Complex Variables & Transforms (20A54302)

LECTURE NOTES

II - B.TECH & I- SEM

Prepared by:

Dr. B. NAGABHUSHANAM REDDY, Professor
Department of H &S

VEMU INSTITUTE OF TECHNOLOGY

(Approved By AICTE, New Delhi and Affiliated to JNTUA, Ananthapuramu)

Accredited By NAAC, NBA(EEE, ECE & CSE) & ISO: 9001-2015 Certified Institution

Near Pakala, P.Kothakota, Chittoor- Tirupathi Highway

Chittoor, Andhra Pradesh-517 112

Web Site: www.vemu.org

Unit – 1

Complex - analysis

- **Function of Complex Variable/ Differentiation:**

If for each value of the complex variable Z= X+iY in a given region 'R' , we have one or more values of w=f(z)=u+iv, Then W is said to be a function of 'Z' , and we have w=f(z)=u+iv.

Where u and v are real and imaginary parts of f(z). z=x+iy

and

f(z)=u(x,y)+iv(x,y) is a complex function.

- **Continuity of a Function:**

Let f(z) is said to be continuous function at z=z if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- **Differentiability of a Function:**

A function f(z) is said to be differentiable at z=z if

$$\lim_{\Delta z \to 0} \left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right)$$

exists. It is donated by $f^{l}(z_0)$

i.e. $f^{l}(z_0) = \lim_{\Delta z \to 0} \left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right)$

- Analytical Function:

The complex function f(z) is said to be analytical function at z=a if the function f(z) has derivative at z=a and neighbourhood of z=a.

Example:

1. Let f(z) = z^2 f'(z) = 2z

At z=0, f'(z) = 2(0) = 0 (finite) f(z) has derivative at z=0

Finally f(z) is called **analytical** function.

2. Let f(z) = $\frac{1}{z}$

f'(z) = $\frac{-1}{z^2}$ At z=0, f'(z) = $\frac{-1}{(0)^2} = \infty$

f(z) has no derivative at z=0

Finally f(z) is called **not analytical** function.

- **Singular Point:**

Let z=a is said to be singular point if the function f(z) is not analytical at z=a.

Example:

f(z) = $\frac{1}{z}$, f'(z) = ∞ z = 0 is called singular point.

- **Cauchy – Riemann Equations in Cartesian co-ordinates:**
- If f(z) is continuous in some neighbourhood of z and differentiable at z then the first order partial derivatives satisfy the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at the point z which are called the Cauchy-Riemann equations.

proof:

Let f(z) = u+iv be an analytical function

By definition of analytical function, f(z) has derivative.

i.e. $f^{I}(z) = \lim_{\triangle z \to 0} \left(\frac{f(z+\triangle z)-f(z)}{\triangle z}\right)$ exists (finite)

1) z = x+iy f(z) = u+iv f(z) = u(x,y)+iv(x,y)

2) z = x+iy $\triangle z = \triangle x + i \triangle y$ ***3)*** $f(z + \triangle z)$ = ?

$(z + \triangle z)$ = x+iy+ △x+ i△ y

$(z + \triangle z)$ = (x+ △x)+i(y+ △ y)

$f(z + \triangle z)$ = u(x+ △x , y+ △ y) + iv(x+ △x , y+ △ y)

$$f^{I}(z) = \lim_{\triangle x + i\triangle y \to 0} \left(\frac{[u(x+\triangle x, y+\triangle y) + iv(x+\triangle x, y+\triangle y)] - [u(x,y) + iv(x,y)]}{\triangle x + i\triangle y}\right) \to ①$$

We know that △ x+i △y = 0+i0 △

x = 0, △ y = 0

Case (1) If $\triangle y = 0$, put $\triangle y = 0$ in ①.

$$f^{I}(z) = \lim_{\triangle x\to 0}\left(\frac{[u(x+\triangle x, y)+iv(x+\triangle x,y)) - [u(x,y)+iv(x,y)]}{\triangle x}\right)$$

$$f^{I}(z) = \left(\lim_{\triangle x\to 0}\frac{[u(x+\triangle x, y)-u(x,y)]}{\triangle x} + \lim_{\triangle x\to 0}\frac{i[v(x+\triangle x, y)-u(x,y)]}{\triangle x}\right)$$

$$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \to ②$$

Case (2) If $\triangle x = 0$, put $\triangle x = 0$ in ①

$$f^{I}(z) = \lim_{\triangle y\to 0}\left(\frac{[u(x,y+\triangle y)+iv(x,y+\triangle y)) - [u(x,y) + iv(x,y)]}{i\triangle y}\right)$$

$$f^{I}(z) = -i\lim_{\triangle y\to 0}\left(\frac{[u(x,y+\triangle y)-u(x,y)]}{\triangle y} + \lim_{x\to 0}\frac{i[v(x,y+\triangle y)-u(x,y)]}{\triangle y}\right)$$

$$f^{I}(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \to ③$$

Equate ② & ③

Compare the real and imaginary parts

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\left\{ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right.$$

$$\left. \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right\}$$

(If $ux = vy$ and $uy = -vx$)

These are **Cauchy – Riemann** Equations in **Cartesian** co-ordinate System.

Cauchy – Riemann Equations in Polar co-ordinates:

Let z=x+iy

We know that x=rcosθ ,

y=rsinθ z =

rcosθ+irsinθ z =

r(cosθ+isinθ) z = $re^{i\theta}$

f(z)=u+iv f($re^{i\theta}$) = u(r, θ)+iv(r,

θ) → ①

Differentiate ① w.r.t 'r',

$f^{l}(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$ →②

Differentiate ① w.r.t 'θ',

f^{l} (→③

Substitute ② in ③ , We get

$$[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}] = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} re^{i\theta}) \ rie^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

ir

ir

Lets compare real and imaginary parts

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

These are **Cauchy – Riemann** Equations in **Polar** co-ordinate System. **Examples**

1) Show that f(z) = xy+iy is not analytical

Solution : Given , f(z) = xy+iy

f(z) = u+iv u= xy

v= y

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial y} = 1$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

It doesn't **not satisfies C-R equations** and hence its **not an analytical function**.

2) Show that f(z) = 2xy+i(x^2- y^2**)** is not analytical function. Solution: Given f(z) = 2xy+i(x^2- y^2**)**

f(z) = u+iv

u=2xy $\quad$ v= x^2- y^2

$\frac{\partial u}{\partial x}$= 2y , $\quad$ $\frac{\partial v}{\partial x}$= 2x

$\frac{\partial u}{\partial y}$= 2x , $\quad$ $\frac{\partial v}{\partial y}$ = -2y

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

It doesn't **not satisfies C-R** equations and hence its **not an analytical** function.

3) Test the analyticity f(z) = e^x(cosy-isiny) and also find the f'(z) Solution: Given f(z) = e^xcosy - ie^xsiny

f(z) = u+iv u = e^xcosy

v = -e^xsiny

$\frac{\partial u}{\partial x}$ = e^x cosy , $\frac{\partial v}{\partial x}$ = - e^xsiny ,

$\frac{\partial u}{\partial y}$ = -e^x siny

$\frac{\partial v}{\partial y}$ = -e^xcosy

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

f(z) is **not analytical** function and the f'(z) does not exist.

4) Show that f(z) = z z^2 is not analytical function

Solution : Given f(z) = z z^2

$f(z) = (x+iy)|(x+iy)|^2 = (x+iy)[\sqrt{x^2+y^2}]^2$

$f(z) = x(x^2+y^2)+iy(x^2+y^2)$ f(z) =

u+iv

$u = x(x^2+y^2) = x^3+xy^2$ $V = y(x^2+y^2) = x^2y+y^3$

$\frac{\partial u}{\partial x} = 3x^2+y^2$, $\frac{\partial v}{\partial x} = 2xy$

$\frac{\partial u}{\partial y} = 2xy$, $\frac{\partial v}{\partial y} = x^2+3y^2$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

f(z) is **not analytical** function

5) Show that w= logz is an analytical function and also find $\frac{dw}{dz}$

Solution : Given w = logz

put $z = re^{i\theta}$

$^{i}\theta$ = log r + log $e^{i}\theta$ w w

= log re

= log r + iθ log e

f(z) = w = log r +iθ = u+iv u

= log r v = θ

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial v}{\partial r} = \theta$$

$$\frac{\partial u}{\partial \theta} = 0, \frac{\partial v}{\partial \theta} = 1$$

$$\frac{\partial u}{r\partial r} = \frac{\partial v}{\partial \theta} \; \& \; \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$r(\frac{1}{r}) = 1$ & $0 = 0$ It is an **analytical** function f(z)

= u+iv

$$f(re^{i\theta}) = u(r, \theta)+iv(r, \theta)$$

differentiate on both sides w.r.t 'r'

$$f^{I}(re^{i\theta})\, e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$f^{I}(z)\, e^{i\theta} = \frac{1}{r} + i(0)$$

$$f^{I}(z) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

6) Show that f(x) = sinz is an analytical function everywhere in the complex plane

Solution : Given f(x) = sinz

f(x) = sin(x+iy) f(x) = sinx

cos(iy) + sin(iy) cosx f(X) = sinx

coshy + isinhy cosx f(x) = u+iv

u = sinx coshy v= sinhy cosx

$\frac{\partial u}{\partial x}$ = cosx coshy , $\frac{\partial v}{\partial x}$ = -sinx sinhy

$\frac{\partial u}{\partial y}$ = sinx sinhy, $\frac{\partial v}{\partial y}$ = coshy cosx & It is an **analytical** function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

7) Test the analyticity of the function f(z) = e^x (cosy+isiny) and find f'(z). Solution : Given , f(z) = e^x (cosy+isiny) = u+iv

u = e^x cosy v = e^x siny

$\frac{\partial u}{\partial x} = e^x$ cosy , $\frac{\partial v}{\partial x} = e^x$ siny

$\frac{\partial u}{\partial y} = -e^x$ siny $\frac{\partial v}{\partial y} = e^x$ cosy

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

& It is an **analytical** function

f(z) = u+iv

f'(z) = $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = e^x$ cosy + i e^x siny

f'(z) = e^x(cosy+isiny)

f'(z) = e^x i e^y = $e^{(x+iy)}$

f'(z) = e^z

8) Determine P such that the function f(z) = $\frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}(\frac{px}{y})$ be an analytical function.

Solution :

Given , f(z) = $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(\frac{px}{y})$

It is an analytical function, It satisfies the C-R equation

) v = $\tan^{-1}(\frac{px}{y})$ u = $\frac{1}{2}\log(x^2+y^2$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}2x,$ $\frac{\partial v}{\partial x} = \frac{1}{1+(\frac{px}{y})^2}\frac{p}{y}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}2y$

$\frac{\partial v}{\partial y} = \frac{1}{1+(\frac{px}{y})^2}px(\frac{-1}{y^2})$

$\frac{\partial v}{\partial y} = \frac{y^2}{y^2+(\frac{px}{y})^2}(\frac{-px}{y^2})$

similarly : $\frac{\partial v}{\partial x} = \frac{py}{p^2x^2+y^2}$ $\frac{\partial v}{\partial y} = \frac{-px}{y^2+p^2x^2}$,

By given f(z) is an analytical function, f(z) satisfies C-R equations.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{x}{x^2+y^2} = \frac{-px}{y^2+p^2x^2}$

Comparing the equations we get:

P = -1

9) Prove that function f(z) defined by f(z) = -R equations are satisfied at the origin, yet f'(0) does not exist.

Solution : Given f(z) = $\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$

i) To show that f(z) is continuous at z=0 let $\lim f(z) = \lim_{x\to 0} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$ (given f(0) = 0) $\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$, z≠0 and f(0) is continues and C

$z\to 0$

$y\to 0$

$\lim_{x\to 0} \frac{x\ (1+i)}{x^2}$

$_3$ f(z) = f(z) =

lim x(1+i) = 0 = f(0)

$x\to 0$ **f(z) is continuous**

ii) To show that C-R equations are satisfied at origin

f(z) = $\frac{x^3+x^3 i-y^3+iy^3}{x^2+y^2} = \frac{x^3-y^3}{x^2+y^2} + \frac{i(x^3+y^3)}{x^2+y^2}$ f(z)

= u+iv

u $\frac{x^3-y}{x^2+y^2}$ ^{3} = , $\frac{x^3+y^3}{x^2+y^2}$

v =

$$\frac{\partial u}{\partial x} = \lim_{x\to0} \frac{u(x,0)-u(0,0)}{x}$$

$$\frac{\partial u}{\partial x} = \lim_{x\to0} \frac{x-0}{x} \Rightarrow \lim_{x\to0}$$

$$\frac{\partial u}{\partial x} = 1$$

= 1

$$\frac{\partial u}{\partial y} = \lim_{y\to0} \frac{u(0,y)-u(0,0)}{y}$$

$$\frac{\partial u}{\partial y} = \lim_{y\to0} \frac{-y-0}{y} \Rightarrow \lim_{y\to0} -1 = -$$

$$\frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to0} \frac{v(x,0)-v(0,0)}{x}$$

$$\frac{\partial v}{\partial x} = \lim_{x\to0} \frac{x-0}{x} = \lim_{x\to0} 1 = 1$$

$$\frac{\partial v}{\partial x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to0} \frac{v(0,y)-v(0,0)}{y}$$

$$\frac{\partial v}{\partial y} = \lim_{y\to0} \frac{y-0}{y} = \lim_{x\to0} 1 = 1$$

$$\frac{\partial v}{\partial y} = 1$$

1

C – $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ **&** $\partial u \quad \partial v \frac{}{\partial y} = - \frac{}{\partial x}$

x

lim 0

R Equations are satisfied at origin **iii)** To show that $f^{I}(z)$ does not exist at origin

$f^{I}(z) = \lim_{y\to0} \frac{f(z)-f(0)}{z}$

( )

x^3 1+i −y2 +^{3} y(1−i) 2 −

$\frac{^3(1+i)-0}{^2+y^2}$

( )

$f^{I}(z) = \lim_{x \to 0, y \to 0} \quad x \quad x + iy$

$\lim \quad x$

$x \to 0 \quad x$

$f^{I}(z) =$

$x\ 1 + i\ ^3\ f^{I}(z)\ \lim_{x \to 0}\ x^3 =$

= 1+i **(Finite)**

$f^{I}(z)$ Exists

At y = mx

$f^{I}(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$ $\quad f^{I}(z) =$

$\lim_{x \to 0} \frac{\frac{x^3(1+i) - m^3x^3(1-i)}{x^2 + x^2m^2}}{x + imx}$

$y \to mx$

$f^{I}(z) = \lim_{x \to 0} \frac{x^3[(1+i) - m^3(1-i)]}{x^2(1+m^2)x(1+im)}$

$y \to mx$

$f^{I}(z) \quad \lim_{x \to 0} \frac{[(1+i) - m^3(1-i)]}{(1+m^2)(1+im)}$

$= \frac{[(1+i) - m^3(1-i)]}{(1+m^2)(1+im)}$

$f^{I}(z) =$ **(Infinite) $f^{I}(z)$ depends upon the 'm' value, so that the $f^{I}(z)$ does not exist at origin**

Part – B

Laplace Equations

the equation of the form $\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0$ **or** $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$

Harmonic Function

The function u and v are said to be harmonic, if it satisfies Laplace Equations

i.e

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Milne – Thomson Method

When u is given find f(z) :

1) To find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

2) To find $f^{I}(z) = u+iv$

Differentiate w.r.t ‘x’ we get

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f^{I}(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ (From C-R equation)

$f^{I}(z) =$

$\frac{\partial u}{\partial x} = \emptyset_1(z_1$

,0)

$\frac{\partial u}{\partial y} = \emptyset_2(z_2$

,0) $f^{I}(z) =$

$\emptyset_1(z_1,0) - i\, \emptyset_2(z_2,0)$

Integrate w.r.t 'z' **f(z) = ₁∅ ?(z₁,0) dz - i ₂∅ ?(z₂,0)**

dz + c When v is given find f(z):

1) To find $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}$

2) To find f(z) = u+iv

Differentiate w.r.t 'x' , we get

$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$f^{I}(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = \emptyset_1(z_1$,0)

$\frac{\partial v}{\partial x} = \emptyset_2(z_2$,0) (From C-R equation)

$f^{I}(z) = \phi_1(z_1,0) + i\,\phi_2(z_2,0)$

Integrate w.r.t 'z' **f(z) = ₁[∅(z₁,0) + i ∅₂(z₂,0)**

]dz + c

1) Construct an analytical function f(z) when $u = x^3 - 3x\,y^2 + 3x + 1$ is given

$$\frac{\partial u}{\partial x} = 3x^2 - 3\,y^2 + 3$$

$$\frac{\partial u}{\partial y} = -6xy$$

Solution:

By Milne Thomson Method

f(z) =u+iv

$$\frac{\partial u}{\partial x} = \phi_1(z,0) = 3\,z^2 + 3$$

$$\frac{\partial u}{\partial y} = \phi_2(z,0) = -6(z)\,(0) = 0$$

$$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \phi_1(z,0) - i\,\phi_2(z,0)$$

$f^{I}(z) = ,0)$

Integrate w.r.t 'z' f(z) =

$\int[\phi_1(z,0) + i\,\phi_2(z,0)]\,dz + c$ f(z)

$= \int(3z^2 + 3 - 0)\,dz + c$ f(z)

$$= \frac{3z^3}{3} + 3z + c$$

f(z) = z^3 + 3z + c

2) Construct an analytical function f(z) when u = sinx coshy is given

Solution: $\frac{\partial u}{\partial x}$ = cosx sinhy

$\frac{\partial u}{\partial y}$ = sinx sinhy

By Milne Thomson Method

f(z) =u+iv

$\frac{\partial u}{\partial x} = \emptyset_1(z$,0) = cosz(1) = cosz

$\frac{\partial u}{\partial y} = \emptyset_2$(z,0) = sinz(0)

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ = 0

f'(z) = $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \emptyset_1(z,0) - i\,\emptyset_2(z$,0)

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ **&** $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

f'(z) =

Integrate w.r.t 'z' f(z) = $\int[\emptyset_1(z,0) - i\,\emptyset_2(z,0)]$ dz + c

f(z) = $\int$cosz dz + c

f(z) = sinz + c

3) Find the analytical function f(z) = u+iv if u+v = $\frac{\sin 2x}{(\cosh 2y - \cos 2x)}$

Solution:

u+v = $\frac{\sin 2x}{(\cosh 2y - \cos 2x)}$

f(z) = u+iv

if(z) = ui-v

(1 + i)f(z) = (u-v)+i(u+v)

f(z) = u+iv

$$\frac{\partial V}{\partial x} = \frac{[coh2y - cos2x]2cos2x - sin2x[0+2sin2x]}{[cosh2y - cos2x]^2}$$

$$\frac{\partial V}{\partial x} = \frac{2cos2x\ coshy - 2cos^2 2x - 2sin^2 2x}{[cosh2y - cos2x]^2}$$

$$\frac{\partial V}{\partial x} = \frac{2cos2x\ coshy - 2}{[cosh2y - cos2x]^2}$$

$$\frac{\partial V}{\partial x} = \emptyset_2(z,0)$$

$$\frac{\partial V}{\partial x} = \frac{2cos2z\ cosh0 - 2}{[cosh0 - cos2z]^2} = \frac{2[cos2z - 1]}{[1 - cos2z]^2} = \frac{-2[1 - cos2z]}{[1 - cos2z]^2}$$

$$\frac{\partial V}{\partial x} = \frac{-2}{2sin^2 z}$$

Where F(z) = (1+i)f(z)

u+v = V

$$\frac{\partial V}{\partial x} = \emptyset_2(z,0) = -cosec^2 z$$

$$\frac{\partial v}{\partial y} = \emptyset_1(z,0) = \frac{[coh2y - cos2x]\ 0 - sin2x[sinh2y(2)]}{[cosh2y - cos2x]^2}$$

$$\emptyset_1(z,0) = \frac{\partial v}{\partial y} = \frac{-2sin2xsinhy}{[cosh2y - cos2x]^2}$$

$$\frac{\partial v}{\partial y} = \frac{-0\ sin2z}{[cosh2y - cos2z]^2} = 0$$

f(z) = u+iv

$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$ $f^{I}(z) =$

f(z) = ∫[$\emptyset_1$(z,0) + i $\emptyset_2$(z,0)] dz + c

f(z) = ∫ $-\text{cosec}^2 z$ (i) dz+ c

f(z) = -i(-cotz) + c = i cotz + c

f(z) = i cot z + c

(1+i) f(z) $\frac{}{1+i}$ + = $\frac{c}{1+i}$ =icotz + c

i

f(z) = $\frac{i(1-i)}{2}$

cotz f(z) =

cotz + c_1

i+1

f(z) = $\frac{}{2}$ cotz + c_1

$\frac{\partial u}{\partial x} = e^x x^2$ cosy + 2x e^x cosy - e y

$\emptyset_1(z,0) = \frac{\partial u}{\partial x} = e^z z^2$

4) Find the analytical function , whose real part is u = $e^x[(x^2 - y^2)(\cos y - 2xy\sin y)]$

$\emptyset_1(z,0) = \frac{\partial u}{\partial x} = e^z z^2 + 2z e^z$

Solution: u = $e^x x^2$ cosy - $e^x y^2$ cosy – 2xy e^x siny

$\frac{\partial u}{\partial y}$ = - $e^x x^2$

x 2

x 2 cosy – 2y e^x siny – 2xy e^x siny

$\emptyset_2(z,0) = \frac{\partial u}{\partial y}$ = 0 + 0 - 0 – 0 = 0

$\cos(0) + 2z\, e^z \cos(0) - 0 - 0 - 0$

$\sin y + e \sin y\, y - 2y\, e^x \cos y - 2x\, e^x \sin y - 2xy\, e^x \cos y$

f(z) = u+iv

$f^{I}(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y}$

$f^{I}(z) =$

f(z) = ,0)] dz + c z(Ø2i -,0) z([Ø1�

f(z) = �(e^z z^2 + 2z e^z - 0) dz + c f(z)

= � e^z (z^2 +2z)dz + c f(z) = � e^z z^2

dz + 2 �ze^z dz

u = z^2 dv = e^z dz du = 2z dz v= e^z f(z) = **e^z**

z^2 - 2 �z dz e^z dz + 2 �z e^z dz + c **f(z) = e^z**

z^2 + c

2

5) The analytical function whose imaginary part is v(x,y) = 2xy **Solution:**

v = 2xy

$\frac{\partial v}{\partial x}$ = 2y = $\emptyset_2$(z,0) = 2(0) = 0

$\frac{\partial v}{\partial y}$ = 2x = $\emptyset_1$(z,0) = 2(z) = 2z f(z)

= f(z) [$\emptyset_1$(z ,0) + i z(,0)] dz + c

= z dz + c

$$f(z) = 2\frac{z^2}{2} + c$$

f(z) = $z^2 + c$

6) Find harmonic conjugate at u = $e^{x^2-y^2}$cos2xy and also find f(z)

Solution : u = $e^{x^2-y^2}$ cos2xy

$\frac{\partial u}{\partial x}$ = $e^{x^2-y^2}$ cos2xy (2x) - $e^{x^2-y^2}$ sin2xy (2y)

$\emptyset_1$(z,0) = e^{z^2-0} cos0 (2z) - $e^{x^2-y^2}$(0)

$\emptyset_1$(z,0) = e^{z^2} 2z $\frac{\partial u}{\partial y}$ = $e^{x^2-y^2}$ cos2xy (-2y) - $e^{x^2-y^2}$ sin2xy (2x)

$\emptyset_2(z,0) = 0 - 0$

$\emptyset_2(z,0) = \mathbf{0}$ f(z)

= u+iv $f^{l}(z)$ =

$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ $f^{l}(z)$ =

$f^{l}(z) = \emptyset_1(z,0) - i\,\emptyset_2(z,0)$

f(z) = ⍰$[\emptyset_1(z,0) - i\,\emptyset_2(z,0)]$ dz + c f(z) = ⍰ $e^{z^2}2z$

dz + c **(put $z^2 = t$ => 2z dz = dt)** f(z) = ⍰ e^t dt +

c = e^t + c

f(z) = e^{z^2} + c f(z) = $e^{(x+iy)^2}$ f(z) =

$e^{x^2-y^2+2xyi}$ + c f(z) = $e^{x^2-y^2}e^{2xyi}$ + c u+iv =

$e^{x^2-y^2}$ [cos2xy+isin2xy] + c u+iv = $e^{x^2-y^2}$

cos2xy + i e $e^{x^2-y^2}$ (sin2xy) + c

v = $e^{x^2-y^2}$ sin2xy + c

7) Find the analytical function f(z) such that Re[$f^{l}(z)$] = 3 x^2 - 4y -3 y^2 and f(1+i) = 0.

Solution : $Re[f^{I}(z)] = 3x^2 - 4y - 3y^2$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$f(z) = u+iv$

$$f^{I}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$Re[f^{I}(z)] = \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 4y - 3y^2 \qquad \frac{\partial v}{\partial y} = 3x^2 - 4y - 3y^2$$

Integrate w.r.t 'x' we get **&** $u = \frac{3x^3}{3} - 4xy - 3y^2x + f(y)$

Integrate w.r.t 'y' we get $v = 3x^2y - \frac{4y^2}{2} - \frac{3y^3}{3} + f(x)$

$$\mathbf{u = x^3 - 4xy - 3y^2x + f(y)} \qquad \mathbf{v = 3x^2y - y^3 - 2y^2 + f(x)}$$

Differentiate w.r.t 'y' we get

$$\frac{\partial u}{\partial y} = -4x - 6xy + f^{I}(y)$$

Differentiate w.r.t 'x' we get

$$\frac{\partial v}{\partial x} = 6xy + f^{I}(x)$$

From C-R equations

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-4x - 6xy + f^{I}(y) = -6xy - f^{I}(x)$$

$$-4x + f^{I}(y) = -f^{I}(x)$$

Compare equation on both sides

i.e $f^{I}(x) = 4x$, $f^{I}(y) = 0$

$f(x) = 4\int x\,dx$ $\qquad f(y) = c\ f(x)$

$= \frac{4x^2}{2} + c$

$f(x) = 2x^2 + c$ **$f(y) = c$**

$f(z) = u+iv$ $f(z) = [x^3 - 4xy - 3y^2x] + i[3x^2y - y^3 - 2y^2] + 2x^2 + c$

given $f(1+i) = 0$ $f(z) = u+iv$

$z = x+iy = (1+i)$

put $x = 1, y = 1$ $f(z) = [1-4-3] + i[3-2-1] +2 +c$ $f(1+i) = 0 = -6 +2i +c$ **$c = 6 - 2i$**

$f(z) = [x^3 - 4xy - 3y^2x] + i[3x^2y - y^3 - 2y^2] + 2x^2 + 6 - 2i$

8) Find the analytic function $f(z) = u+iv$ if $u-v = e^x(\cos y - \sin y)$ **Solution:**

$f(z) = u+iv$ $i\ f(z) = iu-v$

$(1+i)\ f(z) = (u-v) + i\ (u+v)$

$f(z) = u+iv$ **$u = u-v = e^x (\cos y - \sin y)$**

$F(z) = (1+i)\, f(z)$ cosy - e^x siny =

$\frac{\partial u}{\partial x} = e^x$

$\phi_1(z,0) = e^z - 0 = e^z$ siny - e^x

$\frac{\partial u}{\partial y} = - e^x$

cosy = $\phi_2(z,0) = 0 - e^z = - e^z$

$f^I(z) = \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y}$

$f(z) = \int [\phi_1(z,0) - i\, \phi_2(z,0)]\, dz + c$

$f(z) = \int (e^z + i\, e^z)\, dz + c$

$f(z) = (e^z + i\, e^z) + c$ $(1+i)$

$f(z) = e^z + i\, e^z + c$

$f(z) = \frac{e^z(1+i)}{(1+i)} + \frac{c}{1+i}$ **$f(z)$**

$= e^z + c$

Harmonic Conjugate

1) Show that function u= 2xy+3y is harmonic and find harmonic conjugate.

Solution: u= 2xy+3y

$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x+3$

$\frac{\partial^2 u}{\partial x^2} = 0$ $\frac{\partial^2 u}{\partial y^2} = 0$

2 2

$\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$ **u satisfies laplace equation**

'u' is a **Harmonic** function

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

dv = -(2x+3) dx + 2y dy v

= ∫ -(2x+3) dx + 2y dy

$v = -(\frac{2x^2}{2} + 3x) + \frac{2y^2}{2} + c$

$\mathbf{v = -x^2 + y^2 - 3x + c}$

2) Show that u = 2log (x^2+ y^2) is harmonic and find its harmonic conjugate.

Solution: u = 2log (x^2+ y^2)

$$\frac{\partial u}{\partial x} = 2\frac{1}{x^2+y^2}2x \qquad \frac{\partial u}{\partial y} = 2\frac{1}{x^2+y^2}2y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(4)-4x(2x)}{(x^2+y^2)^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(4)-4y(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4x^2+4y^2-8x^2+4x^2+4y^2-8y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$dv = \frac{-4y}{x^2+y^2}dx + \frac{4x}{x^2+y^2}dy$$

$$dv = \frac{-4}{x^2+y^2}(y\,dx - x\,dy)$$

$$v = -4\int\left[\frac{x\,dy - y\,dx}{x^2+y^2}\right]$$

$$= -4\int d\tan^{-1}\left(\frac{y}{x}\right)$$

$$\mathbf{v = -4\tan^{-1}\left(\frac{y}{x}\right) + c}$$

$$d\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left[\frac{x\,dy - y\,dx}{x^2}\right]$$

3) Find f(z) if the imaginary part is $r^2\cos 2\theta + r\sin\theta$ **Solution:**

$$V = r^2\cos 2\theta + r\sin\theta$$

Integrate w.r.t 'r', we get $u = -r^2 \sin 2\theta + \cos\theta\, r + f(\theta)$

$\frac{\partial v}{\partial r} = 2r\cos 2\theta + \sin\theta$

$\frac{\partial v}{\partial \theta} = -2r^2\sin 2\theta + r\cos\theta$

$u = -r^2 \sin 2\theta + r\cos\theta + f(\theta)$

Differentiate w.r.t (θ)

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta - r\sin\theta + f^{\prime}(\theta) \rightarrow ③$

Compare the equations ② & ③

$\frac{\partial v}{\partial r} = 2r\sin 2\theta + \cos\theta \rightarrow ①$

$f^{\prime}(\theta) = 0$

$f(\theta) = c$

$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

$u = -r^2 \sin 2\theta + r\cos\theta + c$

$\frac{\partial u}{\partial \theta} = -r\,[2r\cos 2\theta - \sin\theta]$

$f(z) = (-r^2 \sin 2\theta + r\cos\theta + c) + i\,(r^2 \cos 2\theta + r\sin\theta)$

$\frac{\partial u}{\partial \theta} = -2r^2\cos 2\theta - r\sin\theta \rightarrow ②$

4) Show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right][\text{real } f(z)]^2 = 2\,|f^{\prime}(z)|^2$

Solution: $f(z) = u+iv$

real $f(z) = u$

$[\text{real } f(z)]^2 = u^2$

$\frac{\partial (u^2)}{\partial x} = 2u\frac{\partial u}{\partial x}$

$\frac{\partial^2 (u^2)}{\partial x^2} = 2u\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} \rightarrow ①$

Similarly ,

$\rightarrow ②$

Add equation ②

$$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]\, u^2 = 2\,[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \quad ① \qquad \frac{\partial\ u}{\partial x^2} + \frac{\partial\ u}{\partial y^2}]\quad \frac{\partial^2(u^2)}{\partial y^2} = 2u\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial u}{\partial y} \ \&$$

$$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]\, u^2 = 2\,[(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2] \qquad 2 \qquad 2$$

] + 2u [

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

{ *f(z) = u+iv => f'(z) =*

$$|\quad|_2 = (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \ \}$$

2 f'(z)

$$[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}]\, u^2 \ [\text{real } f(z)]^2 = |\quad| \ 2\, f'(z)^{\,2}$$

5) If f(z) is analytical function with constant modulus ,then show that f(z) is constant.

Solution:

let f(z) is constant modulus

f(z) = u+iv

$|f(z)| = \sqrt{u^2 + v^2}$ = constant

$\sqrt{u^2 + v^2}$ = c

$u^2 + v^2 = c^2 = c_1$

Differentiate w.r.t **'x'**

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \rightarrow ①$$

$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$ Differentiate w.r.t **'y'**

= 0 → ②

By C-R

equations

$2u\frac{\partial v}{\partial y} - 2v\frac{\partial u}{\partial y}$

① ❾ $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}$ = 0 → ③

= 0 → ④ $\frac{\partial v}{\partial y} - v^2\frac{\partial u}{\partial y}$

Multiply ③ * v ❾ uv $u^2\frac{\partial u}{\partial y} + uv\frac{\partial v}{\partial y}$ = 0

④ * u ❾ = 0

Subtract then uv

u = $\frac{\partial v}{\partial y}$ - $v^2 \frac{\partial u}{\partial y}$ - $u^2 \frac{\partial u}{\partial y}$ - uv $\frac{\partial v}{\partial y}$ = 0

u = c

$-\frac{\partial u}{\partial y}(u^2 + v^2) = 0$

Similarly

$u^2 + v^2 \neq 0$

$\frac{\partial u}{\partial y} = 0$

$\int \frac{\partial u}{\partial y} = c$

v = c f(z) is

constant

Conformal Mapping :

A transformation w = f (z) is said to be conformal if it preserves angel between oriented curves in magnitude as well as in orientation.

Bilinear Transformation :

The transformation $w = f(z) = \frac{az+b}{cz+d}$ is called the bilinear transformation or mobius transformation. Where a,b,c,d are complex constants.

The method to find the bilinear transformation if three points and their images are given as follows:

We know that we need four equations to find 4 unknowns. To find a bilinear transformation we need three points and their images.

in cross ration, three are four points $(w, w_1, w_2, w_3,) = (z, z_1, z_2, z_3,)$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Since we have to get $w = \frac{az+b}{cz+d}$, we take one point as ‘z’ and its image as ‘w’

Problems about bilinear transformation:

1) Find the bilinear transformation on which maps the points (-1, 0, 1) into the points (0,i,3i) in w-plane

Solution : In z-plane, $z_1 = -1$, $z_2 = 0$, $z_3 = 1$

In w-plane, $w_1 = 0$, $w_2 = i$, $w_3 = 3i$

In cross ration,

(w,0,i,3i) = (z,-1,0,1)

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} \quad \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i-3i)}{(0-i)(3i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{(w)(-2i)}{(-i)(3i-w)} = \frac{-(z+1)}{-(1-z)}$$

=

-2wi(1-z) = (z+1) [- [i(3i-w)]]

-2wi + 2wiz = -[-3-wi](z+1)

-2wi + 2wiz = 3z + wiz + 3 +wi

$$\frac{(w-0)(0-1)}{(0-i)(1-0)} = \frac{(0-1)(i-0)}{(1-0)(0-z)}$$

$$\frac{-w}{-i} = \frac{-i}{-z} \qquad w = \frac{i^2}{z} = \frac{-1}{z} \qquad \mathbf{w = \frac{-1}{z}}$$

-3wi + wiz = (3z + 3)

w[i(3-z)] = z(z+1) **w**

-

$$= \frac{-3(z+1)}{-i(3-z)}$$

$_1 = \alpha = \frac{1}{z_1{}^{I}} = \frac{1}{0}$ $[z_1{}^{I}$ = 0], z_2 = i, z_3 = 0

2) Find the bilinear transformation which maps the points $(\alpha, i, 0)$ in the z-plane into $(0, i, \alpha)$ in the w-plane.

$w_1 = 0$, $w_2 = i$, $w_3 = \frac{1}{0} = \alpha = \frac{1}{w_3{}^{I}}$ $[w_3{}^{I} = 0]$

$(w_2 - \frac{1}{})$ $(z - \frac{1}{})$ $(z_2 - z_3)$

Solution: In z-plane, z

1 1

$$\frac{{}_1)(w_3' w_2 - 1)}{1 - \quad '} = \frac{(z z_1' - 1)(z_2 - z_3)}{1 \quad '}$$

In w-plane,

$(w - w_1)$

(w −w) (− w) (− z) (z z)$_1$ $_2$ $w_3{}^{I}$ =

$z_1{}^{I}$ $_2$ $_3$– ((ww−w$_1$−w$_2$) (ww_3) (−Z_1z$_2$) (z$_3$−z) $w_3{}^{I}$ $z_1{}^{I}$

3) Find the bilinear transformation that maps the points $(0, i, \alpha)$ respectively into $(0, 1, \alpha)$.

Solution: In z-plane, $z^1 = 0$, $z_2 = i$, $z_3 = = \frac{1}{0} = \frac{1}{\alpha} = \frac{1}{z_3^I}$ $[z_3^I = 0]$

$w_1 = 0$, $w_2 = 1$, $w_3 = \frac{1}{w_3^I} = \frac{1}{0} = \alpha$ $[w_3^I = 0]$

In w-plane,

$$\frac{(w-w_1)\left(w_2-\frac{1}{w_3^I}\right)}{(w_1-w_2)\left(\frac{1}{w_3^I}-w\right)} \; w_3^I = z_3^I \; \frac{(z-z_1)\left(z_2-\frac{1}{z_3^I}\right)}{(z_1-z_2)\left(\frac{1}{z_3^I}-z\right)} \; \frac{(w-w_1)(w_2 w_3'-1)}{(w_1-w_2)(1-w w_3')} = \frac{(z-Z_1)(Z_2 Z_3'-1)}{(\quad)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-0)(i(0)-0)}{(0-i)(1-0)}$$

$$\frac{-w}{-1} = \frac{-z}{-i}$$

$$\mathbf{w = -iz}$$

Fixed point :

The transformation $w = \frac{az+b}{cz+d}$

The roots of this transformation are called fixed points or invariant points.

$z = \frac{az+b}{cz+d}$ (we know that w = f(z)) $z(cz+d) =$

$az+b$ $c\,z^2 + dz = az+b$ $\mathbf{c\,z^2+(d-a)z - b = 0}$

Problems:

1) Find the fixed points of the transformation w =

Solution: The roots of above transformation are called fixed points

$\frac{z-1+i}{z+2}$ **put w** $\frac{z-1}{z+1}$

= z z = $\frac{z-1}{z+1}$ z(z+1)

= z-1 z^2 +z – z +1

= 0 z^2

+1 = 0 z^2 = -1 **z =** $\pm$

i fixed points $\pm$ i

2) The fixed points of the transformation w =

Solution: **put w = z**

z = $\frac{z-1+i}{z+2}$

z(z+2) = (z-i+1) (a =1, b =1, c =1-i)

z^2 +2z = z-i+l

z^2+z+i-l = 0

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1+4(1-i)}}{2}$$

$$z = \frac{-1 \pm \sqrt{1+4-4i}}{2} = \frac{-1 \pm \sqrt{3-4i}}{2}$$

$$\frac{-1+\sqrt{3-4i}}{2} \quad \& \quad \frac{-1-\sqrt{3-4i}}{2}$$

3) Determine the bilinear transformation whose fixed points are 1,-1 **Solution:**

Given fixed points are z = 1,-1

The roots of the transformation is $w = \frac{az+b}{cz+d}$ are called fixed points **put w = z**

$$z = \frac{az+b}{cz+d}$$

$cz^2+(d-a)z - b = 0$ $(z+1)(z-1) = 0$

$z^2 - 1 = 0$ (c =1, d =0, a =0, b =1)

$$\mathbf{w = \frac{0z+1}{1z+0} = \frac{1}{z}}$$

Problems on images:

1) Write the image of the triangle with vertices (i,1+i,1) in the z-plane under the transformation w = 3z+4-2i

Solution:

y

(x,y) = (1,0)

In w-plane:

y

in z-plane Transformation z =i ❾ x+iy = 0+i w= 3z+4-2i (x,y) = (0,1) w=

3(x+iy)+4-2i z= 1+i ❾ x+iy = 1+i u+iv = w

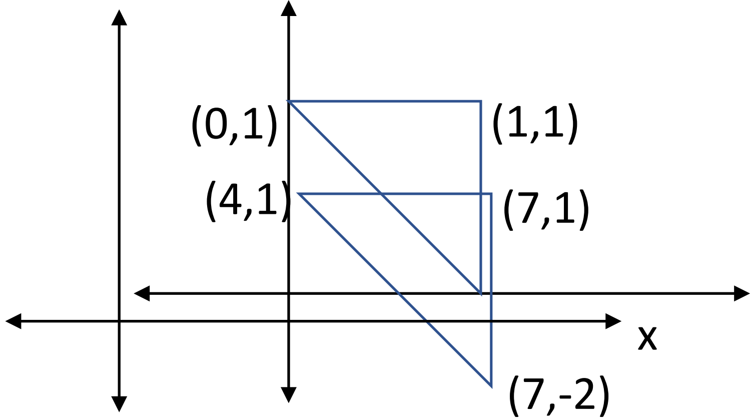
(x,y) = (1,1) u = 3x+4, v= 3y-2

x **z- plane**

(1,0)

z =1 ❾ x+iy = 1

i) (x,y) = (0,1) ❾ (u,v) = (4,1) ii) (x,y) = (1,1) ❾ (u,v) = (7,1) iii) (x,y) = (1,0) ❾ (u,v) = (7,-2)



w-plane

Conclusion:

The image of the triangle whose vertices (i,1+i,1) is mapped as triangle whose vertices (4,1) ,(7,1), (7,-2) in w-plane under the transformation **w=3z+4-2i**

2) Find the image of the infinite strip $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution: **In z –plane**

the infinite strip between the lines y =0, y = .

Transformation:

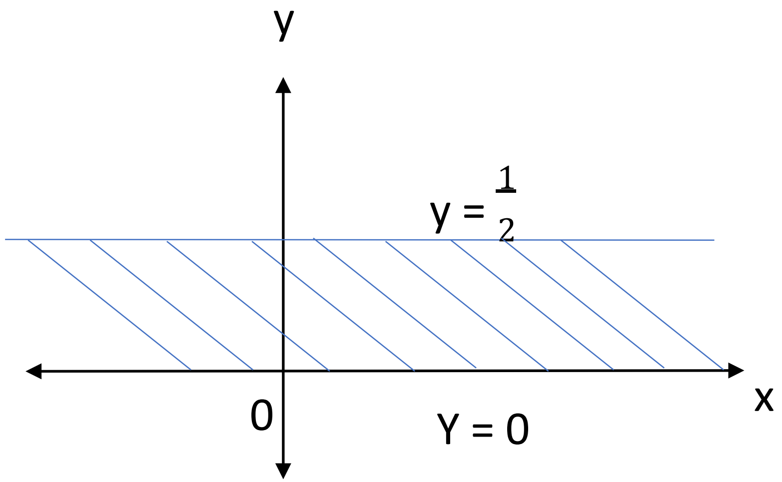
$w = \frac{1}{z}$

$z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv}\;\frac{u-iv}{u-iv}$

$x+iy = \frac{u-iv}{u^2+v^2}$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$



In w –plane **z - plane** $\frac{1}{2}$

i) y = ➔ $0 = \frac{-v}{u^2+v^2}$**0** **ii)** $y = \frac{1}{2}$ ➔ $\frac{1}{2} = \frac{-v}{u^2+v^2}$

$0 = -v$ $u^2 + v^2 = -2v$ **v = 0 Conclusion:** 1

The image of infinite strip 0 < y < is transferred as straight line (v=0) or circle under the transformation w = $\frac{}{2}$

π z

3) Find the image of the region in the z- $\frac{1}{2}$ plane between the lines y = 0 and y = under the transformation w = e 2

Solution: **In z –plane**

The lines are $y = 0,\ y = \frac{\pi}{2}$

Transformation

$w = e^z$

$u+iv = e^{x+iy} = e^x e^{iy}$ y = 0 $u+iv = e^x$

[cosy+isiny] $u = e^x \cos y$ $v = e^x \sin y$

In w-plane

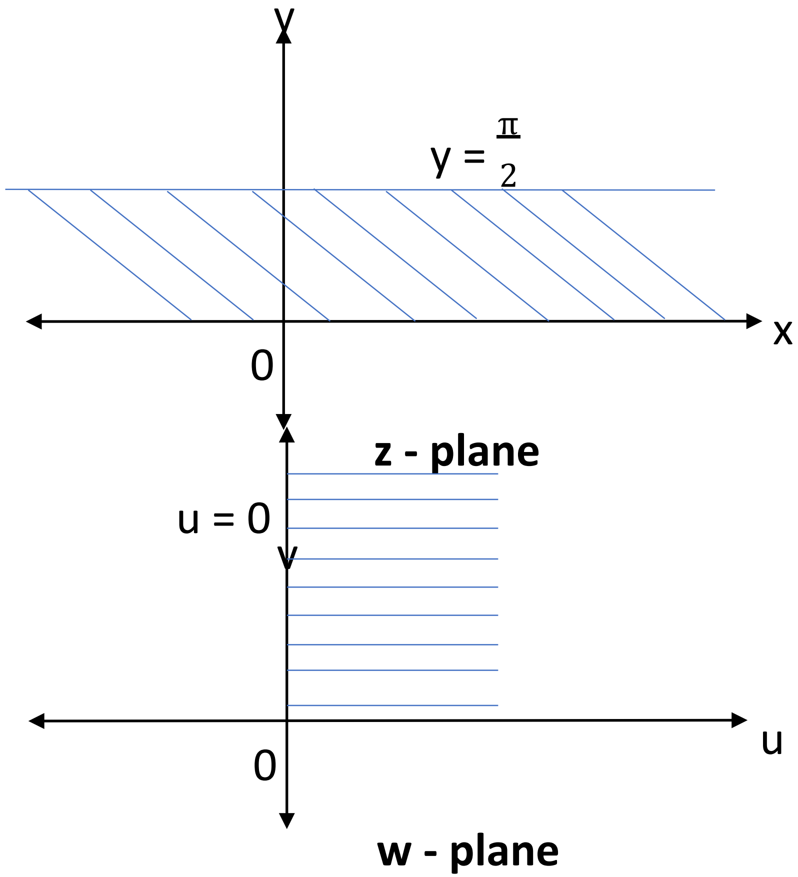
i) y =0 ❾ $u = e^x$**,** $v = 0$

π x

ii) y = ❾ u = 0, v = e

2

v = 0



Conclusion:

The image of the region lines $y = 0$ & $y = \frac{\pi}{2}$ are transferred as first quadrant in the w-plane under the transformation $w = e^z$

4) Show that transformation $w = z + \frac{1}{z}$ maps the circle $|z| = c$ into the eclipse $u = (c + \frac{1}{c}) \cos\theta$, $v = (c - \frac{1}{c}) \sin\theta$. Also discuss the case when c = 1 in detail.

Solution: **Z –plane** **Transformation**

1

The | | circle z = c

| |

w

$\sqrt{\quad}$ = z

+

z

$x + iy = c$ $w = r$ $e^{i\theta} + \frac{1}{re^{i\theta}}$

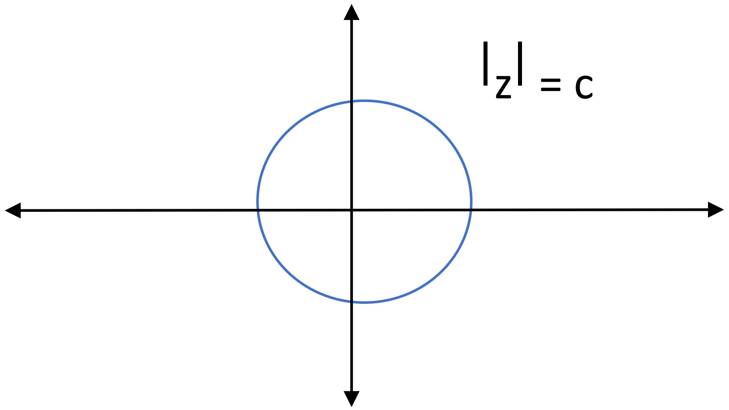
$x^2 + y^2 = c$ u+iv = r(rcos $\mathbf{x^2}$ θ + isinθ)+ $\frac{1}{r}$ (rcosθ - isinθ) $+\mathbf{y^2} = \mathbf{c^2}$ u+iv =

$(r+\frac{1}{r})\cos\theta + i(r-\frac{1}{r})\sin\theta$ $\mathbf{u = (r+\frac{1}{r})\cos\theta}$ $\mathbf{v = (r-\frac{1}{r})\sin\theta}$

w –plane

$|z| = c$

$|z| = r$ (r =c)



we know that $\mathbf{\cos^2\theta}$ **+**

$\mathbf{\sin^2\theta = 1}$ $\frac{\mathbf{u^2}}{(c+\frac{1}{c})^2} + \frac{\mathbf{v^2}}{(c-\frac{1}{c})^2} = 1$

Case: **When c = 1**

$|z| = 1$, $\frac{\mathbf{u^2}}{\mathbf{a^2}} + \frac{\mathbf{v^2}}{\mathbf{b^2}} = \mathbf{1}$ r = 1

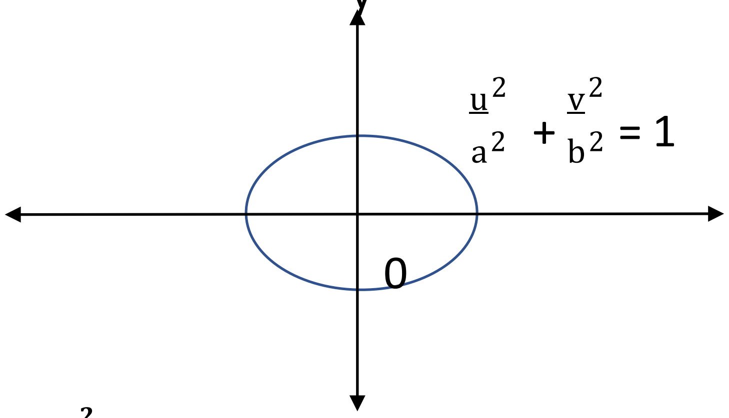
u =2 cosθ , v = 0

u+iv = 2 cosθ + i (0) 2 sinθx **u = 2** **v = 0**

Conclusion:

2

The image of circle z = c is transferred as eclipse **= 1** | | in w – plane and also the image of circle z = 1 when c = 1 is transferred as straight lines u =2 & v = 0 in w – plane under the transformation w = z + **.**



u $\frac{v^2}{2}$

5) Discuss the transformation of w = sinz using example.

$\frac{1}{z}$ **a²**

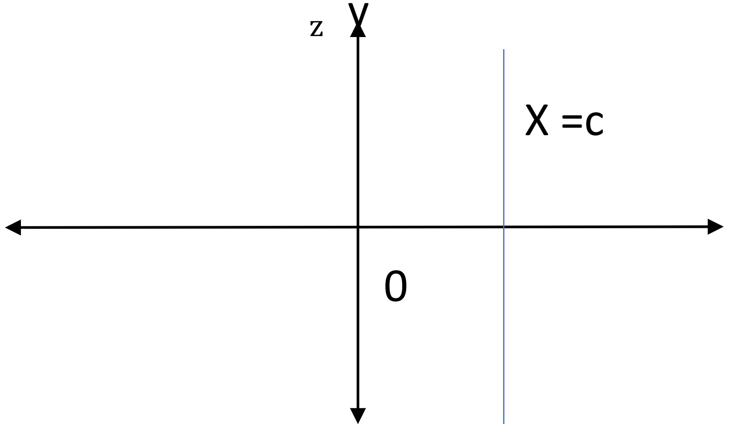
+ b

Solution: Transformation w = sinz

w = sin (x+iy) w = sinxcosiy + cosxsiniy

u+iv = sinxcoshy + icosxsinhyx

u = sinxcoshy v = cosxsinhy



Example: In z –plane **In w –plane**

| | u v z = 1 ,

x = c coshy = ____ , sinhy = ____

sinx cosx

x²+ y² = 1

put x = c $\frac{u^2}{sin^2x}$ - $\frac{v^2}{cos^2x}$

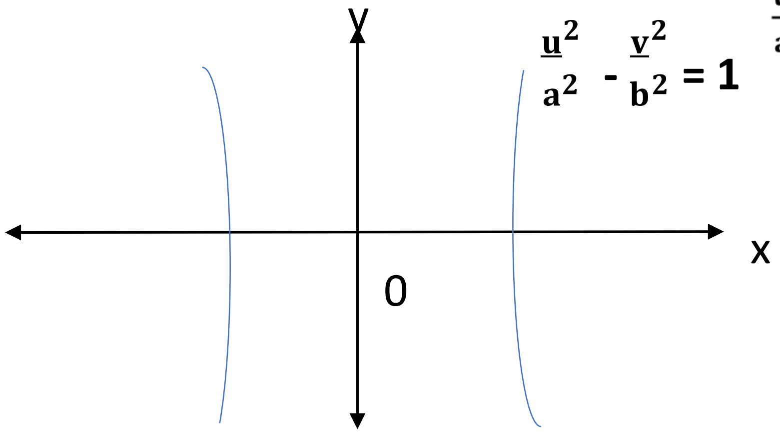
Conclusion: $\frac{u^2}{sin^2c}$ - $\frac{v^2}{cos^2c}$ = 1 cos 2 hy - sinh 2 y = 1

The image line x

= c is transferred as hyperbola = 1

$\frac{u^2}{a^2}$ - $\frac{v^2}{b^2}$ = 1 in w – plane under

the transformation w = sinz.



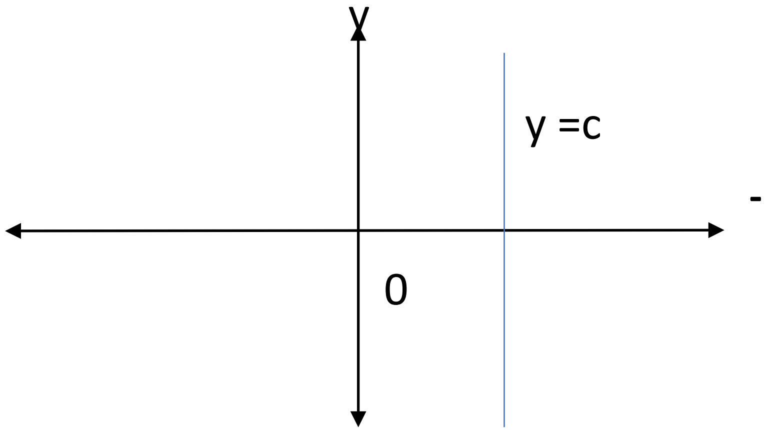
$\frac{u^2}{a^2}$ - $\frac{v^2}{b^2}$ = 1 $\frac{u^2}{a^2}$ - $\frac{v^2}{b^2}$ = 1

6) Discuss the transformation of w = cosz

Solution: Transformation on w = cosz

w = cos(x+iy) w = cosxcosiy – sinxsiniy u+iv = **v =** $\frac{u}{coshy}$,

cosxcoshy – isinxsinhyx **u = cosxcoshy**



sinxsinhy In

z- $\frac{v}{sinhy}$ **plane In w-plane y = c cosx = sinx = -**

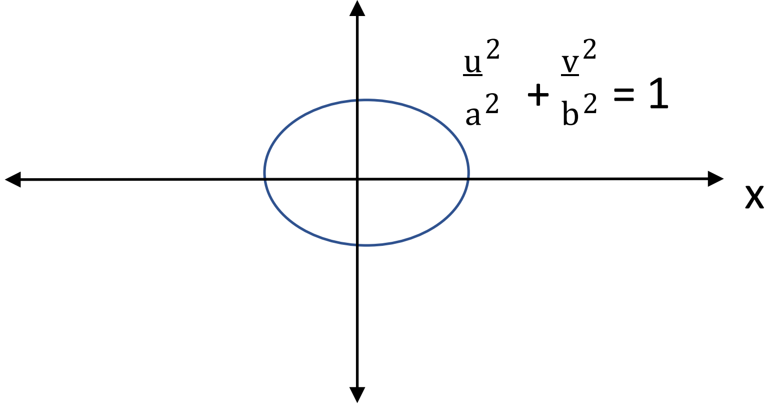
$\cos^2x + \sin^2x = 1$

$\frac{u^2}{cosh^2y} + \frac{v^2}{sinh^2y} = 1$

put y = c y

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$



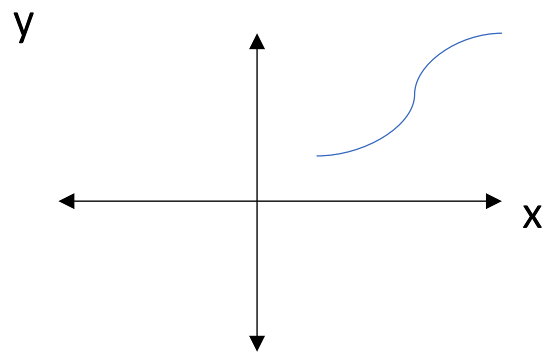
Conclusion:

The image of line y = c is transferred as ellipse $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$ under the transformation w = cosz.

Unit – 2
Complex Integration

Line Integral:

suppose f(z) is a complex function in the region R, and C is a smooth curve in R. Consider an interval (a, b) and a < $x_1 < x_2 \ldots < x_n$ < b are points in (a, b).



$\Delta x_r = x_r - x_{r-1}$ are chord vectors, then

$$\lim_{n\to\infty} \sum_{r=1}^{n} f(z) \, \Delta x_r = \int_a^b f(z)\, dz$$

Where the summation tends to a limit and independent of the points choice. The limit exists if f(z) is continuous along the path.

Evaluation of the integrals: $\int f(z)\, dz = \int (u + iv)(dx + idy) = \int (udx - vdy) + i\int(udy + vdx)$ where u and v are functions of x.

( )

Problems:

1) Evaluate $_c x^2 + ixydz$ from A(1, 1) to B(2, 8) along x = t and y = t^3.

Solution: Along x = t, y = t^3 , dx = dt, dy = 3 t^2 dt , The limits for t are 1 and 2 $_c($

( ))

x^2 +ixy (dx+idy) = $_c$ x^2dx−xydy)+i(xy dx+x^2dy

2 ^{2} dt - 3 t^6 dt + i4 t^4 dt = $\frac{t^3}{3}$-3 $\frac{t^7}{7}$+i4 $\frac{t^5}{5}$(apply the lower

= $_1$ t

and upper limit)

$$= -\frac{1094}{2} + \frac{124i}{5}$$

1+i ^{2} dz along y = x^2

2) Evaluate $_0$ z

1+i ^{2} dz along y = x^2 , dy = 2x dx

Solution: $_0$ z

1+i ^{2}- y^2+2ixy)(dx+ idy)

= $_0$ (x

1 ^{2}- x^4) dx - 2 x^3 2x dx + i(x^2- $x^4$2xdx+2 x^3dx)

= 0 (x
2 2

= - +i
3 3

3) Evaluate $\int_{1-i}^{2+i}(2x + 1 + iy)dz$ along (1-i) to (2+i).

Solution: Along (1-i) to (2+i) is the straight line AB joining (1,-1) to (2,1).

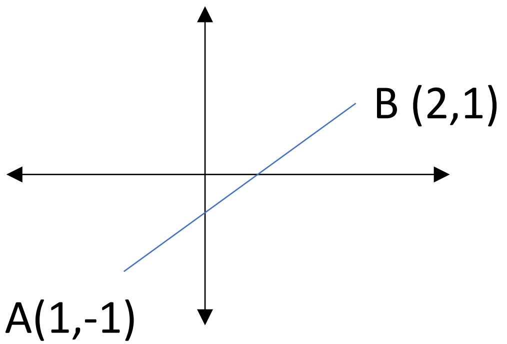
The equation of AB is y-1 = - $\frac{(-1-1)}{(1-2)}$ (x-2) y-2x = -3, y = 2x-3, dy = 2dx

X varies from 1 to 2

$\int_{1-i}^{2+i}(2x+1+iy)dz = \int_1^2 (2x+1\ dx) - (2x-3)2dx + i[2x-3]dx + (2x+1)2dx]$

$= \int_1^2 (-2x+7)dx + i(6x-1)dx$

$= -2\frac{x^2}{2} + 7x + i(6\frac{x^2}{2} - x)$ | (apply the lower and upper limit)



$\int_{1-i}^{2+i}(2x+1+iy)dz$ **= 4+8i**

4) Evaluate $\int_{(0,0)}^{(1,1)} [3x^2 + 5y + i(x^2 - y^2)]dz$ along $y^2 = x$.

Solution: Along $y^2 = x$, $2ydy = dx$, y varies from 0 to 1.

$_{(0}^{(1,0),3)}[3x^2+5y+i(x^2-y^2)][dx+idy] = {}_0^1\, 3y^4 2ydy+5y2y - (y^4-y^2)dy + i[(3y^4+5y)dy+(y^4-y^2)2ydy]$

$= 5\frac{y^6}{6}\ \frac{y^5}{5}\quad - \frac{y^3}{3}\quad + 1\frac{y^6}{6}\ \frac{y^5}{5} + i(2\frac{y^4}{4} + 3\frac{y^2}{2} - 2 + 5\quad)$ (apply the lower and upper limit)

$= \frac{129}{30} + \frac{44i}{15}$

$(1,3)\quad {}^2ydx+(x^2-y^2)dy$ along a) $y = 3x^2$ b) $y = 3x$.

5) Evaluate $(0,0)\ x$

Solution: a) $y = 3x^2$, $dy = 6xdx$, x varies from 0 to 1.

$_{(0}^{(1,0),3)}\, x^2ydx+(x^2-y^2)dy = {}_0^1\, 3x^4dx+(x^2-9x^4)6xdx$

(1,3) 5 4

(0,0) x2ydx +(x2 − y2)dy = 3 __

x5 + 6 x4 -54 x66

69

= - __

10

b) y = 3x , dy = 3dx, x varies from 0 to 1.

(1,3)

(0,0) $x^2ydx + (x^2 - y^2)dy$ =

$_0{}^1\, 3x^3dx + (x^2 - 9x^2)3dx$

x^4 x^3

= 3 __ - 24 __(apply the lower

4 3

and upper limit)

29

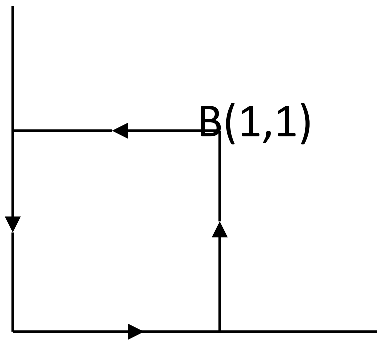
= - __

4

6) Evaluate $_c(3z + 1)dz$ where C is the boundary of the square with vertices at the points z = 0, z = 1, z =1+I, z = i and the orientation of C is anti-clockwise. **Solution: C is the square OABC**

$\int_c (3z+1)dz = \int_{c_1} (3z+1)dz + \int_{c_4} (3z+1)dz$

$+ \int_{c_2} (3z+1)dz + \int_{c_3} (3z+1)dz$



Along C_1 = OA

$y = 0$, $dy = 0$

X varies from 0 to 1 $\int_{c_1} (3z+1)\,dz = \int_0^1 (3x+1)dx = 3\frac{x^2}{2} + x$ (apply the lower and upper limit)

Z=0 0 Z=1

$= \frac{5}{2}$

Along c_2 = AB

x =1, dx =0 **y**

varies from 0 to 1

1 **3** $c_2(3z + 1)dz$ = i $_0$ [3 (1+iy)+1]dy = **4i - 2**

Along c_3= BC y =1, dy=0 x

varies from 1 to 0

0 **3** $c_3(3z + 1)dz$ = $_1$ [3 (x +

i)+1]dx = **-** **_2** -3i-1

Along c_4= CO x =0, dx=0 y

varies from 1 to 1

1 **3**) $c_4(3z + 1$ dz = 1

$[3iy + 1]$idx = **_2 -i**

5 3 **5** **3** $_c$ (‾ ‾)3z+1 dz= =**2** +**4*i***− **2** − **2**

−**3i**−**i**+ **_2** =**0**

$_c(3z+1)dz=0$

(1,1) 2 +4xy+ix2]dz along y = x2 **7)**

Evaluate (0,0) [3 x

Solution: $y = x^2$, $dy = 2xdx$,

(0(1,0),1)[3 x^2 +4xy+ix^2] = $_0{}^1$(3 x^2+4 x^3+i x^2)(dx+i2xdx)

1 2+4 x^3-2 x^3)dx + i(6 x^3+8 x^4+ x^2)dx

= 0 (3 x

$= 1 + 1 - \frac{1}{2} + i(\frac{3}{2} + \frac{8}{5} + \frac{1}{3})$ (apply the lower and upper limit)

$= \frac{3}{2} + \frac{103i}{30}$

8) Evaluate $\int_c (y^2 + 2xy)dx + (x^2 - 2xy)dy$ **,** where is the boundary of the region by $y = x^2$ and $x = y^2$

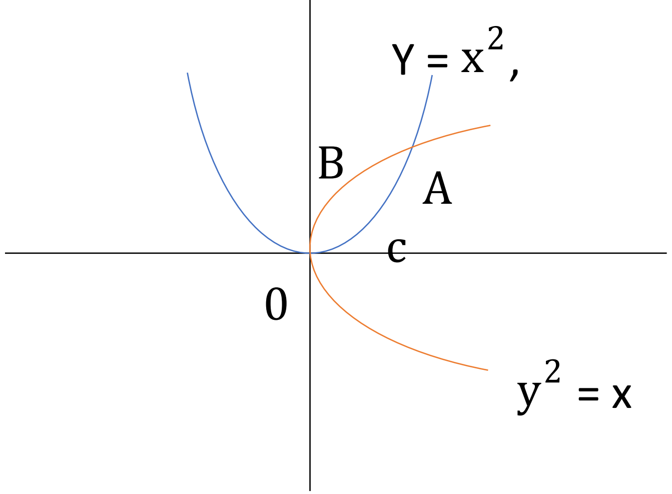
Solution:

C_1: Along OA, $y = x^2$, $dy = 2xdx$ X varies from 0 to 1 $\int_{c_1} (y^2 + 2xy)dx + (x^2 - 2xy)dy = \int_0^1 (x^4+2x^3)dx + (x^2$

$-2x^3)2xdx$ **=** $-\frac{2}{5}$ C_2: Along A**BO**, $x = y^2$, $dx = 2ydy$ y varies from 1 to 0 -

$\int_{c_2} (y^2 + 2xy)dx + (x^2 - 2xy)dy$ =

$$= \int_0^1 (y^2+2y^3)2y\,dy + (y^4 - 2y^3)dy = \mathbf{-1}$$

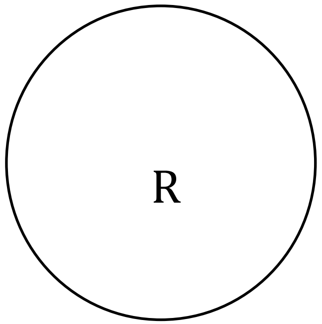


$$\int_c (y^2 + 2xy)dx + (x^2 - 2xy)dy = -1 + \frac{2}{5} = -\frac{3}{5}$$

Cauchy's theorem

If f(z) is analytical and $f'(z)$ is continuous inside and c' on a simple closed curve C, then $\int_c f(z)dz = 0$.

Proof: Suppose **R** is the region bounded by C f(z) = u+iv z = x+iy



Where C

$$\int_c f(z)dz = \int_c (u + iv)(dx + idy) = \int_c (udx - vdy) + i(udy + vdx)$$

Since $f'(z)$ is continuous, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous in R.

According to Green's theorem

$$\frac{\partial v}{\partial}\ \frac{\partial u}{\partial}$$

$$\oint_c udx + vdy = \iint_R \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) dxdy$$

$$\oint_c f(z)\, dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i \iint_R \left(\frac{\partial V}{\partial Y} - \frac{\partial U}{\partial X}\right) dxdy$$

Since f(z) is analytic $\oint_c f(z)\, dz = \iint_R \left(\frac{\partial U}{\partial Y} - \frac{\partial u}{\partial y}\right) dxdy + i \iint_R \left(\frac{\partial V}{\partial Y} - \frac{\partial V}{\partial Y}\right) dxdy$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\oint_c f(z)\, dz = 0$$

Cauchy's Integral Formula

If f(z) is analytical within and on a simple closed curve and c' a is any point inside C, then

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)dz}{(z-a)}$$

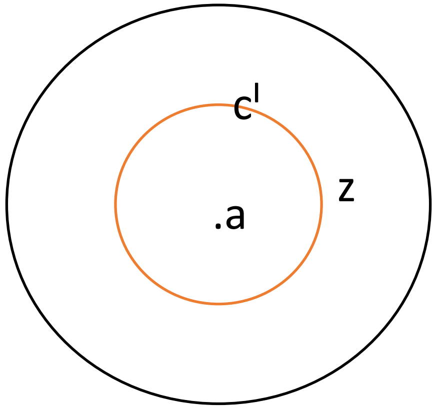
proof: C is a closed curve and a is any point inside C, Enclose a within a circle C whose radius is r and the centre is at a. Now C is inside C.

$\frac{f(z)}{(z-a)}$ is not analytical inside C.

By Cauchy's theorem for multiple connected region $\int_c g(z)\,dz = \int_{c'} g(z)\,dz$

( ) f(z) ( )

g(z) = (z−a) C



Where c' is |z−a| = r

$z - a = re^{i\theta}$, $z = a + re^{i\theta}$

$dz = rie^{i\theta}d\theta$

θ varies from 0 to 2π in c'

$\int_c \frac{f(z)\,dz}{(z-a)} = \int_{c'} \frac{f(z)\,dz}{(z-a)} = \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}}\, rie^{i\theta}\,d\theta = i\int_0^{2\pi} f(a + re^{i\theta})\,d\theta$

As r → 0, c' → 0

$$\int_c \frac{f(z)\,dz}{(z-a)} = i\int_0^{2\pi} f(a)\,d\theta = f(a)\,2\pi i$$

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)}$$

Cauchy's integral formula for the derivatives

f(z) dz 1

f(a) = c (**z**−**a**)

2πi

Differentiating with respect to a successively

f z dz 1
( )

$f^{I}(a)$ = 2πi c (z−a)2

2 f z dz

$f^{II}(a)$ = c (z−a)3

( )

~~2πi~~

( )

f iii (a) = 2.3 c (fz−az dz)4

2πi ( )

f iv (a) = 2.3.42πi c (fz−az dz)5

.

.

.

( )

f n(a) = 2n!πi c (z−af z)dzn+1

We can evaluate easily the integrals of complex functions using this formula.

Problems:

ze^{z}dz

1) Evaluate c ——(z+2)3 where C is **z** = 3. **Solution:**

z = -2 lies inside z = 3 | |

According to Cauchy's integral formula

( )

1 f z dz z a = -2] f^{II}(a) = c (z−a)3 ,

[f(z) = z e

πi2

f^{I}(z) = z e^{z} + e^{z} f^{II}(z) = z e^{z} +

2e^{z} f^{II}(-2) = - 2e^{-2} + 2e^{-2}=0

ze^z dz

c ——(z+2)3 = 0.

dz

2) Evaluate c ——z3(**z+4**) where C is **z** = 2 using Cauchy's integral formula.

Solution: z = 0 lies inside C and z = -4 lies outside.

According to Cauchy's integral formula

1

2 f(z)dz

f^{II}(a) = 2πi c (z−a)3 [a=0 —— and f(z)= (**z+4**)] f^{I}(z)= $-\frac{1}{(z+4)^2}$ f^{II}(z) = $\frac{2}{(z+4)^3}$ and f^{II}(0) = $\frac{1}{32}$

dz iπ

c z^3(z+4) = 32

$\frac{(z^3-\sin 3z)dz}{(z-\frac{\pi}{2})^3}$ | |

3) Evaluate c where C is **z** = 2 using Cauchy's integral formula.

Solution: **A**ccording to Cauchy's integral formula

1 f z dz (π) 3 = sin3z] $f^{II}(a)$ = c (z−a)3 [a= and f(z) = z

πi 2

$\frac{\pi}{2}$ <2, z = $\frac{\pi}{2}$ lies inside C: **z** = 2 | | $f^{I}(z) = 3z^2$-

3cos3z $f^{II}(z)$ = 6z+9 sin3z $f^{II}(\frac{\pi}{2})$ = 3π-9

f( ) z dz

c (z−a)3 = πi(3π-9)

dz

4) Evaluate *c* *ez*(z−1)3 where C is | **z** | = 2 using Cauchy's integral formula.

dz $e^{-z}dz$

Solution: *c* ________*ez*(z−1)3 = *c* ______(z−1)3

z = 1 lies inside C i.e|z|=2

f(z) = e^{-z}

According to Cauchy's integral formula

1 f( ) z dz c (***z***−***a***) =

f(a), [a =1] __ ( )

$2\pi i$

$1\ f\ z\ dz$ $f''(a) = \pi i$ $c\ (z-a)3$

$f'(z) = -e^{-z}$ $f''(z) = e^{-z}$, $f''(1) = e^{-1}$

$$\frac{e^{-z}dz}{c\ (z-1)3} = \frac{i\pi}{e}$$

5) Using Cauchy's integral formula evaluate $\frac{z4dz}{(z+1)(z-i)2}$ where C is ellipse and $9\ x2+4\ y2 = c$ 36.

Solution: $\frac{z^4dz}{c\ (z+1)(z-i)^2}$

$= c\ (z+1)(1+i)^2 - c\ (z-i)(1+i)^2 + (1+i)$ $\frac{z^4dz}{}$ $\frac{z^4dz}{}$ $\frac{1}{}$ $\frac{z^4dz}{}$ c

$(z-i)^2$ Splitting into partial fractions z = -1 and z = i lie inside $9\ x^2+4\ y^2 = 36$

$$\frac{1}{2\pi i} c \frac{f\ z\ dz\ (\ )}{(z-a)}$$

f(a) =

$$\frac{f(z)dz\ 1}{c\ (z-a)2} = f'(a)$$

$2\pi i$

$f(z) = z^4$, a = -1, f(-1) = 1, a=I, f(i) $\frac{1}{(1+i)^2}$ = 1 $\frac{1}{(1+i)}$

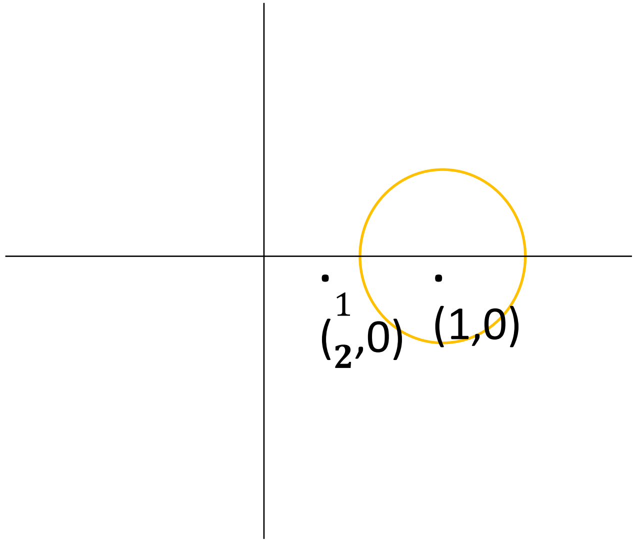
$f'(z) = 4z^3$ and $f'(i) = -4i$

$\oint_c \frac{z^4 dz}{(z+1)(z-i)^2} = \frac{1}{(1+i)^2}\, 2\pi i - 2\pi i + 2\pi i\,(-4i)$

$= \frac{8\pi}{(1+i)} = 4\pi(1-i)$

6) Evaluate $\oint_c \frac{\log z\,dz}{(z-1)^3}$ where C is $|z-1| = \frac{1}{2}$ using Cauchy's integral formula

Solution:



According to Cauchy's integral formula

1 f z dz $f^{II}(a)$ ( )

2 $\oint_c \frac{}{(z-a)^3}$ = 2! [

a =1] πi 1

$|z-1| = \frac{1}{2}$ is a circle whose centre is (1,0)

1 1 and

radius is $\frac{1}{2}$, a=1 lies inside C $\frac{f(z)dz}{}$ $-\frac{1}{z^2}$

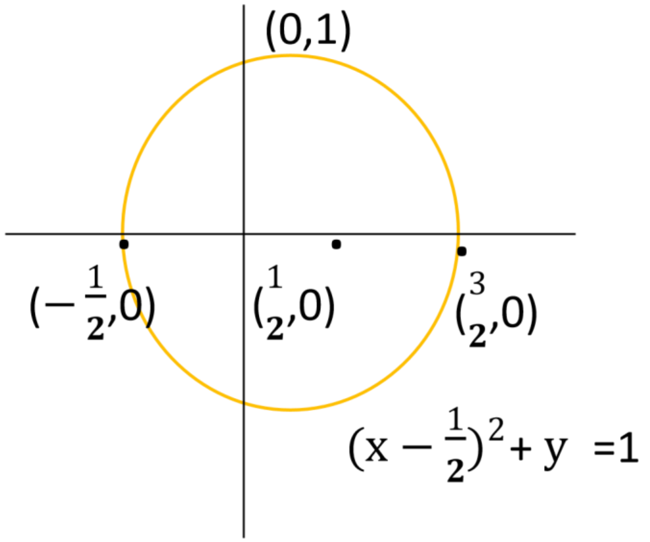
f(z) = logz, $f^I(z)=\frac{1}{z}$, $f^{II}(z) = -\frac{1}{z^2}$, $f^{II}(1) = -1$

$f^{II}(a) = \pi i \int_c \frac{f(z)dz}{(z-a)^3}$

$= -\pi i \int_c \frac{\log z\,dz}{(z-1)^3}$

7) Evaluate $\int_c \frac{(z^2-z-1)dz}{z(z-i)^2}$ where C is $\left|z-\frac{1}{2}\right| = 1$

Solution:



According to Cauchy's integral formula

$\int_c \frac{f(z)dz}{(z-a)} = 2\pi i f(a)$

z =0 inside C and z=i is outside C

f(z) = $\frac{(z^2-z-1)}{(z-i)^2}$, [a=0, f(0) =1]

$\int_c \frac{(z^2-z-1)dz}{z(z-i)^2}$ = **2πi**

$(3z^2+7z+1)dz$

9) If F(a) = c (z−a) using Cauchy's integral formula where C is |z| = 2, F(1), F(3), $f^{II}(1-i)$.

(3z²+7z+1)dz

Solution: Suppose F(a) = c (z−a)

(3z²+7z+1)dz

F(1) = **, [z=1 lies inside C]**

c (z−1)

f(z)dz

c (z−a) = 2πi f(a)

[f(z) = 3z² +7z+1, f(1) = 3+7+1 =11]

(3z +7z+1)

2 c =

2πi 11 = 22 πi = F(1)

(z−3)

2

(3z +7z+1)

F(z) = c ―――― dz, [z=3 is outside C]

(z−3)

(3z +7z+1)

2 c

―――― = 0 = F(3)

(z−3)

a = 1-i is inside C

F(a) = 2πi(3 a²+7a+1)

$F^I(a)$ = 2πi(6a+7)

$F^{II}(a) = 12\pi i$

$F^{II}(1-i) = 12\pi i$

Complex Power Series

Taylor's Theorem:

If f(z) is analytic inside and a simple closed circle C with centre at a, then for z inside C f(z) = f(a) + $f^{I}(a)(z-a) + \frac{f^{II}(a)}{2!}(z-a)^2 + \frac{f^{III}(a)}{3!}(z-a)^3 + \ldots$

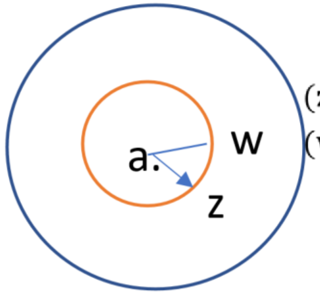
Proof: Let Z be any point inside C, then enclose z with a circle c^{I}, with centre at a , let w be a point on c^{I}, then

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

$$= \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\frac{(z-a)^2}{(w-a)^2}+\frac{(z-a)^3}{(w-a)^3}+\ldots+\frac{(z-a)^n}{(w-a)^n}+\ldots\right]$$

$|z-a| < |w-a|$

$\left|\frac{z-a}{w-a}\right| < 1$



Therefore, this series converges uniformly multiplying both sides by f(w) and integrating with respect to w on c^{I}

$$\int_{c^{I}} \frac{f(w)}{(w-z)}dw = \int_{c^{I}} \frac{f(w)}{(w-a)}dw + (z-a)\int_{c^{I}} \frac{f(w)}{(w-a)^2}dw + (z-a)^2\int_{c^{I}} \frac{f(w)}{(w-a)^3}dw + \ldots + (z-a)^n\int_{c^{I}} \frac{f(w)}{(w-a)^{n+1}}dw$$

f(w) is analytic on c^{I}

$\frac{1}{}$ $\frac{f(w)\,dw}{(w-z)}$ c^{I}

$f(z) = \frac{(f)^{n}}{}$ 2πi $\frac{1}{}$ $\frac{(\ )}{}$

(a) f w dw

and n! = 2πi c^{I} $(w-a)_{n+1}$ Dividing by 2πi

$\frac{1}{}f$ w dw $\frac{1}{}f$ w dw $\frac{(z-a)}{}$ f w dw $\frac{(z-a)}{}^{2}$ f w dw $\frac{(z-a)}{}^{n}$ f w dw $\frac{(\ )}{}$ c^{I} (w−z) = 2πi c^{I} $\frac{(\ )}{(w-a)}$ + 2πi c^{I} $\frac{(\ )}{(w-a)_2}$ + 2πi $(w-a)_3$ +...+ 2πi $\frac{(\ )}{}$ c^{I}

$(w-a)_{n+1}$+...

2πi

(z−a) (z−a)

$2n$ f(z) =

$f(a)+(z-a)\,f^{I}(a)+ \_\_\_\_\_ f^{II}(a)+...+ \quad (f)^{n}(a)+...$

2! ! n

This is Taylor's series of f(z)

if z-a = h

h2 hn

$f(a+h)=f(a) + h\,f^{I}(a)+ f^{II}(a)+...+$ n! (a)+... 2! — — $(f)^{n}$ if

a=0, h=z

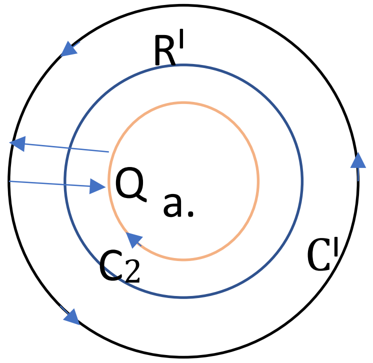
z2 zn n

—

$f(z)=f(0) + z\,f^{I}(0)+$ __2! $f^{II}(a)+...+$ n!

(f) (a)+...

This is a Maclaurin's series of f(z)



Laurent series

If f(z) is analytic in a ring R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2, ($r_1 > r_2$) with centre at a then for all z in R P $f(z) = a_0 + a_1 (z-a)+ a_2 (z-a)^2+\ldots+ \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots$ C1

f w dw 1 ( )

Where $a_n = 2\pi i\ C_1\ (w-a)_{n+1}$ ( )

f w dw 1

and $b_n = 2\pi i\ C_2\ (w-a)_{-n+1}$

Where c' is any curve in R encircling C_2

Proof: Consider cross cut PQ and f(z) is analytic in the region R' bounded by PQ, z is any point in R'.

f w dw 1 ( ) f w dw ( ) f w dw ( ) f w dw

f(z) = [PQ $\frac{1}{2\pi i}$ (w−z) - C_2 (w−z) - QP (w−z) + C_1 (w−z)]

( ) ( )

f w dw f w 1 dw

$f(z) = \frac{1}{2\pi i}$ [C_1 (w−z) - C_2 (w−z)] ❼ **Equation 1**

∞

Where C_1 and C_2 are described anticlockwise

Consider

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-z)} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-a)} + \frac{(z-a)}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-a)^2} + \dots + \frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-a)^{n+1}} + \dots$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-a)^{n+1}}$$

$$= \sum_{n=0}^{\infty}(z-a)^n a_n$$ ❼ **Equation 2**

Where $$a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)\,dw}{(w-a)^{n+1}}$$

Consider $$\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)\,dw}{(w-z)}$$

For C_2, w-a < z-a

$\left|\frac{w-a}{z-a}\right|<1$

$= + +...]$

$\frac{1}{(w-z)} = \frac{1}{w-a-(z-a)} = \frac{1}{(z-a)(1-\frac{w-a}{z-a})}$

f w dw (z−a)³ 1 f w dw f w dw

C 2πi C2 (z−a) + 2πi C2 (w−a)−1 +

c2 2πi (w−a)−3

$= \frac{1}{(z-a)}\left[1-\frac{w-a}{z-a}\right]^{-1}$

$= \sum b \quad \frac{1}{(z-a)}\left[1+\frac{w-a}{z-a} + \frac{(w-a)^2}{(z-a)^2} + \frac{(w-a)^3}{(z-a)^3}\right.$

2πi

$_n (z-a)^{-n}$ ❼ **equation 3** Where

f w dw

$- \frac{1}{} = \frac{f(w)dw}{2\ (w-z)}$ (z−a)²

$b^n = \frac{1}{2\pi i} \int_{C_2} \frac{(\ )}{(w-a)^{-n+1}}$

Substituting equations 2 & 3 in 1, we get $f(z) = \sum_{n=0}^{\infty}(z-a)^n a_n + \sum_{n=1}^{\infty}(z-a)^{-n} b_n$ This is called the Laurent series of f(z)

The first part $\sum_{n=0}^{\infty}(z-a)^n a_n$ is called the analytic part and the second part $\sum_{n=1}^{\infty}(z-a)^{-n} b_n$ is called the principal part. If the principal part is zero, the series reduces to the Taylor's series

Problems

1) Expand log z by Taylor's series about z = 1.

Solution: The given function is f(z) = log z

Taylor's series is

$f(z) = f(a) + f^{I}(a)(z-a) + \frac{f^{II}(a)}{2!}(z-a)^2 + \frac{f^{III}(a)}{3!}(z-a)^3 + ... + \frac{f^n(a)}{n!}(z-a)^n + ...$

a=1, f(1) = 0

$$f^{I}(z) = \frac{1}{z}, \quad f^{I}(1) = 1,$$

$$f^{II}(z) = -\frac{1}{z^2}, \quad f^{II}(1) = -1,$$

$$f^{III}(z) = \frac{2}{z^3}, \quad f^{III}(1) = 2, \qquad f^{iv}(z) = \frac{-3!}{z^4}, \quad f^{iv}(1) = -3!$$

$$\log z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \ldots + \frac{(-1)^{n-1}}{n}(z-1)^n + \ldots$$

2) Obtain all the Laurent series of the function $\frac{7z-2}{(z+1)z(z-2)}$ about $z = -1$

Solution: $f(z) = \frac{7z-2}{(z+1)z(z-2)}$

put $z+1 = u$, $z = u-1$, $z-2 = u-3$

$$\frac{7z-2}{(z+1)z(z-2)} = \frac{7(u-1)-2}{u(u-1)(u-3)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u-3}$$

$7u-9$

A = lim = -3

u→0 u−1 (u−3) 7u−9

B = lim = 1

u→1 u (u−3)

7u−9 C

= lim = 2 u→3 u−1 u

( )

- 3 + 1 + 2 = - 3 - 1 − u −1 (2) 1 − u −1 u u−1

u−3 u3 3

(−)

= - 3 - (1+u+ u^2+ u^3+...) - (1+ u + u2 +...) u 3 9

2/3 − —

= - u3 - 53 -(1 + —322)(z+1) -(1 + — 322) $(z+1)^2$−(1 + —$3^2$4) $(z+1)^3$+...

3) Expand $\frac{1}{(z^2-3z+2)}$ is the region

(i) $0 < |z-1| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

Solution:

(i) $\frac{1}{(z^2-3z+2)}$ $\frac{1}{(z-2)}$ $\frac{1}{(z-1)}$ = -

$|z-1| < 1$

$$\frac{1}{(z-2)} - \frac{1}{(z-1)} = \frac{1}{(z-1-1)} - \frac{1}{(z-1)}$$

$$= -\frac{1}{[1-(z-1)]} - \frac{1}{(z-1)} = -\left(1-(z-1)\right)^{-1} - \frac{1}{(z-1)}$$

$$= -\left(1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right) - \frac{1}{(z-1)}$$

(ii) $1<|z|$, $|z|<2$, $\frac{|z|}{2}<1$, $\frac{1}{|z|}<1$

$$\frac{1}{(z-2)} - \frac{1}{(z-1)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

(iii) $|z|>2$, $2<|z|$, $\frac{2}{|z|}<1$, $\frac{1}{z}<1$

$$\frac{1}{(z-2)} - \frac{1}{(z-1)} = \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

(1+ + ...) - z z z z z z

n−1

$= \frac{1}{z}$ $(2 = \sum_{n=1}^{\infty} \frac{}{n}$ $= \sum_{n=1}$ $\frac{2}{} + \frac{2^2}{2}$ $\frac{1}{}(1 + \frac{1}{} + \frac{1}{2} + \cdots)$

1 $n-1$ **-1)**

∞ $\frac{2}{z^n}$ - ∞ $\frac{1}{}$ $n=1$ z^n

4) Find the Laurent series expansion of the function $\frac{(z^2-1)}{(z+2)(z+3)}$ if $2 < z < 3$.

Solution:

$$f(z) = \frac{(z^2-1)}{(z+2)(z+3)} = 1 - \frac{(5z+7)}{(z^2+5z+6)}$$

$$= 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$$

$$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \ldots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \ldots\right)$$

$$= 1+3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{n-1}}{z^n} + 8\sum_{n=1}^{\infty}\frac{(-1)^n z^{n-1}}{3^n}$$

$$= \mathbf{1}+\sum_{n=1}^{\infty}(-1)^n\left(-\frac{2^{n-1}}{z^n}+\frac{8z^{n-1}}{3^n}\right)$$

5) Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about z=1 as Laurent series. Also indicate the region of convergence of the series.

Solution: $f(z) = \frac{e^{2z}}{(z-1)^3}$

put z-1 =u, z= 1+u

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2(1+u)}}{(u)^3} = \frac{e^2e^{2u}}{(u)^3} = \frac{e^2}{u^3}\left(1+2u+\frac{(2u)^2}{2!}+\ldots\right)$$

$$= \frac{e^2}{(z-1)^3}\left(1+2(z-1)+\frac{(2(z-1))^2}{2!}+\ldots\right)$$

$$= \mathbf{e^2}\left(\frac{1}{(z-1)^3}+\frac{1}{(z-1)^2}+\frac{2}{z-1}+\cdots\right)$$

6) Express $f(z) = \frac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of z-1.

Solution: $f(z) = \frac{z}{(z-1)(z-3)}$

$$\frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$\frac{z}{(z-1)(z-3)} = \frac{A}{(z-1)} + \frac{B}{(z-3)}$$

$$A = \lim_{z\to 1} \frac{z}{(z-3)} = -\frac{1}{2}$$

$$B = \lim_{z\to 3} \frac{z}{(z-1)} = \frac{3}{2}$$

$$f(z) = \frac{3}{2(z-3)} - \frac{1}{2(z-1)} = \frac{3}{2(z-1-2)} - \frac{1}{2(z-1)}$$

$$= \frac{3}{-4\left(1-\frac{z-1}{2}\right)} - \frac{1}{2(z-1)}$$

$$= -\frac{3}{4}\left(1-\frac{z-1}{2}\right)^{-1} - \frac{1}{2(z-1)} - \frac{3}{4}\left(1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\ldots\right) - \frac{1}{2(z-1)}$$

$$= \frac{1}{2(z-1)} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

Contour Integration

Singular points

Singular point: A point at which f(z) ceases to be analytic is called a singular point.

Isolated singular point: Suppose z=a is a singular point of a function f(z) and no other singular point of f(z) exists in a circle with centre at a, then z=a is said to be an isolated singular point.

In such a case f(z) can be expanded by Laurent series around z=a

Pole: If the principal part of f(z) consists of a finite number of terms $b_1, b_2 ... b_n$ $b_n \neq$

0 then (z-a) is said to be a pole of order n.

if n=1, z=a is said to be a simple pole.(note: if f(z) has a pole at z=a, then $\lim_{z \to a} f(z) = \infty$)

Removable singularity: If a single valued function f(z) is not defined at z=a and $\lim_{z \to \infty} f(z)$ exists, then z=a is said to be a removable singularity $f(z) = \frac{\sin z}{z}$, z=0 is a removable singularity.

Essential singularity: If the principal part of f(z) consists of an infinite number of terms, then z=a is said to be an essential singularity

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$ z=0 is an essential singularity.

Singularity at infinity: Suppose we substitute $z = \frac{1}{w}$, $f(\frac{1}{w}) = F(w)$ (say), then the singularity at w=0 of F(w) is called the singularity at infinity. e^z has an essential singularity at z =∞, since $e^{\frac{1}{z}}$ has an essential singularity at z=0.

Entire function: A function which is analytic everywhere in the finite plane is called an entire function or integral function.

Examples: e^z , sin z, cos z are entire functions.

Note: An entire function can be represented by a Taylor series which has an infinite radius of convergence. Conversely, if a power series has an infinite radius of convergence, it represents an entire function.

Liouville's theorem: If f(z) is analytic and bounded, i.e $|f(z)| < m$ for some constant m in the entire complex plane, then f(z) is a constant.

Residue: We know that $\int_c \frac{dz}{(z-a)} = 2\pi i$ where C is $|z-a| = R$ and $\int_c \frac{dz}{(z-a)^n} = 0$, if $n \neq -1$.

$\int_c f(z)\,dz = 2\pi i\, b_1$ where C is the circle with centre at a and f(z) is expanded in Laurent series. b_1 is said to be the residue of f(z) at z=a [the coefficient of $\frac{1}{(z-a)}$ in the principal part of the Laurent series of f(z)].

Cauchy's Residue Theorem:

Statement: If f(z) is an analytic function inside and on a closed curve 'C' except at a finite number of points, inside C, then

$\int_c f(z)\,dz = 2\pi i$ (sum of the residues at the points where f(z) is not analytic and which lie inside C).

If the poles of order one and n then the residues are

$\lim_{z\to a} \frac{d}{dz}[(z-a) f(z)]$, $\lim_{z\to a} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$

1) Find the poles of the function and the corresponding residues at each pole, $f(z) = \frac{e^{iz}}{(z^2+1)}$

Solution: The given function is $f(z) = \frac{e^{iz}}{(z^2+1)}$, f(z) is not analytic at z = i and z = -i

Therefore, the poles of f(z) are i and -i, both are simple poles

If z=a is a simple pole, then the residue at z= a is $\lim_{z\to a}(z-a)f(z)$

Res z=i= $\lim(z-i)f(z) = \lim(z-i)\frac{e^{iz}}{}$ $= -\, i\, e^{-1}$

$z\to i z\to i$ $(z+i)(z-i)$ **2**

e^{iz} **i**

Res z= -i= $\lim(z+i)f\,z = \lim(z+i)$ ______ $\_\,( )$ = **e.** $z\to -i z\to -i$ $(z-i)(z+i)$ **2**

$\sin 2z$

2) Find the poles of the function and the corresponding residues at each pole, f(z) = $\frac{\sin^2 z}{(z-\frac{\pi}{6})^2}$.

$\sin^2 z$ π

Solution: The given function is f(z) = $\frac{\sin^2 z}{(z-\frac{\pi}{6})^2}$, z- $\frac{\pi}{6}$ is a double pole

6

2 π

sin

$\_$ 6π $\lim_{\pi}$– dzd $z(z-\pi\, 6)2$

Res at z = =

$(z-\frac{\pi}{6})^2$

$z\to 6$

$$= \lim_{z\to\frac{\pi}{6}} 2\sin z\cos z = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

z sinz

3) Find the residue of $\frac{z\sin z}{(z-\pi)^3}$ at z = π.

z sinz

Solution: The given function is f(z) = $\frac{z\sin z}{(z-\pi)^3}$, z = π is a pole of order 3

If z = a is a pole of order 3, then residue at z = a is

1

[(z − a)

$\lim_{z\to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}$ $^n f(z)]$ (a = π)

$\frac{1}{2} \frac{d^2}{dz^2}$

Res at z = π = $\lim_{z\to\pi} \frac{d^2}{dz^2}$ (z sinz)

2

1 d

= lim (z 2 =

cosz + sinz) $_{z\to\pi}$ dz lim (cosz 1

$_{z\to\pi}$ 2

– z sinz + cosz) = **-1**.

4) Evaluate $\int_c \frac{(\cos\pi z^2+\sin\pi z^2)dz}{(z-1)^2(z-2)}$ where C is |z| = 3.

Solution: The given function is f(z) = $\frac{(\cos\pi z^2+\sin\pi z^2)}{(z-1)^2(z-2)}$, z = 1 is a double pole and z = 2 is a simple pole, both lie inside C.

Res at z = 1 = $\lim_{z\to 1} \frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to 1} \frac{d}{dz}\left(\frac{(\cos\pi z^2+\sin\pi z^2)}{(z-2)}\right)$

= $\lim_{z\to 1} \frac{(z-2)(-2z\sin\pi z^2+2z\cos\pi z^2)-\cos\pi z^2-\sin\pi z^2}{(z-2)^2}$ = 3

$\lim_{z\to 2} \frac{(\cos\pi z^2+\sin\pi z^2)}{(z-1)^2}$ = **1**

Res at z = 2 = lim z −

2 f(z) = lim z→2

According to residue theorem

(cosπz²+sinπz²)dz

−( ) = 2 πi(sum of the residues) = 2 πi(3+1) =8 πi c z−1

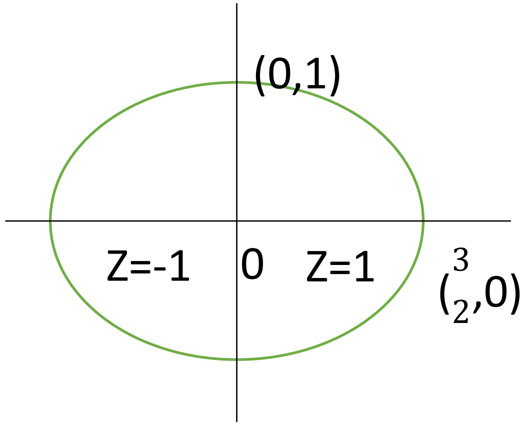
2(z−2)

z secz dz ²+9 y²=9

5) Evaluate c 1−z2 where C is 4 x

z secz **Solution:**

The given function is f(z) = 1−z2 z=1 and -1 are simple poles and $4x^2+9y^2=9$ is a ellipse whose semi minor and major axes are 1 and $\frac{3}{2}$ .1 and -1 both lie inside C.



z secz sec1

Res at z=1 = lim (z-1)f(z) = lim - = -

z→1 z→1 z+1 2 z secz sec1

Res at z= - 1 = lim (z+1)f(z) = lim - = -

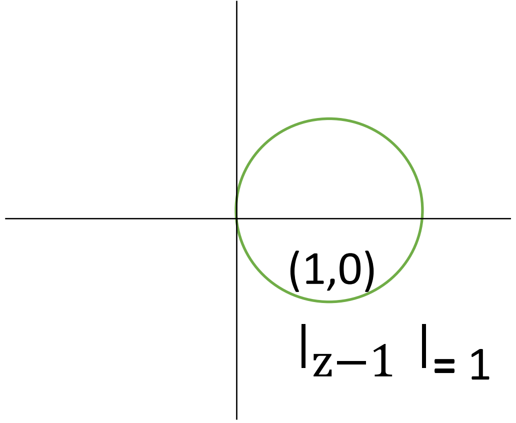
z→−1 z→−1 z−1 2

z secz dz c 1−z2 = 2 πi (sum of the residues, by residue theorem)

= 2 πi (-sec 1) **= - 2 πi (sec 1)**

e^z dz

6) Evaluate $_c$ $\frac{e^z dz}{(z+2)(z-1)}$ Where C is the circle $|z-1| = 1$.



Solution: The given function is f(z) = $_c$ $\frac{e^z}{(z+2)(z-1)}$ **,** z =-2 and 1 are simple poles , z=1 lies inside C and z =-2 lies outside C.

e^z $\quad$ $\underline{e}$ $\quad$ Res at z=1 = lim

(z1)f(z) = lim **=** $\quad z\to1 \quad z\to1 \quad z+2 \quad 3$ $\quad \overline{\quad}$ ( )

$_c$ **f(z) dz = 2 πi**

(sum of residues at the poles which lie inside C)

e^zdz 2 πie

c (z+2)(z−1) = **3**

Evaluation of real integrals in unit circle

2π

We can evaluate the integrals of the type $_0$ f(cos θ, sin θ)dθ where f(cos θ, sin θ) is a rational function, using residue theorem.

iθ, we can write cos θ ________ =

$e^{i\theta}+e^{-i\theta}$ we know that if z = e

2

1 $\quad e^{i\theta}-e^{-i\theta}$ $\quad$ cos θ = $\frac{1}{2}$ (z+ ―) and sin θ = ________

z $\quad$ 2i

$\underline{1}$ $\quad$ 1

$\sin\theta = (z- \;) \; 2i \; z$

i

$e^{i\theta}$

$d\theta =$

dz

and

$d\theta =$

—

dz

iz

By this substitution we can change the integral into a function of z.

We know that $_c f(z)dz = 2\pi i$ (sum of the integrals) We

| |

take C is z =1, then θ varies from 0 to 2π

2π

0 $f(\cos\theta, \sin\theta)d\theta = {}_c g(z)dz$ where C is z =1

| |

1 1 1 dz g(z) = f [2 (z+) , (z-)] — z 2i z iz

We can evaluate using residue theorem

Problems

2π $d\theta$ 2π

1) Show that $0 \quad a+b\sin\theta = \sqrt{a^2-b^2}$ **,** a>b>0 using residue theorem.

Solution: Consider C = |z| =1, z = $e^{i\theta}$

1 1 1

$\cos\theta = \frac{1}{2}(z+ \quad)$, $\sin\theta = \_(z- \quad)$

z 2i z

$2\pi \quad d\theta \qquad dz$

$0 \quad a+b\sin\theta = \quad c \; iz[a+2b\, i(z- 1z)]$

2

f(z) = [$\_\_\_\_\_\_ bz^2+2aiz-b$]

2 dz

$c\, f(z)dz = c \_\_\_\_\_\_ bz^2+2aiz-b\, dz$

$bz^2 + 2aiz - b = b(z-\alpha)(z-\beta)$

2ai

where $(\alpha+\beta) = -$, $\alpha\beta = -1$

b

$-ai+i\sqrt{a^2-b^2} \qquad -ai-i\sqrt{a^2-b^2}$

$\alpha =$ and $\beta =$ b b

$|\alpha| < 1$ and $|\beta| > 1$ α lies in C $\quad c\, f(z)\, dz = 2\pi i$ Res Z = α

2

Res Z = α = lim (Z - α) f(z) = lim $\_\_\_\_$

$z\to\alpha \qquad z\to\alpha \; b(z-\beta)$

$= \frac{2}{b(\alpha-\beta)}$

$$= \frac{2}{b\left[\frac{-ai+i\sqrt{a^2-b^2}}{b} + \frac{ai+i\sqrt{a^2-b^2}}{b}\right]}$$

$$= \frac{1}{i\sqrt{a^2-b^2}}$$

$$\int_c f(z)dz = \frac{1}{i}\int \frac{2\,dz}{bz^2+2aiz-b} = \frac{2\pi i}{i\sqrt{a^2-b^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

2) Evaluate $\int_0^{2\pi} \frac{d\theta}{(6-3\cos\theta)^2}$ using residue theorem

Solution: $\int_0^{2\pi} \frac{d\theta}{(6-3\cos\theta)^2}$

Substitute $z = e^{i\theta}$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$dz = i\,e^{i\theta}d\theta$ and $d\theta = \frac{dz}{iz}$

$2\pi \quad d\theta \quad dz \quad 4z\,dz$

$$\left(z+\frac{1}{z}\right) \quad = \ _c \frac{9i(z^2-4z+1)^2}{}$$

$$0 \quad (6-3\cos\theta)^2 = \ _c\ 32\ iz[6-\ ]^2$$

The poles are α and β where α = 2 - 3 and β = 2 + 3 and both are double poles, among which α lies inside C.

$$\text{Res at } z = \alpha = \lim_{z\to\alpha} \frac{d}{dz}\left[(Z-\alpha)^2 f(z)\right]$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{z}{(Z-\beta)^2}\right] = \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$$

(α + β) = 4, α – β = -2 3 Res at z=α = $\frac{4}{24\sqrt{3}} = \frac{1}{6\sqrt{3}}$ =

$$\int_c \frac{4z\,dz}{9\,i(z^2-4z+1)^2} = \frac{4}{9i}\ 2\pi i\ \frac{1}{6\sqrt{3}} = \frac{\mathbf{4\pi}}{\mathbf{27\sqrt{3}}}$$

3) Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$, a>b>0 using residue theorem

Solution: $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ put z = $e^{i\theta}$, dz = $e^{i\theta}$ dθ $\frac{dz}{}$ =dθ cos θ = $\frac{1}{2}(z+\frac{1}{z})$ izz

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \int_c \frac{4z\,dz}{i(2az+bz^2+b)^2}$$

The poles are α and β, both are double poles

$$\frac{-a+\sqrt{a^2-b^2}}{} \qquad \frac{-a-\sqrt{a^2-b^2}}{}$$

Where $\alpha =$ and β

$= b\ b$

a lies inside C

$\underline{d}$ z

Residue at z = α = $z \to \lim \alpha$ dz $[b2(Z - \beta)2]$

$\underline{1}(\underline{\alpha + \beta})$

= - ( ) 2 2

$b(\alpha - \beta)$

2 $\underline{1} - 2ab^3$ a

$= - b(b8(a2 - b2)\underline{32}) = 4(a2 - b2)\ \frac{3}{2}$

2π dθ

0 (a+bcosθ)2 = 2πi (Res z = α by residue theorem)

$2\pi i a^4$ **2πa**

2 = $\underline{3}$ = **$\underline{3}$**

$4i(a^2 - b^2)(\mathbf{a2 - b2})\mathbf{2}$

Contour integration when the poles lie on imaginary axis

f(x)

We can evaluate integrals of the type

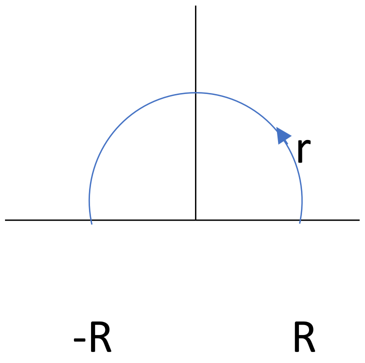
___ = h(x), using residue theorem. g(x)

Consider $_c h(z)\,dz$ when the poles of h(z) lie on imaginary axis. We take positive imaginary axis. Integration is taken over the semicircle and the line – R to R. The poles lie on upper half plane. If the poles lie on real axis

$$_c h(z)\,dz = {}_{-R}^{R} h(z)\,dz + {}_r h(z)\,dz$$

We know that by residue theorem $_c h(z)\,dz$ = 2πi (sum of the residues of h(z) at its poles which lie on upper half plane)

$$_{-R}^{R} h(z)\,dz + {}_r h(z)\,dz = 2\pi i \text{ (sum of the residues)}$$



In the limiting case R → ∞ we get

$$_{-\infty}^{\infty} h(x)\,dx \text{ (if } {}_r h(z)\,dz = 0)$$

Problems:

1) Evaluate by contour integration $_0^{\infty} \frac{dx}{1+x^2}$

Solution: Consider $_c \frac{dz}{1+z^2}$ where C is the contour consisting of semicircle Γ and the line (diameter) from –R to R.

dz dz dz

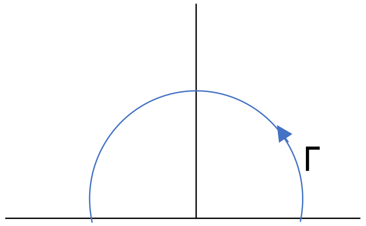
R c

$\frac{}{1+z^2} = \int_{-R}^{R} \frac{}{1+z^2} + \int_r \frac{}{1+z^2}$

r

$\frac{dz}{1+z^2} = 0$

$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_c \frac{dz}{1+z^2}$



The poles of f(z) are ± i, i lie on upper half plane.

1 1

Res at z=i= $\lim_{z \to i}$ (z-i) f(z) = $\lim_{z \to i} \frac{1}{(z+i)} = \frac{1}{(2i)}$

$\int_c \frac{dz}{1+z^2} = 2\pi i$ (residue at z=i)

$= 2\pi i \cdot \frac{1}{(2i)} = \pi$

$2\int_0^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ [f(x) is even]

$= \int_0^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\int_c \frac{dz}{1+z^2} = \frac{\pi}{2}$

2) Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)}$ using residue theorem.

Solution: $\int_{-\infty}^{\infty} f(x)\,dx$

$\int_{-R}^{R} f(z)\,dz = \int_c f(z) + \int_r f(z)\,dz$ $[\int_r f(z)\,dz = 0]$

= $_c$ f Z dz $\int_{-\infty}^{\infty} f(z)\,dz = \int_c f(Z)\,dz$

The poles of f(z) = $\frac{z^2}{(1+z^2)(4+z^2)}$ are i, -i, 2i,-2i.

All are simple poles i and 2i lie on upper half plane.

Res at z=i= lim (z-i)f(z) z → i

$= \lim_{z \to i} \frac{z^2}{(i+z)(4+z^2)} = -\frac{1}{6i}$

Res at z=2i = lim (z-2i)f(z) z → 2i

$= \lim_{z \to 2i} \frac{z^2}{(z+2i)(1+z^2)} = -\frac{4}{4i(-3)} = \frac{1}{3i}$ According to residue theorem $\int_c f(Z)\,dz$

= 2πi (sum of residues)

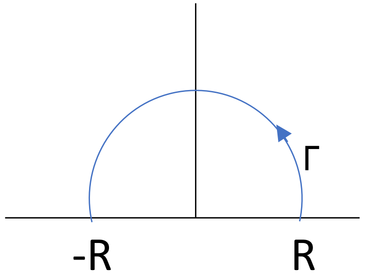
$= 2\pi i\left(-\frac{1}{6i} + \frac{1}{3i}\right) = \frac{\pi}{3}$

$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(4+x^2)} = \frac{\pi}{3}$

∞ x2dx

3) Evaluate $\int_0^{\infty} \frac{dx}{1+x^6}$ using residue theorem.

Solution: $\int_{-\infty}^{\infty} f(x)\,dx$



$$= \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz \qquad \left[\int_{\Gamma} f(z)\,dz = 0\right]$$

$$= \int_{C} f(z)\,dz$$

$$\int_{-R}^{R} f(z)\,dz = \int_{C} f(z)\,dz$$

The poles are , $e^{(2n+1)\pi i/6}$ where $n=0,1,2,3,4,5$

[$-1=\cos\pi+i\sin\pi= e^{\pi i} =\cos(2n+1)\pi+i\sin(2n+1)\pi$

$$(-1)^{\frac{1}{6}}= \frac{\cos(2n+1)\pi}{6} + i\,\frac{\sin(2n+1)\pi}{6} = e^{(2n+1)\pi i/6}$$

When n = 0, 1, 2 i.e , $e^{\frac{\pi i}{6}}$, $e^{\frac{3\pi i}{6}}$, $e^{\frac{5\pi i}{6}}$ lie on upper half plane.

$$\text{Res at } z \to e^{\frac{\pi i}{6}} = \lim_{z \to e^{\frac{\pi i}{6}}} \left(z- e^{\frac{\pi i}{6}}\right) f(z) \qquad \text{form } \frac{0}{0}$$

$$= \lim_{z \to e^{\frac{\pi i}{6}}} \frac{z^2\left(z- e^{\frac{\pi i}{6}}\right)}{(1+z^6)}$$

$$= \lim_{z \to e^{\frac{\pi i}{6}}} \frac{\left(3z^2-2z\, e^{\frac{\pi i}{6}}\right)}{6z^5}$$

$\frac{(3z-2\, e^{\overline{6}})}{6z^4}$ = lim

πi Z →eπi 6

−3πi

$\frac{e\,6}{\frac{2\pi i}{3}}$ 1 ___ 1 π π **i**

6 = = e 6 = (cos -i sin __) = -

6 2 2 **6**

6e

3πi πi

Res at z → e ___6 lim = (z-e__ 2)f(z) $\frac{0}{0}$ form

3πi

Z →e 6 ___πi

$= \lim \frac{z^2(z- e\, 2)}{(1+z^6)}$

πi

Z →e 2 πi

$= \lim \frac{(3z^2-2z\, e^{\overline{2}})}{6z^5}$

πi

Z →e 2 πi

$= \lim \frac{(3z-2\, e^{\overline{2}})}{6z^4}$

πi

Z →e 32π 1 3π 3π **i**

$\frac{1}{6}$ = e $\frac{-}{2}$ = (c6os ___ 2 -i sin) =

2 **6**

5πi 5πi

Res at $z \to e^{\frac{5\pi i}{6}}$ = $\lim_{z \to e^{\frac{5\pi i}{6}}} (z - e^{\frac{5\pi i}{6}}) f(z)$ $\frac{0}{0}$ form

$$= \lim_{z \to e^{\frac{5\pi i}{6}}} \frac{z^2(z - e^{\frac{5\pi i}{6}})}{(1+z^6)}$$

$$= \lim_{z \to e^{\frac{5\pi i}{6}}} \frac{(3z^2 - 2z e^{\frac{5\pi i}{6}})}{6z^5}$$

$$= \lim_{z \to e^{\frac{5\pi i}{6}}} \frac{1}{6} e^{\frac{-15\pi i}{6}} = \frac{1}{6}(\cos\frac{15\pi}{6} - i \sin\frac{15\pi}{6})$$

According to residue theorem

$$\int_c f(Z)\, dz = 2\pi i \text{ (sum of residues)}$$

$$= 2(\ -\ -\ ) = \frac{\pi}{3}$$

$$\int \frac{x^2}{1+x^6} = \frac{\pi}{3}$$

$$\int_0 \frac{x^2}{1+x^6}$$

πi i i i

π

– —

∞ xdx

∞ = 3

∞ xdx

=

6

-i sin) = -

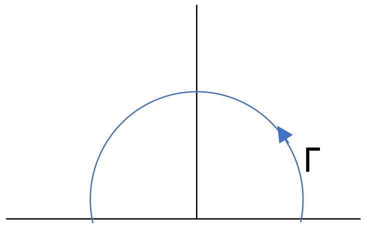
6

6

4) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3}$ using residue theorem.

Solution: $\int_{-\infty}^{\infty} f(x)\,dx$

$= \int_{-R}^{R} f(z)\,dz + \int_{\Gamma} f(z)\,dz$ $\quad [\int_{\Gamma} f(z)\,dz = 0]$



$= \int_{c} f(z)\,dz$

$\int_{-R}^{R} f(z)\,dz = \int_{c} f(z)\,dz$

The function is $f(z) = \frac{1}{(z^2+1)^3}$ -R R

The poles are i and –i of order 3, z=i lies on upper half plan and inside the semicircle

Res at z=i = $\lim_{z \to i} \frac{1}{2} \frac{d^2}{dz^2} [(z-i)^3 f(z)]$

$= \lim_{z \to i} \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{(z+i)^3} \right)$

$= \frac{1}{2} \lim_{z \to i} \frac{12}{(z+i)^5}$

$= \frac{6}{\ }\ \frac{3}{\ } = \mathbf{i}$

$(2i)^5$ **16**

According to residue theorem $\int_c f(Z)\, dz = 2\pi i$ (residue at z = i)

$$= 2\pi i \frac{3}{16i} = \frac{3\pi}{8}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{8}$$

Evaluation of the integrals of the type

$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx$

Jordan's Lemma

If f(z) is a function of z satisfying the following properties:

(i) f(z) is analytic in upper half plane except at a finite number of poles

(ii) f(z) → 0 uniformly as $|z| \to \infty$ with $0 \le \arg z \le \pi$

(iii) a is a positive integer, then

$$\lim_{r \to \infty} \int_c f(z)\, e^{iaz} dz = 0$$

Where C is a semicircle with radius r and centre at the origin

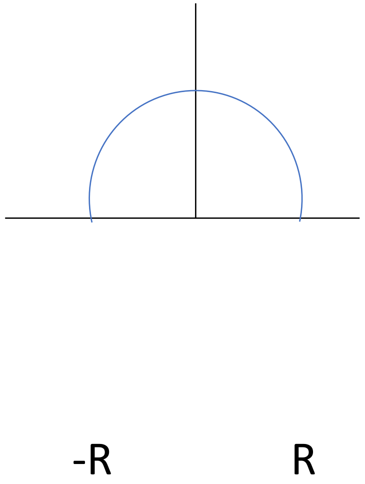
$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx = \int_c e^{imx} f(z)\, dz = 2\pi i$

(sum of the residues which lie on upper half plane)

Problems

1) Evaluate $\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+16)(x^2+9)}$ using residue theorem.

Solution: $\int_c f(z)\, e^{imz} dz = \int_r e^{imx} f(z)\, dz + \lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(z)\, dz$



=> $\int_r e^{imx} f(z)\, dz = 0$ (Jordan's Lemma)

$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx = \int_c e^{imx} f(z)\, dz = 2\pi i$

(sum of the residues which lie on upper half plane)

$\int_c \frac{e^{iz} dz}{(z^2+16)(z^2+9)}$ z=3i , -3i, 4i and -4i are simple poles. 3i and 4i lie on upper half plane.

Res at z = 3i = lim (z-3i)f(z) z → 3i

$$= \lim_{z\to 3i} \frac{e^{iz}}{(z^2+16)(z+3i)}$$

$$= \frac{e^{-3}}{(-9+16)(6i)} = \frac{-ie^{-3}}{42}$$

Res at z = 4i = lim (z-4i)f(z) z → 4i

$$= \lim_{z\to 4i} \frac{e^{iz}}{(z+4i)(z^2+9)}$$

$$= \frac{e^{-4}}{(9-16)(8i)} = \frac{ie^{-4}}{56}$$

$$\int_c \frac{e^{iz}dz}{(z^2+16)(z^2+9)} = 2\pi i\left(\frac{-i}{42e^3} + \frac{i}{56e^4}\right) = \frac{\pi(4e^{-3}-3e^{-4})}{84}$$

$$\text{R.P} \int_c \frac{e^{iz}dz}{(z^2+16)(z^2+9)} = \int_c \frac{\cos z\, dz}{(z^2+16)(z^2+9)}$$

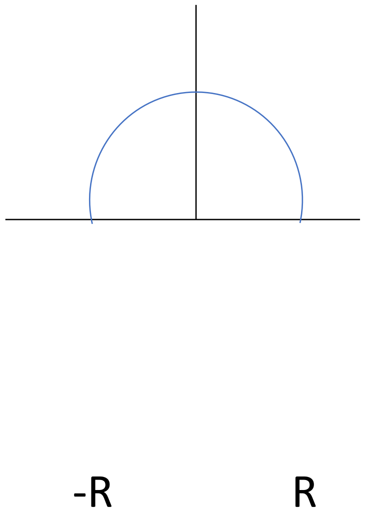
$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+16)(x^2+9)} = \frac{\pi(4e^{-3}-3e^{-4})}{84}$$

∞ $\frac{x\sin x\, dx}{(a^2+x^2)}$
0

2) Evaluate ( )

Solution: : $_c f\ z\ e^{imz}dz =\ _r e^{imx} f$ ( ) $_R \lim_{\to\infty} \int_{-}$ $z\ dz +$ ( )

$^R{}_R\ e^{imx} f(z)\ dz$



=> $_r e^{imx} f\ z\ dz\ = 0$

z

f(z) = ______(a2+z2)

z = ai and –ai are simple poles.

Res at z =

ai = lim (zai)f(z)

$z \to ai$

zeiz

lim
→

= =

$\frac{}{(z+ai)}$ z ai

$\frac{aie^{-a}}{(2ai)}$ $\frac{\mathbf{e}}{\mathbf{2}}$ = $-a$ c e^{-a} $-\mathbf{a}$

$-\mathbf{a}$

$\mathbf{i\,e}$

0 $\frac{}{(a^2+z^2)}$ zsinx dz = $2\pi i \frac{}{2}$ = $\boldsymbol{\pi}$**i e** $\infty \frac{xsinx\,dx}{} = \frac{\pi}{} \mathbf{i\, e2}$ $-\mathbf{a}$

(a^2+x^2)

$$\int_{-\infty}^{\infty} \frac{xsinx\,dx}{(a^2+x^2)} = \boldsymbol{\pi}$$

Unit -3

LAPLACE TRANSFORMS

LAPLACE TRANSFORM

Definition:

Let f(t) be a function of t, defined ∀ t≥0. If the integral

∞ $^{-st}$ **f(t) dt exists, then it is called the Laplace Transform of**

$_{0}$□ e

f(t) and it is denoted by L{f(t)} or f(s).

Here s is parameter, real or complex.L is called Laplace Transform operator.

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

Def: Piece-wise Continuous Function:

Afunction is said to be piece-wise continuous (or) Sectionally Continuous) over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided into a finite number of sub intervals, in each of which f(t) is continuous and both right and left hand limits at every end point if the sub intervals.

Def:Functions of Exponential Order:

A function f(t) is said to be of exponential order as t →∞ if

$$\lim_{t\to\infty} (e)^{-at} f(t) = finite\ quantity$$

(or)

If for a given positive integer T, ∋ a positive number M Such that $|f(t)| < Me^{at} \quad \forall\, t \geq T,$

Sufficient Conditions for existence of Laplace Transform are 1)

f(t) is Piece-wise Continuous Function in [a, b] where $a>0$, 2)

f(t) is of Exponential Order function.

Linear Property:

Theorem: If c_1, c_2 are constants and f_1, f_2 are functions of t, then

$$L[c_1 f_1(t) + c_2 f_2(t)]=c_1 L[f_1(t)]+ c_2 L[f_2(t)]$$

Proof: The definition of Laplace Transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \text{ -----(1)}$$

By definition

$$L[c_1 f_1(t) + c_2 f_2(t)]=\int_0^\infty e^{-st} [c_1 f_1(t)+ c_2 f_2(t)]\, dt$$

$$= \int_0^\infty e^{-st} c_1 f_1(t)\, dt + \int_0^\infty e^{-st} c_2 f_2(t)\, dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$

$=c_1\ L[f_1(t)] + c_2\ L[f(t)]$

Laplace Transform (L.T) of some Standard Functions:

1)Show that $L\{1\} = \frac{1}{s}$

Solution: By definition of L.T $L[f(t)] = f(t) \int_0^\infty e^{-st}$ dt----------(1)

Put f(t)=1 o.b.s $L[1] = \int_0^\infty e^{-st}$ .1. dt

$-1 = = \_\_ (0\text{-}1) = \left[\frac{e^{-st}}{} \right]^\infty$ 1/s

$-s \quad s\ 0$

2) L[c] = L[c.1] =c. L[1]= c.(1/s) = c/s

3) Show that $L[e^{at}] = \frac{1}{s-a}$

Solution: By definition of L.T , L[f(t $)]= \int_0^\infty e^{-st} f(t)$ dt----------(1)

$e^{at}] = \int_0^\infty e^{-st}\ e^{at}$ dt

$= \int_0^\infty e^{-(s-a)T}$ dt

$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$ $(e^{-\infty} = 0)$

$= \frac{1}{s-a}$

Put f(t) = e^{at} o.b.s in (1) L[

Note: L[e^{-at}] $= \frac{1}{s+a}$

4) Show that L[Cos at]= $\frac{s}{s^2+a^2}$ and L[Sin at] = =$\frac{a}{s^2+a^2}$

Solution: W.k.t $e^{i\theta}$ = cos θ + i sinθ

e^{iat}= cos at + i sin at

L[e^{iat}] = L[cos at + i sin at]

L[cos at + i sin at]= L[e^{iat}]

$=\frac{1}{s-ia}$ (L [e^{at}]=$\frac{1}{s-a}$)

$=\frac{s+ia}{(s-ia)(s+ia)}$

$= \frac{s+ia}{s^2+a^2}$

$=\frac{s}{s^2+a^2}$ + i $\frac{a}{s^2+a^2}$

Equte real and imaginary parts we get

L[Cos at]= $\frac{s}{s^2+a^2}$ and L[Sin at] = =$\frac{a}{s^2+a^2}$

5) Find L [Sin hat]

$$\frac{e^{at}-e^{-at}}{2}$$

Solution: L [Sin hat] = L [$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$ **] = ½ [L** $\{e^{at}\}$ **–L** $\{e^{-at}\}$ **]**

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

6) **Find L [Cos hat]**

$$\frac{e \quad +e^{-}}{2}$$

$at \qquad at$

Solution: L [Cos hat] = L [] = ½ [L $\{e^{at}\}$ **+L** $\{e^{-at}\}$ **]**

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

s

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = = \overline{s^2-a^2}$$

7) **Show that** **(i)) L** $[t^n] = \rho(\text{n+1})/s^{n+1}$**,** **n>-1**

(ii) **L** $[t^n] = \text{n!}/s^{n+1}$**,** **n is +ve integer**

Solution: : By definition of L.T

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \text{--------} (1)$$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

put st =x i.e t = x/s

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$dt = \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x}\, x^n\, dx$$

$$= \frac{1}{s^{n+1}}\, \rho(n+1), \quad \text{for } (n+1) > 0$$

$$L[t^n] = \rho(n+1)/s^{n+1}, \quad n > -1$$

$$L[t^n] = n!/s^{n+1}, \quad n \text{ is +ve integer}$$

FORMULAE

1) $L\{1\} = \frac{1}{s}$

2) $L\{c\} = \frac{c}{s}$

3) $L[e^{at}] = \frac{1}{s-a}$, $L[e^{-at}] = \frac{1}{s+a}$

s

4) L[Cos at]= $\overline{s^2+a^2}$

a

5) L[Sin at] $\overline{s^2+a^2}$ =

a

$\overline{s^2-a^2}$

6) L[Sin hat] s =

$\overline{s^2-a^2}$

7) L[Cos hat]=

8) $L(t^n)=\rho(n+1)/s^{n+1}$, n>-1

9) $L(t^n)=$ n!$/s^{n+1}$, n is +ve integer

PROBLEMS

1.Find the Laplace Transformation (L.T) of t^2+2t+3

Solution: L $[t^2+2t+3]$ = L$[t^2]+2L[t]+L[3]$

$= \frac{!}{s^3} + 2.\frac{}{s^2} + \frac{}{s}$ 2 1 3

2. Find $L[t^{\frac{5}{2}} + 4]$

Solution: $L[t^{\frac{5}{2}} + 4] = L[t^{\frac{5}{2}}] + L[4]$

$$= \frac{\rho(\frac{7}{2})}{s^{7/2}} + \frac{4}{s}$$

3. Find $L[e^{3t} + 3e^{-2t}]$

Solution: $L[e^{3t} + 3e^{-2t}] = L[e^{3t}] + 3L[e^{-2t}]$

$$= \frac{1}{s-3} + 3\,\frac{1}{s+2}$$

4. Find $L[\text{Sin } 3t + Cos^2 2t]$

Solution: $L[\text{Sin } 3t + Cos^2 2t] = L[\text{Sin } 3t] + L[Cos^2 2t]$

$$= \frac{3}{s^2+9} + L\left[\frac{1 + Cos\,4t}{2}\right]$$

$$= \frac{3}{s^2+9} + \frac{1}{2}\{ L[1] + L[\text{Cos } 4t] \}$$

$$= \frac{3}{s^2+9} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$$

5. Find L[f(t)] if f(t)= 0, $0 < t < 2$

= 3, $t > 2$

Solution: By definition of L.T

∞ $^{-st}$ f(t) dt

L[f(t)]= $_0$∫ e

= $\int_0^2 e^{-st}$ ∞ $^{-st}$ f(t) dt

f(t) dt + $_2$∫ e

= 0 + ∫$_2$ e^{-st} .3. dt

$= 3\left(\frac{e^{-st}}{-s}\right)^{\infty}$ ∞

2 $e_{-\infty}$= 0

$e{-2s}$

= 3

s

<u>First shifting Theorem (F.S.T):</u>

If L[f(t)]=f (s) then L[e^{at} f(t)]= f(s-a)

Proof : By definition of L.T

L[f(t)]= $\int_0^\infty e^{-st}$ f(t) dt = f(s)--------(1)

L[e^{at}f(t)]= $\int_0^\infty e^{-st}\ e^{at}$ f(t)dt

= $\int_0^\infty e^{-(s-a)t}$ f(t) dt Put s-a=p = $\int_0^\infty e^{-pt}$ f(t)

dt

= f(p) = f(s-a)

Note: L[e^{-at} f(t)] = f(s+a)

Problems:

1) Find L[t³ e^{-3t}]

Solution : let f(t) = t³

L[f(t)] = L[t³] = $\frac{3!}{s^{3+1}}$ = $\frac{6}{s^4}$ = f(s)

By F.S.T , L[e^{-at} f(t)] = f(s+a) a=3 L[e^{-3t}

f(t)] = f(s+3)

L[e^{-3t} t³] = $\frac{6}{(s+3)^4}$

2) **Find L [e^{-t}(3 sin 2t – 5 cosh 2t)]**

Solution : Let f(t) = (3 sin 2t – 5 cosh 2t) L

[f(t)] = L[(3 sin 2t – 5 cosh 2t)]

$$= 3\frac{2}{s^2+4} - 5\frac{s}{s^2-4} = f(s)$$

By F.S.T , L[e^{-at} f(t)] = f(s+a) **a=1**

L[e^{-1t} f(t)] = f(s+1)

$$= \frac{6}{(s+1)^2+4} - \frac{5(s+1)}{(s+1)^2-4}$$

L [e^{-t}(3 sin 2t – 5 cosh 2t)] $= \frac{6}{s^2+2s+5} - \frac{5s+5}{s^2+2s-3}$

Second Shifting Theorem (S.S.T)

STATEMENT:- If L[f(t)]=f(s) and g(t)=f(t-a), t>a

= 0, t<a then L{g(t)}=e^{-as} f(s)

PROOF:- By definition of L.T

L[f(t)]= $\int_0^{\infty} e^{-st}$ f(t) dt = f(s)--------(1)

$\int_0^{\infty} e^{-st}$ L[g(t)]= g(t) dt = $= \int_0^{u} e^{-st}$ g(t)

dt + $\int_a^{\infty} e^{-st}$ g(t) dt

= 0 + $\int_a^{\infty} e^{-st}$ f(t -a) dt put t-a=x $= \int_0^{\infty} e^{-s(a+x)}$ f(x) dx

t=a+x

= $e^{-as} \int_0^{\infty} e^{-sx}$ f(x) dx

dt=dx, (x=0 to ∞)

= e^{-as} f(s)

Example :

Find Laplace Transform of g(t) = $\cos\left(t - \frac{2\pi}{3}\right)$, if $t > \frac{2\pi}{3}$

= 0, if $t < \frac{2\pi}{3}$

Solution: Let f (t) = cos t , a = $\frac{2\pi}{3}$

f (t-a) = cos (t-a) f (t - $\frac{2\pi}{3}$) = cos (t - $\frac{2\pi}{3}$)

L [f(t)] = L [cos t]= $\frac{s}{s^2+1}$ = f(s)

By S.S.T L [g(t)] = e^{-as} f(s)

$$= \left(e^{-\frac{2\pi}{3}s}\right)\frac{s}{s^2+1}$$

<u>Change of scale property</u>:

If L[f(t)] = f(s) then L [f(at)] = $\frac{1}{a}$ f($\frac{s}{a}$)

NOTE: L [f($\frac{t}{a}$)] = a f(as)

Example: If L [f(t)] = $\frac{9s^2-12s+15}{(s-1)^3}$ then find L [f(3t)]

Solution: Given $\frac{9s^2-12s+15}{(s-1)^3}$ = f(s)

L[f(t)] = by Change of scale property, L [f(at)] = $\frac{1}{a}$ f($\frac{s}{a}$)

L [f(3t)] = $\frac{1}{3}$ f($\frac{s}{3}$)

$$= \frac{1}{3}\left[\frac{9(\frac{s}{3})^2-12\left(\frac{s}{3}\right)+15}{(\frac{s}{3}-1)^3}\right]$$

$$= \frac{1}{3}\left[\frac{s^2-4s+15}{(s-3)^3/27}\right]$$

$$= \frac{9(s^2-4s+15)}{(s-3)^3}$$

Laplace transformof the derivative of f(t)

□ If f(t)is continousfor all t □0 and f (t)□is piecewisecontinous, then

L{f (t)}□exists,providedlim e stf(t)□0 □and □□

L{f □(t)} □sL{f(t)}-f(0) sf(□s)-f(0)

L{f ^{n} (t)}□s^{n}f(s)-s^{n-1}f(0)-s^{n-2}f (0□)....f^{n-1}(0)

Example Derivelaplace transformof sin at

Let f(t) sinat thenf'(t) = a cosat and f''(t) -a² sinat

Also f(0) = 0, f' (0) = a from this also f''(0) = 0, also from this

By derivative formula,

$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$-------(1)

$L\{-a^2 \sin at\} = s^2 L(\sin at) - a$

$(-a^2) L(\text{Sin at}) + a = s^2 L(\sin at)$ a =

$(s^2 + a^2) L(\sin at)$

$$L(\sin at) = \frac{a}{s^2 + a^2}$$

Laplace transform of the integration of f(t)

If L[f(t)]=f(s) then $L[\int_0^t f(t)dt] = \frac{f(s)}{s}$

Example:

Find L.T. of $\int_0^t \sin at\, dt$ Solution:

Let f(t) = sin at

L[f(t)] = L[sin at] = $\frac{a}{s^2+a^2}$ = f(s)

L[$\int_0^t f(t)dt] = \frac{f(s)}{s}$

L[$\int_0^t \sin at\, dt$ = $\frac{1}{s}\left(\frac{a}{s^2+a^2}\right)$

Multiplication by t :

If L[f(t)]=f(s) then L[t f(t)] = - $\frac{d}{ds}$ **[f(s)]**

L[t^2 f(t)] = $(-1)^2 \frac{d^2}{ds^2}$ **[f(s)]**

L[t^n f(t)] = $(-1)^n \frac{d^n}{ds^n}$ **[f(s)]**

Example : Find L[t sin^2t]

Solution: Let f(t) = $sin^2 t$

let L[f(t)] = L[$sin^2 t$] = L[$\frac{1-COS\,2t}{2}$]

$\frac{1}{2}$ (L[1] – L[Cos 2t]) = $\frac{1}{2}(\frac{1}{s} - \frac{s}{s^2+4}) = \frac{2}{s(s^2+4)}$ = f(s)

By theorem L[t f(t)]

$$= - \frac{d}{ds}[\,f(s)]$$

$$= - \frac{d}{ds}[\frac{2}{s(s^2+4)}]$$

$$= -2\,[\frac{-1}{\{s(s^2+4)\}^2}]\,\frac{d}{ds}(s(s^2+4))$$

$$= [\frac{2}{\{s(s^2+4)\}^2}]\,\frac{d}{ds}(s^3+4s)$$

$$= [\frac{2}{\{s(s^2+4)\}^2}]\,(3s^2+4)$$

$$= \frac{6s^2+8}{s^2(s^2+4)^2}$$

Division by t:

If L[f(t)]=f(s) then L[$\frac{f(t)}{t}$] = $\int_s^\infty f(s)ds$, provided $\lim_{t\to 0}\frac{f(t)}{t}$ exists.

Problems: (1) Find L[$\frac{e^{-3t} - e^{-4t}}{t}$]

Solution: Let f(t) = $e^{-3t} - e^{-4t}$

L[f(t)] = L[$e^{-3t} - e^{-4t}$] = $\frac{1}{S+3}$ - $\frac{1}{S+4}$ = f(s) w.k.t

, L[$\frac{f(t)}{t}$] = $\int_s^\infty f(s)ds$,

L[$\frac{e^{-3t} - e^{-4t}}{t}$] = $\int_s^\infty (\frac{1}{S+3} - \frac{1}{S+4}) ds$

= $\left[\log (s+3) - \log (s+4)\right]_s^\infty$

= $\left[\log \left(\frac{s+3}{S+4}\right)\right]_s^\infty$ = $\left[\log \frac{s(1+\frac{3}{s})}{s(1+\frac{4}{s})}\right]_s^\infty$

= log 1 - log ($\frac{s+3}{S+4}$)

= 0 - log ($\frac{s+3}{S+4}$) = log ($\frac{s+4}{S+3}$)

(2). Find L.T of $\frac{\cos at - \cos bt}{t}$

Solution: Let f(t) = cos at – cos bt

L[f(t)] = L[cos at – cos bt]

f(s) = $\frac{s}{s^2+a^2}$ - $\frac{s}{s^2+b^2}$

w.k.t , L[$\frac{f(t)}{t}$] = $\int_s^\infty f(s)ds$

L[$\frac{\cos at - \cos bt}{t}$] = $\int_s^\infty (\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}) \, ds$

∞

1

$^2 + a^2) - \log (s^2 + b^2)]$

= [log (s

2

∞ s

1 = $s s$ 2 2 + + a b 2 2

) log (2

s

= $\frac{1}{2} \log (\frac{s^2+b^2}{s^2+a^2})$

Evaluation of integrals by Laplace transforms:

(1). Using L.T. Evaluate $\int_0^\infty [\frac{e^{-t} - e^{-2t}}{t}]\, dt$

Solution: First we will find L$[\frac{e^{-t} - e^{-2t}}{t}]$ let

$f(t) = e^{-t} - e^{-2t}$

$L[f(t)] = L[e^{-t} - e^{-2t}]$

$= \frac{1}{S+1} - \frac{1}{S+2} = f(s)$

w.k.t , $L[\frac{f(t)}{t}] = \int_s^\infty f(s)ds$,

$L[\frac{e^{-t} - e^{-2t}}{t}] = \int_s^\infty (\frac{1}{S+1} - \frac{1}{S+2})\, ds$

$= [\log(s+1) - \log(s+2)]_s^\infty = \log\left(\frac{s+1}{S+2}\right)_s^\infty$

$= \log \ldots$ ∞ s

$$= \log\left(\frac{s\left(1+\frac{1}{s}\right)}{s\left(1+\frac{2}{s}\right)}\right) = \log 1 - \log\left(\frac{s+1}{S+2}\right)$$

$$= 0 - \log\left(\frac{s+1}{S+2}\right) = \log\left(\frac{s+2}{S+1}\right)$$

therefore, $L\left[\frac{e^{-t} - e^{-2t}}{t}\right] = \log\left(\frac{s+2}{S+1}\right)$

The definition of Laplace Transform is

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$L\left[\frac{e^{-t} - e^{-2t}}{t}\right] = \int_0^{\infty} e^{-st}\left[\frac{e^{-t} - e^{-2t}}{t}\right] dt = \log\left(\frac{s+2}{S+1}\right)$$

Put s=0 on both sides

$$\int_0^{\infty} 1\left[\frac{e^{-t} - e^{-2t}}{t}\right] dt = \log\left(\frac{2}{1}\right) = \log 2$$

2. Using LT find $\int_0^{\infty}\left(\frac{\cos at - \cos bt}{t}\right) dt$

Solution: First we find $L\left[\frac{\cos at - \cos bt}{t}\right]$

: Let f(t) = cos at – cos bt

L[f(t)] = L [cos at – cos bt] f(s)

$$= \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

w.k.t , $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} f(s)ds$

$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^{\infty} \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$= \frac{1}{2}\left[\log (s^2+a^2) - \log (s^2+b^2)\right]_s^{\infty}$

$= \frac{1}{2} \left[\log \left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^{\infty}$

$= \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2}\right)$

By definition of LT, $\int_0^{\infty} e^{-st}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2} \log \left(\frac{s^2+b^2}{s^2+a^2}\right)$

Put s=0 o.b.s $\int_0^{\infty}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2} \log \left(\frac{b^2}{a^2}\right)$

$= \log\sqrt{\left(\frac{b^2}{a^2}\right)} = \log (b/a)$

3. ST $\int_0^{\infty}\left(\frac{\cos 5t - \cos 3t}{t}\right) dt = \log (3/5)$ Note: put a=5, b=3 in above problem

Laplace Transform of Periodic Function:

Definition : A function f(t) is said to be periodic with period T , if $\forall\, t$, f(t+T) = f(t) where T is positive constant.

The least value of T > 0 is called the periodic function of f(t).

Example: sin t $= \sin(2\pi + t) = \sin(4\pi + t) = - - - - - -$ Here sint is periodic function with period 2π.

Formula :- If f(t) is periodic function with period T $\forall t$ *then*

$$L[f(t)] = \frac{1}{1-e^{-st}} \int_0^T e^{-st} f(t)\, dt$$

Problem : Find the L. T of the function f(t) = e^t, 0< t <5 and f(t)=f(t+5)

$$\frac{1}{1-e^{-s5}} \int_0^5 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-s5}} \int_0^5 e^{-st}\, e^t\, dt$$

Solution : Here T=5 $L[f(t)= \frac{1}{1-e^{-5s}} [\frac{e^{(1-s)t}}{1-s}] = \frac{1}{1-e^{-5s}} [\frac{e^{5(1-s)}}{1-s}]$

The unit step function or Heaviside's unit function :

It is denoted by u(t-a) or H(t-a) and is defined as H(t-a) = 0, t<a

=1, t>a <u>L.T.</u>

<u>of unit step function:</u>

Prove that L[H(t-a)] = $\frac{e-as}{s}$

Solution : L[H(t- a)] = $\int_0^{\infty} e^{-st}$ H(t -a) dt

$= \int_0^{a} e^{-st}$ H(t -a) dt + $\int_a^{\infty} e^{-st}$ H(t -a) dt

$= \int_0^{a} 0 + \int_a^{\infty} e^{-st} . 1$. dt

$= \left(\frac{e^{-st}}{-s}\right)$

$= \left(\frac{e^{-sa}}{s}\right)$

Inverse Laplace Transform :

Definition : If f(s) is the Laplace Transform of f(t) then f(t) is called the inverse Laplace Transform of f(s) and is denoted by $L^{-1}[f(s)]$. i.e., f(t) = $L^{-1}[f(s)]$

L^{-1} is called inverse Laplace Transform operator, but not reciprocal.

Example : If L$[e^{at}] = \frac{1}{s-a}$ then $e^{at} = L^{-1}[\frac{1}{s-a}]$

Linear Property :

If $f_1(s)$ and $f_2(s)$ are L.T. of $f_1(t)$ and $f_2(t)$ respectively then

$L^{-1}[c_1 f_1(s) + c_2 f_2(s)] = c_1 L^{-1}[f_1(s)] + c_2 L^{-1}[f_2(s)]$ where c_1

, c_2 constants.

Standard Formulae :

1 $\Rightarrow L^{-1}[\frac{1}{s}] = 1$

(1) L

(2) L $[e^{at}] = \frac{1}{s-a}$ [1] = $\Rightarrow L^{-1}[\frac{1}{s-a}] = e^{at}$

(3) L $[e^{-at}] = \frac{1}{s+a}$ s $\Rightarrow L^{-1}[\frac{1}{s+a}] = e^{-at}$

a

(4) L [sin at] = $\overline{s^2+a^2}$ $\Rightarrow L^{-1}[\frac{1}{s^2+a^2}] = \frac{1}{a}$ sin at

(5) L [Cos $\frac{s}{s^2+a^2}$ at $\Rightarrow L^{-1}[\frac{s}{s^2+a^2}]$ = cos at]=

5) L [Sin hat] $\frac{a}{s^2-a^2}$ $\Rightarrow L^{-1}[\frac{1}{s^2-a^2}] = \frac{1}{a}$ sinh = at

6) L [Cos $\overline{s^2-a^2}$ s $\Rightarrow L^{-1}[\frac{s}{s^2-a^2}$ hat]=] = cosh at

n

7) $L(t^n) = \rho(n+1)/s^{n+1}$, $n > -1$ $\Rightarrow L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{\rho(n+1)}$

8) $L(t^n) = n!/s^{n+1}$, n is +ve integer $\Rightarrow L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$

Problems:

(1) Find $L^{-1}[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}]$

solution : $L^{-1}[\frac{1}{s^2}] + L^{-1}[\frac{1}{s+4}] + L^{-1}[\frac{1}{s^2+4}] + L^{-1}[\frac{s}{s^2-9}]$

$= t + e^{-4t} + \frac{1}{2}\sin 2t + \cosh 3t.$

(2)Find $L^{-1}[\frac{1}{s^2+25}]$

solution :

$L^{-1}[\frac{1}{s^2+25}] = L^{-1}[\frac{1}{s^2+5^2}] = \frac{1}{5}\sin 5t$

(3) Find $L^{-1}[\frac{1}{2s-5}]$

solution : $L^{-1}\left[\frac{1}{2s-5}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s-5/2}\right] = \frac{1}{2} e^{\frac{5}{2}t}$ solution :

(4) Find $L^{-1}\left[\frac{2s+1}{s(s+1)}\right]$

$$L^{-1}\left[\frac{2s+1}{s(s+1)}\right] = L^{-1}\left[\frac{s+s+1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s+1} + \frac{1}{s}\right] = e^{-t} + 1$$

(5) Find $L^{-1}\left[\frac{3s-8}{4s^2+25}\right]$

solution: $L^{-1}\left[\frac{3s-8}{4s^2+25}\right] = ¼\, L^{-1}\left[\frac{3s-8}{s^2+25/4}\right]$

$= ¼ \{ 3L^{-1}\left[\frac{s}{s^2+(5/2)^2}\right] - 8\, L^{-1}\left[\frac{1}{s^2+(5/2)^2}\right] \}$

$= ¾ \text{ Cos } \frac{5}{2}t - (¼ \times 8 \times \frac{2}{5}$ Sin $\frac{5}{2}t)$ a=5/2

= ¾ Cos $\frac{5}{2}$t − 4/5 Sin $\frac{5}{2}$t

FIRST SHIFTING THEOREM OF INVERSE L.T:

If $L^{-1}[f(s)] = f(t)$ then $L^{-1}[f(s-a)] = e^{at} f(t)$

$= e^{at} L^{-1}[f(s)]$

PROOF: By definition of L.T

$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = f(s)$--------(1)

$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)dt$

$= \int_0^\infty e^{-(s-a)t}$

$L[e^{at}$

f(t) dt Put s-a=p $= \int_0^\infty e^{-pt}$ f(t)

dt

= f(p) = f(s-a)

$L[e^{at}f(t)]$= f(s-a)

$\Rightarrow$ L^{-1}[f(s-a)] = $e^{at} f(t)$ (or) $L^{-1}[\ f(s-a)\ ] = \ = e^{at} L^{-1}[\ f(s)\ ]$

Note: $L^{-1}[\ f(s+a)\ ] = \ = e^{-at} L^{-1}[\ f(s)\ ]$

PROBLEMS

$L^{-1}[\frac{s+3}{(s+3)^2+8^2}]$ 1) Find

Solution : $L^{-1}[\frac{s+3}{(s+3)^2+8^2}] = e^{-3t} L^{-1}[\frac{s}{s^2+8^2}]$ by F.S.T

$= e^{-3t}$ Cos 8t.

$L^{-1}[\frac{1}{s^2+2s+5}]$

$L^{-1}[\frac{1}{s^2+2s+5}] = L^{-1}[\frac{1}{(s+1)^2+4}] = e^{-t} L^{-1}[\frac{1}{s^2+2^2}] = e^{-t}$ ½ Sin 2t

$L^{-1}[\frac{1}{(s+1)^2}]$

2) Find

Solution : $L^{-1}[\frac{1}{(s+1)^2}] = L^{-1}[\frac{1}{(s+1)^2}] = e^{-t} L^{-1}[\frac{1}{s^2}] = e^{-t}$ t

3) Find

Solution :

4) Find Inverse L.T of $\frac{s}{(s+3)^2}$

$$L^{-1}[\frac{s}{(s+3)^2}] = L^{-1}[\frac{s+3-3}{(s+3)^2}] = e^{-3t} L^{-1}[\frac{s-3}{s^2}]$$

Solution :

$$= e^{-3t} \{ L^{-1}[\frac{1}{s}] - 3 L^{-1}[\frac{1}{s^2}] \} = e^{-3t}(1\text{-}3t)$$

$$L^{-1}[\frac{s+3}{s^2-10s+29}]$$

$$L^{-1}[\frac{s+3}{s^2-10s+29}] = L^{-1}[\frac{s+3}{(s-5)^2+4}] = L^{-1}[\frac{(s-5)+5+3}{(s-5)^2+4}]$$

$$= e^{5t} L^{-1}[\frac{s+8}{s^2+4}$$

$$= e^{5t} \{L^{-1}[\frac{s}{s^2+4}] + 8 L^{-1}[\frac{1}{s^2+4}]\}$$

$$= e^{5t} \{L^{-1}[\frac{s}{s^2+2^2}] + 8 L^{-1}[\frac{1}{s^2+2^2}]\}$$

5) Find

Solution :

]

(By F.S.T)

$= e^{5t}$ [Cos 2t + 8 x ½ x Sin 2t] a=2

$=e^{5t}$

SECOND SHIFTING THEOREM: [Cos 2t + 4 Sin 2t]

If L^{-1}[f(s)] = f(t) then L^{-1}[e^{-as} f(s)] = g (t) *where g(t) = f(t-a), t>a*

=0, t<a

Proof: By S.S.T of L.T , L [g(t)] = e^{-as} f(s) (write proof of SST)

$\Rightarrow L^{-1}$[e^{-as} f(s)] = g $t()$

$\Rightarrow L^{-1}$[e^{-as} f(s)] = *f(t-a), t>a*

=0, t<a ***Note:***

We can also written as L^{-1}[e^{-as} f(s)] = *f(t-a)* H(t-a)

Problem:

Find $L^{-1}[\frac{e^{-\pi s}}{s^2+1}]$

$L^{-1}[\frac{e^{-}}{s^2+1}] = L^{-1}[e^{-\pi s}\frac{1}{s^2+1}]$ πs

Solution:

Let f(s) = $\frac{1}{s^2+1}$

L^{-1} [f(s)] = $L^{-1}[\frac{1}{s^2+1}]$ = Sin t = f(t)

by S.S.T $L^{-1}[\, e^{-as} f(s)\,] = f(t-a),\ t>a$

$= 0, \quad t<a$

So $L^{-1}[\, e^{-\pi s} f(s)\,] = f(t-\pi),\ t>\pi$

$= 0, \quad t<\pi$

$L^{-1}[e^{-\pi s}\frac{1}{s^2+1}]$ = Sin $(t-\pi),\ t>\pi = 0,$

$t<\pi$

Chang of scale property :

If $L^{-1}[f(s)] = f(t)$ then $L^{-1}[f(\frac{s}{a})] = a\, f(at)$

(or) $L^{-1}[f(as)] = \frac{1}{a} f(\frac{t}{a})$

Proof : By the change of scale property,

$$L[f(at)] = \frac{1}{a} f(\frac{s}{a})$$

$$\Rightarrow L^{-1}[f(\frac{s}{a})] = a\, f(at)$$

(or)

$$L^{-1}[f(as)] = \frac{1}{a} f(\frac{t}{a})$$

Problem(1): If $L^{-1}[\frac{s^2-1}{(s^2+1)^2}]$ **= t cost , then find** $L^{-1}[\frac{9s^2-1}{(9s^2+1)^2}]$

Solution : Given $L^{-1}[\frac{s^2-1}{(s^2+1)^2}]$ **= t cost**

i.e., $L^{-1}[f(s)] = f(t)$

, Here $f(s) = \frac{s^2-1}{(s^2+1)^2}$ **f(t) = t cost**

Now $L^{-1}[\frac{9s^2-1}{(9s^2+1)^2}] = L^{-1}[\frac{(3s)^2-1}{\{(3s)^2+1\}^2}]$

$= L^{-1}[f(3s)]$ **By change of scale property ,**

$= \frac{1}{3} f(\frac{t}{3})$

$L^{-1}[f(as)] = \frac{1}{a} f(\frac{t}{a})$ $= \frac{1}{3}\frac{t}{3} \cos\frac{t}{3}$ **a = 3**

Inverse Laplace Transform of partial fractions :

Problems : (1) Find $L^{-1}[\frac{(s^2+1)(s-1)}{s^4}]$

Solution : Given $L^{-1}[\frac{(s^2+1)(s-1)}{s^4}] = L^{-1}[\frac{(s^3-s^2+s-1)}{s^4}]$

$= L^{-1}[\frac{1}{s}] - L^{-1}[\frac{1}{s^2}] + L^{-1}[\frac{1}{s^3}] - L^{-1}[\frac{1}{s^4}]$

$= 1 - t + \frac{1}{2}t^2 - \frac{t^3}{6}$

(2). Find $L^{-1}[\frac{s+5}{s^2-3s+2}]$

Solution : Here $f(s) = \frac{s+5}{s^2-3s+2}$ reduce into partial fractions

$f(s) = \frac{s+5}{s^2-3s+2} = \frac{s+5}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \quad ----(1)$

$\Rightarrow s+5 = A(s-2) + B(s-1)$

put s=1 on both sides $\Rightarrow A = -6$

put s=2 on both sides $\Rightarrow B = 7$

Therefore (1) $\Rightarrow f(s) = \frac{-6}{s-1} + \frac{7}{s-2}$

$L^{-1}[f(s)] = L^{-1}[\frac{-6}{s-1} + \frac{7}{s-2}] = -6e^{t} + 7e^{2t}$

Inverse Laplace Transform of derivatives :-

If $L^{-1}[f(s)] = f(t)$ then $L^{-1}[\frac{d^n}{ds^n} f(s)] = (-1)^n t^n f(t)$

1 1 f′(s) =

Proof : By theorem of L.T. $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$\Rightarrow L^{-1}[\frac{d^n}{ds^n} f(s)] = (-1)^n t^n f(t)$

Note:- $L^{-1}[f'(s)] = -t f(t)$

Problem (1):- Find $L^{-1}[\log(\frac{s+3}{s+4})]$

Solution : Let $f(s) = \log(\frac{s+3}{s+4}) = \log(s+3) - \log(s+4)$

$\overline{s+3}$ − $\overline{s+4}$

L^{-1}[f′(s)] = $L^{-1}[\frac{1}{s+3} - \frac{1}{s+4}]$

= $e{-3t}$ - $e{-4t}$

$L^{-1}[\log(\frac{s+1}{s-1})]$

By theorem, - t f(t) = e^{-3t} - $\frac{e^{-3t} - e^{-}}{-t}$ e^{-4t} H.W. Find $4tt$ so, $\frac{e^{t} - e^{-}}{t}$

f(t) = Ans: L^{-1}[f(s)] =

[replace 3 by ⇒ L^{-1}[f(s)] = $\frac{e^{-4t} - e^{-3t}}{t}$ 1 and 4 by (-1)]

(2) Find $L^{-1}[\frac{s}{(s^2+a^2)^2}$]

Solution: W.K.T $L^{-1}[\frac{1}{(s^2+a^2)}] = \frac{1}{a}$ sin at

i.e L^{-1}[f(s)] = f(t) 1 Let f(s)

= , f(t) $\frac{1}{(s^2+a^2)}$ − = sin at

a

We have L^{-1}[f'(s)] = - t f(t)

$L^{-1}[\frac{d}{ds}(\frac{1}{(s^2+a^2)})]$ = - t $\frac{1}{a}$ sin at

$L^{-1}[\frac{-2s}{(s^2+a^2)^2}]$ = - $\frac{t}{a}$ sin at

$\Rightarrow L^{-1}[\frac{s}{(s^2+a^2)^2}] = \frac{t}{2a}$ sin at

Inverse L.T. of integrals :-

If L^{-1}[f(s)] = f(t) then $L^{-1}[\int_s^\infty f(s)\,ds] = \frac{f(t)}{t}$

Proof : We have L$[\frac{f(t)}{t}] = \int_s^\infty f(s)\,ds$ provided exist

$\Rightarrow L^{-1}[\int_s^\infty f(s)\,ds] = \frac{f(t)}{t}$

Multiplication by powers of s :-

If L^{-1}[f(s)] = f(t) and f(0) = 0, then L^{-1}[s f(s)] = f'(t) Proof :

W.K.T. L[f'(t)] = s L[f(t)] – f(0)

= s f(s) – 0

⇒ L^{-1}[s f(s)] = f'(t)

In general we have, ⇒ $L^{-1}[s^n$ f(s)] = f^n(t) if = f^n(0) = 0

Problems :

$$\boldsymbol{L^{-1}[\frac{s^2}{(s^2+a^2)^2}]}$$

(1) Find $\boldsymbol{L^{-1}[\frac{s}{(s^2+a^2)^2}] = L^{-1}[s. \frac{s}{(s^2+a^2)^2}]}$

2

solution :

$\boldsymbol{s}$

Let f(s) =

$L^{-1}[f(s)] = \overline{(s^2+a^2)^2}$ f(t) = f'(t) =

$\frac{1}{2a}$ [sin $L^{-1}[\frac{s}{(s^2+a^2)^2}]$ at + t a cos at]

We have L^{-1}[s f(s)] = f'(t)

$\Rightarrow L^{-1}[\frac{s^2}{(s^2+a^2)^2}] = \frac{1}{2a}$

(2) Find $L^{-1}[\frac{s^2}{(s-1)^4}]$ (sin at + at cos at)

$\frac{s}{(s-1)^4}$

Solution

: [f(s)] = $L^{-1}[\frac{s}{(s-1)^4}]$

$= L^{-1}[\frac{s-1+1}{(s-1)^4}]$

$= e^t\, L^{-1}[\frac{s+1}{s^4}$

$= e^t\, L^{-1}[\frac{1}{s^3} + \frac{1}{s^4}]$

$= e^t\, (\frac{t^2}{2} + \frac{t^3}{6}) = \mathbf{f(t)}$

Let f(s) = L_{-1}

] by F.S.T.

$$e^t \left(\frac{t^2}{2} + \frac{t^3}{6}\right) + e^t\left(t + \frac{t^2}{2}\right)$$

Now f'(t) = $= e^t\left(t + t^2 + \frac{t^3}{6}\right)$

By theorem L^{-1}[s f(s)] = f'(t)

$L^{-1}\left[s \frac{s}{(s-1)^4}\right] = e^t\left(t + t^2 + \frac{t^3}{6}\right)$ **Division**

by power of S :

Theorem: If $L^{-1} f\ s\ [\ (\ )]\quad (\ )\quad (\ )\quad = f$ t , then $L^{-1}\left[\frac{f(s)}{}\right]$

$s = {}_0 ⍰^t f\ t\ dt$

Prof: we have by LT,

$\int_0^t f$ (t) dt] = $\frac{f(s)}{s}$ L [

$\Rightarrow L{-1}\left[\frac{f(s)}{s}\right] = {}_0\int^t f\ t\ dt$

$L^{-1}\left[\frac{f(s)}{s^2}\right] = {}_0\int^t [{}_0\int^t f\ t\ dt]$dt

Note: ( )

Problem:

1) Find $L^{-1}[\frac{1}{s(s+3)}]$

solution: Let f (s) = $\frac{1}{s+3}$

$L^{-1}[f(s)] = L^{-1}$ [$\frac{1}{s+3}$] = e^{-3t}= f(t)

By theorem , $L^{-1}\left[\frac{1}{s} . f(s)\right] = {}_0\int^t f$(t)dt

$\Rightarrow L^{-1}$ [$\frac{1}{s(s+3)}$] = $\int_0^t e^{-3t}dt = \frac{e^{-3t}}{-3}]\int_0^t = \frac{1-e^{-3t}}{3}$

2) Find $L^{-1}[\frac{1}{s(s^2+a^2)}]$

Solution : let f(s) $\frac{1}{s^2+a^2}$, L^{-1} = [f(s)] = sinat = f(t) $\frac{1}{a}$

By theorem $L^{-1}[\frac{1}{s}$ f(s)] = $\int_0^t f$(t)dt

$\Rightarrow L^{-1}[\frac{1}{s(s^2+a^2)}] = \int_0^t \frac{1}{a} sinat = \frac{1}{a}$ (- $\frac{cosat}{a}$

= $\frac{1}{a^2}$(1 -cos at)

)

3) Find $L^{-1}[\frac{1}{s^2(s^2+a^2)}]$

$\overline{s^2+a^2}$

1 1 solution : let f(s)

= , f (t) = _ sin at

a

by theorem, $L^{-1}[\frac{1}{s^2}$ f(s)] = $\int_0^t \int_0^t f$(t)dt = ?$_0^t$ [?$_0$ a sin at dt]dt

= $\int_0^t \frac{1}{a^2}$(1- cos at) dt = $\frac{1}{a^2}$ (t - $\frac{sin\ at}{a}$) t

1

Convolution : -

If f(t) and g(t) are two functions defined for $t \geq 0$, then the convolution of f(t) and g(t) is defined as, f(t) * g(t) = $\int_0^t f(u)\, g(t-u)du$.

f(t) * g(t) can also be written as (f * g)(t). Note:- The convolution operation is commutation

i.e., (f * g) (t) = (g * t) (t)

$$\Rightarrow \int_0^t f(u)\, g\,(t-u)du = \int_0^t f(t-u)\, g\,(u)du$$

Convolution theorem :-

If L[f(t)] = f(s) and L[g(t)] = g(s) then L[f(t) * g(t)] = L[f(t)] . L[g(t)]

(or)

= f(s). g(s)

So, L[(f * g) (t)] = f(s) . g(s)

Corollary :-L^{-1}[f(s). g(s)] = (f * g) t

$$= \int_0^t f(u)\, g(t-u)du$$

$$= \int_0^t f(t-u)\, g\,(u)du.$$

Problems:

(1). Find $L^{-1}[\frac{1}{(s-2)(s^2+1)}]$ by using convolution theorem.

1 1 solution: Let f(s) =

$\overline{s-2}$, g(s) = $\overline{s^2+1}$

L^{-1} [f(s)] = $L^{-1}[\frac{1}{s-2}]$ = e^{2t} , L^{-1} [g(s)] = $L^{-1}[\frac{1}{s^2+1}]$ = sin t

By convolution theorem ,

L^{-1}[f(s). g(s)] = $\int_0^t f(t-u)\, g(u)du$

$\Rightarrow L^{-1}[\frac{1}{(s-2)(s^2+1)}$ = $\int_0^t e^{2(t-u)} \sin u\, du$

= $e^{2t} \int_0^t e^{-2u} \sin u\, du$

= e^{2t} $[\frac{e^{-2u}}{(-2)^2+1^2}$ (-

2

sin u –

cos u)]

cos t) - $\frac{e^0}{5}$ (-1)]

= $e^{2t}[\frac{e^{-2t}}{5}$ (-2 sin t – e^{2t}

= $\frac{1}{5}$ (-

2 sin t – cos t) +

5

$= \frac{1}{5}[e^{2t}$ - 2 sin t – cos t]

2) Find $L^{-1}[\frac{1}{s(s^2-a^2)}]$ by convolution theorem

Solution : Let f(s) = $\frac{1}{s}$, g(s) = $\frac{1}{s^2-a^2}$

L^{-1}[f(s)] = $L^{-1}[\frac{1}{s}]$ = 1 = f(t) , L^{-1}[g(s)] = $L^{-1}[\frac{1}{s^2-a^2}]$

$= \frac{1}{a}$ sinh at = g(t) By convolution theorem ,

L^{-1}[f(s). g(s)] = $\int_0^t f(t-u)\, g(u)du$

$\Rightarrow L^{-1}[\frac{1}{s(s^2-a^2)}] = \int_0^t 1\ \frac{1}{a}\sinh au\, du$

$= \frac{1}{a}[\frac{\cosh au}{a}]$, (apply limits o to t)

$= \frac{1}{a^2}$ (cosh at – 1)

Application of L . T to Ordinary Differential Equations :

The L . T method is easier , time – saving and excellent tool for solving O.D.Es

Working rule for finding solution of D . E by L . T:

1) **Write down the given equation and apply L . T O . B . S**
2) **Use the given conditions**
3) **Re arrange the given equation to given transformation of the solution**
4) **Take inverse L.T O. B. S to obtain the desireds obesve Sali stying the given conditions**

The formulae to be used in this process are:

$L[f^1(t)] = s f(s) - f(0)$

$L[f^{11}(t)] = s^2 f(s) - s f(0) - f^1(0)$

$L[f^{111}(t)] = s^3 f(s) - s^2 f(0) - sf(0) - f^{11}(0)$

Note : let f(t) = y (t) and f (s) = y (s) **Problems :**

1) Solve $4y^{11} + \pi^2 y = 0$, $y(0) = 2$, $y^1(0) = 0$

Solution : Here y = y (t)

Given D . E $4\,y^{11}(t) + \pi^2 y(t) = 0$ Let L . T O.B.S

$4\,L[\,y^{11}(t)] + \pi^2 L[\,y(t)$

$\Rightarrow 4\,[\,s^2 L(y)] - s\,y(0) - y^1(0)] + \pi$] = L [0]²

$\Rightarrow L[y]\,[4s^2 + \pi^2\,]$ - L [y]= 0

$\Rightarrow L[y] = \frac{8s}{4s^2+\pi^2}$ 4s(2) – 0 = 0

Let L^{-1} O . B . S, we get y (t) $L^{-1}[\frac{s}{4(s^2+\pi^2/4)}]$ = 8

$= \frac{8}{4}\,L^{-1}\,[\frac{s}{s^2+(\pi^2/2)^2}$] $= 2.\cos \pi/2t$

$\Rightarrow y(t) = 2.\cos^{\pi}/2t$ is solution of gven D.E

3) Solve $y^{111}+2y^{11} - y^1 - 2y = 0$ with $y(0) = y^1(0) = 0$, $y^{11}(0) = 6$

Solution : given D . E

Let L . T On Both Sides

$L[y^{111}] + 2 \, L[y^{11}] - L[y^{1}] - 2 \, L[y] = 0$

- – – - s y (0)

$y^{1}(0)]$

$- s \, L[y] - y(0) - 2 \, L[y] = 0$

$\Rightarrow L \, [y] \, (s^3 + 2s^2 - s - 2) - 6 = 0$

$\Rightarrow L[y] = \frac{6}{s^3+2s^2-s-2}$

$\Rightarrow s^3 \, L[y] s^2 \, y(0) s \, y^{1}(0) y^{11}(0) + 2[s^2 \, L[y]$

$L[y] = \frac{6}{(s-1)(s+1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$ _ _ _ _ _ (1) -

6= A (s + 1) (s +2) + B (s - 1) (s + 2) + C (s – 1) (s+1) _ _ _ _(2) Put s = 1 in _ _ _ (2) 6 = A (2) (3) $\Rightarrow A = 1$

Put s = - 1 in (2)

$\Rightarrow 6 = B(-2)(1)$ $\Rightarrow B = -3$

Put s = -2 in (2)

$\Rightarrow 6 = C(-3)(-1)$ $\Rightarrow C = 2$

Substitute A , B , C in (1)

$$\Rightarrow L[y] = \frac{1}{S-1} - \frac{3}{S+1} + \frac{2}{S+2}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}\right]$$

$$\Rightarrow y(t) = e^{t} - 3e^{-t} + 2e^{-2t}$$

is the solution of given D . E

HW: Solve the D.E $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$

Ans: $y(t) = \frac{e^{-t}}{3}(\sin t - 2\sin 2t)$

UNIT – IV

FOURIER SERIES

Periodic Function :

Definition : **A function f(x) is said to be periodic with period T , if ∀ x , f(x+T) = f(x) where T is positive constant.**

The least value of T > 0 is called the periodic function of f(x).

Example: sin x = sin (2$\pi + x$) $= \sin(4\pi + x) = ----- -$

Here sinx is periodic function with period 2π. ***Def:***

Piecewise Continuous Function:

A function is said to be piece-wise continuous (or) Sectionally Continuous) over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided into a finite number of sub intervals, in each of which f(x) is continuous and both right and left hand limits at every end point if the sub intervals.

Dirichlet Conditions:

A function f(x) satisfies Dirichlet conditions if

(1) f(x) is well defined and single valued except at a finite no. of points in (-l,l)

(2) f(x) is periodic function with period 2l

(3) f(x) and f'(x) are piece wise continuous in (-l,l)

Fourier Series: If f(x) satisfies Dirichlet conditions , then it can be represented by an infinite series called Fourier Series in an interval (-l,l) as

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an\ \cos\frac{n\pi x}{l} + + \sum_{n=1}^{\infty} bn\ \sin\frac{n\pi x}{l}$ ------------ (1) where

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$, $an = \frac{1}{l}\int_{-l}^{l} f(x)\ \cos\frac{n\pi x}{l}$ dx

bn $= \frac{1}{l}\int_{-l}^{l} f(x)\ \sin\frac{n\pi x}{l}$ dx

Here a_0 , an and bn are called Fourier coefficients.

These are also calle Euler's formula.

Note (1): If x∈ $(-\pi, \pi)$ $(i.e., inteval\ is\ (-\pi, \pi))$

Then f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty}(an\ cosnx + bn \sin nx)$

Where a_0 = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, an = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

bn $= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

Note (2): In interval $(0,2\pi)$, f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty}(an\cos nx + bn\sin nx)$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, an $= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

bn $= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Note (3): The Fourier Series in (-l,l) , $(-\pi,\pi)$, $(o, 2\pi)$, $(c, c+2\pi)$ are called Full range expansion series

Note (4): The above series (1) converges to f(x) if x is a point of continuity

The above series (1) converges to $\frac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Note (5) : At x= $\pm\pi$, f(x) = $\frac{f(\pi-0)+f(-\pi+0)}{2}$ here $x \in (-\pi,\pi)$

Even and odd functions:

Case (1): If the function f(x) is an even function in the interval (-l,l)

i.e., f(-x) = f(x) then $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

an = $\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}$ dx (since f(x) & $\cos\frac{n\pi x}{l}$ are even functions)

bn = $\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx \Rightarrow bn=0$ (since f(x) . $\sin\frac{n\pi x}{l}$ is odd function)

Therefore, in this case we get (only) Fourier cosine series only.

Case (2): If function f(x) is odd i.e., f(-x) = - f(x) then

an = 0 (since f(x) $\cos\frac{n\pi x}{l}$ is odd) (a_0=0 also)

And bn = $\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

In this case we get fourier sine series only.

[only for intervals (-l,l) , ($-\pi,\pi$)] **Problems**

:

1)Find Fourier series for the function f(x) = e^{ax} in (0,2π) Solution : Given function f(x) = e^{ax} in (0,2π)

$$\frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\,dx = \frac{1}{\pi}\left(\frac{e^{ax}}{a}\right)$$

a_0 =) apply limits 0 to 2π

$$= \frac{1}{a\pi}\left(e^{2\pi a} - 1\right)$$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} \cos nx\, dx$ an =

$= \frac{1}{\pi}[\frac{e^{ax}}{a^2+n^2}$ (a cos nx + n sin nx)] apply limits 0 to 2π

$= \frac{1}{\pi}[\frac{e^{2\pi a}}{a^2+n^2}$ (a cos 2nπ + 0) - $\frac{e^0}{a^2+n^2}$

$= \frac{1}{\pi}\frac{1}{a^2+n^2}[e^{2\pi a}$ a – 1. a]

$= \frac{a}{\pi(a^2+n^2)}(e^{2\pi a} - 1)$

$= \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} \sin nx\, dx$

$= \frac{1}{\pi}[\frac{e^{ax}}{a^2+n^2}$ (a sin nx + n cos nx)] apply limits 0 to 2π

$= \frac{1}{\pi}[\frac{e^{2a\pi}}{a^2+n^2}$ (0 – n cos 2nπ) - $\frac{e^0}{a^2+n^2}$ (0 - n)]

$= \frac{1}{\pi}\frac{n}{a^2+n^2}(1 - e^{2\pi a}) = \frac{-n}{\pi(a^2+n^2)}(e^{2\pi a} - 1)$

$\frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$

(a + 0)] apply limits 0 to 2π

bn

Now the fourier series is f(x) =

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} an \ \cos nx + \sum_{n=1}^{\infty} bn \ \sin nx$$

$$= \frac{\frac{1}{a\pi}(e^{2\pi a} - 1)}{2} + \sum_{n=1}^{\infty} \frac{a}{\pi(a^2+n^2)} (e^{2\pi a} - 1)$$

$$\frac{-n}{\pi(a^2+n^2)} (e^{2\pi a} - 1) \ \sin nx$$

cosnx $+ \sum_{n=1}^{\infty}$

(2): Find Fourier series for the function f(x) = e^x **in** $(0,2\pi)$

Solution : Given function f(x) = e^x **in** $(0,2\pi)$ a_0 **=**

apply $\frac{1}{\pi}\int_0^{2\pi} f(x)dx$ $=\frac{1}{\pi}\int_0^{2\pi} e^x\, dx$ limits 0 to 2π

$= \frac{1}{\pi}(e^x)$ apply limits 0 to 2π

$= \frac{1}{\pi}\ (e^{2\pi} - 1)$

an = $\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$ bn

$= \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\, dx$

$= \frac{1}{\pi}[\frac{e^x}{1+n^2}$ (1 cos nx + n sin nx)]

$= \frac{1}{\pi}[\frac{e^{2\pi}}{1+n^2}$ (cos 2nπ + 0) - $\frac{e^0}{1+n^2}$ (cos 0 + 0)]

$= \frac{1}{\pi}\frac{1}{1+n^2}[e^{2\pi} - 1]$

$= \frac{1}{\pi(1+n^2)}(e^{2\pi} - 1)$

$= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\, dx$

$= \frac{1}{\pi}[\frac{e^x}{1+n^2}$ (sin nx + n cos nx)] apply limits 0 to 2π

$= \frac{1}{\pi}[\frac{e^{2\pi}}{1+n^2}$ (0 – n cos 2nπ) - $\frac{e^0}{1+n^2}$ (0 - n)]

$= \frac{1}{\pi}\frac{n}{1+n^2}(1 - e^{2\pi}) = \frac{-n}{\pi(1+n^2)}(e^{2\pi} - 1)$

Now the fourier series is f(x) =

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} an \ \cos nx + \sum_{n=1}^{\infty} bn \ \sin nx$$

$$= \frac{\frac{1}{\pi}(e^{2\pi}-1)}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi(1+n^2)} (e^{2\pi} - 1) \ \cos nx + \sum_{n=1}^{\infty} \frac{-n}{\pi(1+n^2)} (e^{2\pi} - 1) \ \sin nx$$

Problem (3): H.W

Find Fourier series for the function f(x) = e^{-x} in $(0,2\pi)$

(Hint:- put a = - 1 in problem (1) we get the solution.)

(4) Express f(x) = x - π as Fourier Series in the interval - π < x < π Solution:

Given function f(x) = x - π a_0

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - \pi) \ dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x \ dx - \frac{1}{\pi} \int_{-\pi}^{\pi} \pi \, dx$$

= 0 – [x] with limits - π to π

= 0 – [π + π] = 2π an =

dx $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - \pi)\cos nx\,dx$ (since even)

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$ - $\frac{1}{\pi}$(- π) $\int_{-\pi}^{\pi}\cos nx\,dx$

$= \frac{1}{\pi}$(0) (since x cosnx is odd) + 2$\int_0^{\pi}\cos nx$

= 0 + 2 [$\frac{\sin nx}{n}$] 0 to π limits apply we get an =

0+0 = 0

bn $= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - \pi)\sin nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$ - $\frac{1}{\pi}$(- π) $\int_{-\pi}^{\pi}\sin nx\,dx$

(even) (odd)

$= \frac{1}{\pi}\ 2\int_0^{\pi} x\sin nx$ dx – 0 (since sin nx is odd)

$= \frac{2}{\pi}$ [{ x(- $\frac{\cos nx}{n}$)} - $\int_0^{\pi}\frac{-\cos nx}{n}$ dx]

$= \frac{2}{\pi}$ [-$\pi\frac{\cos n\pi}{n}$ + 0 + $\frac{1}{n}$ ($\frac{\sin nx}{n}$)] apply limits 0 to π

$= \frac{2}{\pi}$ [-$\pi\frac{\cos n\pi}{n}$ + 0 + $\frac{1}{n}$ (0)] = - $\frac{2}{n}$ cos nπ = - $\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$, n=1,2,3...........

Now the Fourier Series of f(x) is f(x)

$= \frac{a_0}{2} + \sum_{n=1}^{\infty}(an\ cosnx + bn \sin nx)$ f(x)

$= \frac{2\pi}{2} + \sum_{n=1}^{\infty}[(0)\ cosnx + \frac{2}{n}(-1)^{n+1}\sin nx]$

$= \pi + \sum_{n=1}^{\infty}[\frac{2}{n}(-1)^{n+1}\sin nx]$

(5)Obtain the Fourier series for $f(x) = x - x^2$ in the interval $[-\pi, \pi]$ $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{n}{1^2}$ 2

Hence show that (or)

$\pi 2$

$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots =$

12

Solution : Given function is $f(x) = x - x^2$ in $[-\pi, \pi]$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\ dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\ dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ dx$

$= 0$ (odd) $- \frac{1}{\pi}[\frac{x^3}{3}] = -2\pi^2/3$

an = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

= $\frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\, dx$

= $\frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\, dx$ - $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$

(odd) (even)

= 0 - $\frac{1}{\pi} 2 \int_{0}^{\pi} x^2 \cos nx\, dx$

u = x^2 , dv = cos nx dx

= - $\frac{2}{\pi}$ [($\frac{x^2 \sin nx}{n}$) - $\frac{2}{n}\int_{0}^{\pi} x \sin nx\, dx$]

du = 2x dx , dv = ∫ $\cos nx\, dx$

apply limits 0 to π

= - π2 [0 - π2 {($\overline{-x \cos n\, nx}$) + 0∫π $\underline{\cos n}$ $\underline{nx}$ = $\frac{}{n}$ $\sin nx$ dx

apply limits 0 to π

= $\frac{4}{\pi n}$ [-$\pi \frac{(-1)^n}{n}$ + $\frac{1}{n^2}$

= $\frac{4}{n^2}(-1)^{n+1}$

4 (sin nx)] ∫ $udv = \frac{4}{1^2} = 4$ $uv - \int vdu$

an = if n is odd a1 =

n^2

$-\frac{4}{n^2}$ if n is even

a2 = $\frac{4}{2^2}$ =1

a3 = $\frac{4}{3^2}$ = 4/9

bn = $\frac{1}{\pi}\int_{-\pi}^{\pi}$ (x – x²) $\cos nx\, dx$

= $\frac{1}{\pi}$ [$\int_{-\pi}^{\pi}$ x $\sin nx\, dx$ - $\int_{-\pi}^{\pi}$ x² $\sin nx\, dx$

(even) (odd)

= $\frac{2}{\pi}$ [($\frac{-\text{x}\cos nx}{n}$) + $\frac{1}{n}\int_{0}^{\pi}$ $\cos nx\, dx$]

= $\frac{2}{n}$ [$-\pi\frac{(-1)^n}{n}$ + $\frac{1}{n^2}$ (

sin nx)] b1 = 2/1 = 2 = $\frac{2}{n}(-1)^{n+1}$ = $\frac{2}{n}$ if n is odd

b2 = - 2/2 = -1

= - $\frac{2}{n}$ if n is even

b3 = 2/3

Now, f(x) = $\frac{a_0}{2}$ + $\sum_{n=1}^{\infty}(an\ cosnx + bn\sin nx)$------(1) substitute in (1)

$\Rightarrow$ f(x) = $\frac{-\pi^2}{3}$ + 4 ($\frac{\cos x}{1^2}$ - $\frac{\cos 2x}{2^2}$ + $\frac{\cos 3x}{3^2}$ − …… . .)

+ 2 ($\frac{\sin x}{1}$ − $\frac{\sin 2x}{2}$ + $\frac{\sin 3x}{3}$ +…….)--------(2)

put x = 0 in (2)

f(0) = 0 = $\frac{-\pi^2}{3}$ + 4($\frac{1}{1^2}$ - $\frac{1}{2^2}$ + $\frac{1}{3^2}$.......)

$\Rightarrow$ $\frac{1}{1^2}$ - $\frac{1}{2^2}$ + $\frac{1}{3^2}$ + ...= $\frac{\pi^2}{12}$

Half range series

(1) The half range cosine series in (0,l) is f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \ \cos\frac{n\pi x}{l}$

$a_0 = \frac{2}{l}\int_0^l f(x)dx$, $an = \frac{2}{l}\int_0^l f(x) \ \cos\frac{n\pi x}{l}$ dx

(2)The half range sine series in (0,l) is f(x) = $\sum_{n=1}^{\infty} b \ \sin\frac{n\pi x}{l}$

where bn = $\frac{2}{l}\int_0^l f(x) \ \sin\frac{n\pi x}{l}$ dx

Note :1) The half range cosine series in (0,π) is f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos nx$ where

$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx$, $an = \frac{2}{\pi}\int_0^\pi f(x) \ \cos n\pi$ dx

Note :2) The half range sine series in (0,π) is f(x) = $\sum_{n=1}^{\infty} bn \sin nx$ where

bn = $\frac{2}{\pi}\int_0^\pi f(x)$ sin nx dx

(1)Express f(x) = π-x as Fourier cosine and sine series in (0, π)

Solution :

The half range cosine series for f(x) is f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos nx$ ....(1)

where a₀= $\frac{2}{\pi}\int_0^{\pi} f(x)$ dx = $\frac{2}{\pi}\int_0^{\pi} \pi$ −x dx

= $\frac{2}{\pi}$ [πx - $\frac{x^2}{2}$] apply limits o to π

= $\frac{2}{\pi}$ [π^2- $\frac{\pi^2}{2}$ - (0-0)] = $\frac{2}{\pi}$ ($\frac{\pi^2}{2}$) = π

an = $\frac{2}{\pi}\int_0^{\pi} f(x) \cos n\pi$ dx

= $\frac{2}{\pi}\int_0^{\pi}(\pi - x) \cos n\pi$ dx

= $\frac{2}{\pi}$ [{(π−x) $\frac{sinnx}{n}$} + $\int_0^{\pi} \frac{sinnx}{n}$ dx]

(apply o to π)

= $\frac{2}{\pi}$ [(0-0) + $\frac{1}{n}$ (- $\frac{\cos nx}{n}$)] apply o to π

= - $\frac{2}{\pi n^2}$ [$\cos n\pi$ - cos 0]

= - $\frac{2}{\pi n^2}$ [$\cos n\pi$ - cos 0]

= - $\frac{2}{\pi n^2}$ [[$(-1)^n$ −1] = $\frac{2}{\pi n^2}$ [[$1-(-1)^n$]

Now (1) ⇒ f(x) = $\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}$ [[$1-(-1)^n$] $\cos nx$.

H.W.) Express f(x) = π-x as fourier sine series in (o, π Ans : 2 $\sum_{n=1}^{\infty} \frac{sinnx}{n}$ (bn = $\frac{2}{n}$)

2) Find the half range sine series of f(x) = x in the range 0 < x < π

$\pi 2$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ \ldots\ldots\ldots\ldots =$

8

Solution : The half range cosine series for f(x) is f(x)

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos nx$ ....(1)

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)$ dx = $\frac{2}{\pi}\int_0^{\pi}$ x dx = $\frac{2}{\pi}\left[\frac{x^2}{2}\right]$ apply limits o to π

$= \pi$

an = $\frac{2}{\pi}\int_0^{\pi} f(x)\ \cos nx\ \mathrm{dx}$

= $\frac{2}{\pi}$?$_0^{\pi}$ (x) $\cos nx\ \mathrm{dx}$

= $\frac{2}{\pi}$ **[**$\{(x)\ \frac{\sin\ nx}{n}\}$ **-** ?$_0^{\pi}$ $\frac{\sin\ nx}{n}$ dx]

(apply o to π)

= $\frac{2}{\pi}$ **[** (0-0) - $\frac{1}{n}$ (- $\frac{\cos nx}{n}$

= $\frac{2}{\pi n^2}$ **[**$\cos n\pi$ - cos 0]

= $\frac{2}{\pi n^2}$ **[** $[(-1)^n - 1]$)] apply o to π

an = 0 if n is even

$= -\dfrac{4}{\pi n^2}$ if n is odd

Now (1) $\Rightarrow$: $f(x) = \dfrac{\pi}{2} + \sum_{n=1}^{\infty} \dfrac{-4}{\pi n^2} \cos nx$ if n is odd

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \ldots\ldots..\right)$$

Put x=0 on both sides

$$\Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \ldots\ldots..\right)$$

$$\Rightarrow \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \ldots\ldots..\right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$$

3) Express f(x) = $\cos x$, 0 < x < π in half range sine series

π $\qquad \sum_{n=1}^{\infty} bn \sin nx$ -------(1)

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \ \sin \text{nx dx}$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos x \ \sin \text{nx dx}$$

$$= \frac{2}{\pi} \frac{1}{2} \int_0^{\pi} [\sin(\text{n} + 1)\text{x} + \sin(\text{n} - 1)\text{x}] \text{ dx}$$

$$= \frac{1}{\pi} \left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]$$ apply limits o to π

$$= \frac{1}{\pi} \left[\frac{-\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right.$$

$$= \frac{1}{\pi} \left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^n}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right.$$

$$= \frac{1}{\pi} \left[\left(\frac{(-1)^2(-1)^n}{n+1} + \frac{(-1)^n}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right] \right.$$

$$= \frac{1}{\pi} \left[(-1)^n \left\{ \frac{1}{n+1} + \frac{1}{n-1} \right\} + \left\{ \frac{1}{n+1} + \frac{1}{n-1} \right\} \right]$$

$$= \frac{1}{\pi} \left[\{(-1)^n + 1\} \left(\frac{1}{n+1} + \frac{1}{n-1} \right) \right]$$

$$= \frac{2n}{\pi} \left[\frac{1+(-1)^n}{n^2-1} \right]$$ (n not equal to 1)

Solution : The half range sine series in (0,) is f(x) = where bn

]

] , n is not equal to 1

bn = 0 if n is odd.

$= \frac{4n}{\pi(n^2-1)}$ if n is even

b1 = b3 = b5 = -------- = 0

(1) ⇒ f(x) = $\sum_{n=2}^{\infty} \frac{4n}{\pi(n^2-1)} \sin nx$, for n is even

4)Find half range sine series for f(x) = x(π −x) , in 0 < x < π

Deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}$ **+…….=** $\frac{\pi^3}{32}$

Solution : Fourier series is f(x) = $\sum_{n=1}^{\infty} bn \sin nx$...(1)bn

$\frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx$

$= \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx\, dx$

$= \frac{2}{\pi}\pi \int_0^{\pi} x \sin nx\, dx - \frac{2}{\pi} \int_0^{\pi} x^2 \sin nx\, dx$

$= \; = 2 \left[\left(\frac{-x\, cosnx}{n}\right) - \int_0^{\pi} \frac{-\cos nx}{n} dx\right] - \frac{2}{\pi} \left[\left(\frac{-x^2\, cosnx}{n}\right) - \int_0^{\pi} \frac{-\cos nx}{n} 2x\, dx\right]$

(apply 0 to π)

$= 2 \left[\left(\frac{-\pi\, cosn\pi}{n}\right) + 0 + \frac{1}{n}\left(\frac{\sin nx}{n}\right) 0 \text{ to } \pi\right] - \frac{2}{\pi} \left[\left(\frac{-\pi^{2}\, cosn\pi}{n}\right) + \mathbf{0} + \frac{2}{n} \int_0^{\pi} x \cos nx\, dx\right]$

(apply o to π)

$= 2 \left[-\pi \frac{(-1)^n}{n} + 0\right] + \frac{2}{\pi}.\pi^2 \frac{(-1)^n}{n} - \frac{4}{\pi n} \left[\left(\frac{x \sin nx}{n}\right) 0 \text{ to } \pi - \int_0^{\pi} \frac{\sin nx}{n} dx\right.$

$= 2 \left[-\pi \frac{(-1)^n}{n}\right] + 2\pi \frac{(-1)^n}{n} + \frac{4}{\pi n^2} \left(\frac{-\, cosnx}{n}\right) 0 \text{ to } \pi$

$= \frac{4}{\pi n^3} [- \cos n\pi + \cos 0]$

$= \frac{4}{\pi n^3} [[1-(-1)^n]$ sub in (1) bn

(1) ⇒ f(x) = $\sum_{n=1}^{\infty} \frac{4}{\pi n^3}$ [$[1-(-1)^n]$] $\sin nx$

(1) ⇒ f(x) = b1 sin x + b2 sin 2x + b3 sin 3x + ...

= $\frac{4}{\pi}$ $(2)\sin x + 0 +$ $\frac{4}{\pi.3^3}$

⇒ $x(\pi - x)$ = $\frac{8}{\pi}$ [$\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3}$ + ...](2) sin 3x + ... Put

x = π/2 on both sides

$\pi\ \pi$ [− ^{3} + ...] ⇒

(2) =

3

2

$\frac{8}{\pi}$ $\frac{1}{1}$ $\frac{1}{3}$

⇒ $\pi 4 \frac{2}{}(\pi 8)$ = [^{3} − ^{3} + 5^3 .....]

$\frac{1}{1}$ $\frac{1}{3}$ $\frac{1}{}$

3

⇒ [$\frac{1}{1}$ − $\frac{1}{3^3}$ + $\frac{1}{5^3}$ .....] = $\pi 32$

3

- **<u>FOURIER SERIES IN AN ARBITRARY INTERVAL I,e in (-l,l</u>) & (0,2l)**

- **Problem** : **1) Obtain the half range sine series for e^x in 0<x<1 Solution** : Given f(x) = e^x in (0,l)

The half range sine series for f(x) in (0,l) is f(x)= $\sum_{n=1}^{\infty} bn \ \sin\frac{n\pi x}{l}$ ......(1)

l=1 Where bn $= \frac{2}{l} \int_0^l f(x) \ \sin\frac{n\pi x}{l}$ dx

$= \frac{2}{1} \int_0^1 f(x) \ \sin n\pi x$ dx

bn

$= 2 \int_0^1 e^x \sin(n\pi x)$ dx

$=2 \frac{e^x}{(1)^2+(n\pi)^2}$ (sin nπx - nπ . cos nπx) apply limits 0 to 1

$= \frac{2}{1+n^2\pi^2}$ [e^1(0 - nπ . cos nπ) - e^0(0 - nπ . cos 0)]

$= \frac{2}{1+n^2\pi^2}$ [-n$\pi . e$. cos n$\pi + n\pi$]

$= \frac{2}{1+n^2\pi^2} [-n\pi e(-1)^n + n\,\pi]$

$= \frac{2n\pi}{1+n^2\pi^2} [1 - e(-1)^n]$

bn

(1) $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2} [1 - e(-1)^n] \ \sin n\pi x$

f(x)=

2) $\sum_{n=1}^{\infty} bn \ \sin\frac{n\pi x}{l}$ ....(1) $= \frac{2}{l} \int_0^l f(x) \ \sin\frac{n\pi x}{l} \ dx$ **Find the half range sine series of f(x) = 1 in (0,l) Solution** : The half range sine series in (0,l) is f(x) =

where bn

$$= \frac{2}{l} \int_0^l 1 \sin\frac{n\pi x}{l} \ dx$$

$$= \frac{2}{l} \left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}} \right]$$ apply limits o to l

$$= -\frac{2}{l} \cdot \frac{l}{n\pi} [\cos n\pi - \cos 0]$$

$$= -\frac{2}{n\pi} [(-1)^n - 1]$$

bn = 0 if n is even

if n is odd

$\Rightarrow \sum_{n=1}^{\infty} \frac{}{n\pi} \sin \frac{}{l}$

Now (1) , if n is odd

3)Find the half range cosine series of f(x) = x(2-x) in the range $0 \leq x \leq 2$

Hence find sum of series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots$

Solution : Given function f(x) = x(2-x) = 2x - x^2

The half range cosine series for f(x) is f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos\frac{n\pi x}{l}$ ....(1)

where $a_0 = \frac{2}{2}\int_0^2 f(x))\, dx = \frac{2}{2}\int_0 \int_0^2 f(x)\; 2x - x^2)\, dx$

$= \frac{2}{2}\left[\frac{2x^2}{2} - \frac{2x^3}{3}\right]$ apply 0 to 2 $= -\frac{4}{3}$

$an = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$= \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$ (l=2)

$= \int_0^2 (2x - x^2)\cos\frac{n\pi x}{2}\,dx$ (using integration by parts)

$= [(2x - x^2)\frac{2}{n\pi}\{\sin\frac{n\pi x}{2} + (2\text{-}2x)\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2} + (2)\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}]$

apply limits 0 to 2

$= \frac{-8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2} = \frac{-8}{n^2\pi^2}[1 - (-1)^n]$

$\frac{-16}{n^2\pi^2}$ when n is even an =

= 0 when n is odd

Substitute the values of a_0 and an in (1) we get

$$(1) \Rightarrow 2x - x^2 = \frac{2}{3} - \frac{16}{\pi^2}\sum_{n=2,4,6}^{\infty}\left(\frac{1}{n^2}\cos\frac{n\pi x}{2}\right)$$

$$= \frac{2}{3} - \frac{16}{\pi^2}\left(\frac{1}{2^2}\cos \pi x + \frac{1}{4^2}\cos 2\pi x + \frac{1}{6^2}\cos 3\pi x + \text{------}\right)$$

$$= \frac{2}{3} - \frac{16}{\pi^2}\cdot\frac{1}{2^2}\left(\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \text{-------}\right)$$

$$\Rightarrow 2x - x^2 = \frac{2}{3} - \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \text{-------}\right)\text{-------}(2)$$

Putting x = 1

in (2) we get

$$2 - 1 = \frac{2}{3} - \frac{4}{\pi^2}\left(\cos \pi + \frac{1}{2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \frac{1}{4^2}\cos 4\pi + \text{-------}\right)$$

$$\Rightarrow 1 - \frac{2}{3} = -\frac{4}{\pi^2}\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\ldots\ldots\ldots\right)$$

$$\Rightarrow \frac{1}{3} = \frac{4}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots\ldots\right)$$

$\pi 2$

$+ \Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}$ ....) =

12

(4) Expand f(x) = e^{-x} *as Fourier series* in (-1,1)

Solution : Here l = 1

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$= \frac{1}{1}\int_{-1}^{1} e^{-x}dx$ $=(\frac{e^{-x}}{-1})$apply limits -1 to 1

$= - e^{-1} + e^{1}$ = e - $\frac{1}{e}$ = 2 sinh 1

an = $\frac{1}{l}\int_{-l}^{l} f(x)\ \cos\frac{n\pi x}{l}$ dx

= 1 $\int_{-1}^{1} e^{-x}\cos(n\pi x)$ dx

= $\frac{e^{-x}}{(-1)^2+(n\pi)^2}$ (- cos nπx + nπ . sin nπx) apply limits -1 to 1

$= \frac{1}{1+n^2\pi^2}$ $[e^{-1}\{-(-1)^n + 0\} - e^1\{-(-1)^n + 0\}]$

$= \frac{1}{1+n^2\pi^2}$ $(-1)^n$ $(e - e^{-1})$

$= \frac{1}{1+n^2\pi^2}$ $(-1)^n$ 2sinh1

bn $= \frac{1}{l}$ $\int_{-l}^{l} f(x)$ $\sin\frac{n\pi x}{l}$ dx bn

$= \frac{1}{l}$ $\int_{-l}^{l} f(x)$ $\sin\frac{n\pi x}{l}$ dx bn

$= \frac{1}{l}$ $\int_{-l}^{l} f(x)$ $\sin\frac{n\pi x}{l}$ dx (l=1)

$= \int_{-1}^{1} e^{-x} \, s\,in(n\pi x)$ dx

$= \frac{e^{-x}}{(-1)^2+(n\pi)^2}$ (

- sin nπx -

nπ . cos

nπx)

apply

limits -1

to 1

$= \frac{1}{1+n^2\pi^2}$ $[e^{-1}$(0 - nπ . cos nπ) - e^1(0 - nπ . cos nπ)]

$= \frac{1}{1+n^2\pi^2}$ nπ . cos nπ $(e - e^{-1})$

$= \frac{1}{1+n^2\pi^2}$ n$\pi(-1)^n$ 2sinh1

Now Fourier series of f(x) in (-l,l) is

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos n\,\pi x + + \sum_{n=1}^{\infty} bn\sin n\pi x$ ....(1)

$f(x) = \frac{2\sinh 1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}(-1)^n\, 2\sinh 1 \cos n\pi x + \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}\, n\pi(-1)^n\, 2\sinh 1 \sin n\pi x$

$\Rightarrow f(x) = 2\sinh 1 + \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}(-1)^n \{\cos n\pi x + n\pi \sin n\pi x\}\right]$

- Functions having points of discontinuity : Problems:

(1) If f(x) is a function with period 2π is defined by f(x) =

0 , for - π < x ≤ 0

$= x$, for $0 \leq x < \pi$ then write the fourier series for f(x)

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$

Solution : The Fourier series in $(-\pi, \pi)$ is **f(x) =**

$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ ---------(1)

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$

$= \frac{1}{\pi}\left[0 + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left(\frac{x^2}{2}\right)$ 0 to π $= \frac{\pi}{2}$

an = $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$= \frac{1}{\pi}[\,0 + \int_0^{\pi} x\cos nx\, dx\,]$ $\quad$? $u\,dv$ = uv - ? $v\,du$

$= \frac{1}{\pi n^2}[(-1)^n - 1]$ $\quad$ u = x , $\quad$ dv = cos nx dx = 0 , if n is even

$= -\frac{2}{\pi n^2}$, if n is odd

bn $= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$= \frac{1}{\pi}[\,0 + \int_0^{\pi} x\sin nx\, dx\,]$

$= \frac{1}{\pi}[\,(\frac{-x\cos nx}{n}) - \int_0^{\pi}\frac{-\cos nx}{n}\,dx]$ $\quad$ (apply 0 to π)

$= \frac{1}{\pi}[\,(\frac{-\pi\cos n\pi}{n}) + 0 + \frac{1}{n}(\frac{\sin\ nx}{n})$ 0 to π]

$= \frac{1}{\pi}[\,\frac{-\pi(-1)^n}{n}$ + 0 + 0 = - $\frac{(-1)^n}{n}$

bn $= \frac{1}{n}$, if n is odd

$= -\frac{1}{n}$, if n is even

(1) ⇒ $f(x) = \frac{1}{2}\frac{\pi}{2} - \frac{2}{\pi}[(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \text{-----}) + (\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\text{ ------})\text{------}(2)$

Put x = 0 on both sides $\quad$ f(0) = 0

$(2) \Rightarrow 0 = \frac{\pi}{4}$ $- \frac{2}{\pi}(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}$ ------

$\frac{2}{\pi}(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}$ ------$) = \frac{\pi}{4}$

$\Rightarrow (\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}$ ------$) = \frac{\pi^2}{8}$ **) + 0**

Problem (2) : Find Fourier series to represent the function f(x) given by

f(x) = - k , for $-\pi < x < 0$

k , for $0 < x < \pi$ hence show

that $1 \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ ------ $= \frac{\pi}{4}$ Solution : In

$-\pi < x < 0$

i.e., $x \in (-\pi, 0)$, f(x) = - k

f(-x) = - f(x) in $(0, \pi)$

In $0 < x < \pi$ i.e., $x \in (0, \pi)$ f(x)

= k f(-x) = k = -

(-k)

= - f(x) in $(-$

π,0) There fore f(x) is odd function in $(-\pi, \pi)$

so $a_0 = 0$, an = 0

bn $= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi}\int_0^{\pi} k \sin nx\, dx$

$= \frac{2k}{\pi} \left(\frac{-\cos nx}{n}\right.$

$= \frac{2k}{\pi n} [(-1)^n - 1]$

bn

) apply limits 0 to π

= 0 , if n is even

$= \frac{4k}{\pi n}$, if n is odd

Now f(x) = $\sum_{n=1}^{\infty} bn \sin nx$

= $b_1 \sin 1x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x$ --------f(x)

$= \frac{4k}{\pi} \sin x + 0 + \frac{4k}{\pi} \frac{\sin 3x}{3} + 0 +$ ---------(1)

Deduction : put x = $\frac{\pi}{2}$ on both sides in (1)

$$(1) \Rightarrow k = \frac{4k}{\pi}(1) + \frac{4k}{\pi}(-\frac{1}{3}) + \frac{4k}{\pi}(\frac{1}{5}) + \text{--------}$$

$$\Rightarrow k = \frac{4k}{\pi}[1 - \frac{1}{3} + \frac{1}{5} \text{---------}$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \text{---------}$$

<u>Parseval's Formula</u> :-

Prove That $\int_{-l}^{l}[f(x)]^2$ dx = l [$\frac{a_0{}^2}{2} + \sum_{n=1}^{\infty}(an^2 + bn^2)$

Proof :- We know that the Fourier series of f(x) in (-l,l) is **f(x)**

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} an \ \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} bn \ \sin\frac{n\pi x}{l}$ ------------(1)

Multiplying on both sides of (1) by f(x) and integrate term by term from -l to l we get $\int_{-l}^{l}[f(x)]^2$ dx =

$$\frac{a_0}{2}\int_{-l}^{l} f(x)dx + \sum_{n=1}^{\infty} an \int_{-l}^{l} f(x) \ \cos\frac{n\pi x}{l} \ dx$$

$$+ \sum_{n=1}^{\infty} bn \int_{-l}^{l} f(x) \ \sin\frac{n\pi x}{l} \ dx \text{ --------(2)}$$

Now $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx \Rightarrow \int_{-l}^{l} f(x)dx = l\, a_0$

an = $\frac{1}{l}\int_{-l}^{l} f(x) \ \cos\frac{n\pi x}{l} \ dx \Rightarrow \int_{-l}^{l} f(x) \ \cos\frac{n\pi x}{l} \ dx$ **= l an**

and bn $= \frac{1}{l}\int_{-l}^{l} f(x) \ \sin\frac{n\pi x}{l} \ dx \Rightarrow \int_{-l}^{l} f(x) \ \sin\frac{n\pi x}{l} \ dx$ **= l bn**

Substitute these in (2)

$$\frac{a_0}{2} . l \; a_0 + \sum_{n=1}^{\infty} an . l \; an + \sum_{n=1}^{\infty} bn . l \; bn$$

$$(2) \Rightarrow \int_{-l}^{l} [f(x)]^2 \, dx = l \left[\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} (an^2 + bn^2) \right]$$

This is called parseval's formula.

Note 1): In (0,2l) the parseval's formula is

$$\int_{0}^{2l} [f(x)]^2 \, dx = l \left[\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} (an^2 + bn^2) \right]$$

Note :2) If 0 < x < l (for half range cosine series of f(x)) parsevel's formula is

$$\int_{0}^{l} [f(x)]^2 \, dx = \frac{l}{2} \left[\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty} an^2 \right]$$

Note :3) If 0 < x < l (for half range sine series of f(x)) parsevel's formula is

$$\int_{0}^{l} [f(x)]^2 \, dx = \frac{l}{2} \left[\sum_{n=1}^{\infty} bn^2 \right]$$

Problem : prove that in 0 < x < l, $x = \frac{l}{2} - \frac{4l}{\pi^2} \left(\frac{\cos\frac{\pi x}{l}}{1^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} + \text{--------} \right)$ **and hence deduce that**

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \text{------} = \frac{\pi^4}{96}$$

Solution : Let f(X) = x , 0 < X < l

The Fourier cosine series for f(x) in (0,l) is

f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty} an \cos\frac{n\pi x}{l}$ ------- (1)

Here $a_0 = \frac{2}{l}\int_0^l f(x)$) dx

$= \frac{2}{l}\int_0^l x$ dx

$= \frac{2}{l}[\frac{x^2}{2}]$ apply limits 0 to l

=

$= \frac{2}{l}[\frac{l^2}{2}] = l$

an $\frac{2}{l}\int_0^l f(x) \cos\frac{n\pi x}{l}$ dx

$= \frac{2}{l}\int_0^l x \cos\frac{n\pi x}{l}$ dx

u = x, dv = $\cos\frac{n\pi x}{l}$ dx

$n\pi x$

$= \frac{2}{l}[\{\frac{x \sin\frac{}{l}}{\frac{n\pi}{l}}\}$ 0 to l - $\int_0^l \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$ dx]

$= \frac{2}{l} \cdot \frac{l}{n\pi}[(0-0) - \{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\}$ 0 to l]

$= \frac{2}{n\pi} \cdot \frac{l}{n\pi}[\cos n\pi - \cos 0]$

$= \frac{2l}{n^2\pi^2}[[(-1)^n - 1]$

$-4l$ $-4l$ an = 0 ,

n is even $a_1 = \overline{\pi^2.1^2}$, $a_3 = \overline{\pi^2.3^2}$

$= \frac{-4l}{n^2\pi^2}$, n is odd $a_2 = 0$, $a_4 = 0$ …………

Substitute a_0,an in (1)

$$(1) \Rightarrow \frac{l}{2} . \frac{-4l}{\pi^2} \left(\frac{\cos\frac{\pi x}{l}}{1^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} + \text{--------}\right)$$

Now $a_0 = l$, $a_1 = \frac{-4l}{\pi^2.1^2}$, $a_3 = \frac{-4l}{\pi^2.3^2}$ -------

From parseval's formula , we have

$\int_0^l [f(x)]^2$ dx = $\frac{l}{2}$ [$\frac{{a_0}^2}{2}$

$\Rightarrow \int_0^l x^2$ $\frac{l}{2}$ [$\frac{l}{2}$ + $\frac{16l}{\pi^4.1^4}$+ a_1^2 + $\frac{16l}{\pi^4.3^4}$ + ---------] $a_2^2 + a_3^2$ + -------]

2 2 2

dx = + 0^2 +

$\Rightarrow (\frac{x^3}{3}$ l $\frac{1}{2} + \frac{16}{\pi^4.1^4} + \frac{16}{\pi^4.3^4}$ + -------]

) 0 to l = .

l^2 [2

$\Rightarrow \frac{1}{3}(2l^3).\ \frac{2}{l^3} = \frac{1}{2} + \frac{16}{\pi^4.1^4} + \frac{16}{\pi^4.3^4} + \text{-------}$

$\Rightarrow \frac{2}{3} - \frac{1}{2} = \frac{16}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \text{------}\right)$

$\Rightarrow \frac{1}{6}\ .\frac{\pi^4}{16} = \frac{1}{1^4} + \frac{1}{3^4} + \text{------}$

There fore $\frac{1}{1^4} + \frac{1}{3^4} + \text{------} = \frac{\pi^4}{96}$

COMPLEX FOURIER SERIES in (-l,l) or (0,2l):-

The complex form of Fourier series of a periodic function f(x) of period 2l is defined by

f(x) = $\sum_{n=-\infty}^{\infty} cn\ e^{\frac{in\pi x}{l}}$ --- (1) where cn $= \frac{1}{2l}\int_{-l}^{l} f(x)\ e^{\frac{-in\pi x}{l}}$ dx , n=0,-1,1,2….

Note (1) : If period of function is 2π, i.e., in (- π , π) or (0,2 π) then complex fourier series is f(x) = $\sum_{n=-\infty}^{\infty} cn\ e^{inx}$ -----(2)

Where cn $= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}$ dx , n = 0,-1,1,-2,2 ………

Problem : Find complex fourier series of f(x) = e^x if - π < x < π and f(x) = f(x + 2 π)

Solution : Complex fourier series of f(x) = e^x is f(x) = $\sum_{n=-\infty}^{\infty} cn\ e^{inx}$ ----(1)

When cn = $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ e^{-inx}$ dx

cn = $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}\ e^{-inx}$ dx = $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}$

= $\frac{1}{2\pi}$ [$\frac{e^{(1-in)x}}{1-in}$] limits ($-\pi, \pi$) = $\frac{1}{2\pi(1-in)}$ [$e^{(1-in)\pi}$ - $e^{(1-in)(-\pi)}$]

= $\frac{1}{2\pi(1-in)}$ [$e^{\pi}.\ e^{-in\pi}$ - $e^{-\pi}.\ e^{in\pi}$]

= $\frac{1}{2\pi}.\ \frac{1}{(1-in)}$ [$e^{\pi}.\ (-1)^{n}$ - $e^{-\pi}.\ (-1)^{n}$

= $\frac{1}{2\pi(1-in)}\ (-1)^{n}(e^{\pi}$- $e^{-\pi})$

$\pm in\pi$ = $(-1)^{n}$+-0

1 $1+in$

] e = cos n π + l

sin n π ____ ____

$(1\ -in$) $1\ +in$ *

$= \frac{(-1)^n}{2\pi} . \frac{1+in}{(1+n^2)}$. (2 sin h π)

$= (-1)^n \cdot \frac{1+in}{\pi(1+n^2)}$

Therefore cn

$(1) \Rightarrow f(x) = \sum_{n=-\infty}^{\infty} (-1)^n . \frac{1+in}{\pi(1+n^2)}$

(sin h π) sub in (1)

(sin h π) e^{inx}

Problem : Find the complex form of the fourier series of f(x) = e^{-x} ,-1 $\leq$ x $\leq$ 1

here(l=1)

Solution : The complex fourier series of f(x) in (-1,1) is

f(x) = $\sum_{n=-\infty}^{\infty} cn\, e^{\frac{in\pi x}{l}}$ -----(1)

Where cn = $\frac{1}{2}\int_{-1}^{1} e^{-x}\ e^{-in\pi x}$ dx = $\frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}$ dx

$= \frac{1}{2} [\frac{e^{-(1+in\pi)x}}{-(1+in\pi)x}$] limits(-1,1)

$= -\frac{1}{2} . \frac{1}{1+in\pi} [e^{-(1+in\pi)} - e^{(1+in\pi)}$

$= \frac{1}{2} [\frac{1-in\pi}{1+\pi^2 n^2}] [e^{(1+in\pi)} - e^{-(1+in\pi)}]$

$= \frac{1}{2} [\frac{1-in\pi}{1+\pi^2 n^2}]$ [e. $e^{in\pi} - e^{-1} . e^{-in\pi}$]

$= \frac{1}{2} [\frac{1-in\pi}{1+\pi^2 n^2}] [(-1)^n$ (e - e^{-1})]

$= \frac{1}{2} (-1)^n ([\frac{1-in\pi}{1+\pi^2 n^2}]$ 2 sin h)

$(1) \Rightarrow f(x) = \sum_{n=-\infty}^{\infty} (-1)^n [\frac{1-in\pi}{1+\pi^2 n^2}]$ sin h . $e^{-in\pi x}$

UNIT V

FOURIER TRANSFORMS

&

Z- TRANSFORMS

•

• **FOURIER TRANSFORMS**

Fourier Integral Theorem:-

Statement : If f(x) is a given function defined in (-l,l) and satisfies Dirichlet's condition then f(x) = $\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt\, d\lambda$.

The representation of f(x) is known as Fourier Integral of f(x)

Problems on integral theorem:

(1) Express the function f(x) = 1 , $|x| \leq 1$

$= 0$, $-\infty < x < -1$ =

0 , 1 < x < ∞

as fourier integral and hence evaluate (i) $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda$

(ii) $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

- **Solution:** The Fourier Integral theorem is given by f(x)

$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t-x)\, dt\, d\lambda.$

$= \frac{1}{\pi}\int_0^\infty \left[\int_{-1}^{1} 1 .\cos\lambda(t-x)\, dt \right] d\lambda$

$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda}\right] d\lambda$ limits (-1 to 1) for t

$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(1-x) - \sin\lambda(-1-x)}{\lambda}\right] d\lambda$

$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin(\lambda-\lambda x)+\sin(\lambda+\lambda x)}{\lambda}\right] d\lambda$

$= \frac{1}{\pi}\int_0^\infty 2. \left[\frac{\sin\lambda .\cos\lambda x}{\lambda}\right] d\lambda$

therefore $f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{\sin\lambda .\cos\lambda x}{\lambda}\right] d\lambda$ -----(1)

Deduction :

(I) $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda = \frac{\pi}{2}$ f(x)

$= \frac{\pi}{2}$, $|x| \leq 1$

$= 0$, $|x| > 1$ -------- (2)

Put x = 0

(2) $\Rightarrow \int_0^\infty \frac{\sin\lambda\cos 0}{\lambda}\,d\lambda = \frac{\pi}{2}$

$\Rightarrow \int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2}$

$\Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Fourier cosine & sine Integrals:

1) Fourier cosine Integral of f(x) is

$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x \int_0^\infty f(t)\cos\lambda t\,dt\,d\lambda$

2) Fourier sine Integral of f(x) is

$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \int_0^\infty f(t)\sin\lambda t\,dt\,d\lambda$

Problems:-

2) Express $f(x) = 1$, $0 \leq x \leq \pi$

$0 , x > \pi$ as a fourier sine integral and

Hence evaluate $\int_0^\infty (\frac{1 - \cos \lambda\pi}{\lambda}) \sin \lambda x \, d\lambda$

Solution : Fourier sine integral of f(x) is given by

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \, [\int_0^\infty f(t) \sin \lambda t \, dt] \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \, [\int_0^\pi \sin \lambda t \, dt] \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \, (\frac{- \cos \lambda t}{\lambda}) \, (0 \text{ to } \pi) \, d\lambda$$

$$\frac{2}{\pi} \int_0^\infty (\frac{1 - \cos \lambda\pi}{\lambda}) \sin \lambda x \, d\lambda \qquad f(x) =$$

$$\Rightarrow \int_0^\infty (\frac{1 - \cos \lambda\pi}{\lambda}) \sin \lambda x \, d\lambda \qquad \pi = \qquad f(x) .$$

2

$$= \frac{\pi}{2} . 1 , 0 \leq x \leq \pi$$

$$0 \quad , x > \pi$$

Problem : 3) Using Fourier Integral show that

$$\int_0^\infty \frac{1 - \cos \lambda\pi}{\lambda} \sin x \lambda \, d\lambda = \frac{\pi}{2} , 0 < x < \pi$$

$$0 \; , \; x > \pi$$

Solution : Let f(x) =1 , $0 \leq x \leq \pi$

0 , $x > \pi$

then write above solution (problem.(2) solution).

Problem :4) Using Fourier Integral , show that $e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos \lambda x}{\lambda^2+a^2} d\lambda$

Solution : Let f(x) = e^{-ax}

The Fourier Cosine Integral is given by f(x)

$= \frac{2}{\pi} \int_0^\infty \cos \lambda x \ [\int_0^\infty f(t) \cos \lambda t \, dt] \, d\lambda$

Now f(t) = e^{-at}

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \cos \lambda x \ [\int_0^\infty e^{-at} \cos \lambda t \, dt] \, d\lambda \text{ ----(1)}$$

at

$$\int_0^\infty e^{-at} \cos \lambda t \, dt = [\frac{e^{-}}{a^2+\lambda^2}($$

$$= 0 - \frac{e^0}{a^2+\lambda^2}($$

Therefore

Now -a cos λt + λ sin λt)(0 to ∞)]

a

$$-a.1 + 0) = \overline{a^2+\lambda^2}$$

sub in (1)

$$(1) \Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos \lambda x \, . \, \frac{a}{a^2+\lambda^2}\, d\lambda$$

$$= \frac{2a}{\pi}\int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2}\, d\lambda$$

Problem 5): Prove that $\frac{\pi}{2} e^{-x} = \int_0^\infty \frac{\cos \lambda x}{a^2+\lambda^2}\, d\lambda$, put a = 1 in above problem(4)

Solution : Let f(x) = e^{-x}

Problem 6): Using Fourier Integral , show that

$$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\, d\lambda \quad (a,b > 0)$$

Solution : Let f(x) = e^{-ax}

The Fourier Sine integral is given by f(x) =

$$\frac{2}{\pi}\int_0^\infty \sin \lambda x \left[\int_0^\infty f(t) \sin \lambda t \, dt \right] d\lambda$$

f(x) =

$$\frac{2}{\pi}\int_0^\infty \sin \lambda x \left[\int_0^\infty e^{-at} \sin \lambda t \, dt \right] d\lambda \text{ ----(1)}$$

$$\int_0^\infty e^{-at} \sin \lambda t \, dt = \left[\frac{e^{-at}}{a^2+\lambda^2}(-a \sin \lambda t - \lambda \cos \lambda t)(0 \text{ to } \infty)\right]$$

$$= 0 - \frac{1}{a^2+\lambda^2}(-\lambda) = \frac{\lambda}{a^2+\lambda^2}$$

sub in (1)

$$(1) \Rightarrow f(x) = \frac{2}{\pi}\int_0^{\infty} \sin \lambda x \,.\, \frac{\lambda}{a^2+\lambda^2}\, d\lambda$$

$$\Rightarrow e^{-ax} = \frac{2}{\pi}\int_0^{\infty} \frac{\lambda \sin \lambda x}{\lambda^2+a^2}\, d\lambda \text{ -----------(2)}$$

similarly , $$e^{-bx} = \frac{2}{\pi}\int_0^{\infty} \frac{\lambda \sin \lambda x}{\lambda^2+b^2}\, d\lambda \text{ --------(3)}$$

$$(2) - (3) = e^{-ax} - e^{-bx} = \frac{2}{\pi}\int_0^{\infty} \lambda \sin \lambda x \left(\frac{1}{\lambda^2+a^2} - \frac{1}{\lambda^2+b^2}\right) d\lambda$$

$$= \frac{2}{\pi}\int_0^{\infty} \lambda \sin \lambda x\left[\frac{b^2-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)}\right] d\lambda$$

$$= \frac{2}{\pi}(b^2 - a^2)\int_0^{\infty} \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\, d\lambda$$

There fore , $$e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^{\infty} \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\, d\lambda$$

FOURIER TRANSFORMATION:

Definition : 1)The fourier transform of f(x) , $-\infty < x < \infty$ is denoted by f(s) or F{f(x)} and is defined as ,

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx = f(s) \text{ -------(1)}$$

The inverse fourier transform is given by

$$f(x) = F^{-1}\{f(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds \text{ -----(2)}$$

F{f(x)} = f(s)

Note 2): Some authors also defined as

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

and inverse fourier transform as $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$

Def : 3) : $F\{f(x)\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\, dx$ and

Inverse Fourier Transform as $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} f(s)\, ds$

Def: ***Fourier Sine Transform***:-

The Fourier Sine Transform of f(x), $0 < x < \infty$ is denoted by fs(s) or Fs{f(x)} and defined by

$$Fs\{f(x)\} = \int_{0}^{\infty} f(x) \sin sx\, dx = fs(s) \text{ -----(3)}$$

$$Fs\{f(x)\} = \int_{0}^{\infty} f(x) \sin sx\, dx = fs(s) \text{ -----(3)}$$

The inverse Fourier Sine Transform is given by

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} fs(s) \sin sx\, ds \text{ -------(4)}$$

Note : Some authors also defined as

$$Fs\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx = fs(s)$$

and inverse fourier sine transform as f(x) = $\sqrt{\frac{2}{\pi}} \int_0^{\infty} fs(s) \sin sx \, ds$

Def : ***Fourier Cosine Transform :-***

The Fourier Cosine Transform of f(x) , 0 < x < ∞ is denoted by fc(s) or Fc{f(x)} and defined by

$Fc\{f(x)\} = \int_0^{\infty} f(x) \cos sx \, dx = fc(s)$ -----(5) and

The inverse Fourier Cosine Transform is given by,

$f(x) = \frac{2}{\pi} \int_0^{\infty} fc(s) \cos sx \, ds$ -------(6)

Note : Some authors also defined as

$Fc\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$

and inverse fourier cosine transform as f(x) = $\sqrt{\frac{2}{\pi}} \int_0^{\infty} fc(s) \cos sx \, ds$

Linear Property: If f(s) , g(s) are Fourier Transform of f(x) & g(x) then

$F\{c_1 f(x) + c_2 g(x)\} = c_1 F\{f(x)\} + c_2 F\{g(x)\}$

$= c_1 f(S) + c_2 g(s)$

Proof:- The definition of Fourier Transform is

$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx = f(s)$ -----(1)

By definition $F\{c_1 f(x) + c_2 g(x)\} = \int_{-\infty}^{\infty} e^{isx} [c_1 f(x) + c_2 g(x)]\, dx$

$= c_1 \int_{-\infty}^{\infty} e^{isx} f(x)\, dx + c_2 \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$

$= c_1 f(s) + c_2 g(s)$ by (1) ***Note:-***

Linear Property:

(I) $Fs\{c_1 f(x) + c_2 g(x)\} = c_1 fs(s) + c_2 gs(s)$

(II) $Fc\{c_1 f(x) + c_2 g(x)\} = c_1 fc(s) + c_2 gc(s)$

Proof:- (I) The definition of Fourier Sine Transform is

$Fs\{f(x)\} = \int_{0}^{\infty} f(x) \sin sx\, dx = fs(s)$ -----(1)

By the definition , $Fs\{c_1 f(x) + c_2 g(x)\} = \int_{0}^{\infty} [c_1 f(x) + c_2 g(x)] \sin sx\, dx$

$= c_1 \int_{0}^{\infty} f(x) \sin sx\, dx + c_2 \int_{0}^{\infty} g(x) \sin sx\, dx$

$= c_1 fs(s) + c_2 gs(s)$ by (1) ***Change***

of scale property:

Statement : If $F\{f(X)\} = f(s)$ then $F\{f(ax)\} = \frac{1}{a} f(\frac{s}{a})$

Proof :- The definition of Fourier Transform of f(x) is

$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx = f(s)$ --------(1)

By definition F{f(ax)} $= \int_{-\infty}^{\infty} e^{isx}$ f(ax) dx let ax =t x = t/a

$= \int_{-\infty}^{\infty} e^{is\frac{t}{a}}$ 1 1

f(t) dt dx = dt

$= \frac{1}{a}\int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)t}$

a a f(t) dt

$= \frac{1}{a}\int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)x}$

$= \frac{1}{a}$ f($\frac{s}{a}$)

f(x) dx (by property of def. integral)

{f(ax)} = $\frac{1}{a}$ fs($\frac{s}{a}$)

Note : 1) If Fs{f(x)} = fs(s) then Fs

ax)} = $\frac{1}{a}$ fc($\frac{s}{a}$)

2) If Fc{f(x)} = fc(s) then Fc{f(

Proof: (I) The definition of Fourier Sine Transform is

Fs{f(x)} $\int_0^{\infty} f(x)$ sin sx dx = fs(s) -----(1)

= {f(ax)} = $\int_0^{\infty} f(ax)$ sin

By definition $= \int_0^{\infty} f(t)$ sin $\frac{t}{a}$). $\frac{1}{a}$ Fssx dx let ax =t

1

s(dt dx = dt

a

)t. dt $= \frac{1}{a}\int_0^\infty f(t)\sin(\frac{s}{a}$

$= \frac{1}{a}\int_0^\infty f(x)\sin(\frac{s}{a}$)x. dx = $\frac{1}{a}$ fs($\frac{s}{a}$) by (1)

Shifting Property:-

If F{f(x)} = f(s) then F{f(x-a)} = $e^{isa} f(s)$

Proof : F{f(x)} $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx =f(s)----(1)

= $\int_{-\infty}^{\infty} e^{isx}$ f(x

By definition $= \int_{-\infty}^{\infty} e^{is(t+a)}$ F{f(x-a)} = -a) dx let

x-a=t f(t) dt $= \int_{-\infty}^{\infty} e^{ist}\, e^{isa}$ x=t+a

$= e^{isa} \int_{-\infty}^{\infty} e^{isx}$ f(t) dt dx= dt

f(x) dx

= e^{isa} f(s) by (1)

Modulation Theorem :-

If F{f(x)} = f(s) then F{f(x)cos ax} = $\frac{1}{2}$ {f(s-a) + f(s+a)}

$= \frac{1}{2}[\int_{-\infty}^{\infty} e^{i(s+a)x}$ $\int_{-\infty}^{\infty} e^{i(s-a)x}$

Proof: The defination of Fourier Transform is $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx}$ f(x) dx =f(s)----(1) By definition F{f(x) cos ax} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) cos ax dx

$= \int_{-\infty}^{\infty} e^{isx} \frac{e^{iax}+e^{-iax}}{2}$ f(x) dx

f(x) dx + f(x) dx

$= \frac{1}{2}$ {f(s-a) + f(s+a)}

Note: If Fs(s) & Fc(s) are Fourier Sine & Cosine Transform of f(x) respectively

Then (i) Fs{f(x) cos ax} = $\frac{1}{2}$ {Fs(s+a) + Fs(s-a)}

(ii) Fs{f(x) sin ax} = $\frac{1}{2}$ {Fs(s+a) - Fs(s-a)}

(iii) Fs{f(x) sin ax} = $\frac{1}{2}$ {Fc(s+a) - Fc(s-a)}

Proof: The definition of Fourier Sine Transform of f(x) is

Fs{f(x)} = $\int_{0}^{\infty} f(x)$ sin sx dx = fs(s) -----(1)

By definition Fs{f(x) cos ax} = $\int_{0}^{\infty} f(x)$ cos ax sin sx dx

sx. Cos ax) dx

= $\int_0^\infty f(x) \cdot \frac{1}{2} \cdot$(2.sin sin (s-a)x dx]

= $\frac{1}{2}$ f(x) $\int_0^\infty [\sin(sx + ax) + \sin(sx-ax)]dx$

= $\frac{1}{2}$ [$\int_0^\infty f(x)$ sin (s+a)x dx + $\int_0^\infty f(x)$

= $\frac{1}{2}$ [Fs(s+a) + Fs(s-a)]

Similarly we get (ii) & (iii) Problems:

1) Find Fourier Transform of f(x) = e^{ikx}, a < x < b

0 , x <a , x > b

Solution : By definition , F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx -∞ ∞

= $\int_a^b e^{isx} e^{ikx}$ dx

= $\int_a^b e^{i(s+k)x}$ dx

= $[\frac{e^{i(s+k)x}}{i(s+k)}]$ (apply limits a to b)

= $\frac{e^{i(s+k)b} - e^{i(s+k)a}}{i(s+k)}$

2) Find , F{f(x)} if f(x) = x, |x| < a

0 , |x| > a |x| < a means –a < x < a

Solution : By definition , F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

= $\int_{-a}^{a} e^{isx}$ x dx

use = $\int_{-a}^{a} x.e^{isx}$ dx , integration by parts ,

dx ? $udv =$ = $(\frac{xe^{isx}}{is})$ - $\frac{1}{is}\int_{-a}^{a} e^{isx}$ $uv -$? vdu

(apply –a to a) u=x, dv= e^{isx}dx

= $\frac{1}{is}$(a. e^{ias} + a. e^{-ias}) - $\frac{1}{is}$ $(\frac{e^{isx}}{is}$

= $\frac{2a\cos as}{is} + \frac{1}{s^2}$ $(e^{ias} - e^{-ias})$ = $\frac{e^{isx}}{is}$

= $\frac{-2ia\cos as}{s} + \frac{2i\sin as}{s^2}$) (apply –a to a) du=dx, v= ? . e^{isx} dx

3) If f(x) = 1 , |x| < a

0 , |x| > a , Find Fourier Transform of f(x)

Deduce that $\int_{-\infty}^{\infty} \frac{\sin as \cos sx}{s}\, ds$ (ii) $\int_{-\infty}^{\infty} \frac{\sin s}{s}\, ds$

(i) $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

Solution : F{f(X)} = |x| < a means –a < x < a

= ? $_{-a}$ e_{isx} .1. dx

= $\frac{e^{isx}}{is}$ (-a to a)

$= \frac{1}{is}(e^{ias} - e^{-ias})$

$= \frac{1}{is}$

$2 \sin as$(2i sin as)

f(s) = F{f(x)} = f(s)

s

Deduction :

Inverse Fourier Transform is defined by f(x) = $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}$ f(s) ds

$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\cos sx - i \sin sx) \frac{2 \sin as}{s}$ ds f(x) =

$= \frac{2}{2\pi}[\int_{-\infty}^{\infty}(\cos sx) \frac{\sin as}{s}$ ds – i $\int_{-\infty}^{\infty}(\sin sx) \frac{\sin as}{s}$

ds]

(even) (odd)

$\Rightarrow f(x) = \frac{1}{\pi}[2 \int^{\infty}(\cos sx) \frac{\sin as}{s}$ ds - 0]

(i) $\int_0^{\infty} \frac{\sin as \cos sx}{s} ds = \frac{\pi}{2} . f(x)$

$= \frac{\pi}{2}$ 1, |x| < a

0, |x| > a

(ii) Put a = 1 , x = 0 in (i) we get

$$\int_0^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}.1$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

4) Find Fourier Transform of f(x) = 1 - x^2 , |x| ≤ 1

0, |x| > 1

$$\int_0^{\infty} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$$

Evaluate

Solution:- F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

= $\int_{-1}^{1} e^{isx}$ (1 - x^2) dx

= $\int_{-1}^{1}$ (1 – x^2)e^{isx} dx

$\int udv = uv - \int vdu$

=[{(1 - x^2). $\frac{e^{isx}}{is}$ } - $\int_{-1}^{1} \frac{e^{isx}}{is}$ (-2x)dx]

(limits -1 to 1)

u= (1 – x^2) dv= e^{isx}dx

0 + $\frac{2}{is} \int_{-1}^{1} x.e^{isx}$

= [0 – dx du =-2x

dx, v= $\int$. e^{isx} dx

= $\frac{e^{isx}}{is}$

$\int_{-1}^{1} \frac{e^{isx}}{is}$ dx]

= $\frac{2}{is}$ [($\frac{xe^{isx}}{is}$) (-1 to 1) -

= $is2$ [1.(e is $+ise$ $-is$) - $is1$ e $isisx$] (-1 to 1) is - 2i sin s)

= $\frac{4}{s^3}$

[sin s – s = $is2$ [$2\cos is\, s$ - $is1$ (e is $-ise$ $-is$)] cos s] = f(s)

Deduction : = $is2$. $is1$ (2 cos s 1 Inverse

Fourier = - $s2^2$. 2[cos s - $\sin s\ s$] $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}$

Transform is defined by f(x) = f(s) ds

. $\frac{4}{s^3}$ [sin s – s cos s] ds

$$= \frac{1}{2\pi} . 4 \int_{-\infty}^{\infty} (\cos sx - i \sin sx) \frac{(\sin s - s\cos s)}{s^3} ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}$$

$$= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \cos sx \frac{(\sin s - s\cos s)}{s^3} ds - i \int_{-\infty}^{\infty} \sin sx \frac{(\sin s - s\cos s)}{s^3} ds \right]$$

(even function) (odd function)

$$f(x) = \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \cos sx \frac{(\sin s - s\cos s)}{s^3} ds - 0 \right.$$

$$\Rightarrow \int_{-\infty}^{\infty} \cos sx \frac{(\sin s - s\cos s)}{s^3} ds = \pi\, f(x)$$

2

$= \frac{\pi}{2}(1 - x^2)\ ,\ |x| \preceq 1$

$0,\ |x| > 1$

At $x = \frac{1}{2}$, $\Rightarrow \int_{-\infty}^{\infty} \cos\frac{s}{2}\, \frac{(\sin s - s\cos s)}{s^3}\, ds = \frac{\pi}{2}(1 - \frac{1}{4})$ put

$s = x$

$\Rightarrow \int_{-\infty}^{\infty} \cos\frac{x}{2}\, \frac{(\sin x - x\cos x)}{x^3}\, dx = \frac{\pi}{2}(1 - \frac{1}{4}) = \frac{3\pi}{8}$

$\Rightarrow 2\int_{0}^{\infty} \cos\frac{x}{2}\, \frac{(\sin x - x\cos x)}{x^3}\, dx = \frac{3\pi}{8}$

$\int_{0}^{\infty} \cos\frac{x}{2}\, [\frac{(x\cos x - \sin x)}{x^3}]\, dx = -\frac{3\pi}{16}$

5) Find Fourier Transform of $f(x) = \frac{1}{2a}$ if $|x| \preceq a$

$0,\ \text{if}\ |x| > a$

Solution : By definition,

$$F\{f(x)\} = f(s) = \int_{-\infty}^{\infty} e^{isx} \, f(x)\, dx$$

$$= \int_{-\infty}^{-a} e^{isx} f(x)dx + \int_{-a}^{a} e^{isx} f(x)dx + \int_{a}^{\infty} e^{isx} f(x)dx$$

$$= \int_{-a}^{a} \frac{1}{2a} e^{isx}\, dx = \frac{1}{2a}\frac{e^{isx}}{is} \quad (apply\ limits) = \frac{1}{2a}\frac{(e^{isa} - e^{-isa})}{is}$$

$$= \frac{\sin as}{ias}$$

6) Find Fourier Transform of f(x) = sin x , if $0 < x < \pi$

0 , otherwise

Solution : By definition,

$$F\{f(x)\} = f(s) = \int_{-\infty}^{\infty} e^{isx} \, f(x)\, dx$$

$$= \int_{-\infty}^{0} e^{isx} f(x)dx + \int_{0}^{\pi} e^{isx} f(x)dx + \int_{\pi}^{\infty} e^{isx} f(x)dx$$

$$= \int_{0}^{\pi} e^{isx} \sin x \, dx$$

$$= \frac{e^{isx}}{(is)^2 + 1^2} [is \sin x - 1.\cos x] \quad \text{apply 0 to } \pi$$

$$= \frac{1}{1-s^2}[e^{is\pi}(0 - \cos\pi) - e^{0}(0 - 1)]$$

$$= \frac{1}{1-s^2}[e^{is\pi}(1) - 1(0 - 1)]$$

$$= \frac{e^{is\pi} + 1}{1-s^2}$$

7) Find Fourier Transform of f(x) = xe^{-x}, $0 < x < \infty$

Solution : By definition,

F{f(x)} = f(s) = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

$= \int_{0}^{\infty} e^{isx}\, xe^{-x}\, dx$

$= \int_{0}^{\infty} x\, e^{(is-1)x}\, dx$

$= \left[\frac{x\, e^{(is-1)x}}{is-1} - 1.\frac{e^{(is-1)x}}{(is-1)^2}\right]$ (0 to ∞)

$= \left[\frac{x\,\{e^{isx} - e^{-x}\}}{is-1}\right]$(0 to ∞) $- \frac{1}{(is-1)^2}\left(e^{isx} - e^{-x}\right)$

$= [(0\text{-}0) - \frac{1}{(is-1)^2}(0 - 1)$

$= \frac{1}{(is-1)^2}$

$= \frac{1}{(is-1)^2} \cdot \frac{(is+1)^2}{(is+1)^2}$

$= \frac{(1+is)^2}{(1+s)^2}$

$-x^2$ $-x^2$

8) Find Fourier Transform of $e^{\frac{-x^2}{2}}$. Show that $e^{\frac{-x^2}{2}}$ is reciprocal Solution : By definition,

$$F\{f(x)\} = f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} e^{\frac{-x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2isx)} dx$$

$(x-is)^2/2 = y^2$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}[(x-is)^2+s^2} dx$$

$x-is = \sqrt{2}y$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-is)^2} dx$$

$dx = \sqrt{2}dy$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-s^4}{2}} \int_{-\infty}^{\infty} e^{-y^2} \sqrt{2}\, dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{2}} 2\int_{0}^{\infty} e^{-y^2} dy$$

$$= e^{\frac{-s^2}{2}} \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= e^{\frac{-s^2}{2}} = f(s)$$

Therefore Function is self reciprocal

9) Find the inverse Fourier Transform of f(x) of f(s) = $e^{-|s|y}$

Solution : We have |s| = -s , if s < 0

s , if s > 0

From inverse Fourier Transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-isx} f(s)\, ds + \int_{0}^{\infty} e^{-isx} f(s)\, ds\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{-isx}\, e^{sy}\, ds + \int_{0}^{\infty} e^{-isx}\, e^{-sy}\, ds\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(y-ix)s}\, ds + \frac{1}{2\pi}\int_{0}^{\infty} e^{-(y+ix)s}\, ds$$

$$= \frac{1}{2\pi}\left[\frac{e^{(y-ix)s}}{y-ix}\right](-\infty\ to\ 0) + \frac{1}{2\pi}\left[\frac{e^{-(y-ix)s}}{-(y+ix)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{y-ix}\right] + \frac{1}{2\pi}\left[\frac{1}{y+ix}\right]$$

$$= \frac{1}{2\pi}\left[\frac{y+ix+y-ix}{(y-ix)(y+ix)}\right] = \frac{1}{2\pi}\frac{2y}{y^2-i^2x^2}$$

$$= \frac{1}{\pi}\frac{y}{y^2+x^2}.$$

] (0 to ∞)

Problems on sine and cosine Transform:-

1) Find Fourier cosine Transform of f(x) defined by f(x) = cos x , 0 < x < a

= 0 , x > a

Solution : Fc{f(x)} = $\int_0^{\infty} f(x)\cos$ sx dx

= $\int_0^{a} \cos x \cos$ sx dx = $\frac{1}{2}\int_0^{a} 2\cos x \cos$ sx dx

= $\frac{1}{2}\int_0^{a}[\cos(x+sx)$

= $\frac{1}{2}[\int_0^{a} \cos(1+s)x$ dx + $\int_0^{a} \cos(1-s)x$

= $\frac{1}{2}[\frac{\sin(1+s)x}{1+s} + \frac{\sin(1-s)x}{1-s}]$ (apply 0 to a)

= $\frac{1}{2}[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}]$

2cosAcosB=cos(A+B)+cos(A-B)

+ cos (x-sx)] dx

A=x, B=sx

dx]

2) Find Fourier cosine Transform of f(x) defined by f(x) = x , 0 < x < 1

2-x , 1 < x < 2

0 , x > 2

Solution : $F_c\{f(x)\} = \int_0^\infty f(x)\cos sx\,dx$

$= \int_0^1 f(x)\cos sx\,dx + \int_1^2 f(x)\cos sx\,dx + \int_2^\infty f(x)\cos sx\,dx$

$= \int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx + 0$

$= [\,x\frac{\sin sx}{s} - 1\,(-\frac{\cos sx}{s^2})]$ (apply 0 to 1) + $[(2-x)\frac{\sin sx}{s} - (-1)\,(-\frac{\cos sx}{s^2})]$

$= (\frac{\sin s}{s} + \frac{\cos s}{s^2} - 0 - \frac{1}{s^2}) + (\,0 - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2})$

$= \frac{2\cos s - \cos 2s - 1}{s^2}$

$= \frac{2\cos s - (2\cos^2 s - 1) - 1}{s^2}$

$= \frac{1}{s^2}(2\cos s - 2\cos^2 s)$

$= \frac{2}{s^2}\cos s(1 - \cos s)$

)] (1to 2)

3)Find Fourier sine & cosine Transform of $2e^{-5x} + 5e^{-2x}$

Solution : Given f(x) = $2e^{-5x} + 5e^{-2x}$

Fs{f(x)} = $\int_0^{\infty} f(x)$ sin sx dx

= $\int_0^{\infty} (2e^{-5x} + 5e^{-2x})$ sin

= [2 $\int_0^{\infty} e^{-5x}$ sin sx dx + 5 $\int_0^{\infty} e^{-2x}$ sin sx dx

=[2 { $\frac{e^{-5x}}{25+s^2}$ (- 5 sin sx – s cos sx)} (apply 0 to ∞)} sx

dx

+ 5 $\{\frac{e^{-2x}}{4+s^2}$ (- 2 sin sx – s cos sx)} (apply 0 to ∞)}

0)} + 5 {0 $\frac{e^{0}}{25+s^2}$ (0 – s cos $\frac{e^{0}}{-4+s^2}$(-s)}]

= [2 { 0 -

= $[\frac{2s}{25+s^2} + \frac{5s}{4+s^2}]$

Similarly $\frac{10}{s^2+25} + \frac{10}{s^2+4}$] (ii) Fc{f(x)} = [

4) Find Fourier cosine Transform of (i) e^{-ax} cos ax , (ii) e^{-ax} sin ax Solution

: Given f(x) = e^{-ax} cos ax (i)

Fc{f(x)} = $\int_0^\infty f(x)\cos sx\,dx$

$= \int_0^\infty e^{-ax} \cos ax \cos sx\,dx$

$= \frac{1}{2}\int_0^\infty e^{-ax}\, 2\cos ax \cos sx\,dx$

$= \frac{1}{2}\left[\int_0^\infty e^{-ax} \cos (a+s)x\,dx + \int_0^\infty e^{-ax} \cos (a-s)x\,dx\right]$

$= \frac{1}{2} \cdot \frac{e^{-ax}}{a^2+(a+s)^2}$ {-a cos (a+s)x + (a+s) sin (a+s)x} (apply 0 to ∞)

$+ \frac{e^{-ax}}{a^2+(a-s)^2}$ {-a cos (a-s)x + (a-s) sin (a-s)x} (apply 0 to ∞)

$= \frac{1}{2}$[{0 - $\frac{e^0}{a^2+(a+s)^2}$ (– a cos 0)} + {0 - $\frac{e^0}{a^2+(a-s)^2}$ (– a cos 0)}]

$= \frac{1}{2}\left[\frac{a}{a^2+(a+s)^2} + \frac{a}{a^2+(a-s)^2}\right]$

(ii) Similarly Fs{f(x)} = Fs{(e^{-ax} sin ax) = $\frac{1}{2}\left[\frac{a}{a^2+(s-a)^2} - \frac{a}{a^2+(a+s)^2}\right]$

5) Find Fourier cosine & sine Transform of e^{-ax}, a > 0 hence

deduce (i) $\int_0^\infty \frac{\cos sx}{a^2+s^2}$ ds (ii) $\int_0^\infty \frac{s \sin sx}{a^2+s^2}$ ds

Solution : Let f(x) = e^{-ax}

Fc{f(x)} = $\int_0^\infty f(x)$ cos sx dx

= $\int_0^\infty e^{-ax}$ cos sx dx

= [$\frac{e^{-ax}}{a^2+s^2}$ (-a cos sx + s sin sx)] (apply 0 to ∞)

= [0 - $\frac{e^0}{a^2+s^2}$ (– a + 0)] = $\frac{a}{a^2+s^2}$= Fc(s)-----------(1)

Fs{f(x)} = $\int_0^\infty f(x)$ sin sx dx

= $\int_0^\infty e^{-ax}$ sin sx dx

= [$\frac{e^{-ax}}{a^2+s^2}$ (-a sin sx - s cos sx)] (apply 0 to ∞)

Fs{f(x)} = $\frac{s}{a^2+s^2}$ -------(2)

By Inverse cosine Transform

f(x) = $\frac{2}{\pi} \int_0^\infty fc(s)$ cos sx ds

= $\frac{2}{\pi} \int_0^\infty \frac{a}{a^2+s^2}$ cos sx ds

$\Rightarrow \int_0^\infty \frac{1}{a^2+s^2} \cos$ sx ds = $\frac{e^{-ax}}{a} \cdot \frac{\pi}{2}$

By inverse sine Transform ,

f(x) = $\frac{2}{\pi} \int_0^\infty fs(s) \sin$ sx ds

= $\frac{2}{\pi} \int_0^\infty \frac{s}{a^2+s^2} \sin$ sx ds

$\Rightarrow \int_0^\infty \frac{s}{a^2+s^2} \sin$ sx ds = $\frac{\pi}{2} \cdot e^{-ax}$ 1

6) Find Fourier sine Transform of f(x) = $\frac{1}{x}$

Solution : Fs{f(x)} = $\int_0^\infty f(x) \sin$ sx dx

= $\int_0^\infty \frac{\sin sx}{x}$ dx ----(1)

= $\frac{\pi}{2}$

7) Find Fourier sine Transform of $\frac{e^{-ax}}{x}$, hence deduce that

Solution : Fs{f(x)} = $\int_0^\infty f(x)$ sin sx dx

= $\int_0^\infty \frac{e^{-ax}}{x}$ sin sx dx = I ---(1)

$\frac{dt}{ds}$ = $\int_0^\infty \frac{e^{-ax}}{x}$ .x.cos sx dx

= $\int_0^\infty e^{-ax}$ cos sx dx

= [$\frac{e^{-ax}}{a^2+s^2}$ (-a cos sx + s sin sx)] (apply 0 to ∞)

= [0 - $\frac{e^0}{a^2+s^2}$ (– a + 0)]

⇒ $\frac{dt}{ds}$ = $\frac{a}{a^2+s^2}$

Integrate on both sides w.r.t. s we get

I= a $\int \frac{1}{a^2+s^2}$ ds = a .$\frac{1}{a}$. $Tan^{-1}\frac{s}{a}$ + c

= $Tan^{-1}(\frac{s}{a})$ + c ------(2)

put s = 0 on both sides we get {in (1) & (2)}

0 = Tan^{-1}(0) + c ⇒ 0 = 0 + c ⇒ c=0

$I = Tan^{-1}(\frac{s}{a})$ = Fs{f(x)}

8)Find Fourier cosine Transform of $\frac{1}{1^2+x^2}$, and

(ii) Fourier sine Transform of $\frac{x}{1^2+x^2}$

Solution : Let f(x) = $\frac{1}{1^2+x^2}$, We will find Fc{f(x)} = Fc{ $\frac{1}{1^2+x^2}$}

Fc{f(x)} $= \int_0^\infty f(x) \cos sx\, dx$

$= \int_0^\infty \frac{1}{1^2+x^2} \cos sx\, dx = I$ -------(1)

Differentiate on both sides w.r.t s

$$\frac{dI}{ds} = \int_0^\infty - \frac{x \sin sx}{1+x^2} \, dx \text{ ----(2)}$$

$$= - \int_0^\infty \frac{x^2 \sin sx}{x(1+x^2)} \, dx$$

$$= - \int_0^\infty \frac{(1+x^2-1) \sin sx}{x(1+x^2)} \, dx$$

$$= -[\int_0^\infty \frac{\sin sx}{s} \, dx - \int_0^\infty \frac{\sin sx}{x(1+x^2)}$$

$$\frac{dI}{ds} = - \frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)} \, dx \text{ ----(3)}$$

dx Diff

on both sides w.r.t 's'

We get $\frac{d^2 I}{ds^2} = \int_0^\infty \frac{x \; cos \; sx}{x(1+x^2)}$ dx

$\Rightarrow \frac{d^2I}{ds^2} = I$ by (1) $\Rightarrow \frac{d^2I}{ds^2} - I = 0$

$\Rightarrow (D^2-1)I = 0$ This is D.E

A.E. is $m^2-1 = 0$

$m = \pm 1$

solution is $I = c_1 e^{s} + c_2 e^{-s}$ ----- (4)

$\frac{dI}{ds} = c_1 e^{s} - c_2 e^{-s}$ ------(5)

From (1) & (4) , $c_1 e^{s} + c_2 e^{-s} = \int_0^{\infty} \frac{1}{1+x^2} . \cos sx \, dx$

Put $s = 0$ on both sides

$\Rightarrow c_1 + c_2 = \int_0^\infty \frac{1}{1+x^2}$ dx

= $(tan^{-1})(0\ to\ \infty)$ = $tan^{-1}\infty$ - tan^{-1} 0

= $\frac{\pi}{2}$ - 0

there fore , $c_1 + c_2 = \frac{\pi}{2}$ ------(6)

From (3) & (5) , $c_1 e^{s}$ - $c_2\ e^{-s}$ = - $\frac{\pi}{2}$ + $\int_0^\infty \frac{\sin sx}{x(1+x^2)}$ dx

$\Rightarrow c_1$ - $c_2 =$ - $\frac{\pi}{2}$ -----(7)

solve (6) & (7) we get $c_1 = 0$, $c_2 = \frac{\pi}{2}$ sub in (4)

(4) $\Rightarrow$ $\mathrm{I} = \frac{\pi}{2} . e^{-s}$

i.e., Fc {f(x)} = Fc$\{\frac{1}{1+x^2}\}$ = $\frac{\pi}{2}$. e^{-s}

Now I = $\frac{\pi}{2}$. e^{-s}

$\frac{dI}{ds}$ = - $\frac{\pi}{2}$. e^{-s} ----------(8)

From (2) & (8) , we have

- $\int_0^\infty \frac{x\ \sin sx}{1+x^2}$ dx = - $\frac{\pi}{2}$. e^{-s}

$\Rightarrow \int_0^\infty \left(\frac{x}{1+x^2}\right) \sin sx$ dx = $\frac{\pi}{2} . e^{-s}$

There fore Fs $\{\frac{x}{1+x^2}\} = \frac{\pi}{2} . e^{-s}$

9) Find the Inverse Fourier Cosine Transform of f(x) of fc(s) = $\frac{1}{2a}$ (a - $\frac{s}{2}$) , s < 2a

0 , s $\geq$ 2a

Solution : From the inverse Fourier Cosine Transform , we have

f(X) = $\frac{2}{\pi} \int_0^\infty fc(x)$ cos sx ds

= $\frac{2}{\pi}$ [$\int_0^{2a} fc(x)$ cos sx ds + $\int_{2a}^\infty fc(x)$ cos sx ds]

= $\frac{2}{\pi} \frac{1}{2a} \int_0^{2a}$ (a – $\frac{s}{2}$) cos sx ds

= $\frac{1}{\pi a}$ [{(a - $\frac{s}{2}$) . $\frac{\sin sx}{x}$ }(0 to 2a) - $\int_0^{2a} \frac{\sin sx}{x}$ (- $\frac{1}{2}$) ds]

= $\frac{1}{\pi a}$ [(0-0) + $\frac{1}{2} . \frac{1}{x^2}$ (- cos sx)(0 to 2a)]

= $\frac{1}{2\pi a x^2}$ (- cos 2ax + cos 0)

= $\frac{1-\cos 2ax}{2\pi a x^2} = \frac{\sin^2 ax}{\pi a x^2}$

10) Find f(x) if its Fourier Sine Transform is e^{-as}

Solution : Given f(s) = e^{-as}

By definition of inverse sine transform

f(x) = $\frac{2}{\pi}\int_0^\infty fs(x)$ sin sx ds

= $\frac{2}{\pi}\int_0^\infty e^{-as}$ sin sx ds

= $\frac{2}{\pi}$ [$\frac{e^{-as}}{a^2+x^2}$ (-a sin sx – x cos sx)(0 to ∞)

= $\frac{2}{\pi}$ [0 - $\frac{1}{a^2+x^2}$ (-x)]

= $\frac{2x}{\pi(a^2+x^2)}$

11) Find the Inverse Fourier Sine Transform f(x) of Fs(s) = $\frac{s}{1+s^2}$

(or)

Find f(x) if its Fourier sine Transform is $\frac{s}{1+s^2}$

Solution : By Fourier Inverse sine Transform f(x) = f(x) = $\frac{2}{\pi}\int_0^\infty fs(x)$ sin sx ds = I

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\, ds = I \text{ -------(1)}$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{1}{s} - \frac{1}{s(s^2+1)}\right) \sin sx \text{ as}$$

$$= \frac{2}{\pi}\left[\int_0^\infty \frac{\sin sx}{s}\, ds - \int_0^\infty \frac{\sin sx}{s(s^2+1)}\, ds\right.$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2} - \int_0^\infty \frac{\sin sx}{s(s^2+1)}\, ds\right]$$

$$f(x) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(s^2+1)}\, ds = I \text{ --------(2)}$$

diff on both sides w.r.t. X

We get $\frac{dI}{dx} = -\frac{2}{\pi}\int_0^\infty \frac{s\cos sx}{s(s^2+1)}\, ds$ ---------(3)

Diff w.r.t. x

$$\frac{d^2I}{dx^2} = -\frac{2}{\pi}\int_0^\infty -s\,\frac{\cos sx}{(s^2+1)}\, ds$$

$$= \frac{2}{\pi}\int_0^\infty s\,\frac{\cos sx}{(s^2+1)}\, ds$$

$\frac{d^2I}{dx^2} = I$ from (1) $\Rightarrow (D^2 - 1)I = 0$ ------(4) is D.E.

Solution of (4) is $I = c_1 e^x + c_2\, e^{-x}$-----------(5)

$\frac{dI}{dx} = c_1 e^x - c_2\, e^{-x}$------------(6)

From (2) & (5)

If x = 0 ,I = 1 ,

$\Rightarrow c_1 + c_2 = 1$ (5)

Substitute in (3) & (6)

(5) $\Rightarrow f(x) = c_2 e^{-x}$

$\Rightarrow f(x) = e^{-x}$

From (5)

$I = 0 +$

$\Rightarrow \frac{dI}{dx} = -\frac{2}{\pi}\int_0^{\infty} \frac{1}{1+s^2}\, ds$

If x = 0 , (3)

if $x = 0$, (6) $\Rightarrow c_1 - c_2 = -\frac{2}{\pi}(\tan^{-1} s)(0 \text{ to } \infty)$

$= -\frac{2}{\pi}\frac{\pi}{2} = -1$

Now solve $c_1 + c_2 = 1$ &

$c_1 - c_2 = -1$ we get $c_1 = 0$ & $c_2 = 1$

Convolution : The convolution of two functions f(x) & g(x) over the interval $(-\infty, \infty)$ *is defined as* $f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, g(x-u)\, du$

CONVOLUTION THEOREM: If F{f(x)} and F{g(x)} are Fourier Transform of functions f(x) and g(x) , then

$$F\{f(x) * g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \{f(x) * g(x)\}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)du\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, g(x-u)dx\right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(u+y)}\, g(y)dy\right] du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isu} f(u)du \,.\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isy}\, g(y)dy$$

$$= F\{ f(x)\} * F\{g(x)\}$$

Relation between Fourier and Laplace Transform:

Statement: If $f(t) = e^{-xt} g(t)$, $t > 0$ then $F\{f(t)\} = L\{g(t)\}$

$\quad\quad 0 \quad , \quad t < 0$

Proof : $F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist}\, f(t)\, dt$

$$= \int_{-\infty}^{0} e^{ist} \ f(t)\, dt + \int_{0}^{\infty} e^{ist} \ f(t)\, dt$$

$$= 0 + \int_{0}^{\infty} e^{ist} \ e^{-xt} \ g(t)\, dt$$

$$= \int_{0}^{\infty} e^{-(x-is)t} \ g(t)\, dt$$

$$= \int_{0}^{\infty} e^{\rho t} \ g(t)\, dt$$

$$= L\{g(t)\}$$

***Fourier Transform of derivatives of a function*:**

Statement : If $F\{(f(x)\} = f(s)$ then $F\{f^{n}(x)\} = (-is)^{n} f(s), if$ the 1^{st} (n-1) derivatives of f(x) vanish identically as $x \to \pm\infty$

Proof : By definition $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} \ f(x)\, dx$ -------(1)

$$F\{f'(x)\} = F\{\frac{d}{dx} f(x)\}$$

$$= \int_{-\infty}^{\infty} e^{isx} \ f'(x)\, dx$$

$$= [e^{isx} \ f(x)](-\infty \ to \ \infty) - \int_{-\infty}^{\infty} f(x).is. \ e^{isx}\, dx$$

$$= 0 - is \int_{-\infty}^{\infty} e^{isx} \ f(x)\, dx$$

There fore F{f'(x)} = - is F{f(x)}

F{f'(x)} = - is f(x) --------(2)

Now F{f''(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f''(x) dx

= [e^{isx} f'(x)](- ∞ to ∞) - $\int_{-\infty}^{\infty} f'(x). is.\ e^{isx}$ dx

= 0 – is $\int_{-\infty}^{\infty} e^{isx}$ f'(x) dx

= - is . F{f'(x)}

= -is (-is) f(s) by (2)

There fore F{f''(x)} = $(-is)^2$ f(s)

Similarly we can show that F{f^n(x)} = $(-is)^n\ f(s)$

Finite Fourier Transforms :-

Definition : The Finite Fourier sine Transform of f(x) , 0 < x < l is defined by

Fs{f(x)} = (s) = $\int_0^l f(x)$ sin $\frac{s\pi x}{l}$ dx fs

If 0 < x < π , (s) = $\int_0^{\pi} f(x)$ sin Fs{f(x)} = fs sx dx

The function f(x) is called the inverse finite Fourier sine transform of fs(s) and is

given by f(x) = ds

If $0 < x < \pi$, f(x) $\frac{2}{l} \sum_{s=1}^{\infty} fs(s) \sin \frac{s\pi x}{l}$ = sx

Definition : The finite Fourier sine Transform of f(x) , 0 < x < l is $\frac{2}{\pi} \sum_{s=1}^{\infty} fs(s) \sin$ defined by

$$Fc\{f(x)\} = fc(s) = \int_0^l f(x) \cos \frac{s\pi x}{l} dx$$

If $0 < x < \pi$, $Fc\{f(x)\} = \int_0^\pi f(x) \cos sx \, dx$

The function f(x) is called inverse finite Fourier cosine transform of f(x) and is given

by $f(x) = Fc^{-1}\{fc(s)\} = \frac{1}{l} fc(0) + \frac{2}{l} \sum_{s=1}^{\infty} fc(s) \cos \frac{s\pi x}{l}$ ds f(x)

$= Fc^{-1}\{fc(s)\} = {}^{1} fc(0) + \frac{2}{\pi}\ \sigma_{s=1}^{\infty} fc(s) \cos$ sx , $(0, \pi)$

π

Problem :

1) Find the Fourier Finite cosine transform of f(x) = x , $0 < x < \pi$ **Solution** : Fc{f(x)}

$= fc(s) = \int_0^\pi f(x) \cos sx \, dx$

$= \int_0^\pi x \cos sx \, dx = \left(\frac{x \sin sx}{s}\right)(0 \text{ to } \pi) - \frac{1}{s} \int_0^\pi \sin sx \, dx$

$= (0 - 0) - \frac{1}{s} \left(\frac{-\cos sx}{s}\right)(0 \ to \ \pi)$

s = 1,2,3,........ $= \frac{1}{s^2}$ [cos sπ – 1]

If s = 0 , fc(s) = $= \frac{1}{s^2} [(-1)^s - 1]$

Therefore $\int_0^\pi x$ dx = $\frac{x^2}{2}$ (0 to π) $= \frac{\pi^2}{2}$ fc(s) = $\frac{1}{s^2} [[(-1)^s - 1]$, s > 0 ; $\frac{\pi^2}{2}$, s = 0

2) Find the Fourier Finite sine transform of f(x) = $\frac{x}{\pi}$, 0 < x < π

Solution : Fs(n) = $\int_0^\pi \frac{x}{\pi}$ sin nx dx $= \frac{1}{\pi} \int_0^\pi x$ sin nx dx

$\frac{1}{\pi}$ [x($\frac{-\cos nx}{n}$)] (0 to π) - 1($\frac{-\sin nx}{n^2}$) (0 to π)

$= \frac{1}{\pi}$ [- $\frac{\pi}{n}$ cos nπ + 0 – 0 – 0] = - $\frac{1}{n}$ cos nπ = - $\frac{1}{n} (-1)^n = \frac{(-1)^{n+1}}{n}$

3) Find the Fourier Finite sine transform of f(x) = x^3 in (0 , π) Solution : By definition the finite Fourier sine Transform is

Fs{f(x)} = $\int_0^\pi f(x)$ sin sx dx

= $\int_0^\pi x^3$ sin sx dx

u = x^3 $3x^2$ 6x 6 0 dv= sin nx dx $\frac{-\cos nx}{n}$ $\frac{-\sin nx}{n^2}$ $\frac{\cos nx}{n^3}$ $\frac{\sin nx}{n^4}$

= [- x^3 $\frac{\cos nx}{n}$ - $3x^2$($\frac{-\sin nx}{n^2}$) + 6x ($\frac{\cos nx}{n^3}$) – 6($\frac{\sin nx}{n^4}$)] (0 to π)

= [$-\pi^3$ $\frac{\cos n\pi}{n}$ - 0 + 6π $\frac{\cos n\pi}{n^3}$ - 0] – 0

=$\frac{-\pi^3}{n}(-1)^n + \frac{6\pi}{n^3}(-1)^n$

= $(-1)^n \frac{\pi}{n}[\frac{6}{n^2} - \pi^2]$, n = 1,2,3……

4) Find Finite sine Transform of f(x) = x in 0 < x < 4

Solution : Let f(x) is Fs{f(x)} = $\int_0^4 f(x) \sin \frac{n\pi x}{4}$ dx

= [x ($\frac{-\cos \frac{n\pi x}{4}}{\frac{n\pi}{4}}$)(0 to 4) – ($\frac{-\frac{n\pi x}{4}\sin}{\frac{n^2\pi^2}{16}}$)(0 to 4)]

= - $\frac{4}{n\pi}$. 4 . Cos nπ – 0 + $\frac{16}{n^2\pi^2}$ (0 – 0)

= - $\frac{16}{n\pi}$ cos nπ = - $\frac{16}{n\pi}(-1)^n$

Similarly Fc{f(x)} = $\frac{16}{n^2\pi^2}[(-1)^n - 1]$ = fc(n) if n = 0 , fc(0) = $\int_0^4 x$ dx = $(\frac{x^2}{2})$ (0 to 4) = 8

<u>Parseval's Identity for Fourier Transforms</u> :-

Statement : If f(s) & g(s) are Fourier Transform of f(x) & g(x) respectively then

(i) $\frac{1}{2\pi}$ = $\int_{-\infty}^{\infty} f(x)$ g(x) dx

(ii) $\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(s)|_2$ ds = $\int_{-\infty}^{\infty} |f(x)|_2$ dx

Now (iii) $\frac{2}{\pi} \int_{-\infty}^{\infty} fc(s)$ gc(s) ds = $\int_0^{\infty} f(x)$ g(x) dx

Proof : By the inverse Fourier Transform we have

g(x) = $\frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, e^{-isx}$ ds ------(1)

Taking cojugate Complex on both sides in (1)

(1) $\Rightarrow \mathrm{g(x)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, e^{isx}$ ds

$\int_{-\infty}^{\infty} f(x)$ g(x) dx = $\int_{-\infty}^{\infty} f(x)\, [\frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, e^{isx}$ ds

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, [\int_{-\infty}^{\infty} f(x)\, e^{isx}$]

= $\frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)$ f(s) ds

$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)$ g(s) ds = $\int_{-\infty}^{\infty} f(x)$ g(x) dx ---------(2)

dx dx]

ds

(ii) Putting g(x) = f(x) in (2) we get

$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)$ f(s) ds = $\int_{-\infty}^{\infty} f(x)$ f(x) dx

$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2$ ds = $\int_{-\infty}^{\infty} |f(x)|^2$ dx --------- Therefore (3)

For Sine Transform:

(2) ⇒ $\frac{2}{\pi}\int_{0}^{\infty} fs(s)$ gs(s) ds = $\int_{0}^{\infty} f(x)$ g(x) dx

$\frac{2}{\pi}\int_{0}^{\infty} |fs(s)|^2$ ds = $\int_{0}^{\infty} |f(x)|^2$ dx

Similarly for Cosine

Problem 1):**)** If f(x) = 1 , |x| < a

0 , |x| > a , Find Fourier Transform of f(x)

$\int_{0}^{\infty} \frac{\sin\ ax}{x^2}\ dx = \frac{\pi a}{2}$ 2

Deduce that

Solution : F{f(X)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx |x| < a means –a < x < a

= $\int_{-a} e^{isx}$ .1. dx

$= \frac{e^{isx}}{is}$ (-a to a)

$= \frac{1}{is}(e^{ias} - e^{-ias}) = \frac{1}{is}$ (2i sin as)

$= \frac{2 \sin as}{s} =$ f(s)

F{f(x)} = f(s)

By parseval's identity for Fourier Transform

$\int_{-\infty}^{\infty} |f(x)|_2$ dx = $\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(s)|_2$ ds

$\Rightarrow \int_{-a}^{a} 1$ dx = $\frac{1}{2\pi} \int_{-\infty}^{\infty} (\frac{2 \sin as}{s})_2$ ds

$\Rightarrow$ x(-a to a) $= \frac{1}{2\pi} 2^2 \int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2}$ ds

$\Rightarrow 2a = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2}$ ds

$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 as}{s^2}$ ds = aπ

$\Rightarrow \int_{-\infty}^{\infty} (\frac{\sin as}{s})_2$ ds = a π

$\Rightarrow 2 . \int_{0}^{\infty} \frac{\sin^2 as}{s^2}$ ds = aπ

$\int_{0}^{\infty} \frac{\sin \ as}{s^2}$ $\frac{2}{2}$ aπ

Therefore ds =

2)Find Fourier Transform of f(x) = 1 - x^2 , |x| $\precsim$ 1

0, |x| > 1 is $\frac{4}{s^3}$ [sin s – s cos s]

Using Parseval's Identity Prove That

$\int_0^\infty [\frac{(\sin x - x\cos x)}{x^3}]^2 dx = \frac{\pi}{15}$

F{f(x)} = $\int_{-\infty}^{\infty} e^{isx}$ f(x) dx

Solution : :- $= \int_{-1}^{1} e^{isx} (1 - x^2)\, dx$

$= \int_{-1}^{1} (1 - x^2) e^{isx}\, dx$

$\int udv = uv - \int vdu$ $=[\{(1 - x^2). \frac{e^{isx}}{is}\} - \int_{-1}^{1} \frac{e^{isx}}{is} (-2x)dx]$

(limits -1 to 1)

u= (1 – x²) dv= e^{isx}dx

du =-2x dx, v= $\int . e^{isx}$ dx

$= [\, 0 - 0 + \frac{2}{is} \int_{-1}^{1} x.e^{isx}$ dx]

By parseval's identity for Fourier Transform

$= \frac{e^{isx}}{is}$

$= \frac{2}{is} [(\frac{xe^{isx}}{is})$ (-1 to 1) $- \int_{-1}^{1} \frac{e^{isx}}{is}$ dx]

$= \frac{2}{is}[1.(\frac{e^{is}+e^{-is}}{is}) - \frac{1}{is}\frac{e^{isx}}{is}]$ (-1 to 1)

$= \frac{2}{is} [\frac{2\cos s}{is} - \frac{1}{is} (\frac{e^{is}-e^{-is}}{is})]$

$= \frac{2}{is}.\frac{1}{is}$ (2 cos s - $\frac{1}{is}$ 2i sin s)

$= -\frac{2}{s^2}$ .2[cos s - $\frac{\sin s}{s}$

$= \frac{4}{s^3}$ [sin s – s cos s] = f(s)

$\int_{-\infty}^{\infty} |f(x)|_2$ dx = $\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|_2$ ds $\int_{-1}^{1}(1-x^2)^2$ dx =

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ [\frac{4}{s^3}$ (sin s – s cos $s)]^2$ ds $2.\int_0^1(1-x^2)^2$ dx = $\frac{1}{2\pi}$ $\Rightarrow\int_0^{\infty}[\frac{(\sin s - s\cos s)}{s^3}]^2$

. 2. 16 $\int_0^{\infty}[\frac{(\sin s - s\cos s)}{s^3}]^2$ ds

ds =

15

8

$\Rightarrow\frac{16}{\pi}\int_0^{\infty}[\frac{(\sin s - s\cos s)}{s^3}$

$]^2$ ds = 2 . —

15

$\Rightarrow\int_0^{\infty}[\frac{(\sin x - x\cos x)}{x^3}$

π

π

$]^2$ dx =

15

Shifting Properties:-

1.Shifting f(n) to the right : -

If Z[f(n)]=F(Z) then Z[f(n-k)]=Z^{-k}F(Z)

0 k k+1 k+2 k+3 - - - - - - - ∞

Proof: we know that

Z[f(n)]= $\sum_{n=0}^{\infty} f(n)Z^{-n}$ consider Z[f(n-k)]=$\sum_{n=0}^{\infty} f(n-k)Z^{-n}$ (k,n are different forms)

$= \sum_{n=k}^{\infty} f(n-k)Z^{-n}$ (since we are shifting f(n) to right)

=f(0)z^{-k}+ f(1)$z^{-(k+1)}$+ f(2)$z^{-(k+2)}$+ - - - - - - - -

= $Z^{-k}[f(0) +$ f(1)z^{-1}+f(2)z^{-2}+ − − − − − −]

=$Z^{-k} \sum_{n=0}^{\infty} f(n)Z^{-n}$

=$Z^{-k}F(Z)$

-

Z[f(n-k)]=$Z^{-k}F(Z)$

NOTE :- Z[f(n-k)]=Z^{-k}F(Z) putting k=1 ,we have

Z[f(n-1)]=$Z^{-1}F(Z)$ putting k=2 ,we have Z[f(n-2)]=$Z^{-2}F(Z)$

putting k=3 ,we have

Z[f(n-3)]=$Z^{-3}F(Z)$

2.Shifting f(n) to left :-

If Z[f(n)]=F(Z) then Z[f(n+k)]=Z^k[F(Z)-f(0)-f(1)$Z^{-1} - f(2)Z^{-2} - \text{-------} -$ f(k-1)$Z^{-(k-1)}$]

Proof: we know that Z[f(n)]=$\sum_{n=0}^{\infty} f(n)Z^{-n}$ $[Z^{-n} = Z^k . Z^{-(n+k)}]$

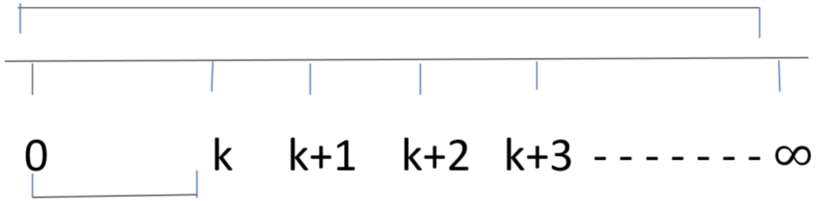
consider Z[f(n+k)]=$\sum_{n=0}^{\infty} f(n+k)Z^{-n}$

$=Z^k \sum_{n=k}^{\infty} f(n+k)Z^{-(n+k)}$

$=Z^k \sum_{n=k}^{\infty} f(n)Z^{-n}$ (replace (n+k) by n)

$= Z^k [\sum_{n=0}^{\infty} f(n)Z^{-n} - \sum_{n=0}^{(k-1)} f(n)Z^{-n}]$

$=Z^k [Z[f(n)] - \sum_{n=0}^{(k-1)} f(n)Z^{-n}]$



Z[f(n+k)]=Z^k[F(Z)-f(0)-f(1)$Z^{-1} - f(2)Z^{-2} - \text{------} -$f(k-1)$Z^{-(k-1)}$] which is Recurrence formula ∴

In particular

(a)If k=1 then Z[f(n+1)]=Z[F(Z)-f(0)]

(b) If k=2 then Z[f(n+2)]=Z^2[F(Z)-f(0)-f(1)Z^{-1}]

(c) If k=3 then Z[f(n+3)]=Z^3[F(Z)-f(0)-f(1)Z^{-1}-f(2)Z^2] - - - - - - - - - - - - - - - and so on.

Problems:1.Prove Z($\frac{1}{(n+1)}$)=Zlog($\frac{Z}{Z-1}$)

Solution- let f(n)=Z($\frac{1}{n+1}$)

we know that Z[f(n)]=$\sum_{n=0}^{\infty} f(n)Z^{-n}$

Z[$\frac{1}{n+1}$]=$\sum_{n=0}^{\infty} \frac{1}{n+1} Z^{-n}$

=$\sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{1}{Z^n}$

=$\frac{1}{1} \cdot \frac{1}{1} + \frac{1}{2} \cdot \frac{1}{Z} + \frac{1}{3} \cdot \frac{1}{Z^2} +$ - - - - - - - -

expansion needs 'Z' in denominator's, for this,multiply &divide with 'Z'

]

[x + $\frac{x^2}{2} + \frac{x^3}{3} +$ - - - - - - - =-log(1 -x)]

$=Z[\frac{1}{Z} + \frac{1}{2}\cdot\frac{1}{Z^2} + \frac{1}{3}\cdot\frac{1}{Z^3} + \frac{1}{4}\cdot\frac{1}{Z^4} + ------$

$=Z[\frac{\frac{1}{Z}}{1} + \frac{(\frac{1}{Z})^2}{2} + \frac{(\frac{1}{Z})^3}{3} + ------ --]$

$=Z[-\log(1-\frac{1}{Z})]$

$=Z[\log(1-\frac{1}{Z})^{-1}]$

$=Z\log(\frac{Z-1}{Z})^{-1}$

$=Z\log(\frac{Z}{Z-1})$

∴ hence proved

2.Find $Z[\frac{1}{n!}]$ and using shifting theorem evaluate (a)$Z(\frac{1}{(n+1)!})$ and (b)$Z(\frac{1}{(n+2)!})$

Solution- we know that $Z[f(n)]=\sum_{n=0}^{\infty} f(n)Z^{-n}$

let $f(n)=\frac{1}{n!}$ for n=0,1,2,3 - - -

$Z[\frac{1}{n!}] = \sum_{n=0}^{\infty}\frac{1}{n!}Z^{-n}$

$=1 + \frac{1}{1!}Z^{-1} + \frac{1}{2!}Z^{-2} + \frac{1}{3!}Z^{-3} + -------$

$=1+\frac{1}{Z} + \frac{(\frac{1}{Z})^2}{2!} + \frac{(\frac{1}{Z})^3}{3!} + -------------$

$=e^{\frac{1}{Z}}$ - $[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ------]$

=F(Z) (say) By shifting theorem

-> Z[f(n+1)]=Z[F(Z)-F(0)]

2 [( ) ( ) -(1])

(1) $Z[\frac{1}{(n+1)!}] = Z[e^{\frac{1}{Z}} - 1]$ [f(0)=$\frac{1}{0!} = 1]$

(2) $Z[\frac{1}{(n+2)!}] = Z^2[e^{\frac{1}{Z}} - 1 - \frac{1}{1!}Z^{-1}]$

$= Z^2[e^{\frac{1}{Z}} - 1 - Z^{-1}]$

- f(n)= 1 / $n!$ >Z[f(n+2)]=$Z\ F\ Z - F\ 0 - F\ 1\ Z$

f(n+ 1)=$\frac{1}{(n+1)}$

! f(n+ 2)=$\frac{1}{(n+2)}$

!

(n)]=-Z$\frac{d}{dZ}[F(Z)]$

Proof:- we know that Z[f(n)]=$\sum_{n=0}^{\infty} f(n)Z^{-n}$

∴ Z[nf(n)]=−Z $\frac{d}{dZ}[F(Z)]$

Z[nf(n)] =$\sigma_{n=0}^{\infty} nf(n)Z^{-n}$

=-Z $\sigma_{n=0}^{\infty} f(n)(-n)Z^{-n-1}$

=-Z $\sigma_{n=0}^{\infty} \frac{d}{dz}[f(n)Z^{-n}]$

= -Z $\frac{d}{dz}$ [$\sigma_{n=0}^{\infty}$ $f(n)Z^{-n}$]

= -Z$\frac{d}{dz}[Zf(n)]$

F(Z)

pb)If F(Z)=__________ $(Z-1)_4$ then find the values of f(2) and f(3)

= -Z$\frac{d}{dz}$

$(Z-1)^4$

=$\frac{Z^2(5+3Z^{-1}+12Z^{-2})}{Z^4(1-Z^{-1})^4}$

=$\frac{1}{Z^2}\frac{(5+3Z^{-1}+12Z^{-2})}{(1-z^{-1})^4}$

Solution:-Given F(Z)=

By Intial value theorem we have

<u>Multiplication by ‘n’:</u>If Z[f(n)]=F(Z) then

Z[nf

$[\, Z^{-n} = Z^1.Z^{-n-1} \,]$

$[\frac{d}{dz}(Z^{-n}) = (-n)Z^{-n-1}]$

$5Z^2+3Z+12$

$5Z^2+3Z+12$

f(0)= $\lim_{Z\to\infty} F(Z) = 0 \quad (\frac{1}{\infty} = 0)$ ⟶ 1

f(1)= $\lim_{Z\to\infty} Z[f(Z)$ -f(0)] = 0

f(2)= $\lim_{Z\to\infty} Z^2[F(Z) - f(0) - f(1)Z^{-1}]$

=5 - 0 – 0

=5

f(3)= $\lim_{Z\to\infty} Z^3$[F(Z)-f(0)-f(1)Z^{-1} –f(2)Z^{-2}]

= $\lim_{Z\to\infty} Z^3$[F(Z)- (0) –(0.Z^{-1}) $- 5Z^{-2}$]

= $\lim_{Z\to\infty} Z^3[\frac{5Z^2+3Z+12}{(Z-1)^4} - \frac{5}{Z^2}]$

= $\lim Z^3 5Z^4+3Z^3+^2(12Z-Z1^2)-^45\ Z-1\ ^4$

$z\to\infty$ $\left[\frac{Z}{(\quad)}\right]$

$= \lim Z^3 5Z^4+3Z^3+12Z^2-{}^2 5((ZZ-{}^4 1-)4^4 Z^3+6Z^2-4Z+1)$

$z\to\infty$ $\left[\frac{}{Z}\right]$

$= \lim Z^3 5\left[Z^4+3Z^3+12Z^2-5^2 Z(Z^4+-201)^4 Z^3-30Z^2+20Z\right]-5$

$z\to\infty$ $\left[\frac{Z}{(\quad)}\right]$

$= \lim Z^3 2\left[3Z2^3.\ Z184\ 1Z\ {}^2Z+\ 201\ Z4\right]-5$

$(\quad)$

$z\to\infty$ Z

$= \lim Z^3 23-18Z3^-[^1 1+-20Z-Z1^{-2} 4-5Z^{-3}$ $z\to\infty$ z

= 23

$\to (Z-1)^4 = (z-1)^2.(z-1)^2$

$=(Z^2+1-2Z)(Z^2+1-2Z)$

$=Z^4+Z^2-2Z^3+Z^2+1-2Z-2Z^3-2Z+4Z^2 = Z^4+$

$6Z^2-4Z^3-4Z+1$

INVERSE Z-TRANSFORM

$[g(0) + g(1)Z^{-1} + g(2)Z^{-2} + g(3)Z^{-3} + ----- + g(n)Z^{-n} + -----]$

$= \sum_{n=0}^{\infty}[f(0)g(n) + f(1)g(n-1) + f(2)g(n-2) + ---- + f(n)g(0)]Z^{-n}$

∗We have Z[f(n)]=F(Z) which can be also written as f(n)=$Z^{-1}[F(Z)]$.

Then f(n) is called inverse Z-transform of F(Z)

∗Thus finding the sequence {f(n)} from F(Z) is defined as <u>Inverse Z-Transform.</u>

∗The symbol Z^{-1} is the Inverse Z – Transform.

<u>CONVOLUTION THEOREM(v.v.imp):-</u>

If $Z^{-1}[F(Z)] = f(n)$ and $Z^{-1}[G(Z)] = g(n)$ then

$$Z^{-1}[F(Z).G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$$

[where * is convolution operator]

Proof:- We have F(Z)=$\sum_{n=0}^{\infty} f(n)Z^{-n}$ and $G(Z) = \sum_{n=0}^{\infty} g(n)Z^{-n}$ then

F(Z).G(Z) = $[f(0) + f(1)Z^{-1} + f(2)Z^{-2} + f(3)Z^{-3} + ----- + f(n)Z^{-n} + -----]$

=Z[f(0)g(n)+f(n)g(n-1)+ ------- +f(n)g(0)] Z^{-1}[F(Z).G(Z)]

=f(0)g(n)+f(n)g(n-1)+ ------- +f(n)g(0)

$$= \sum_{m=0}^{n} f(m) g(n-m)$$

$$\therefore Z^{-1}[F(Z).G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m) g(n-m)$$

Problems:-

1.Evaluate (a) $Z^{-1}\left[\left(\frac{Z}{Z-a}\right)^2\right]$ (b) $Z^{-1}\left[\frac{Z^2}{(Z-a)(Z-b)}\right]$

Solution:-

(a) $Z^{-1}\left[\left(\frac{Z}{Z-a}\right)^2\right]$

$= Z^{-1}\left[\frac{Z}{Z-a} . \frac{Z}{Z-a}\right]$

$F(Z) = \frac{Z}{Z-a} \Rightarrow f(n) = Z^{-1}\frac{Z}{Z-a} = a^n$

$G(Z) = \frac{Z}{Z-a} \Rightarrow g(n) = Z^{-1}\frac{Z}{Z-a} = a^n$

by convolution theorem, $Z^{-1}[F(Z) . G(Z)] = Z^{-1}\left[\frac{Z}{Z-a} . \frac{Z}{Z-a}\right] = \sum_{m=0}^{n} f(m) g(n-m)$

$= \sum_{m=0}^{n} a^m . a^{n-m}$

$Z^{-1}[F(Z).G(Z)] = f(n) * f(m)g(n-m) = \sum_{m=0}^{n} a^n$

$= a^n \sum_{m=0}^{n} 1$

$= a^n[1 + 1 + 1 + \text{- - - - -} + 1]$ (n+1)times

$= (n+1)a^n$

(b) $Z^{-1}\left[\frac{Z^2}{(Z-a)(Z-b)}\right]$

$= Z^{-1}\left[\frac{Z}{Z-a} . \frac{Z}{Z-b}\right]$

F(Z)= $\frac{Z}{Z-a}$ => f(n) $= Z^{-1}\left[\frac{Z}{Z-a}\right] = a^n$

G(Z)= $\frac{Z}{Z-b}$ => g(n) $= Z^{-1}\left[\frac{Z}{Z-b}\right] = b^n$

by convolution theorem ,

$$Z^{-1}[F(Z).G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$$

$Z^{-1}[F(Z).G(Z)] = Z^{-1}\left[\frac{Z}{Z-a} . \frac{Z}{Z-b}\right]$

$= \sum_{m=0}^{n} a^m . b^{n-m}$

$$= \sigma_{nm=0}\, b_n . \left(\frac{a}{b}\right)^m$$

$$= b^n \sigma_{nm=0} \left(\frac{a}{b}\right)^m$$

$$= b^n\left[\left(\frac{a}{b}\right)^0 + \left(\frac{a}{b}\right)^1 + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \text{- - - - - - - - -} + \left(\frac{a}{b}\right)^n\right]$$

this is in geometric progression,

$$a + ar^2 + ar^3 + \text{- - - -} + ar^{n-1} + \text{- - - -} = \frac{a(1-r^n)}{1-r},\ r<1$$

$$= \frac{a(r^n-1)}{1-r},\ r>1$$

$$= \frac{b^n\left[1-\left(\frac{a}{b}\right)^{n+1}\right]}{1-\frac{a}{b}}$$

$$\frac{b^n\left[1-\frac{a^{n+1}}{b^{n+1}}\right]}{}$$

$= \quad b-a$

b

Put a=3 and b=4 we get

$$Z^{-1}\left[\frac{Z^2}{(\ )(\ )}\right] = \frac{b^n\left[\frac{b^{n+1}-a^{n+1}}{n+1}\right]}{\frac{b-a}{b}}$$

$$4^{n+1}-3^{n+1} = 4^{n+1} - 3^{n+1} \quad Z-3 \quad Z-4 \quad 4-3$$

$$= b^n.\frac{b^{n+1}-a^{n+1}}{b^{n+1}}.\frac{b}{b-a}$$

2.Using Convolution theorem

$$\frac{b^{n+1}-a^{n+1}}{b-a}$$

$=$

show that $Z^{-1}\left[\frac{Z^2}{(Z-a)(Z-b)}\right] = \frac{b^{n+1}-a^{n+1}}{b-a}$

$Z \therefore Z^{-1}\left[\frac{1}{n!} * \frac{1}{n!}\right] = \frac{2^n}{n!}$ where $*$ is convolution operator

Solution: f(n)= $\frac{1}{n!}$ g(n)= $\frac{1}{n!}$

$$f(n) * g(n) = \sum_{m=0}^{n} f(m) g(n-m)$$

$$= \sum_{m=0}^{n} \frac{1}{m!} . \frac{1}{(n-m)!}$$

$$= 1.\frac{1}{n!} + \frac{1}{1!} . \frac{1}{(n-1)!} + \frac{1}{2!} . \frac{1}{(n-2)!} + - - - - + \frac{1}{n!} . \frac{1}{(0)}$$

$$= \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{2!} . \frac{1}{(n-2)!} + - - - - - - + \frac{1}{n!}$$

$$\left[\frac{1}{(n-1)!} = \frac{n}{n(n-1)!} = \frac{1}{(n-1)!}\right]$$

$$= \frac{1}{n!} + \frac{1}{1!}\frac{n}{n!} + \frac{1}{2!}\frac{n(n-1)}{n!} + - - - - - - + \frac{1}{n!}$$

!

$$= \frac{1}{n!}\left[1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + - - -\right.$$

$$= \frac{1}{n!}(1+1)^n$$

$$= \frac{2^n}{n!}$$

- to (n+1) terms]

3. Evaluate $Z^{-1}\left[\frac{Z^2}{(Z-4)(Z-5)}\right]$

Z

$Z-4$

Z

$Z-5$

Solution- Given $Z^{-1}\left[\frac{Z}{Z-4}\cdot\frac{Z}{Z-5}\right]$

F(Z)=—— => $f(n) = Z^{-1}\left[\frac{Z}{Z-4}\right] = 4^n$ [$G(Z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)g(n-m)$

[(G(Z)=( )] => $g(n) = Z^{-1}\left[\frac{Z}{Z-5}\right] = 5^n$ $Z{-1}\,F$

$Z-5$

by convolution theorem , $Z^{-1}[F(Z)$. =σ$nm=0$ $4m$. $5n-m$

$Z\ .G\ Z\ = Z^{-1}\left[\frac{Z}{Z-4}\cdot\frac{Z}{}\right]$

=σ$nm=0$ $5n$. $(45)m$

= $5n$ σ$nm=0$ $(45)m$

$=5^n[(\frac{4}{5})^0 + (\frac{4}{5})^1 + (\frac{4}{5})^1 + (\frac{4}{5})^3 + \text{- - - - - - - - -} + (\frac{4}{5})^n]$

$=5^n\,[1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3 + \text{- - - - - - - - - -} + (\frac{4}{5})^n]$

this is in geometric progression,

^{1} +ar^3+ - - - -+ar^{n-1}+ - - - - ______ $=\frac{a(1-r^n)}{1-r}$, r<1 a+ar

$a(r^n-1)$

$= \qquad\qquad , r>1$

$1-r$

$= \frac{5^n\left[1-\left(\frac{4}{5}\right)^{n+1}\right]}{1-\frac{4}{5}}$

$= \frac{5^n\left[1-\frac{4^{n+1}}{5^{n+1}}\right]}{\frac{5-4}{5}}$

$= \frac{5^n\left[\frac{5^{n+1}-4^{n+1}}{5^{n+1}}\right]}{\frac{5-4}{5}}$

$= 5^n . \frac{5^{n+1}-4^{n+1}}{5^{n+1}} . \frac{5}{1}$

$\frac{5^{n+1}-4^{n+1}}{1}$

$-1\left[\frac{Z^2}{(Z-4)(Z-5)}\right]$

$=$

$= \qquad \therefore Z5_{n+1} - 4_{n+1}$

Partial Fractions Method:-

$\left[\frac{}{Z^{2}+11Z+24}\right]$ (non repeated linear factors) $(v. imp)$

1.Find Z^{-1} $\left[\frac{}{}\right]$ $\frac{}{}$

Solution:- let F(Z) = $Z^{-1}\ Z2+11ZZ+24$ = $(Z+3)Z(Z+8)$

then $\frac{F(Z)}{Z} = \frac{1}{(Z+3)(Z+8)} = \frac{A}{(Z+3)} + \frac{B}{(Z+8)} \rightarrow 1$

$= \frac{1}{(Z+3)(Z+8)} = \frac{A(Z+8)+B(Z+3)}{(Z+3)(Z+8)}$

= 1 =A(Z+8)+B(Z+3) → 2

put Z=-8 ⇒ $1 = A(-8+8) + B(-8+3)$

1 = B(-5)

B= $\frac{-1}{5}$

put Z=-3 ⇒ $1 = A(-3+8) + B(-3+3)$

1 =A(5)

A= $\frac{1}{5}$

$\{Z+8=0 \Rightarrow Z=-8 \ \& \ Z+3=0 \Rightarrow z=-3\}$

now substitute A and B values in equation -1 we get

$$\left[\frac{F(Z)}{Z}\right] = \frac{1}{5(Z+3)} - \frac{1}{5(Z+8)}$$

$$F(Z) = \frac{Z}{5(Z+3)} - \frac{Z}{5(Z+8)}$$

$$= Z^{-1}\left[\frac{Z}{5(Z+3)} - \frac{Z}{5(Z+8)}\right]$$

$$= \frac{1}{5}\left[Z^{-1}\left[\frac{Z}{Z+3}\right] - Z^{-1}\left[\frac{Z}{Z+8}\right]\right]$$

$$= \frac{1}{5}[(-3)^n - (-8)^n]$$

$$\therefore Z^{-1}\left[\frac{Z}{Z^2+11Z+24}\right] = \frac{1}{5}[(-3)^n - (-8)^n]$$

$$Z^{-1}[F(Z)] \quad \left(Z^{-1}\left[\frac{Z}{Z-a}\right] = a^n \Rightarrow Z^{-1}\left[\frac{Z}{Z-a}\right] = a^n\right)$$

2.Find the Inverse Z-Transform of $\frac{Z}{(Z-1)(Z-2)}$

Solution:- let $F(Z) = \frac{Z}{(Z-1)(Z-2)}$ here we can resolve F(Z) into partial fractions directly as follows

$$F(Z) = Z\left[\frac{1}{(Z-1)(Z-2)}\right] = Z\left[\frac{1}{Z-2} - \frac{1}{Z-1}\right]$$

$$F(Z) = \frac{Z}{Z-2} - \frac{Z}{Z-1}$$

hence $Z^{-1}[F(Z)] = Z^{-1}\left[\frac{Z}{Z-2}\right] - Z^{-1}\left[\frac{Z}{Z-1}\right]$

$$= 2^n - 1^n$$

$$\left[\frac{}{(5Z-1)(5Z+2)}\right]$$

Solution:- let F(Z) = $\frac{Z(3Z+1)}{(5Z-1)(5Z+2)}$ then

$$\frac{F(Z)}{Z} = \frac{3Z+1}{(5Z-1)(5Z+2)} = \frac{A}{5Z-1} + \frac{B}{5Z+2} \rightarrow 1 \text{ (by partial fractions)}$$

$$\frac{3Z+1}{(5Z-1)(5Z+2)} = \frac{A(5Z+2)+B(5Z-1)}{(5Z-1)(5Z-2)}$$

3Z+1 = A(5Z+2)+B(5Z-1)

put Z= $\frac{-}{5} \Rightarrow A = \frac{}{152}$ 8

put Z= $\frac{1}{5} \Rightarrow B = \frac{1}{15}$ substituting A and B values in equation-1 we get

$$\frac{F(Z)}{Z} = \frac{8}{15}\frac{1}{5Z-1} + \frac{1}{15}\frac{1}{5Z+2}$$

$$\frac{F(Z)}{Z} = \frac{8}{15}\frac{1}{\left(Z-\frac{1}{5}\right)} + \frac{1}{15}\frac{1}{\left(Z+\frac{2}{5}\right)}$$

3.Find Z^{-1} $3Z^2+Z$

hence F(Z) $= \frac{8}{75} \cdot \frac{Z}{\left(Z-\frac{1}{5}\right)} + \frac{1}{75} \cdot \frac{Z}{\left(Z+\frac{2}{5}\right)}$

Z^{-1}[F(Z)] $= Z^{-1}\left[\frac{8}{75}\left(\frac{Z}{Z-0.2}\right) + \frac{1}{75}\left(\frac{Z}{Z+0.4}\right)\right]$

$\frac{8}{75} Z^{-1}\left(\frac{Z}{Z-0.2}\right) + \frac{1}{75} Z^{-1}\left(\frac{Z}{Z-(-0.4)}\right)$

$\frac{8}{75}(0.2)^n + \frac{1}{75}(-0.4)^n$

$\therefore Z^{-1}\left[\frac{3Z^2+Z}{(5Z-1)(5Z+2)}\right] = \frac{8}{75}(0.2)^n + \frac{1}{75}(-0.4)^n$

=

=

<u>Geometric Progression</u>:a)

Finite –

$a + ar + ar^2 + ar^3 + \text{- - - - - - - -} + ar^{n-1} + ar^n = \frac{a(1-r^{n+1})}{1-r}$

b) Infinite –

$a + ar + ar^2 + ar^3 + \text{- - - - - - - -} + ar^{n-1} + ar^n + \text{- - -} = \frac{a}{a-r}$

$+ r + r^2 + r^3 + \text{- - - - - - - - - -} + r^n + \text{- - - - - - - -} = \frac{1}{1-r}$

eg; 1

4.Find $Z^{-1}\left[\frac{Z}{(Z+3)^2(Z-2)}\right]$ (repeated Linear factor of form (ax + b)2 times)

Solution:-let F(Z) = $\frac{Z}{(Z+3)^2(Z-2)}$

$$\frac{F(Z)}{Z} = \frac{1}{(Z+3)^2(Z-2)}$$

$\frac{F(Z)}{Z} = \frac{1}{(Z+3)^2(Z-2)} = \frac{A}{Z-2} + \frac{B}{Z+3} + \frac{c}{(Z+3)^2}$ $\rightarrow 1$

$$\frac{1}{(Z+3)^2(Z-2)} = \frac{A(Z+3)^2+B(Z-2)(Z+3)+c(Z-2)}{(z-2)(z+3)^2}$$

1 =A$(Z+3)^2$+$B(Z-2)(Z+3)$+ $c(Z-2)$ { $Z-2=0 \Rightarrow Z=2$ & Z+3 =0 => Z=-3 } put Z=2 => 1=A(2 + 3)^2

1 =A(25)

A = $\frac{1}{25}$

put Z=-3 =>1=c(-3-2)

1 = -5c c= $\frac{-1}{5}$

now comparing the co-efficients of Z^2 on both sides

0=A+B

$B = \frac{-1}{25}$ substituting A,B and C

values in equation-1,we get

$$\frac{F(Z)}{Z} = \frac{1}{(Z+3)^2(Z-2)} = \frac{1}{25}\cdot\frac{1}{Z-2} - \frac{1}{25}\cdot\frac{1}{Z+3} - \frac{1}{5}\cdot\frac{1}{(Z+3)^2}$$

$$F(Z) = \frac{1}{25}\cdot\frac{Z}{Z-2} - \frac{1}{25}\frac{z}{Z+3} - \frac{1}{5}\cdot\frac{z}{(Z+3)^2}$$

$$Z^{-1}\left[\frac{Z^2}{(Z+3)^2(Z-2)}\right] = Z^{-1}\left[\frac{1}{25}\cdot\frac{Z}{Z-2} - \frac{1}{25}\frac{z}{Z+3} - \frac{1}{5}\cdot\frac{z}{(Z+3)^2}\right]$$

$$= \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^n$$

$$\therefore Z^{-1}\left[\frac{Z^2}{(Z+3)^2(Z-2)}\right] = \frac{1}{25}2^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^n$$

Solutions Of Difference Equations

Difference Equations:-

Just as the Differential equations are used for dealing with continuous process in nature , the difference equations are used for dealing of discrete process.

Definition:-

A difference equation is a relation between the difference of an unknown function at one (or) more general value of the argument.

thus $\Delta y_n + 2y_n = 0$ *and*

$\Delta^2 y_n + 5\Delta y_n + 6y_n = 0$ are difference equations

Solution:-

The solution of a difference equation is an expression for y_n which satisfies the given difference equation

General Solution:-

The general solution of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.

Linear Difference Equation:-

The Linear difference equation is that in which $y_{n+1}, y_{n+2}, y_{n+3}$ – – – – – – – – etc occur to the 1^{st} degree only and are not multiplied together.

The difference equation is called Homogeneous if f(n)=0,Otherwise it is called as NonHomogeneous equation (i.e:-f(n)$\neq 0$)

Working rule (or) Working Procedure:-

To solve a given linear difference equation with constant co-efficient by Z-transforms.

Step-1 :- Let Z(y_n)=Z[y(n)] =Y(Z)

Step-2 :-Take Z-Transform on bothsides of the given difference equation.

Step-3 :-Use the formulae $Z(y_n) = Y(Z)$

$Z[y_n + 1] = Z[Y(Z) - y_0]$

$Z[y_n + 2] = Z^2[Y(Z) - y_0 - y_1 Z^{-1}]$

Step-4:-Simplify and find Y(Z) by transposing the terms to the right and dividing by the co-efficient of y(Z).

Step-5:-Take the Inverse Z-Transform of Y(Z) and find the solution y_n

This gives y_n as a function of n which is the desired solution. Problems:-

1.Solve $y_{n+1} - 2y_n = 0$ using Z –Transforms.

Solution:-let $Z[y_n] = Y(Z)$

taking Z-Transform of the given equation we get $Z[y_{n+1}] - 2Z[y_n] = 0$

$Z[y_{n+1}] = Z[Y(Z) - y_0]$

$Z[Y(Z) - y_0] - 2Y(Z) = 0$

$[Z(a^n) = \frac{Z}{Z-a}]$

$Y(Z)[Z-2] = Z(y_0)$

$Y(Z) = \frac{Z}{Z-2} y_0$

$Z^{-1}[Y(Z)] = Z^{-1}\left[\frac{Z}{Z-2}\right] y_0$

=> $Z[Y(n)] = Y(Z)$

$Z^{-1}[Y(Z)] = y_n$

$y_n = 2^n y_0$

2.Solve the difference equation using Z-Transforms

$\mu_{n+2} - 3\mu_{n+1} + 2\mu_n = 0$ Given that

$\mu_0 = 0$, $\mu_1 = 1$

Solution:-let $Z(\mu_n) = \mu\, Z(\ )$

$Z(\mu_{n+1}) = Z[\mu(Z\ ) - \mu_0]$

$Z(\mu_{n+2}) = Z^2 \mu\left[Z(\ ) - \mu_0 - \frac{\mu_1}{Z}\right]$ now taking Z-Transform on both sides of

the given equation we get

$Z(\mu_{n+2}) - 3Z(\mu_{n+1}) + 2Z(\mu_n) = 0\ Z^2 - \mu_0 - \frac{\mu_1}{Z}$

$\left[\mu(Z)\right.$ - $3Z[\mu\, Z - \mu_0]$ +2$\mu\, Z = 0$ using the given

conditions it reduces to

$Z^2 \left[\mu(Z) - 0 - \frac{1}{Z}\right] - 3Z\mu[\mu Z\ Z( )] - 2\ 0] - 3 + 2\, Z\mu + Z\, 2\ ] = 0$

μ

$Z = \frac{Z}{Z^2 - 3Z + 2}$ = Z (or)

$\frac{1}{-2}$ $\frac{1}{-1}$

= $\frac{Z}{(Z-1)(Z-2)}$

= Z [Z $- Z$]

$$\frac{Z}{Z-2} - \frac{Z}{Z-1}$$

on taking Inverse Z-Transform on both sides we get

$$Z^{-1}[\mu(Z)] = Z^{-1}\left[\frac{Z}{Z-2} - \frac{Z}{Z-1}\right]$$

$$\mu^n = Z^{-1}\left|\frac{Z}{Z-2}\right| - Z^{-1}\left|\frac{Z}{Z-1}\right|$$

$$\mu_n = 2_n - 1$$

3.Solve the difference equation using Z-Transform

$$y_{n+2} - 4y_{n+1} + 3y_n = 0$$

Given that $y_0 = 2\ and\ y_1 = 4$

Solution:- let Z[y_n] $= Y(Z)$

Z[y_{n+1}] $= Z[Y(Z) - y_0]$ Z[y_{n+2}] $= Z^2[Y(Z) - y_0 - y_1Z^{-1}]$

taking Z-Transform of the given equation we get

Z(y_{n+2}) $- 4Z(y_{n+1}) + 3Z(y_n) = 0$

$Z^2[Y(Z) - y_0 - y_1Z^{-1}]$ - 4 $Z[Y(Z) - y_0]$+3Y(Z) = 0 using the given conditions it reduces to

$Z^2[Y(Z) - 2 - 4Z^{-1}]$ - 4 $Z[Y(Z) - 2]$+3Y(Z) = 0 i.e:- $Y(Z)[Z^2 - 4Z + 3] - 2Z^2 - 4Z + 8Z = 0$

$$Y(Z)[Z^2 - 4Z + 3] = Z(2Z-4)$$

$$\frac{Y(Z)}{Z} = \frac{2Z-4}{[Z^2-4Z+3]}$$

$$= \frac{2Z-4}{(Z-1)(Z-3)}$$

$$\frac{Y(Z)}{Z} = \frac{1}{Z-1} + \frac{1}{Z-3} \text{ (reducing by partial fractions)}$$

$Y(Z) = \frac{Z}{Z-1} + \frac{Z}{Z-3}$ on taking Inverse Z-Transform on both sides we obtain

$$Z^{-1}[Y(Z)] = Z^{-1}\left[\frac{Z}{Z-1}\right] + Z^{-1}\left[\frac{Z}{Z-3}\right]$$

$y_n = 1 + 3^n$