,3,\dots,n$ ) take values 1 and 0 according as the  $i^{\text{th}}$  experiment is a Success or a Failure. Then,  $X_i$  is a Bernoulli variate with parameter  $p$ . It denotes the number of success in the  $i^{\text{th}}$  experiment.

Let  $X = X_1 + X_2 + \dots + X_n$ . Then,  $X$  denotes the number of success in these  $n$  repetitions.

Example:

1. Let coin be tossed 3 times. Let  $X_i$  ( $i=1,2,3$ ) be a variate which takes values 1 and 0 according as the  $i^{\text{th}}$  toss result in "Head" or "Tail". Then,  $X = X_1 + X_2 + X_3$  denotes "the number of head" obtained in the 3 tosses.

Result:

If  $X_1, X_2, \dots, X_n$  are independently and identically distributed (i.i.d) Bernoulli variates with common parameter  $p$ , their sum  $X = X_1 + X_2 + \dots + X_n$  is a Binomial variate with parameters  $n$  and  $p$

## 5.2 BINOMIAL DISTRIBUTION

A Probability distribution which has the following probability mass function (p.m.f) is called Binomial distribution.

$$p(x) = {}^n C_x p^x q^{n-x}, x=0, 1, 2, \dots, n.$$

$$0 < p < 1; q = 1 - p$$

Here  $n, p$  are parameters the variable  $X$  is discrete and it is called Binomial variate.

Note1: Binomial p.m.f. has two independent constants, namely,  $n$  and  $p$ . These two constants are the parameters of binomial distribution

Note2: A binomial distribution with parameter  $n$  and  $p$  is denoted by  $b(x;n,p)$  or  $B(n,p)$ .

### EXAMPLES FOR BINOMIAL VARIATE

1. Number of heads obtained in 3 tosses of a coin.
2. Number of male children in a family of 5 children
3. Number of bombs hitting a bridge among 8 bombs which are dropped on it.
4. Number of defective articles in a random sample of 5 articles drawn from a manufactured lot
5. Number of seeds germinating among 10 seeds which were sown

## MEAN & VARIANCE OF BINOMIAL DISTRIBUTION

Mean is  $E(X) = np$

Variance is  $\text{Var}(X) = np(1-p) = npq$

Standard deviation is  $S.D.(X) = \sqrt{npq}$

## RECURRENCE RELATION BETWEEN SUCCESSIVE TERMS

We have  $p(x) = {}^n C_x p^x q^{n-x}$

and  $p(x-1) = {}^n C_{x-1} p^{x-1} q^{n-x+1}$

Therefore,

$$\frac{p(x)}{p(x-1)} = \frac{{}^n C_x p^x q^{n-x}}{{}^n C_{x-1} p^{x-1} q^{n-x+1}} = \frac{(n-x+1)}{x}$$
$$p(x) = \frac{(n+1-x)}{x} \cdot \frac{p}{q} \cdot p(x-1)$$

Thus, this is the recurrence relation between success probabilities of binomial distribution when  $p(x-1)$  is known, this relation can be used to obtain  $p(x)$ .

1. The incidence of an occupational disease in an industry is such that the worker has 25% chance of suffering from it. What is the probability that out of 5 workers, at the most two contract that disease?

Solutions:

X: number of workers contracting the diseases among 5 workers

Then, X is a binomial variate with parameter  $n=5$  and

$P = P[\text{a worker contracts the disease}] = 25/100 = 0.25$

The probability mass function (p.m.f) is –

$P(x) = {}^5 C_x (0.25)^x (0.75)^{5-x}, x=0, 1, 2, \dots, 5$

The probability that at the most two workers contact the disease is ---

$$\begin{aligned} P[X \leq 2] &= p(x=0) + p(x=1) + p(x=2) \\ &= {}^5 C_0 (0.25)^0 (0.75)^5 + {}^5 C_1 (0.25)^1 (0.75)^4 + {}^5 C_2 (0.25)^2 (0.75)^3 \\ &= 0.2373 + 0.3955 + 0.2637 \\ &= 0.8965 \end{aligned}$$

2. In a large consignment of electric lamps, 5% are defective. A random sample of 8 lamps is taken for inspection. What is the probability that it has one or more defectives?

Solution:

X: number of defective lamps

Then, X is B ( $n=8$ ,  $p=5/100 = 0.05$ )

The p.m.f. is –

$$p(x) = {}^8C_x (0.05)^x (0.95)^{8-x}, x = 0, 1, 2, \dots, 8$$

$$\begin{aligned} P[\text{sample has one or more defectives}] &= 1 - P[\text{no defectives}] \\ &= 1 - p(x=0) \\ &= 1 - {}^8C_0 (0.05)^0 (0.95)^8 \\ &= 1 - 0.6634 = 0.3366 \end{aligned}$$

3. In a Binomial distribution the mean is 6 and the variance is 1.5. Then, find (i)  $P[X=2]$  and (ii)  $P[X \leq 2]$ .

Solution:

Let n and p be the parameters. Then,

$$\text{Mean} = np = 6$$

$$\text{Variance} = npq = 1.5$$

$$\frac{\text{Variance}}{\text{Mean}} = \frac{npq}{np} = \frac{1.5}{6} = \frac{1}{4}$$

Therefore,  $q = 1/4$  and  $p = 3/4$

$$\text{Therefore, Mean} = n \cdot 3/4 = 6$$

$$\text{That is, } n = 24/3 = 8$$

The p.m.f. is -----

$$p(x) = {}^8C_x (3/4)^x (1/4)^{8-x}, x=0, 1, 2, \dots, 8$$

$$(i) \quad P[X=2] = {}^8C_2 (3/4)^2 (1/4)^6 = 252/65536 = 0.003845$$

$$\begin{aligned} (ii) \quad P[X \leq 2] &= p(x=0) + p(x=1) + p(x=2) \\ &= {}^8C_0 (3/4)^0 (1/4)^8 + {}^8C_1 (3/4)^1 (1/4)^7 + {}^8C_2 (3/4)^2 (1/4)^6 \\ &= 277/65536 = 0.004227 \end{aligned}$$

### 5.3 POISSON DISTRIBUTION

A probability distribution which has the following probability mass function (p.m.f) is called Poisson distribution

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots, \dots, \dots$$

Here, the variable X is discrete and it is called Poisson variate.

Note 1:  $\lambda$  is the parameter of Poisson, Poisson distribution has only one parameter

Note 2: Poisson distribution may be treated as limiting form of binomial distribution under the following conditions. (Binomial distribution tends to Poisson distribution under the following conditions.)

- (i) p is very small ( $p \rightarrow 0$ )
- (ii) n is very large ( $n \rightarrow \infty$ ) and
- (iii)  $np = \lambda$  is fixed

#### EXAMPLES FOR POISSON VARIATE:

Many variables which occur in nature vary according to the Poisson law. Some of them are

1. Number of deaths occurring in a city in a day
2. Number of road accidents occurring in a city in a day
3. Number of incoming telephone calls at an exchange in one minute
4. Number of vehicles crossing a junction in one minute.

1. The number of accidents occurring in a city in a day is a Poisson variate with mean 0.8. Find the probability that on a randomly selected day
  - (i) There are no accidents
  - (ii) There are accidents

#### Solution:

Let X: number of accidents per day.

Then, X is  $P(\lambda=0.8)$ .

The p.m.f. is –

$$p(x) = \frac{e^{-0.8}(0.8)^x}{x!}, x = 0,1,2,3,\dots$$

(i) Probability that a particular day there are no accidents is -----

$$P[\text{no accidents}] = P[X=0]$$

$$= \frac{e^{-0.8}(0.8)^0}{0!} = e^{-0.8} = 0.449$$

(ii)  $P[\text{accidents occur}] = 1 - P[\text{no accidents}]$

$$= 1 - p(0) = 1 - 0.449 = 0.551$$

2. The number of persons joining a cinema queue in a minute has Poisson distribution with parameter 5.8. Find the probability that (i) no one joins the queue in a particular minute (ii) 2 or more persons join the queue in the minute.

**Solution:**

Let X: number of persons joining the queue in a minute Then,  $\lambda = 5.8$ .

The p.m.f is -----

$$p(x) = \frac{e^{-5.8}(5.8)^x}{x!}, x = 0,1,2,3,\dots$$

(i)  $P[\text{no one joins the queue}] = P[X=0] = p(0)$

$$= \frac{e^{-5.8}(5.8)^0}{0!}$$

$$= e^{-5.8} = 0.003$$

(ii)  $P[\text{two or more join}] = P[X \geq 2]$

$$= 1 - P[X < 2]$$

$$= 1 - \{p(0) + p(1)\}$$

$$= 1 - \left\{ \frac{e^{-5.8}(5.8)^0}{0!} + \frac{e^{-5.8}(5.8)^1}{1!} \right\}$$

$$= 1 - e^{-5.8}\{1 + 5.8\}$$

$$= 1 - 0.003 \times 6.8$$

$$= 1 - 0.0204 = 0.9796$$

3. The average number of telephone calls booked at an exchange between 10-00 A.M. and 10-10 A.M. is Find the probability that on a randomly selected day 2 or more calls are booked between 10-00 A.M. and 10-10 A.M. On how many days of a year, would you expect booking of 2 or more calls during those times gap.

Solution:

Let X: number of telephone calls booked at the exchange during 10-00 A.M. to 10-10 A.M. Then, X is  $P(\lambda=4)$ .

The p.m.f is ---

$$p(x) = \frac{e^{-4}4^x}{x!}, x = 0,1,2,3,\dots$$

$$\begin{aligned} P [2 \text{ or more calls}] &= 1 - P [\text{less than 2 calls}] \\ &= 1 - [p(0) + p(1)] \\ &= 1 - e^{-4}[(4^0)/0! + (4^1)/1!] \\ &= 1 - 0.0183[1+4] \\ &= 1 - 0.0915 = 0.9085 \end{aligned}$$

A year has 365 days. Out of these  $N = 365$  days, the number of days on which there will be 2 or more calls is ---

$$N * P [2 \text{ or more calls}] = 365 * 0.9085 = 332$$

4. 2 percent of the fuses manufactured by a firm are expected to be defective, Find the probability that a box containing 200 fuses contains
- defective fuses
  - 3 or more defective fuse

Solution:

2 percent of the fuses are defective. Therefore, probability that a fuse is defective is  $p = 2/100 = 0.02$

Let X denote the number of defective fuses in the box of 200 fuses.

Here, p is very small and n is very large. Therefore, X can be treated as Poisson variate with parameter  $\lambda=np = 200 * 0.02 = 4$ .

The p.m.f. is ----

$$p(x) = \frac{e^{-4}4^x}{x!}, x = 0,1,2,3,\dots$$

$$\begin{aligned} P [\text{box has defective fuses}] &= 1 - P [\text{no defective fuses}] \\ &= 1 - p(0) \\ &= 1 - e^{-4}[(4^0)/0!] \\ &= 1 - 0.0183 = 0.9817 \end{aligned}$$

$$\begin{aligned} P [3 \text{ or more defective fuses}] &= 1 - P [\text{less than 3 defective fuses}] \\ &= 1 - [p(0) + p(1) + p(2)] \\ &= 1 - e^{-4}[1 + 4 + 8] \\ &= 1 - 0.0183 \times 13 \\ &= 1 - 0.2379 = 0.7621 \end{aligned}$$

5. The probability that a razor blade manufactured by a firm is defective is 1/500. Blades are supplied in packets of 5 each. In a lot of 10,000 packets, how many packets would
- (i) Be free defective blades?
  - (ii) Contains exactly one defective blade? ( $e^{-0.01} = 0.99$ )

Solution:

Let X be the number of defective blades in a packet of 5 blades.  $p = 1/500$

Since p is very small and n is sufficiently large, X is treated as Poisson variate with parameter  $\lambda = np = 5 \times (1/500) = 0.01$

$$p(x) = \frac{e^{-0.01}(0.01)^x}{x!}, x = 0,1,2,3,\dots$$

$$\begin{aligned} \text{(i) } P [\text{no defective blades}] &= p(0) \\ &= e^{-0.01}[(0.01)^0/0!] = 0.99 \end{aligned}$$

The number of packets which will be free of defective blades is ----

$$N * P [\text{no defective blades}] = 10000 * 0.99 = 9900$$

$$(ii) P [\text{one defective blade}] = p (1) \\ = e^{-0.01}[(0.01)^1/1!] = 0.0099$$

The number pf packets which will have one defectives blade is ----

$$N * P [\text{one defective blade}] = 10000*0.0099 = 99$$

6. On an average, a typist mistakes while typing one page. What is the probability that a randomly observed page in free of mistakes? Among 200 pages, in how many pages would you expect mistakes?

Solution:

Let X: number of mistakes in a page.

Then, X is  $P(\lambda=3)$ .

The p.m.f. is ---

$$p(x) = \frac{e^{-3}3^x}{x!}, x = 0,1,2,3,\dots$$

$$P [\text{page is free of mistakes}] = p(0)$$

$$= \frac{e^{-3}3^0}{0!} = e^{-3} = 0.0498$$

$$P [\text{page has mistakes}] = 1-P [\text{Page has no mistakes}]$$

$$= 1-0.0498=0.9502$$

Among 200 pages, the expected number of pages containing mistakes is ---

$$N*P [\text{page has mistakes}] = 200*0.9502=190$$

7. In a Poisson distribution  $P[X=2] = P[X=3]$ . Find  $P[X=4]$ .

Solution:

Let  $\lambda$  be the parameter

Here,  $P [X = 2] = P[X=3]$

$$\Rightarrow 3 \times \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}\lambda^3}{3!}$$

$$\Rightarrow 1 = \frac{\lambda}{3}$$

And so,  $\lambda=3$

The p.m.f. is ----

$$p(x) = \frac{e^{-3}3^x}{x!}, x = 0,1,2,\dots$$

$$\begin{aligned} P[X=4] &= p(4) = \frac{e^{-3}3^4}{4!} \\ &= \frac{0.0498 \times 81}{24} = 0.1681 \end{aligned}$$

8. For a Poisson variable  $3 * P[X=2] = P[X=4]$ . Find standard deviation.

Solution:

Here,  $3 * P[X=2] = P[X=4]$

$$\Rightarrow 3 \times \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}\lambda^4}{4!}$$

$$\Rightarrow 3 = \frac{\lambda^2}{3 \times 4}$$

And so,  $\lambda^2=36$

That is,  $\lambda=6$

Thus, the parameter is  $\lambda=6$

The standard deviation S.D.(X) =  $\sqrt{\lambda} = \sqrt{6} = 2.449$

## 5.4 NORMAL DISTRIBUTION

A probability distribution which has the following probability density function (p.d.f) is called Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty, \quad \sigma > 0$$

Here, the variable X is continuous and it is called Normal variate.

Note 1: The distribution has two parameters, namely,  $\mu$  and  $\sigma$ .

(Here,  $\pi=3.14$  and  $e=2.718$ )

Note 2: This normal distribution has Mean  $E(X) = \mu$  and

Variance =  $V(X) = \sigma^2$ . S.D.(X)= $\sigma$ .

Note 3: A normal variate with parameters  $\mu$  and  $\sigma$  is denoted by  $N(\mu, \sigma^2)$

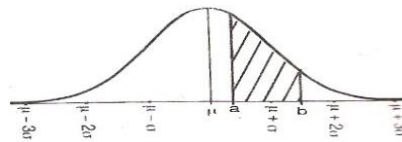
## EXAMPLES FOR NORMAL VARIATE

Many of the variables which occur in nature have normal distribution. Some examples are –

1. Height of students of a college
2. Weight of apples grown in an orchard
3. I.Q(Intelligence Quotient) of a large group of children.
4. Marks scored by students in an examination

## PROPERTIES OF NORMAL DISTRIBUTION (Properties of normal Curve):

A normal distribution with parameters  $\mu$  and  $\sigma$  has the following properties.



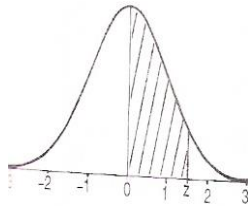
1. The curve is Bell –shaped
  - a. It is symmetrical (Non-skew).  
That is  $\beta_1 = 0$
  - b. The mean, media and mode are equal
2. The curve is asymptotic to the X-axis. That is, the curve touches the X-axis only at  $-\infty$  and  $+\infty$ .
3. The curve has points of inflexion at  $\mu - \sigma$  and  $\mu + \sigma$ .
4. For the distribution ....
  - a. Standard deviation =  $\sigma$
  - b. Quartile deviation =  $2/3 \sigma$  (approximately)
  - c. Mean deviation =  $4/5 \sigma$  (approximately)
5. The distribution is mesokurtic. That is,  $\beta_2 = 3$ .
6. Total area under the curve is unity.  
 $P [a < X \leq b] =$  Area bounded by the curve and the ordinates at a and b
  - a.  $P [\mu - \sigma < X \leq \mu + \sigma] = 0.6826 = 68.26\%$
  - b.  $P [\mu - 2\sigma < X \leq \mu + 2\sigma] = 0.9544 = 95.44\%$
  - c.  $P [\mu - 3\sigma < X \leq \mu + 3\sigma] = 0.9974 = 99.74\%$

### STANDARD NORMAL VARIATE:

A normal variate with mean  $\mu=0$  and standard deviation  $\sigma =1$  is called Standard Normal Variate. It is denoted by  $Z$ . Its probability density function is –

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$

The graph of standard normal distribution is shown in the figure



The shaded area in the figure represents the probability that the variate takes a value between 0 and  $z$ . This area can be read from the table of areas under Standard Normal Curve. Corresponding to any positive  $z$ , the area from 0 to  $z$  can be read from this table.

Let  $X$  be a normal variate with mean  $\mu$  and standard deviation  $\sigma$ . Then  $Z = \frac{X-\mu}{\sigma}$  is a Standard Normal Variate

Therefore, to find any probability regarding  $X$ , the Standard Normal Variate can be made use of.

Note: The Standard Normal Variate is denoted by  $N(0,1)$ .

1.  $X$  is a normal variate with mean 42 and standard deviation 4. Find the probability that a value taken by  $X$  is

- |                       |                         |
|-----------------------|-------------------------|
| (i) less than 50      | (ii) greater than 50    |
| (iii) less than 40    | (iv) greater than 40    |
| (v) between 40 and 44 | (vi) between 37 and 41. |

Solution:

$X$  is a normal variate with parameters  $\mu=42$  and  $\sigma=4$

Therefore,

$z = \frac{x-\mu}{\sigma} = \frac{x-42}{4}$  is a Standard Normal Variate.

- (i)  $p(x < 50) = p\left(\frac{x-42}{4} < \frac{50-42}{4}\right)$   
= P[Z < 2]  
= area from  $(-\infty)$  to 2  
= [area from  $(-\infty)$  to 0] + [area from 0 to 2]  
= 0.5 + 0.4772(from the table)  
= 0.977200
- (ii)  $p(x > 50) = p\left(\frac{x-42}{4} > \frac{50-42}{4}\right)$   
= P[Z < 2]  
= area from 2 to  $\infty$   
= [area from 0 to  $\infty$ ] - [area from 0 to 2]  
= 0.5 - 0.4772(from the table)  
= 0.0228
- (iii)  $p(x < 40) = p\left(\frac{x-42}{4} < \frac{40-42}{4}\right)$   
= P [Z < -0.5]  
= area from  $(-\infty)$  to (-0.5)  
= area from 0.5 to  $\infty$   
= [area from 0 to  $\infty$ ] - [area from 0 to 0.5]  
= 0.5 - 0.1915  
= 0.3085
- (iv)  $p(x > 40) = p\left(\frac{x-42}{4} > \frac{40-42}{4}\right)$   
= P [Z > -0.5]  
= area from (-0.5) to  $\infty$   
= area from (-0.5) to 0] + [area from 0 to  $\infty$ ]  
= [area from 0 to 0.5] + [area from 0 to  $\infty$ ]  
= 0.1915 + 0.5  
= 0.6915

$$\begin{aligned}
\text{(v)} \quad p(40 < x < 44) &= p\left(\frac{40-42}{4} < \frac{x-42}{4} < \frac{44-42}{4}\right) \\
&= P[-0.5 < Z < 0.5] \\
&= \text{area from } -0.5 \text{ to } 0.5 \\
&= \text{area from } -0.5 \text{ to } 0] + [\text{area from } 0 \text{ to } 0.5] \\
&= \text{area from } 0 \text{ to } 0.5] + [\text{area from } 0 \text{ to } 0.5] \\
&= 0.1915 + 0.1915 \\
&= 0.3830
\end{aligned}$$

$$\begin{aligned}
\text{(vi)} \quad p(37 < x < 41) &= p\left(\frac{37-42}{4} < \frac{x-42}{4} < \frac{41-42}{4}\right) \\
&= P[-1.25 < Z < -0.25] \\
&= \text{area from } -1.25 \text{ to } 0.25 \\
&= \text{area from } 0.25 \text{ to } 1.25 \\
&= \text{area from } 0 \text{ to } 1.25] - [\text{area from } 0 \text{ to } 0.25] \\
&= 0.3944 - 0.0987 \\
&= 0.2957.
\end{aligned}$$

2. Height of students is normally distributed with mean 165 cms and standard deviation 5 cms. Find the probability that height of a students is

- (i) more than 177 cms.
- (ii) less than 162 cms.

Solution:

Let X denote height. Then, X is a normal variate with parameters  $\mu = 165$  cms and  $\sigma = 5$

$$\text{Then } z = \frac{x - \mu}{\sigma} = \frac{x - 165}{5}$$

(i) Probability that the student is more than 177 cms tall is –

$$\begin{aligned}
p(x > 177) &= p\left(\frac{x-165}{5} > \frac{177-165}{5}\right) \\
&= P[Z > 2.4]
\end{aligned}$$

$$\begin{aligned}
&= \text{area from } 2.4 \text{ to } \infty \\
&= [\text{area from } 0 \text{ to } \infty] - [\text{area from } 0 \text{ to } 2.4] \\
&= 0.5 - 0.4918 \\
&= 0.0082
\end{aligned}$$

(ii) Probability that the student is less than 162 cms. Tall is ----

$$\begin{aligned}
p(x < 162) &= p\left(\frac{x-165}{5} < \frac{162-165}{5}\right) \\
&= P [Z < -0.6] \\
&= \text{area from } (-\infty) \text{ to } (-0.6) \\
&= \text{area from } 0.6 \text{ to } \infty \\
&= [\text{area from } 0 \text{ to } \infty] - [\text{area from } 0 \text{ to } 0.6] \\
&= 0.5 - 0.2258 \\
&= 0.2742
\end{aligned}$$

3. Mean life of electric bulbs manufactured by a firm is 1200 hrs. The standard deviation is 200 hrs.

(i) In a lot of 10,000 bulbs, how many bulbs are expected have life 1050 hrs. or more?

(ii) What is the percentage of bulbs which are expected to find before 1500 hrs. of service?

Solution:

Let X denotes the life of the bulbs. Then, X is a normal variate with parameters

$\mu=1200\text{hrs}$   $\sigma=200$  hrs

$$\text{Then } z = \frac{x-\mu}{\sigma} = \frac{x-1200}{200}$$

(i) Probability that life of a bulb is 1050 hrs. or more is ---

$$\begin{aligned}
p(x \geq 1050) &= p\left(\frac{x-1200}{200} \geq \frac{1050-1200}{200}\right) \\
&= P [Z \leq -0.75] \\
&= 0.2734 + 0.5 \\
&= 0.7734
\end{aligned}$$

In a lot of  $n=10,000$  bulbs, expected number of bulbs with life 1050 hrs. or more is ---

$$N * P[X \geq 1050] = 10000 * 0.7734 = 7734$$

(ii) Probability that life of a bulb is before 1500 hrs is ---

$$\begin{aligned} p(x \leq 1050) &= p\left(\frac{x - 1200}{200} \leq \frac{1500 - 1200}{200}\right) \\ &= P [ Z < 1.5] \\ &= 0.5 + 0.4332 \\ &= 0.9332 \end{aligned}$$

The percentage of bulbs with life less than 1500 hrs is ---

$$100 * P[X < 1500] = 100 * 0.9332 = 93.32$$

4. The mean and standard deviation of marks scored by a group of students in an examination are 47 and 10 respectively. If only 20% of the students have to be promoted, which should be the marks limits for promotion?

Solution:

Let X denotes marks. Then, X is a normal variate with parameters  $\mu=47$  and  $\sigma=10$ .

$$\text{Then } z = \frac{x - \mu}{\sigma} = \frac{x - 47}{10}$$

Let a be the marks above which if a student scores he would be promoted. Then, since only 20% of the students have to be promoted the probability of a student getting promotion should be  $20/100=0.2$

Therefore,

$$\begin{aligned} p(x \geq a) &= 0.2 \\ p\left(\frac{x - 47}{10} \geq \frac{a - 47}{10}\right) &= 0.2 \\ p\left(z \geq \frac{a - 47}{10}\right) &= 0.2 \end{aligned}$$

And so,  $P[Z \geq z] = 0.2$  where  $z = \frac{a - 47}{10}$

That is, [area from z to  $\infty$ ] = 0.2

That is, [area from 0 to z] = 0.3

From the table of areas, the value of  $z$  for which [area from 0 to  $z$ ] = 0.3 is  $z=0.84$

Therefore,  $z=0.84$ .

And so,  $\frac{a-47}{10} = 0.84$

$$a - 47 = 8.4$$

$$a = 55.4$$

Thus, the marks limit for promotion is  $a = 55.4$

## 9. Practice Quiz

1. The mean of binomial distribution is..... [ B ]  
A) 0                      B)  $np$                       C)  $npq$                       D)  $np(1 + q)$
2. The probability of getting 2 heads in tossing 5 coins [ A ]  
A)  $5C_2 \left(\frac{1}{2}\right)^5$                       B)  $5C_2 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)$                       C)  $5C_3 \left(\frac{1}{2}\right)^2$                       D) None
3. The mean of binomial distribution is 4 and variance is 2 then  $P =$  [ B ]  
A) 1                      B)  $\frac{1}{2}$                       C)  $\frac{1}{3}$                       D) None
4. If  $n = 400$   $P = 0.2$  of binomial distribution then standard deviation is [ A ]  
A) 8                      B) 9                      C) 10                      D) None
5. The condition for binomial distribution approaches to poison distribution when. [ C ]  
A)  $P \rightarrow$  Very small                      B)  $n \rightarrow$  Very large value,  $np = \lambda$  constant  
C) Both A and B                      D) None
6. In ..... distribution mean, variance are equal [ C ]  
A) Normal                      B) Binomial                      C) Poisson                      D) None
7. If  $X$  is a Poisson variate with  $\lambda = 3$  then  $P(x = 0) =$ ..... [ A ]  
A)  $e^{-3}$                       B)  $\frac{e^{-3}}{-3}$                       C)  $-3e^{-3}$                       D) None
8. The shape normal curve is [ B ]  
A) parabola                      B) bell shape                      C) Hyperbola                      D) None
9. The area under the whole normal curve is [ B ]  
A) 0                      B) 1                      C) 0.5                      D) None

10. If  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$  then mean, variances are [ A ]
- A) (0,1)            B) (1,0)            C) (0,0)            D) None
11. The mean, mode, median of a normal distribution are ..... [ A ]
- A) Equal            B) unequal            C) Both A and B    D) None
12. The mean of Poisson distribution is 8, then its variance is---- [ D ]
- A) 2            B) 1            C) 6            D) 8
13. If  $P(1) = P(2)$ , then the mean of the Poisson distribution is---- [ C ]
- A) 1            B) 0            C) 2            D) -1
14. If x is a normal variate find  $P(-0.8 \leq z \leq 1.53)$  [ C ]
- A) 0.5            B) 1            C) 0.7251            D) None
15. If  $\mu = 1, \sigma = 3$  find  $P(3.43 \leq x \leq 6.19)$  [ D ]
- A) 1.5            B) 0.5            C) 0.75            D) 0.1672

## 10. Assignments

S.No	Question	BL	CO
1	Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) 5 girls (c) either 2 or 3 boys, assume equal probabilities for boys and girls.	L3	CO5
2	2% of the items of a factory are defective. The items are packed in boxes. What is the probability that there will be (i) 2 defective items (ii) at least three defective items in a box of 100 items.	L3	CO5
3	If X is a Poisson variate such that $3P(x = 4) = \frac{1}{2}P(x = 2) + P(x = 0)$ find (i) the mean of X (ii) $P(x \leq 2)$	L3	CO5
4	Suppose the weights of 800 male students are normally distributed with $\mu = 140$ pounds and standard deviation 10 pounds find the number of students whose weight are (i) between 138 and 148 pounds (ii) more than 152 pounds.	L3	CO5
5	In a normal distribution 7% of the items are under 35, 89% are under 63. Determine the mean and variance of the distribution.	L3	CO5

## 11. Part A- Question & Answers

S.No	Question& Answers	BL	CO
1	<p>Define Binomial distribution?</p> <p>Sol: A Probability distribution which has the following probability mass function (p.m.f) is called Binomial distribution.</p> $p(x) = nC_x p^x q^{n-x}, x=0,1, 2,\dots,n.$	L1	5
2	<p>What are the mean and variance of binomial distribution?</p> <p>Sol: Mean is <math>E(X) = np</math></p> <p>Variance is <math>Var(X) = np(1-p) = npq</math></p>	L1	5
3	<p>If a coin is tossed 5 times, find the probability of exactly 2 heads?</p> <p>Sol: Number of trials <math>n=5</math></p> <p>Probability of head <math>p= 1/2</math> and hence the probability of tail <math>q =1/2</math></p> <p>For exactly two heads: <math>x=2</math></p> $P(x=2) = 5C_2 p^2 q^{5-2} = 10 \times (1/2)^{2 \times} (1/2)^3=5/16$	L1	5
4	<p>If 75% of all purchases at a certain store are made with a credit card and X is the number among ten randomly selected purchases made with a credit card, then find mean and variance.</p> <p>Sol: Here <math>n=10</math> <math>p=75%=0.75</math>, <math>q=0.25</math>. It follows Binomial distribution.</p> $E(X) = np = (10) (.75) = 7.5, V(X) = npq = 10(.75) (.25) = 1.875$	L3	5
5	<p>Write the probability mass function for poison distribution?</p> <p>Sol: A probability mass function for Poisson distribution</p> $p(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2 \dots \dots \dots$	L1	5
6	<p>A random variable X has a Poisson distribution with parameter <math>\lambda</math> such that <math>P(X = 1) = (0.2) P(X = 2)</math>. Find <math>\lambda</math> value.</p> <p>Sol: For the Poisson distribution, the probability function is defined as: <math>p(x) = \frac{e^{-\lambda} \lambda^x}{x!}</math> where <math>\lambda</math> is a parameter.</p> <p>Given that, <math>P(x = 1) = (0.2) P(X = 2)</math></p> $\frac{e^{-\lambda} \lambda^1}{1!} = (0.2) \frac{e^{-\lambda} \lambda^2}{2!} \Rightarrow \lambda = \lambda^2 / 10 \Rightarrow \lambda = 10$	L3	5

7	<p>Define standard normal distribution?</p> <p>Sol: A normal variate with mean <math>\mu=0</math> and standard deviation <math>\sigma =1</math> is called Standard Normal Variate. It is denoted by Z. Its probability density function is <math>f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty &lt; z &lt; \infty</math></p>	L1	5
8	<p>Define Normal distribution.</p> <p>Sol: A probability distribution which has the following probability density function (p.d.f) is called Normal distribution</p> $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty < x < \infty, \quad \sigma > 0$	L1	5
9	<p>X is a normal variate with mean 42 and standard deviation 4. Find the probability that a value taken by X is greater than 40.</p> <p>Sol: <math>p(x &gt; 40) = p\left(\frac{x-42}{4} &gt; \frac{40-42}{4}\right)</math>  <math>= P [Z &gt; -0.5]</math>  <math>= [\text{area from 0 to 0.5}] + [\text{area from 0 to } \infty]</math>  <math>= 0.1915 + 0.5 = 0.6915</math></p>	L3	5
10	<p>Write down any two properties of Normal distribution.</p> <p>Sol: 1. The curve is Bell –shaped  2. The curve is asymptotic to the X-axis. That is, the curve touches the X-axis only at <math>-\infty</math> and <math>+\infty</math></p>	L1	5

## 12. Part B- Questions

S.No	Question	BL	CO
1	a) Find mean, variance of binomial distributions b) The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively find $P(x \geq 1)$	L2	5
2	a) Find mean, variance of a Poisson distribution b) If the variance of a Poisson variate is 3, then find the probability that (i) $x = 0$ , (ii) $0 < x \leq 3$ (iii) $1 < x < 4$	L3	5
3	Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) 5 girls (c) either 2 or 3 boys, assume equal probabilities for boys and girls.	L3	5
4	The marks obtained in statistics in a certain examination found to be normally distributed. If 15% of the students' marks $\geq 60$ , 40% $\leq 30$ marks, Find the mean and standard deviation.	L3	5
5	The heights of 1000 students are normally distributed with a mean of 174.5 cm and a standard deviation of 6.9 cm. Assuming that the heights are recorded to the nearest half-cm, how many of these students would you expect to have heights: (i) Less than 160.0 cms? (ii) Between 171.5 and 182.0 cms inclusive? (iii) Greater than or equal to 188.0 cms.	L3	5

### 13. Supportive Online Certification Courses

1. Introduction to Probability and Statistics by Prof. G. Srinivasan  
IIT Madras – 4 weeks

### 14. Real Time Applications

S. No	Application	CO
1	Probability Theory used in the earthquake risk assessment and its consequent possible reduction	4

### 15. Contents Beyond the Syllabus

1. Probability theory with applications to engineering problems such as the reliability of circuits and systems.

### 16. Prescribed Text Books & Reference Books

#### Text Book

1. B. S. Grewal, Higher Engineering Mathematics, Khanna Publishers.
2. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons.

#### References:

1. B. V. Ramana, Higher Engineering Mathematics, McGraw Hill Education
2. Alan Jeffrey, "Advanced Engineering Mathematics", Elsevier Publishers