DEVC LECTURE NOTES

(20A54201)

I – BTECH

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Course Code	**DEVC**	**L**	**T**	**P**	**C**
20A54201		**3**	**0**	**0**	**3**

Course Objectives

- To enlighten the learners in the concept of differential equations and multivariable calculus.
- To furnish the learners with basic concepts and techniques at plus two level to lead them into advanced level by handling various real world applications.

Course outcomes (CO) : After completion of the course, the student can able to

CO-1: Identify the essential characteristics of linear differential equations with constant coefficients.

CO-2: Solve the differential equations related to various engineering fields.

CO-3: Identify solution methods for partial differential equations that model physical processes.

CO-4: Interpret the physical meaning of different operators such as gradient, curl and divergence.

CO-5: Estimate the work done against a field, circulation and flux using vector calculus.

Syllabus

Unit-1: Linear differential equations of higher order (Constant Coefficients)

Definitions, homogenous and non-homogenous, complimentary function, general solution, particular integral, Wronskean, method of variation of parameters. Simultaneous linear equations, Applications to L-C-R Circuit problems and Mass spring system.

UNIT 2: Partial Differential Equations

Introduction and formation of Partial Differential Equations by elimination of arbitrary constants and arbitrary functions, solutions of first order equations using Lagrange's method.

UNIT -3 Applications of Partial Differential Equations

Classification of PDE, method of separation of variables for second order equations. Applications of Partial Differential Equations: One dimensional Wave equation, One dimensional Heat equation.

UNIT-4 Vector differentiation

Scalar and vector point functions, vector operator del, del applies to scalar point functions-Gradient, del applied to vector point functions-Divergence and Curl, vector identities.

UNIT -5 Vector integration

Line integral-circulation-work done, surface integral-flux, Green's theorem in the plane (without proof), Stoke's theorem (without proof), volume integral, Divergence theorem (without proof) and applications of these theorems.

Text Books:

1. Erwin Kreyszig, Advanced Engineering Mathematics, 10/e, John Wiley & Sons, 2011.
2. B.S. Grewal, Higher Engineering Mathematics, 44/e, Khanna publishers, 2017.

Reference Books:

1. Dennis G. Zill and Warren S. Wright, Advanced Engineering Mathematics, Jones and Bartlett, 2011.
2. Michael Greenberg, Advanced Engineering Mathematics, 2/e, Pearson, 2018
3. George B.Thomas, Maurice D. Weir and Joel Hass, Thomas Calculus, 13/e, Pearson Publishers, 2013.

4. R.K.Jain and S.R.K.Iyengar, Advanced Engineering Mathematics, 3/e, Alpha Science International Ltd., 2002.
5. Glyn James, Advanced Modern Engineering Mathematics, 4/e, Pearson publishers, 2011.
6. Micheael Greenberg, Advanced Engineering Mathematics, 9th edition, Pearson edn
7. Dean G. Duffy, Advanced engineering mathematics with MATLAB, CRC Press
8. Peter O'neil, Advanced Engineering Mathematics, Cengage Learning.
9. R.L. GargNishu Gupta, Engineering Mathematics Volumes-I &II, Pearson Education
10. B. V. Ramana, Higher Engineering Mathematics, McGraw Hill Education.
11. H. k Das, Er. RajnishVerma, Higher Engineering Mathematics, S. Chand.
12. N. Bali, M. Goyal, C. Watkins, Advanced Engineering Mathematics, Infinity Science Press.

UNIT-1

CHAPTER 11

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

11.1 Introduction

A differential equation of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right)=0$ in which the dependent variable $y(x)$ and its derivatives viz. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc occur in first degree and are not multiplied together is called a Linear Differential Equation.

11.2 Linear Differential Equations (LDE) with Constant Coefficients

A general linear differential equation of n^{th} order with constant coefficients is given by:

$$k_0 \frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + - - - - - + k_{n-1} \frac{dy}{dx} + k_n y = F(x)$$

where $k's$ are constant and $F(x)$ is a function of x alone or constant.
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Or $f(D)y = F(x)$, where $D^n \equiv \frac{d^n}{dx^n}$, $D^{n-1} \equiv \frac{d^{n-1}}{dx^{n-1}}$, ….., $D \equiv \frac{d}{dx}$ are called differential operators.

11.3 Solving Linear Differential Equations with Constant Coefficients

Complete solution of equation $f(D)y = F(x)$ is given by $y =$ C.F + P.I.

where C.F. denotes complimentary function and P.I. is particular integral.

When $F(x) = 0$, then solution of equation $f(D)y = 0$ is given by $y =$ C.F

11.3.1 Rules for Finding Complimentary Function (C.F.)

Consider the equation $f(D)y = F(x)$
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Step 1: Put $D = m$, auxiliary equation (A.E) is given by $f(m) = 0$

$\Rightarrow k_0 m^n + k_1 m^{n-1} + - - - - - + k_{n-1} m + k_n = 0$ ……③

Step 2: Solve the auxiliary equation given by ③

I. If the n roots of A.E. are real and distinct say $m_1, m_2, \dots m_n$
C.F. = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

II. If two or more roots are equal i.e. $m_1 = m_2 = \dots = m_k,\ k \le n$
C.F. = $(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_1 x} + \cdots + c_n e^{m_n x}$

III. If A.E. has a pair of imaginary roots i.e. $m_1 = \alpha + i\,\beta,\ m_2 = \alpha - i\,\beta$

C.F. = $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

IV. If 2 pairs of imaginary roots are equal i.e. $m_1 = m_2 = \alpha + i\,\beta$,
$m_3 = m_4 = \alpha - i\,\beta$
C.F. = $e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + \cdots + c_n e^{m_n x}$

Example 1 Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution: $\Rightarrow (D^2 - 8D + 15)y = 0$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3,5$

C.F. = $c_1 e^{3x} + c_2 e^{5x}$

Since $F(x) = 0$, solution is given by y = C.F

$\Rightarrow y = c_1 e^{3x} + c_2 e^{5x}$

Example 2 Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Solution: $\Rightarrow (D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$ …….①

By hit and trial $(m-2)$ is a factor of ①

∴① May be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

C.F. = $c_1e^x + c_2e^{2x} + c_3e^{3x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x}$

Example 3 Solve $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$ ....①

By hit and trial $(m - 1)$ is a factor of ①

∴① May be rewritten as

$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$

$\Rightarrow m^3(m-1) - 9m^2(m-1) + 26m(m-1) - 24(m-1) = 0$

$\Rightarrow (m-1)(m^3 - 9m^2 + 26m - 24) = 0$ ......②

By hit and trial $(m - 2)$ is a factor of ②

∴② May be rewritten as

$(m-1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$

$\Rightarrow (m-1)[m^2(m-2) - 7m(m-2) + 12(m-2)] = 0$

$\Rightarrow (m-1)(m^2 - 7m + 12)(m-2) = 0$

$\Rightarrow (m-1)(m-3)(m-4)(m-2) = 0$

$\Rightarrow m = 1,2,3,4$

C.F. = $c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Example 4 Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = 0$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m + 1)^2 = 0$

$\Rightarrow \; m = 0, -1, -1$

C.F. = $c_1 + \; (c_2 + c_3 x)e^{-x}$

Since $F(x) = 0,$ solution is given by $y =$ C.F

$\Rightarrow y \; = \; c_1 + \; (c_2 + c_3 x)e^{-x}$

Example 5 Solve the differential equation: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + \; y = 0$

Solution: $\Rightarrow (D^4 - 2D^2 + 1)y = 0$

Auxiliary equation is: $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2 - 1)^2 = 0$

$\Rightarrow (m + 1)^2(m - 1)^2 = 0$

$\Rightarrow \; m = -1, -1, 1, 1$

C.F. = $(c_1 + c_2 x)e^{-x} + \; (c_3 + c_4 x)e^{x}$

Since $F(x) = 0,$ solution is given by $y =$ C.F

$\Rightarrow y \; = \; (c_1 + c_2 x)e^{-x} + \; (c_3 + c_4 x)e^{x}$

Example 6 Solve the differential equation: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

Solution: $\Rightarrow (D^3 - 2D + 4)y = 0$

Auxiliary equation is: $m^3 - 2m + 4 = 0$ …….①

By hit and trial $(m + 2)$ is a factor of ①

∴① May be rewritten as

$m^3 + 2m^2 - 2m^2 - \; 4m + 2m + \; 4 = 0$

$\Rightarrow m^2(m + 2) - 2m(m + 2) + 2(m + 2) = 0$

$\Rightarrow (m + 2)(m^2 - 2m + 2) = 0$

$\Rightarrow \; m = -2, 1 \pm i$

C.F. = $c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Example 7 Solve the differential equation: $(D^2 - 2D + 5)^2 y = 0$

Solution: Auxiliary equation is: $(m^2 - 2m + 5)^2$…….①

Solving ①, we get

$\Rightarrow m = 1 \pm 2i, 1 \pm 2i$

C.F. = $e^x[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = e^x[(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x]$

Example 8 Solve the differential equation: $(D^2 + 4)^3 y = 0$

Solution: Auxiliary equation is: $(m^2 + 4)^3$…….①

Solving ①, we get

$\Rightarrow m = \pm 2i, \pm 2i, \pm 2i$

C.F. = $(c_1 + c_2 x + c_3 x^2) \cos 2x + (c_4 + c_5 x + c_6 x^2) \sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2 x + c_3 x^2) \cos 2x + (c_4 + c_5 x + c_6 x^2) \sin 2x$

11.3.2 Shortcut Rules for Finding Particular Integral (P.I.)

Consider the equation $(D)y = F(x)$, $F(x) \neq 0$
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - + k_{n-1} D + k_n)y = F(x)$

Then P.I = $\frac{1}{f(D)} F(x)$, Clearly P.I. = 0 if $F(x) = 0$

Case I: When $F(x) = e^{ax}$

Use the rule P.I $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, $f(a) \neq 0$

In case of failure i.e. if $f(a) = 0$

P.I = $x\frac{1}{f'(D)}e^{ax}= \quad x\frac{1}{f'(a)}e^{ax}$, $f'(a) \neq 0$

If $f'(a) = 0$, P.I. = $x^2\frac{1}{f''(a)}e^{ax}$, $f''(a) \neq 0$ and so on

Example 9 Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^{2x}$

Solution: $\Rightarrow (D^2 - 2D + 10)y = e^{2x}$

Auxiliary equation is: $m^2 - 2m + 10 = 0$

$\Rightarrow m = 1 \pm 3i$

C.F. = $e^x(c_1 \cos 3x + c_2 \sin 3x)$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)}e^{2x} = \frac{1}{f(2)}e^{2x}$, by putting $D = 2$

$= \frac{1}{2^2 - 2(2)+10}e^{2x} = \frac{1}{10}e^{2x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10}e^{2x}$

Example 10 Solve the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Solution: $\Rightarrow (D^2 + D - 2)y = e^x$

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2,1$

C.F. = $c_1e^{-2x} + c_2e^x$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)}e^x$, putting $D = 1$, $f(1) = 0$

$\therefore$P.I = $x\frac{1}{f'(D)}e^x$ $\quad\quad$ $\because$ P.I. = $x\frac{1}{f'(a)}e^{ax}$ if $f(a) = 0$

$\Rightarrow$ P.I. = $x\frac{1}{2D+1}e^x = \frac{1}{f'(1)}e^x$, $f'(1) \neq 0$

$\Rightarrow$ P.I. = $\frac{xe^x}{3}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-2x} + c_2 e^{x} + \frac{xe^x}{3}$

Example 11 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = \sinh(2x+1) + 4^x$

Solution: $\Rightarrow (D^2 - 4)y = \sinh(2x+1) + 4^x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. = $c_1 e^{2x} + c_2 e^{-2x}$

P.I. = $\frac{1}{f(D)} F(x)$

$= \frac{1}{f(D)} (\sinh(2x+1) + 4^x)$

$= \frac{1}{D^2-4}\left(\frac{e^{(2x+1)} - e^{-(2x+1)}}{2}\right) + \frac{1}{D^2-4}\left(e^{x\log 4}\right)$

$\because \sinh x = \frac{e^x - e^{-x}}{2}$ and $4^x = e^{x\log 4}$

$= \frac{e}{2}\frac{1}{D^2-4}e^{2x} - \frac{e^{-1}}{2}\frac{1}{D^2-4}e^{-2x} + \frac{1}{D^2-4}e^{x\log 4}$

Putting $D = 2, -2$ and $\log 4$ in the three terms respectively

$f(2) = 0$ and $f(-2) = 0$ for first two terms

$\therefore$ P.I. $= \frac{e}{2} x \frac{1}{2D} e^{2x} - \frac{e^{-1}}{2} x \frac{1}{2D} e^{-2x} + \frac{1}{(log 4)^2 - 4} e^{x\log 4}$

$\because$ P.I. $= x\frac{1}{f'(a)}e^{ax}$ if $f(a) = 0$

Now putting $D = 2, -2$ in first two terms respectively

$\Rightarrow$P.I. $= \frac{ex}{8}e^{2x} + \frac{e^{-1}x}{8}e^{-2x} + \frac{4^x}{(log 4)^2 - 4}$ $\quad\because e^{x\log 4} = 4^x$

$\Rightarrow$P.I. $= \frac{x}{4}\left(\frac{e^{(2x+1)} + e^{-(2x+1)}}{2}\right) + \frac{4^x}{(log 4)^2 - 4}$

$\Rightarrow$P.I. $= \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log 4)^2 - 4}$ $\quad\because \cosh x = \frac{e^x + e^{-x}}{2}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2-4}$

Case II: When $F(x) = Sin\ (ax+b) or\ Cos\ (ax+b)$

If $F(x) = Sin\ (ax+b)\ or\ Cos\ (ax+b)$, put $D^2 = -a^2$,

$D^3 = D^2D = -a^2D, D^4 = (D^2)^2 = a^4$,……

This will form a linear expression in D in the denominator. Now rationalize the denominator to substitute $D^2 = -a^2$. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

In case of failure i.e. if $f(-a^2) = 0$

P.I. $= x\frac{1}{f'(-a^2)}\ Sin\ (ax+b)\ or\ Cos\ (ax+b), f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, P.I.= $x^2\frac{1}{f''(-a^2)}\ Sin\ (ax+b)\ or\ Cos\ (ax+b), f''(-a^2) \neq 0$

Example 12 Solve the differential equation: $(D^2 + D - 2)y = \sin x$

Solution: Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m+2)(m-1) = 0$

$\Rightarrow m = -2,1$

C.F. = $c_1e^{-2x} + c_2e^x$

P.I. = $\frac{1}{f(D)}\ F(x) = \frac{1}{f(D)}\sin x = \frac{1}{D^2+D-2}\sin x$

putting $D^2 = -1^2 = -1$

P.I = $\frac{1}{D-3}\sin x = \frac{D+3}{D^2-9}\sin x$, Rationalizing the denominator

$= \frac{(D+3)\sin x}{-10}$, Putting $D^2 = -1$

$\therefore \text{P.I.} = \frac{-1}{10}(D\ sin\ x + 3\ \sin x)$

$= \frac{-1}{10}(\cos x + 3\ \sin x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-2x} + c_2 e^{x} - \frac{1}{10}(\cos x + 3 \sin x)$

Example 13 Solve the differential equation: $(D^2 + 2D + 1)y = cos^2 x$

Solution: Auxiliary equation is: $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$\Rightarrow m = -1, -1$

C.F. = $e^{-x}(c_1 + c_2 x)$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)} cos^2 x = \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{D^2+2D+1} e^{0x} + \frac{1}{2}\frac{1}{D^2+2D+1}\cos 2x$

Putting $D = 0$ in the 1st term and $D^2 = -2^2 = -4$ in the 2nd term

P.I = $\frac{1}{2} + \frac{1}{2}\frac{1}{2D-3}\cos 2x$

$= \frac{1}{2} + \frac{1}{2}\frac{2D+3}{4D^2-3^2}\cos 2x$, Rationalizing the denominator

$= \frac{1}{2} + \frac{1}{2}\frac{(2D+3)\cos 2x}{-25}$, Putting $D^2 = -4$

$\therefore$ P. I. $= \frac{1}{2} - \frac{1}{50}(-4 \sin 2x + 3 \cos 2x)$

Now $y =$ C.F. + P.I

$\Rightarrow y = e^{-x}(c_1 + c_2 x) + \frac{1}{2} - \frac{1}{50}(-4 \sin 2x + 3 \cos 2x)$

Example 14 Solve the differential equation: $(D^2 + 9)y = \sin 2x \cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

C.F. = $c_1 \cos 3x + c_2 \sin 3x$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin 2x \cos x = \frac{1}{2}\frac{1}{D^2+9}(\sin 3x + \sin x)$

$= \frac{1}{2}\frac{1}{D^2+9}\sin 3x + \frac{1}{2}\frac{1}{D^2+9}\sin x$

Putting $D^2 = -9$ in the 1st term and $D^2 = -1$ in the 2nd term

We see that $f(D^2 = -9) = 0$ for the 1st term

$\therefore$ P.I. $= \frac{1}{2}\, x\frac{1}{2D}\sin 3x + \frac{1}{2}\frac{1}{8}\sin x$

$\because$P.I. $= x\frac{1}{f'(-a^2)}\, Sin\,(ax+b)\,, f'(-a^2) \neq 0$

$\Rightarrow$P.I. $= -\frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1\cos 3x + c_2\sin 3x - \frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Case III: When $F(x) = x^n$, n is a positive integer

P.I $= \frac{1}{f(D)}F(x) = \frac{1}{f(D)}x^n$

1. Take the lowest degree term common from $f(D)$ to get an expression of the form $[1 \pm \phi(D)]$ in the denominator and take it to numerator to become $[1 \pm \phi(D)]^{-1}$
2. Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to n^{th} degree as $(n+1)^{th}$ derivative of x^n is zero
3. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

Following expansions will be useful to expand $[1 \pm \phi(D)]^{-1}$in ascending powers of D

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Example 15 Solve the differential equation: $\frac{d^2y}{dx^2} - y = 5x - 2$

Solution: $\Rightarrow (D^2-1)y = 5x - 2$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. = $c_1 e^{x} + c_2 e^{-x}$

P.I. = $\frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-1}(5x-2)$

$= \frac{1}{-(1-D^2)}(5x-2)$

$= -(1-D^2)^{-1}(5x-2)$

$= -[1 + D^2 + \cdots](5x-2)$

$= -(5x-2)$

$\therefore$P.I = $-5x+2$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ c_1 e^{x} + c_2 e^{-x} - 5x + 2$

Example 16 Solve the differential equation: $(D^4 + 4D^2)y = x^2 + 1$

Solution: Auxiliary equation is: $m^4 + 4m^2 = 0$

$\Rightarrow\ m^2(m^2+4) = 0$

$\Rightarrow m = 0, 0, \pm 2i$

C.F. = $(c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x)$

P.I. = $\frac{1}{f(D)}\ F(x)\ = \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}(x^2+1)$

$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1}(x^2+1)$

$= \frac{1}{4D^2}\left[1-\frac{D^2}{4}+\cdots\right](x^2+1)$

$= \frac{1}{4D^2}\left(x^2+1-\frac{1}{2}\right)$

$$= \frac{1}{4D^2}\left(x^2 + \frac{1}{2}\right)$$

$$= \frac{1}{4D}\int \left(x^2 + \frac{1}{2}\right) dx$$

$$= \frac{1}{4D}\left(\frac{x^3}{3} + \frac{x}{2}\right)$$

$$= \frac{1}{4}\int \left(\frac{x^3}{3} + \frac{x}{2}\right) dx$$

$\therefore$ P.I $= \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x) + \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Example 17 Solve the differential equation: $(D^2 - 6D + 9)y = 1 + x + x^2$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3,3$

C.F. = $e^{3x}(c_1 + c_2 x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 6D + 9}(1 + x + x^2)$

$$= \frac{1}{9\left(1 - \frac{2D}{3} + \frac{D^2}{9}\right)}(1 + x + x^2)$$

$$= \frac{1}{9}\left(1 - \left(\frac{2D}{3} - \frac{D^2}{9}\right)\right)^{-1}(1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \left(\frac{2D}{3} - \frac{D^2}{9}\right) + \left(\frac{2D}{3} - \frac{D^2}{9}\right)^2 + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \frac{2D}{3} - \frac{D^2}{9} + \frac{4D^2}{9} + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left(1 + x + x^2 + \frac{2}{3} + \frac{4x}{3} + \frac{2}{3}\right)$$

$\therefore$ P.I = $\frac{1}{9}\left(\frac{7}{3}+\frac{7x}{3}+\ x^2\right)$

Complete solution is: $y\ =\ $ C.F. + P.I

$\Rightarrow y\ =\ e^{3x}(c_1+c_2x)+\frac{1}{9}\left(\frac{7}{3}+\frac{7x}{3}+x^2\right)$

Case IV: When $F(x)=e^{ax}g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)}\ e^{ax}g(x)=e^{ax}\left(\frac{1}{f(D+a)}\ g(x)\right)$

Example 18 Solve the differential equation: $(D^2+2)y=x^2e^{3x}$

Solution: Auxiliary equation is: $m^2+2=0$

$\Rightarrow m^2=-2$

$\Rightarrow m=\pm\sqrt{2}i$

C.F. = $(c_1\cos(\sqrt{2}\,x)+c_2\sin(\sqrt{2}\,x))$

P.I. $=\frac{1}{f(D)}\ F(x)\ =\frac{1}{D^2+2}x^2e^{3x}$

$$=e^{3x}\frac{1}{(D+3)^2+2}x^2$$

$$=e^{3x}\frac{1}{D^2+6D+11}x^2$$

$$=\frac{e^{3x}}{11}\frac{1}{\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)}x^2$$

$$=\frac{e^{3x}}{11}\left(1+\left(\frac{6D}{11}+\frac{D^2}{11}\right)\right)^{-1}x^2$$

$$=\frac{e^{3x}}{11}\left[1-\left(\frac{6D}{11}+\frac{D^2}{11}\right)+\left(\frac{6D}{11}+\frac{D^2}{11}\right)^2+\cdots\right]x^2$$

$$=\frac{e^{3x}}{11}\left[1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}+\cdots\right]x^2$$

$$=\frac{e^{3x}}{11}\left[1-\frac{6D}{11}+\frac{25D^2}{121}+\cdots\right]x^2$$

$$=\frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

$\therefore\ P.I\ =\ \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ (c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x)) + \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Example 19 Solve the differential equation: $(D^3 + 1)y = e^{2x} \sin x$

Solution: Auxiliary equation is: $m^3 + 1 = 0$

$\Rightarrow m^3 = -1$

$\Rightarrow m = -1, \frac{1 \pm \sqrt{3}i}{2}$

C.F. = $c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^3+1} e^{2x} \sin x$

$= e^{2x} \frac{1}{(D+2)^3+1} \sin x$

$= e^{2x} \frac{1}{D^3+6D^2+12D+9} \sin x$

$= e^{2x} \frac{1}{-D-6+12D+9} \sin x$, Putting $D^2 = -1$

$= e^{2x} \frac{1}{11D+3} \sin x$

$= e^{2x} \frac{11D-3}{121D^2-9} \sin x$, Rationalizing the denominator

$= -\frac{e^{2x}}{130}(11D - 3) \sin x$, Putting $D^2 = -1$

$\therefore$ P.I $= -\frac{e^{2x}}{130}(11 \cos x - 3 \sin x)$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ = c_1 e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$- \frac{e^{2x}}{130}(11 \cos x - 3 \sin x)$

Example 20 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = x\ \sinh x$

Solution: $\Rightarrow (D^2 - 4)y = x\ \sinh x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. = $c_1 e^{2x} + c_2 e^{-2x}$

P.I. = $\frac{1}{f(D)} F(x)$

$$= \frac{1}{f(D)} (x\ \sinh x)$$

$$= \frac{1}{D^2 - 4}\left(x\frac{e^x - e^{-x}}{2}\right) \qquad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{D^2 - 4}\left(x\frac{e^x}{2} - x\frac{e^{-x}}{2}\right)$$

$$= \frac{e^x}{2}\frac{1}{(D+1)^2 - 4}x - \frac{e^{-x}}{2}\frac{1}{(D-1)^2 - 4}x$$

$$= \frac{e^x}{2}\frac{1}{(D^2+2D-3)}x - \frac{e^{-x}}{2}\frac{1}{D^2-2D-3}x$$

$$= \frac{e^x}{2}\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{2D}{3}\right)}x - \frac{e^{-x}}{2}\frac{1}{-3\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)}x$$

$$= -\frac{e^x}{6}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1}x + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]^{-1}x$$

$$= -\frac{e^x}{6}\left(1+\frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1-\frac{2D}{3}\right)x$$

$$= -\frac{e^x}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right)$$

$\therefore$ P.I. $= -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Example 21 Solve the differential equation: $(D^2+1)y = x^2 \sin 2x$

Solution: Auxiliary equation is: $m^2+1=0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

C.F. = $c_1 \cos x + c_2 \sin x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} x^2 \sin 2x$

= Imaginary part of $\frac{1}{D^2+1} x^2 e^{i2x}$

Now $\frac{1}{D^2+1} x^2 e^{i2x} = e^{i2x} \frac{1}{(D+2i)^2+1} x^2$

$$= e^{i2x} \frac{1}{D^2+4i^2+4iD+1} x^2$$

$$= e^{i2x} \frac{1}{D^2+4iD-3} x^2$$

$$= e^{i2x} \frac{1}{-3\left(1-\frac{D^2}{3}-\frac{4iD}{3}\right)} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\left(\frac{D^2}{3}+\frac{4iD}{3}\right)\right]^{-1} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)^2\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\frac{D^2}{3}+\frac{4iD}{3}+\frac{16i^2D^2}{9}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\frac{13D^2}{9}+\frac{4iD}{3}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$$= -\frac{1}{3}(\cos 2x + i \sin 2x)\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$\therefore$ P.I. = Imaginary part of $\frac{1}{D^2+1} x^2 e^{i2x} = -\frac{1}{3}\left(\frac{8x}{3}\cos 2x + \left(x^2-\frac{26}{9}\right)\sin 2x\right)$

$$= -\frac{8x}{9}\cos 2x + \frac{1}{27}(26-9x^2)\sin 2x$$

Complete solution is: $y\ =\ $ C.F. + P.I

$$\Rightarrow y\ = c_1 \cos x +\ c_2 \sin x\ - \frac{8x}{9} \cos 2x + \frac{1}{27} (26 - 9x^2) \sin 2x$$

Example 22 Solve the differential equation: $(D^2 - 4D + 4)y = x^2 e^{2x} \sin 2x$

Solution: Auxiliary equation is: $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2$

$\Rightarrow m = 2,2$

C.F. = $(c_1 + c_2 x)e^{2x}$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-4D+4} x^2 e^{2x} \sin 2x$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D} \int x^2 \sin 2x\, dx$$

$$= e^{2x} \frac{1}{D} \left[(x^2) \left(\frac{-\cos 2x}{2} \right) - (2x) \left(\frac{-\sin 2x}{4} \right) + (2) \left(\frac{\cos 2x}{8} \right) \right]$$

$$= e^{2x} \frac{1}{D} \left[-\frac{1}{2} x^2 \cos 2x +\ \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]$$

$$= e^{2x} \left[-\frac{1}{2} \int x^2 \cos 2x +\ \frac{1}{2} \int x \sin 2x\, dx + \frac{1}{4} \int \cos 2x\, dx \right]$$

$$= e^{2x} \left[-\frac{1}{2} \left[(x^2) \left(\frac{\sin 2x}{2} \right) - (2x) \left(\frac{-\cos 2x}{4} \right) + (2) \left(\frac{-\sin 2x}{8} \right) \right] + \right.$$

12x−cos2x2−1−sin2x4+14sin2x2

$$\therefore \text{P.I.} = e^{2x} \left[\frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x \right]$$

Complete solution is: $y\ =\ $ C.F. + P.I

$$\Rightarrow y\ = (c_1 + c_2 x) e^{2x} + e^{2x} \left[\frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x \right]$$

Case V: When $F(x) = x\, g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)} \big(x\, g(x) \big) = x \frac{1}{f(D)} g(x) + \left(\frac{d}{dD} \frac{1}{f(D)} \right)\ g(x)$

Example 23 Solve the differential equation: $(D^2+9)y = x\cos x$

Solution: Auxiliary equation is: $m^2+9=0$

$\Rightarrow m^2 = -9$

$\Rightarrow m = \pm 3i$

C.F. = $(c_1\cos 3x + c_2\sin 3x)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2+9}\ x\cos x$

$= \ x\frac{1}{D^2+9}\ \cos x + \ \frac{-2D}{(D^2+9)^2}\ \cos x$

$= \ x\frac{1}{-1+9}\ \cos x + \ \frac{-2D}{(-1+9)^2}\ \cos x,$ Putting $D^2 = -1$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$\therefore$ P.I. = $\frac{x\cos x}{8} + \frac{\sin x}{32}$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ = c_1\cos 3x + c_2\sin 3x + \ \frac{x\cos x}{8} + \frac{\sin x}{32}$

Example 24 Solve the differential equation:

$$(D^2-1)y = x\sin x + \ (1+x^2)e^x$$

Solution: Auxiliary equation is: $m^2-1=0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^x + c_2e^x$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-1}\ (x\sin x + \ (1+x^2)e^x)$

$= \frac{1}{D^2-1}x\sin x + \ \frac{1}{D^2-1}(1+x^2)e^x$

Now $\frac{1}{D^2-1}x\sin x = = \ x\frac{1}{D^2-1}\ \sin x + \ \frac{-2D}{(D^2-1)^2}\ \sin x$

$$= x\frac{1}{-1-1}\sin x + \frac{-2D}{(-1-1)^2}\sin x, \qquad \text{Putting } D^2 = -1$$

$$= -\frac{1}{2}(x\sin x + \cos x)$$

Also $\frac{1}{D^2-1}(1+x^2)e^x = e^x\frac{1}{(D+1)^2-1}(1+x^2)$

$$= e^x\frac{1}{D^2+2D}(1+x^2)$$

$$= e^x\frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= e^x\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= e^x\frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{4}\right](1+x^2)$$

$$= e^x\frac{1}{2D}\left[1+x^2-x+\frac{1}{2}\right]$$

$$= e^x\frac{1}{2D}\left[x^2-x+\frac{3}{2}\right]$$

$$= \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$$

$\therefore$ P.I. $= -\frac{1}{2}(x\sin x+\cos x)+\frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^x + c_2e^x - \frac{1}{2}(x\sin x+\cos x)+\frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Case VI: When $F(x)$ is any general function of x not covered in shortcut methods I to V above

Resolve $f(D)$ into partial fractions and use the rule:

$$\frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$$

Example 25 Solve the differential equation: $(D^2+3D+2)y = e^{e^x}$

Solution: Auxiliary equation is: $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0$

$\Rightarrow m = -1, -2$

C.F. = $c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+3D+2} e^{e^x}$

$$= \frac{1}{(D+1)(D+2)} e^{e^x}$$

$$= \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{e^x}$$

$$= e^{-x} \int e^x e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} dx$$

$$= e^{-x} \int D e^{e^x} dx - e^{-2x} \int e^x D e^{e^x} dx$$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int e^x e^{e^x} dx\right]$, Integrating 2nd term by parts

$$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int D e^{e^x} dx\right]$$

$$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - e^{e^x}\right]$$

$\therefore$ P.I. $= e^{-2x} e^{e^x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

Example 26 Solve the differential equation: $(D^2 + 4)y = \tan 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F. = $c_1 \cos 2x + c_2 \sin 2x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+4} \tan 2x$

$$= \frac{1}{(D-2i)(D+2i)} \tan 2x$$

$$= \frac{1}{4i}\left(\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right) \tan 2x$$

P.I. $= \frac{1}{4i}\left(\frac{1}{D-2i} \tan 2x\right) - \frac{1}{4i}\left(\frac{1}{D+2i} \tan 2x\right)$ …….①

$$\text{Now } \frac{1}{D-2i}\tan 2x = e^{2ix}\int e^{-2ix}\tan 2x\,dx$$

$$= e^{2ix}\int(\cos 2x - i\sin 2x)\tan 2x\,dx$$

$$= e^{2ix}\int(\sin 2x - i\frac{sin^2 2x}{cos2x})\,dx$$

$$= e^{2ix}\int\left(\sin 2x - i\frac{1-cos^2 2x}{cos2x}\right)dx$$

$$= e^{2ix}\int(\sin 2x - i\sec 2x + i\cos 2x\,)\,dx$$

$$= e^{2ix}\left(-\frac{1}{2}\cos 2x - \frac{i}{2}log|\sec 2x + \tan 2x| + \frac{i}{2}\sin 2x\right)$$

$$\therefore \frac{1}{D-2i}\tan 2x = e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right) \ldots②$$

Replacing i by – i

$$\frac{1}{D+2i}\tan 2x = e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right) \ldots③$$

Using ②and ③ in ①

$$\text{P.I.} = \frac{1}{4i}\left[e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$

$$-\frac{1}{4i}\left[e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$

$$=\frac{1}{4i}\left[-\frac{1}{2} - \frac{i}{2}e^{2ix}log|\sec 2x + \tan 2x| + \frac{1}{2} - \frac{i}{2}e^{-2ix}log|\sec 2x + \tan 2x|\right]$$

$$= \frac{1}{4i}\left[-i\frac{e^{2ix}+e^{-2ix}}{2}log|\sec 2x + \tan 2x|\right]$$

$$\therefore \text{P.I} = -\frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Complete solution is: $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Exercise 11A

Solve the following differential equations:

1. $(D^3 + D^2 - 5D + 3)y = 0$ Ans. $y = (c_1x + c_2)e^x + c_3e^{-3x}$

2. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$ Ans. $y = c_1e^{2x} + c_2e^{3x} + e^{3x}(x-1)$

3. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x \cosh 2x$

Ans. $y = c_1e^{-3x} + c_2e^{2x} + \frac{1}{12}e^{3x} - \frac{1}{12}e^{-x}$

4. $(D-1)^2(D^2+1)^2y = e^x$

Ans. $y = (c_1x + c_2)e^x + (c_3x + c_4)\cos x + (c_5x + c_6)\sin x + \frac{x^2}{8}e^x$

5. $(D^2 - 6D + 9)y = x^2 + 2e^{2x}$

Ans. $y = (c_1x + c_2)e^{3x} + \frac{1}{9}\left(x^2 + \frac{4x}{8} + \frac{2}{3}\right) + 2e^{2x}$

6. $(D^2 + D - 2)y = x + \sin x$

Ans. $y = c_1e^{-2x} + c_2e^x - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$

7. $(D^2 + D)y = (1+e^x)^{-1}$

Ans. $y = c_1 + c_2e^{-x} + x - (1+e^{-x})\log(1+e^x)$

8. $(D^2 + 5D + 6)y = e^{-2x}sec^2x\,(1+2\tan x)$

Ans. $y = c_1e^{-2x} + c_2e^{-3x} + e^{-2x}(\tan x - 1)$

9. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = (x-1)e^{2x}$

Ans. $y = c_1e^{2x} + c_2e^{-6x} + \frac{e^{2x}}{8}\left(\frac{x^2}{2} - \frac{9x}{8}\right)$

10. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}$, given $y = 1, \frac{dy}{dx} = -1$ when $x = 0$

Ans. $y = -\frac{1}{2}e^x - 2e^{2x} + 2x + 3 + \frac{e^{3x}}{2}$

11.4 Differential Equations Reducible to Linear Form with Constant Coefficients

Some special type of homogenous and non homogeneous linear differential equations with variable coefficients after suitable substitutions can be reduced to linear differential equations with constant coefficients.

11.4.1 Cauchy's Linear Differential Equation

The differential equation of the form:

$$k_0\,x^n\frac{d^ny}{dx^n} + k_1\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + - - - - + k_{n-1}\,x\frac{dy}{dx} + k_n\,y = F(x)$$

is called Cauchy's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$x = e^t \quad \Rightarrow \log x = t$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} = Dy$, where $D \equiv \frac{d}{dt}$

Similarly $x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Example 27 Solve the differential equation:

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 13\cos(\log x), x > 0 \qquad \text{.......①}$$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$ and $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

∴① May be rewritten as

$(D(D-1)(D-2) + 3D(D-1) + D + 8)y = 13\cos t$

$\Rightarrow (D^3 + 8)y = 13\cos t \;, D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^3 + 8 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm \sqrt{3}i$

C.F. = $c_1e^{-2t} + e^t\left(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t\right)$

$= \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)\right)$

P.I. = $\frac{1}{f(D)}F(x) = 13\frac{1}{D^3+8}\cos t$

$= 13\frac{1}{-D+8}\cos t$, Putting $D^2 = -1$

$= 13\frac{(8+D)}{64-D^2}\cos t = 13\frac{(8+D)}{65}\cos t \qquad$ Putting $D^2 = -1$

$\therefore$ P.I. $= \frac{1}{5}(8\cos t + D\cos t)$

$= \frac{1}{5}(8\cos t - \sin t)$

$= \frac{1}{5}(8\cos(\log x) - \sin(\log x))$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)\right) + \frac{1}{5}(8\cos(\log x) - \sin(\log x))$$

Example 28 Solve the differential equation:

$$x^2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + x\frac{\mathrm{dy}}{\mathrm{dx}} - y = \frac{x^3}{1+x^2} \qquad \ldots\ldots① $$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = \mathrm{D}(\mathrm{D}-1)\mathrm{y}$

$\therefore$① May be rewritten as

$(\mathrm{D}(\mathrm{D}-1) + \mathrm{D} - 1)y = \frac{e^{3t}}{1+e^{2t}}$

$\Rightarrow (D^2 - 1)y = \frac{e^{3t}}{1+e^{2t}},\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. $= c_1e^{-t} + c_2e^t$

$= \frac{c_1}{x} + c_2x$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2-1}\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{(D-1)(D+1)}\frac{e^{3t}}{1+e^{2t}} = \frac{1}{2}\left(\frac{1}{(D-1)} - \frac{1}{(D+1)}\right)\frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{2}\left(\frac{1}{(D-1)}\frac{e^{3t}}{1+e^{2t}} - \frac{1}{(D+1)}\frac{e^{3t}}{1+e^{2t}}\right)$

$= \frac{1}{2}\left(e^t \int e^{-t}\frac{e^{3t}}{1+e^{2t}}dt - e^{-t}\int e^t \frac{e^{3t}}{1+e^{2t}}dt\right) \because \frac{1}{D+a}\, F(x) = e^{-ax}\int e^{ax}\, F(x)\, dx$

$= \frac{1}{2}\left(e^t \int \frac{e^{2t}}{1+e^{2t}}dt - e^{-t}\int \frac{e^{4t}}{1+e^{2t}}dt\right)$

Put $e^{2t} = u \Rightarrow 2e^{2t}dt = du$

$\therefore \text{P.I} = \frac{1}{4}\left(e^t \int \frac{1}{1+u}du - e^{-t}\int \frac{u}{1+u}du\right)$

$= \frac{1}{4}\left(e^t \log(1+u) - e^{-t}\int \frac{1+u-1}{1+u}du\right)$

$= \frac{1}{4}\left(e^t \log(1+u) - e^{-t}\int \left(1 - \frac{1}{1+u}\right)du\right)$

$= \frac{1}{4}(e^t \log(1+u) - e^{-t}(u - \log(1+u)))$

$= \frac{1}{4}(e^t \log(1+e^{2t}) - e^{-t}(e^{2t} - \log(1+e^{2t})))$

$= \frac{1}{4}\left(x\log(1+x^2) - \frac{1}{x}(x^2 - \log(1+x^2))\right)$

$= \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = \frac{c_1}{x} + c_2 x + \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

$\Rightarrow y = \frac{c_1}{x} + c_3 x + \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1+x^2)\ , c_3 = c_2 - \frac{1}{4}$

Example 29 Solve the differential equation:

$x^2D^2 - 2x\text{D} - 4y = x^2 + 2\log x\ , \quad x > 0$ ....①

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow xD = \delta y, x^2D^2 = \delta(\delta - 1)\text{y}\ , \delta \equiv \frac{d}{dt}$

∴① May be rewritten as

$(\delta(\delta-1)-2\delta-4)y = e^{2t}+2t$

$\Rightarrow (\delta^2-3\delta-4)y = e^{2t}+2t$

Auxiliary equation is: $m^2-3m-4=0$

$\Rightarrow (m+1)(m-4)=0$

$\Rightarrow m=-1,4$

C.F. = $c_1e^{-t}+c_2e^{4t}$

$= \frac{c_1}{x}+\frac{c_2}{x^4}$

P.I. = $\frac{1}{f(\delta)}F(x) = \frac{1}{\delta^2-3\delta-4}(e^{2t}+2t)$

$= \frac{1}{\delta^2-3\delta-4}e^{2t}+\frac{1}{\delta^2-3\delta-4}2t$

$= \frac{1}{-6}e^{2t}+2\frac{1}{-4\left(1-\frac{\delta^2}{4}+\frac{3\delta}{4}\right)}t$ Putting $\delta = 2$ in the 1st term

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left(1-\left(\frac{\delta^2}{4}-\frac{3\delta}{4}\right)\right)^{-1}t$

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left[1+\frac{\delta^2}{4}-\frac{3\delta}{4}+\cdots\right]t$

$= \frac{-e^{2t}}{6}-\frac{1}{2}\left[t-\frac{3}{4}\right]$

$\therefore$ P.I. $= \frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = \frac{c_1}{x}+\frac{c_2}{x^4}\frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

11.4.2 Legendre's Linear Differential Equation

The differential equation of the form: $k_0(ax+b)^n\frac{d^ny}{dx^n}+$

$$k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+k_{n-1}(ax+b)\frac{dy}{dx}+k_n y = F(x)$$

is called Legendre's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$(ax + b) = e^t \quad \Rightarrow t = \log(ax + b)$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{a}{ax+b}$

$\Rightarrow (ax + b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy$, where $D \equiv \frac{d}{dt}$

Similarly$(ax + b)^2 \frac{d^2y}{dx^2} = a^2 D(D - 1)y$

$(ax + b)^3 \frac{d^3y}{dx^3} = a^3 D(D - 1)(D - 2)y$ and so on.

Example 30 Solve the differential equation:

$$(3x + 2)^2 \frac{d^2y}{dx^2} + 3(3x + 2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1 \quad \text{.......①}$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(3x + 2) = e^t \quad \therefore t = \log(3x + 2)$

$\Rightarrow (3x + 2)\frac{dy}{dx} = 3Dy$, $(3x + 2)^2 \frac{d^2y}{dx^2} = 3^2 D(D - 1)y$

Also $3x^2 + 4x + 1 = \frac{1}{3}(9x^2 + 12x + 3)$

$= \frac{1}{3}((3x)^2 + 2.3.2x + 4 - 4 + 3)$

$= \frac{1}{3}((3x + 2)^2 - 1)$

$= \frac{1}{3}(e^{2t} - 1)$

∴① May be rewritten as

$(9D(D - 1) + 9D - 36)y = \frac{1}{3}(e^{2t} - 1)$

$\Rightarrow 9(D^2 - 4)y = \frac{1}{3}(e^{2t} - 1)$

Auxiliary equation is: $9(m^2 - 4) = 0$

$\Rightarrow m = \pm 2$

C.F. $= c_1 e^{-2t} + c_2 e^{2t}$

$= \frac{c_1}{(3x+2)^2} + c_2(3x + 2)^2$

P.I. = $\frac{1}{f(D)}F(x) = \frac{1}{9(D^2-4)}\frac{1}{3}(e^{2t}-1)$

$$= \frac{1}{27}\left(\frac{1}{(D^2-4)}e^{2t} - \frac{1}{(D^2-4)}e^{0t}\right)$$

$= \frac{1}{27}\left(\frac{t}{2.2}e^{2t} - \frac{1}{(0-4)}e^{0t}\right)$, Putting $D = 2$ in 1st term, it is a case of failure $\therefore \frac{1}{(D^2-4)}e^{2t} = t\frac{1}{f'(2)}e^{2x}$, also $D = 0$ in the 2nd term.

$\therefore$ P.I. = $\frac{1}{27}\left(\frac{t}{4}e^{2t} + \frac{1}{4}\right)$

$$= \frac{1}{27}\left(\frac{\log(3x+2)}{4}(3x+2)^2 + \frac{1}{4}\right)$$

$$= \frac{1}{108}[(3x+2)^2\log(3x+2) + 1]$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2 + \frac{1}{108}[(3x+2)^2\log(3x+2) + 1]$$

Example 31 Solve the differential equation:

$$(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1)), \quad x > -1 \text{ ....①}$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(x+1) = e^t$ $\therefore$ t = $\log(x+1)$

$$\Rightarrow (x+1)\frac{dy}{dx} = Dy, (x+1)^2\frac{d^2y}{dx^2} = 1^2 D(D-1)y$$

$\therefore$① May be rewritten as

$$(D(D-1) + D + 1)y = 2\sin t$$

$$\Rightarrow (D^2+1)y = 2\sin t$$

Auxiliary equation is: $(m^2+1) = 0$

$$\Rightarrow m = \pm i$$

C.F. = $c_1\cos t + c_2\sin t$

$$= c_1\cos(\log(x+1)) + c_2\sin(\log(x+1))$$

$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{D^2+1} 2\sin t$

$=2t \frac{1}{2D} \sin t$, Putting $D^2 = -1$, case of failure

$\therefore \frac{1}{(D^2+1)} \sin t = t \frac{1}{f'(D)} \sin t$

$= t \int \sin t \, dt = -\text{t} \cos t$

$\therefore$ P.I$= -\log(x+1) \cos(\log(x+1))$

Complete solution is: $y =$ C.F. + P.I

$y = c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) - \log(x+1)\cos(\log(x+1))$

11.5 Method of Variation of Parameters for Finding Particular Integral

Method of Variation of Parameters enables us to find the solution of 2nd and higher order differential equations with constant coefficients as well as variable coefficients.

Working rule

Consider a 2nd order linear differential equation:

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x)$ ....①

1. Find complimentary function given as: C.F. = $c_1y_1 + c_2y_2$,

   where y_1 and y_2 are two linearly independent solutions of ①

2. Calculate $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, W is called Wronskian of y_1 and y_2

3. Compute $u_1 = -\int \frac{y_2F(x)}{W} dx$, $u_2 = \int \frac{y_1F(x)}{W} dx$
4. Find P.I. = $u_1y_1 + u_2y_2$
5. Complete solution is given by: $y =$ C.F. + P.I

Note: Method is commonly used to solve 2nd order differential but it can be extended to solve differential equations of higher orders.

Example 32 Solve the differential equation: $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2+1)y = \operatorname{cosec} x$

Auxiliary equation is: $(m^2+1) = 0$

$\Rightarrow\ m = \pm i$

C.F. = $c_1\cos x + c_2\sin x = c_1y_1 + c_2y_2$

$\therefore y_1 = \cos x$ and $y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$u_1 = -\int \frac{y_2F(x)}{W}dx = -\int \sin x\ \operatorname{cosec} x\ dx = -\int 1dx = -x$

$u_2 = \int \frac{y_1F(x)}{W}dx = \int \cos x\ \operatorname{cosec} x\ dx = \int \cot x\, dx = log|\sin x|$

$\therefore$ P.I= $u_1y_1 + u_2y_2$

= $-x\cos x + \sin x\ log|\sin x|$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1\cos x + c_2\sin x - x\cos x + \sin x\ log|\sin x|$

Example 33 Solve the differential equation: $(D^2-2D+1)y = e^x$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2-2m+1) = 0$

$\Rightarrow\ m = 1,1$

C.F. = $(c_1 + c_2\,x)e^x = c_1e^x + c_2x\,e^x = c_1y_1 + c_2y_2$

$\therefore y_1 = e^x$ and $y_2 = x\,e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2F(x)}{W}dx = -\int \frac{xe^xe^x}{e^{2x}}\,dx = -\int xdx = -\frac{x^2}{2}$

$u_2 = \int \frac{y_1F(x)}{W}dx = \int \frac{e^xe^x}{e^{2x}}\,dx = \int 1dx = x$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$$= -\frac{x^2}{2} e^x + x^2 e^x = \frac{x^2}{2} e^x$$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = (c_1 + c_2\, x) e^x + \frac{x^2}{2} e^x$$

Example 34 Solve the differential equation: $\frac{d^2y}{dx^2} + 4y = x \sin 2x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2 + 4)y = x \sin 2x$

Auxiliary equation is: $(m^2 + 4) = 0$

$\Rightarrow m = \pm 2i$

C.F. = $c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\frac{1}{2} \int x \sin^2 2x \; dx = -\frac{1}{4} \int x(1 - \cos 4x)\, dx$$

$$= -\frac{1}{4}\left[\frac{x^2}{2} - \left[(x)\left(\frac{\sin 4x}{4}\right) - (1)\left(-\frac{\cos 4x}{16}\right)\right]\right]$$

$$= \left[-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64}\right]$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2} \int x \sin 2x \cos 2x \; dx = \frac{1}{4} \int x \sin 4x \; dx$$

$$= \frac{1}{4}\left[(x)\left(-\frac{\cos 4x}{4}\right) - (1)\left(-\frac{\sin 4x}{16}\right)\right]$$

$$= \left[-\frac{x \cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$$= \cos 2x \left[-\frac{x^2}{8} + \frac{x \sin 4x}{16} + \frac{\cos 4x}{64}\right] + \sin 2x \left[-\frac{x \cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$= \frac{x}{16}(\sin 4x \cos 2x - \cos 4x \sin 2x) + \frac{1}{64}(\cos 4x \cos 2x + \sin 4x \sin 2x)$

$- \frac{x^2}{8}\cos 2x \quad = \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Example 35 Solve the differential equation: $(D^2 - D - 2)y = e^{(e^x + 3x)}$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - m - 2) = 0$

$\Rightarrow m = -1,2$

C.F. = $c_1 e^{-x} + c_2\, e^{2x} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{-x}$ and $y_2 = e^{2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{e^{2x} e^{(e^x + 3x)}}{3e^x}\, dx = -\int \frac{e^{2x} e^{e^x} e^{3x}}{3e^x} dx$

$= -\frac{1}{3}\int e^{4x} e^{e^x} dx$, Putting $e^x = t \Rightarrow e^x dx = tdt$

$u_1 = -\frac{1}{3}\int t^3 e^t dt = -\frac{1}{3}[(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t)]$

$\Rightarrow u_1 = -\frac{e^{e^x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{-x} e^{(e^x + 3x)}}{3e^x}\, dx = \int \frac{e^{-x} e^{e^x} e^{3x}}{3e^x} dx\ \frac{1}{3}\int e^x e^{e^x} dx = \frac{e^{e^x}}{3}$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= -\frac{e^{e^x} e^{-x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6] + \frac{e^{e^x} e^{2x}}{3}$

$= \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = c_1 e^{-x} + c_2\, e^{2x} + \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$$

Example 36 Given that $\therefore\, y_1 = x$ and $y_2 = \frac{1}{x}$ are two linearly independent solutions of the differential equation: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x, x \neq 0$

Find the particular integral and general solution using method of variation of parameters.

Solution: Rewriting the equation as: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$

Given that $\therefore\, y_1 = x$ and $y_2 = \frac{1}{x}$

$\therefore$C.F. = $c_1 y_1 + c_2 y_2 = c_1 x + \frac{c_2}{x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = \int \frac{1}{x}.\frac{1}{x}.\frac{x}{2}\, dx = \frac{1}{2}\int \frac{1}{x} dx = \frac{1}{2}\log x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = -\int x.\frac{1}{x}.\frac{x}{2}\, dx = -\frac{x^2}{4}$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= \frac{x}{2}\log x - \frac{x}{4}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 x + \frac{c_2}{x} + \frac{x}{2}\log x - \frac{x}{4}$

Example 37 Solve the differential equation: $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2 \log x$

using method of variation of parameters.

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Given differential equation may be rewritten as

$(D(D-1)-4D+6)y = te^{2t}$

$\Rightarrow (D^2-5D+6)y = te^{2t}$

Auxiliary equation is: $(m-2)(m-3)=0$

$\Rightarrow\ m = 2,3$

C.F. = $c_1e^{2t} + c_2e^{3t} = c_1y_1 + c_2y_2$

$\therefore y_1 = e^{2t}$ and $y_2 = e^{3t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$u_1 = -\int \frac{y_2F(t)}{W}dt = -\int \frac{e^{3t}te^{2t}}{e^{5t}}\ dt = -\int tdt = -\frac{t^2}{2}$

$u_2 = \int \frac{y_1F(t)}{W}dt = \int \frac{e^{2t}te^{2t}}{e^{5t}}\ dt = \int te^{-t}dt = [(t)(-e^{-t}) - (1)(e^{-t})]$

$= -te^{-t} - e^{-t}$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$= -\frac{t^2}{2}e^{2t} - (te^{-t} + e^{-t})e^{3t}$

$= -\frac{t^2}{2}e^{2t} - te^{2t} - e^{2t} = -e^{2t}\left(\frac{t^2}{2} + t + 1\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1e^{2t} + c_2e^{3t} - e^{2t}\left(\frac{t^2}{2} + t + 1\right)$

or $y = c_1x^2 + c_2\,x^3 - x^2\left(\frac{(\log x)^2}{2} + \log x + 1\right)$

$\Rightarrow y = c_3x^2 + c_2\,x^3 - \frac{x^2}{2}(\log x)^2 - x^2\log x,\ c_3 = c_1 - 1$

11.6 Solving Simultaneous Linear Differential Equations

Linear differential equations having two or more dependent variables with single independent variable are called simultaneous differential equations and can be of two types:

Type 1: $f_1(D)x + f_2(D)y = F(t)$, $g_1(D)x + g(D)y = G(t), D \equiv \frac{d}{dt}$

Consider a system of ordinary differential equations in two dependent variables x and y and an independent variable t:

$$f_1(D)x + f_2(D)y = F(t)\ ,\ \ g_1(D)x + g(D)y = G(t), D \equiv \frac{d}{dt}$$

Given system can be solved as follows:

1. Eliminate y from the given system of equations resulting a differential equation exclusively in x.
2. Solve the differential equation in x by usual methods to obtain x as a function of t .
3. Substitute value of x and its derivatives in one of the simultaneous equations to get an equation in y.
4. Solve for y by usual methods to obtain its value as a function of t.

Example 38 Solve the system of equations: $\frac{dx}{dt} + y = e^t$, $\frac{dy}{dt} - x = e^{-t}$

Solution: Rewriting given system of differential equations as:

$Dx + y = e^t$……①
$Dy - x = e^{-t}$ …..②, $D \equiv \frac{d}{dt}$
Multiplying ① by D
$\Rightarrow D^2x + Dy = e^t$ …..③
Subtracting ② from ③, we get
$(D^2 + 1)x = e^t - e^{-t}$ …….④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F. = $c_1 \cos t + c_2 \sin t$

P.I. = $\frac{1}{f(D)} F(t) = \frac{1}{D^2+1}(e^t - e^{-t}) = \frac{1}{D^2+1} e^t - \frac{1}{D^2+1} e^{-t}$

$= \frac{1}{2} e^t - \frac{1}{2} e^{-t}$, Putting $D = 1$ and $D = -1$ in 1st and 2nd terms respectively

$\therefore x = c_1 \cos t + c_2 \sin t + \frac{1}{2} e^t - \frac{1}{2} e^{-t}$…….⑤

Using ⑤in ① $\Rightarrow D\left[c_1\cos t + c_2\sin t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}\right] + y = e^t$

$\Rightarrow \left[-c_1\sin t + c_2\cos t + \frac{1}{2}e^t + \frac{1}{2}e^{-t}\right] + y = e^t$

$\Rightarrow y = c_1\sin t - c_2\cos t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ ……⑥

⑤ and ⑥ give the required solution.

Example 39 Solve the system of equations: $t\frac{dx}{dt} + y = 0, \ \frac{dy}{dt} + x = 0$

given that $x(1) = 1, \ y(-1) = 0$

Solution: Given system of equations is:

$t\frac{dx}{dt} + y = 0$……①

$t\frac{dy}{dt} + x = 0$ …...②,

Multiplying ① by $t\frac{d}{dt}$

$t\frac{d}{dt}(t\frac{dx}{dt} + y) = 0$

$\Rightarrow t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$ ……③

Subtracting ② from ③, we get

$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$…….④

which is Cauchy's linear differential equation in x with variable coefficients.

.Putting $t = e^k \quad \therefore \log t = k$

$\Rightarrow t\frac{dx}{dt} = Dx, t^2\frac{d^2x}{dt^2} = D(D-1)x \ , D \equiv \frac{d}{dk}$

$\therefore$ ④ may be rewritten as

$(D(D-1) + D - 1)x = 0$ ……⑤

$\Rightarrow (D^2 - 1)x = 0$

To solve ⑤ for x, Auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^k + c_2e^{-k} = c_1t + \frac{c_2}{t}$

$\therefore x = c_1 t + \frac{c_2}{t}$.......⑥

Using ⑥in ① $\Rightarrow t\frac{d}{dt}\left(c_1 t + \frac{c_2}{t}\right) + y = 0$

$\Rightarrow c_1 t - \frac{c_2}{t} + y = 0$

$\Rightarrow y = -c_1 t + \frac{c_2}{t}$ ....⑦

Also given that at $t = 1, x = 1$ and at $t = -1, y = 0$

Using in ⑥and ⑦ $c_1 + c_2 = 1,\ \ c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2}$

Using $c_1 = c_2 = \frac{1}{2}$ in ⑥and ⑦, we get

$x = \frac{1}{2}\left(t + \frac{1}{t}\right),\ y = \frac{1}{2}\left(\frac{1}{t} - t\right)$

Example 40 Solve the system of equations:

$\frac{\mathrm{d}^2 x}{\mathrm{dt}^2} + y = \sin t,\ \ \frac{\mathrm{d}^2 y}{\mathrm{dt}^2} + x = \cos t$

Solution: Rewriting given system of differential equations as:

$D^2 x + y = \sin t$......①
$D^2 y + x = \cos t$ .....②, $D \equiv \frac{d}{dt}$
Multiplying ① by D^2
$D^2(D^2 x + y) = D^2 \sin t$
$\Rightarrow D^4 x + D^2 y = -\sin t$ .....③
Subtracting ② from ③, we get
$(D^4 - 1)x = -\sin t - \cos t$ ....④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^4 - 1 = 0$

$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow m = \pm 1, \pm i$

C.F. = $c_1 e^t + c_2 e^{-t} + (c_3 \cos t + c_4 \sin t)$

P.I. = $\frac{1}{f(D)} F(t) = \frac{1}{D^4 - 1}(-\sin t - \cos t) = -\frac{1}{D^4 - 1}\sin t - \frac{1}{D^4 - 1}\cos t$

Putting $D^2 = -1$ i.e. $D^4 = 1$ in 1st and 2nd terms, it is a case of failure

$\therefore$ P.I. $= -t\frac{1}{4D^3}\sin t - t\frac{1}{4D^3}\cos t$

$= \frac{t}{4}\frac{1}{D}\sin t + \frac{t}{4}\frac{1}{D}\cos t$ putting $D^2 = -1$

$= -\frac{t}{4}\cos t + \frac{t}{4}\sin t$

$\therefore x = (c_1e^t + c_2e^{-t}) + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)$.......⑤

Using ⑤in ①

$\Rightarrow D^2\left[c_1e^t + c_2e^{-t} + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)\right] + y = \sin t$

$$\Rightarrow D\left[c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)\right] + y = \sin t$$

$$\Rightarrow \left[c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)\right] + y = \sin t$$

$\Rightarrow y = -(c_1e^t + c_2e^{-t}) + (c_3\cos t + c_4\sin t) + \left(\frac{t}{4} + \frac{1}{2}\right)(\sin t - \cos t)$ ...⑥

⑤ and ⑥ give the required solution.

Type II: Symmetric simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Simultaneous differential equations in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be solved by the method of grouping or the method of multipliers or both to get two independent solutions: $u = c_1, v = c_2$; where c_1 and c_2 are arbitrary constants.

Method of grouping: In this method, we consider a pair of fractions at a time which can be solved for an independent solution.

Method of multipliers: In this method, we multiply each fraction by suitable multipliers (not necessarily constants) such that denominator becomes zero.

If a, b, c are multipliers, then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{adx+bdy+cdz}{aP+bQ+cR}$

Example 41 Solve the set of simultaneous equations:

$$\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$$

Taking x, y, z as multipliers, each fraction equals

$$\frac{xdx+ydy+zdz}{(xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2)} = \frac{xdx+ydy+zdz}{0}$$

$\Rightarrow \; xdx + ydy + zdz = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c'_1$

1st independent solution is: $u = x^2 + y^2 + z^2 = c_1$……①

Now for 2nd independent solution, taking last two members of the set of equations: $\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$

$\Rightarrow (y - z)dy = (y + z)dz$

$\Rightarrow \; ydy - (zdy + ydz) - zdz = 0$

$\Rightarrow \; ydy - d(yz) - zdz = 0$

Integrating, we get

$\frac{y^2}{2} - yz - \frac{z^2}{2} = c'_2$

$\Rightarrow \quad v = y^2 - 2yz - z^2 = c_2 \quad$ ……….②

① and ② give the required solution.

Exercise 11B

Q1. Solve the following differential equations:

i. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Ans. $\langle y = (c_1 + c_2 \log x)x^2 + \frac{1}{4} + 2x + \frac{x^2}{2}(\log x)^2 \rangle$

ii. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

Ans. $\langle y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2} \rangle$

iii. $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Ans. $\langle y = c_1(2x+3)^{\frac{3+\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3-\sqrt{57}}{4}} - \frac{3}{14}(2x+3) + \frac{3}{4} \rangle$

iv. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos(\log(x+1))$

Ans.$\langle y = c_1 \cos(\log(x+1)) + c_2\, sin(\log(x+1)) + 2\log(x +$ $1. sin \log x+1$

Q2. Solve the following differential equations using method of variation of parameters

i. $\frac{d^2y}{dx^2} + y = xsinx$

Ans. $\langle y = c_1 \cos x + c_2 \sin x + \frac{1}{8}\cos x + \frac{x}{4}\sin x - \frac{x^2}{4}\cos x \rangle$

ii. $(D^2 - 1)y = e^{-2x} \sin e^{-x}$

Ans. $\langle y = c_1 e^x + c_2 e^{-x} - \sin e^{-x} - e^x \cos e^{-x} \rangle$

iii. $(D^2 - 2D)y = e^x \sin x$

Ans. $\langle y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x \rangle$

iv. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \log x$

Ans: $\langle y = c_1 + c_2 e^{2x} + \frac{x^2}{4} e^x (2\log x - 3) \rangle$

Q2. Solve the following set of simultaneous differential equations

i. $\frac{dx}{dt} - 7x + y = 0, \quad \frac{dy}{dt} - 2x - 5y = 0$

Ans:$\langle x = e^{6t}(c_1 \cos t + c_2 \sin t), y = e^{6t}(c_1 - c_2)\cos t + (c_1 + c_2)\sin t) \rangle$

ii. $(D+1)x + (2D+1)y = e^t \,, (D-1)x + (D+1)y = 1$

Ans:$\langle x = c_1e^t + c_2e^{-2t} + 2e^{-t}, y = 3c_1e^t + 2c_2e^{-2t} + 3e^{-t}\rangle$

iii. $\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$

Ans:$\langle xy - z = c_1 ,\ x^2 - y^2 + z^2 = c_2\rangle$

11.7 Previous Years Solved Questions

Q1. Solve $(D^2 + D + 1)^2(D - 2)y = 0$

$\langle \boldsymbol{Q1(h), GGSIPU,\ December\ 2012}\rangle$

Solution: Auxiliary equation is: $(m^2 + m + 1)^2(m - 2)y = 0$…….①

Solving ①, we get

$\Rightarrow\ m = 2, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

C.F. = $c_1e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3\,x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5x)\sin\frac{\sqrt{3}}{2}x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3\,x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5x)\sin\frac{\sqrt{3}}{2}x$

Q2. Solve $(D^2 - 1)y = \cosh x \cos x$

$\langle \boldsymbol{Q8(b), GGSIPU,\ December\ 2012}\rangle$

Solution: Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. = $c_1e^x + c_2e^{-x}$

P.I. = $\frac{1}{f(D)}\ F(x)$

$= \frac{1}{D^2-1}\left(\frac{e^x+e^{-x}}{2}\cos x\right)$ $\qquad \because \cosh x = \frac{e^x+e^{-x}}{2}$

$= \frac{1}{D^2-1}\left(\frac{e^x}{2}\cos x + \frac{e^{-x}}{2}\cos x\right)$

$= \frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x + \frac{e^{-x}}{2}\frac{1}{(D-1)^2-1}\cos x$

$$= \frac{e^x}{2}\frac{1}{(D^2+2D)}\cos x + \frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x$$

$$= \frac{e^x}{2}\frac{1}{2D-1}\cos x + \frac{e^{-x}}{2}\frac{1}{-2D-1}\cos x \qquad \text{Putting } D^2 = -1$$

$$= \frac{e^x}{2}\frac{2D+1}{4D^2-1}\cos x - \frac{e^{-x}}{2}\frac{2D-1}{4D^2-1}\cos x$$

$$= -\frac{e^x}{10}(2D+1)\cos x + \frac{e^{-x}}{10}(2D-1)\cos x \quad \text{Putting } D^2 = -1$$

$$= -\frac{e^x}{10}(-2\sin x + \cos x) + \frac{e^{-x}}{10}(-2\sin x - \cos x)$$

$$\therefore \text{P.I.} = \frac{e^x}{10}(2\sin x - \cos x) - \frac{e^{-x}}{10}(2\sin x + \cos x)$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = c_1e^x + c_2e^{-x} + \frac{e^x}{10}(2\sin x - \cos x) - \frac{e^{-x}}{10}(2\sin x + \cos x)$$

Q3. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ by the method of variation of parameters.

$\langle \boldsymbol{Q9(a), \ GGSIPU, \ December\ 2012} \rangle$

Solution: $\Rightarrow (D^2+4)y = 4\tan 2x$

Auxiliary equation is: $(m^2+4) = 0$

$\Rightarrow m = \pm 2i$

C.F. = $c_1\cos 2x + c_2\sin 2x = c_1y_1 + c_2y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2F(x)}{W}dx = -\frac{4}{2}\int \sin 2x \tan 2x \, dx = -2\int \frac{\sin^2 2x}{\cos 2x}\, dx$$

$$-2\int \frac{1-\cos^2 2x}{\cos 2x}\, dx = -2\int (\sec 2x - \cos 2x)\, dx$$

$$= -2\left[\frac{1}{2}\log|\sec 2x + \tan 2x| - \frac{1}{2}\sin 2x\right]$$

$= [\sin 2x - \log|\sec 2x + \tan 2x|]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2}\int 4 \tan 2x \cos 2x \; dx = \; 2\int \sin 2x \; dx$

$= -\cos 2x$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= \cos 2x\,[\sin 2x - \log|\sec 2x + \tan 2x|] - \sin 2x \cos 2x$

$= -\cos 2x \log|\sec 2x + \tan 2x|$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log|\sec 2x + \tan 2x|$

Q4. Solve the system of equations: $\frac{\mathrm{d}x}{\mathrm{dt}} + x = y + e^t, \;\; \frac{\mathrm{d}y}{\mathrm{dt}} + y = x + e^t$

$\langle \boldsymbol{Q9(b)},\; \boldsymbol{GGSIPU},\; \boldsymbol{December\ 2012} \rangle$

Solution: Rewriting given system of differential equations as:

$(D+1)x - y = e^t \ldots\ldots①$
$(D+1)y - x = e^t \ldots..②, \; D \equiv \frac{d}{dt}$
Multiplying ① by $(D+1)$
$\Rightarrow (D+1)^2 x - (D+1)y = (D+1)e^t$
$(D^2+2D+1)x - (D+1)y = 2e^t \ldots..③$
Adding ② and ③, we get
$(D^2+2D)x = 3e^t \ldots\ldots.④$

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^2 + 2m = 0$

$\Rightarrow m = 0, -2$

C.F. $= c_1 + c_2 e^{-2t}$

P.I. $= \frac{1}{f(D)} F(t) = 3\frac{1}{D^2+2D} e^t$

$= e^t$ Putting $D = 1$

$\therefore x = c_1 + c_2 e^{-2t} + e^t \ldots\ldots.⑤$

Using ⑤in ① ⇒ $D[c_1 + c_2e^{-2t} + e^t] + c_1 + c_2e^{-2t} + e^t - y = e^t$

$\Rightarrow -2c_2e^{-2t} + e^t + c_1 + c_2e^{-2t} - y = 0$

$\Rightarrow y = c_1 - c_2e^{-2t} + e^t$ ……⑥

⑤ and ⑥ give the required solution.

Q5. Solve by method of variation of parameters $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$\langle \boldsymbol{Q8(a), GGSIPU, December\ 2013} \rangle, \langle \boldsymbol{Q3(b), GGSIPU, 2^{nd} term\ 2014} \rangle$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$(m - 3)^2 = 0$

$\Rightarrow m = 3,3$

C.F. = $(c_1 + c_2\, x)e^{3x} = c_1e^{3x} + c_2x\, e^{3x} = c_1y_1 + c_2y_2$

$\therefore y_1 = e^{3x}$ and $y_2 = xe^{3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$$

$u_1 = -\int \frac{y_2F(x)}{W} dx = -\int \frac{xe^{3x}e^{3x}}{x^2e^{6x}} dx = -\int \frac{1}{x} dx = -\log x$

$u_2 = \int \frac{y_1F(x)}{W} dx = \int \frac{e^{3x}e^{3x}}{x^2e^{6x}} dx = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$\therefore$ P.I= $u_1y_1 + u_2y_2$

$= -e^{3x}\log x - e^{3x} = -e^{3x}(1 + \log x)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1 + c_2x)e^{3x} - e^{3x}(1 + \log x)$

Q6. Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2013} \rangle$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m + 1)^2 = 0$

$\Rightarrow\ m = 0, -1, -1$

C.F. = $c_1 +\ e^{-x}(c_2 + c_3x)$

P.I. = $\frac{1}{f(D)}\ F(x)\ = \frac{1}{D^3 + 2D^2 + D}(e^{2x} + \sin 2x)$

$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{-4D - 8 + D} \sin 2x$, putting $D = 2$ in 1st term, $D^2 = -4$ in the 2nd term

$= \frac{1}{18} e^{2x} - \frac{3D - 8}{(3D + 8)(3D - 8)} \sin 2x = \frac{1}{18} e^{2x} - \frac{3D - 8}{(9D^2 - 64)} \sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{100}(3\text{D} - 8)\sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ = c_1 +\ e^{-x}(c_2 + c_3x) +\ \frac{1}{18} e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Q7. Solve $(D^2 - 2D + 1)y = xe^x \cos x$

$\langle \boldsymbol{Q8(a), GGSIPU, December\ 2014} \rangle$

Solution: Auxiliary equation is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m - 1)^2$

$\Rightarrow m = 1{,}1$

C.F. = $(c_1 + c_2x)e^x$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2 - 2D + 1} xe^x \cos x$

$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \cos x$

$= e^x \frac{1}{D^2} x \cos x$

$= e^x \frac{1}{D} \int x \cos x \, dx$

$= e^x \frac{1}{D} [(x)(\sin x) - (1)(-\cos x)]$

$= e^x \frac{1}{D} [x \sin x + \cos x]$

$= e^x [\int x \sin x \, dx + \int \cos x \, dx]$

$= e^x \big[[(x)(-\cos x) - (1)(-\sin x)] + \sin x\big]$

$\therefore$ P.I. $= e^x[-x \cos x + 2 \sin x]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x)e^x + e^x[-x \cos x + 2 \sin x]$

Q8. Solve by M.O.V.P. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2014} \rangle$

Solution: Given differential equation may be rewritten as

$(D^2 - 2D + 1)y = e^x \log x$

: Auxiliary equation is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2$

$\Rightarrow m = 1,1$

C.F. = $(c_1 + c_2 x)e^x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^x$ and $y_2 = xe^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{x\,e^x\,e^x \log x}{e^{2x}}\, dx = -\int x \log x \, dx$

$\int x \log x \, dx = I = \left[(x)(x \log x - x) - (1)\left(I - \frac{x^2}{2}\right)\right]$

$\because \int \log x \, dx = x \log x - x$

$\Rightarrow 2I = x^2 \log x - x^2 + \frac{x^2}{2}$

$$\Rightarrow I = \int x \log x \, dx \;=\; \frac{x^2}{2}\log x - \frac{x^2}{4}$$

$$\therefore u_1 = \frac{x^2}{4} - \frac{x^2}{2}\log x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x \log x}{e^{2x}}\, dx = \int \log x \, dx = x\log x - x$$

$$\therefore \text{P.I} = \left(\frac{x^2}{4} - \frac{x^2}{2}\log x\right) e^x + (x\log x - x) x e^x$$

$$= e^x \left(\frac{x^2}{4} - \frac{x^2}{2}\log x + x^2 \log x - x^2\right)$$

$$= \frac{x^2}{2} e^x \left(\log x - \frac{3}{2}\right)$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x)e^x + \frac{x^2}{2} e^x \left(\log x - \frac{3}{2}\right)$$

Q9. Solve $(D-1)^2(D+1)^2 = \sin^2\frac{x}{2} + e^x + x$

$$\langle \boldsymbol{Q1(a), GGSIPU, December\ 2015} \rangle$$

Solution: Auxiliary equation is: $(m-1)^2(m+1)^2 = 0$

$\Rightarrow m = 1,1,-1,-1$

C.F. = $(c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x}$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{((D-1)(D+1))^2}\left(\sin^2\frac{x}{2} + e^x + x\right)$

$$= \frac{1}{2}\frac{1}{D^4 - 2D^2 + 1}(1 - \cos x) + \frac{1}{D^4 - 2D^2 + 1} e^x + \frac{1}{D^4 - 2D^2 + 1} x$$

$$= \frac{1}{2}\frac{1}{D^4 - 2D^2 + 1} e^{0x} - \frac{1}{2}\frac{1}{D^4 - 2D^2 + 1}\cos x + \frac{1}{D^4 - 2D^2 + 1} e^x + \frac{1}{D^4 - 2D^2 + 1} x$$

Now $\frac{1}{2}\frac{1}{D^4 - 2D^2 + 1} e^{0x} = \frac{1}{2}$, putting $D = 0$

Also $\frac{1}{2}\frac{1}{D^4 - 2D^2 + 1}\cos x = \frac{1}{8}\cos x$ putting $D^2 = -1$

Again $\frac{1}{D^4 - 2D^2 + 1} e^x = x\frac{1}{4D^3 - 4D} e^x$ as $f(1) = 0$, a case of failure 2 times

$= x^2 \frac{1}{12D^2-4}\ e^x = \frac{x^2}{8}\ e^x$, putting $D = 1$

And $\frac{1}{D^4-2D^2+1} x = \frac{1}{1+(D^4-2D^2)} x = [1 + (D^4 - 2D^2)]^{-1} x = x$

$\therefore$ P.I.$= \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}\ e^x + x$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x} - \frac{1}{8}\cos x + \frac{x^2}{8}\ e^x + x + \frac{1}{2}$

Q.10 Solve $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y\ = x^4 \sin x$

$\langle \boldsymbol{Q3(b), GGSIPU, December\ 2015} \rangle$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Equation may be rewritten as

$(D(D-1) - 4D + 6)y = e^{4t}\sin e^t$

$\Rightarrow (D^2 - 5D + 6)y = e^{4t}\sin e^t\ ,\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow\ m = 2,3$

C.F. = $c_1 e^{2t} + c_2 e^{3t}$

$= c_1 x^2 + c_2 x^3$

P.I. = $\frac{1}{f(D)}\ F(x) =\ \frac{1}{D^2-5D+6} e^{4t}\sin e^t$

$= e^{4t}\frac{1}{(D+4)^2-5(D+4)+6}\sin e^t$

$= e^{4t}\frac{1}{D^2+3D+2}\sin e^t =\ e^{4t}\frac{1}{(D+1)(D+2)}\sin e^t$

$=e^{4t}\left[\frac{1}{(D+1)}-\frac{1}{(D+2)}\right]\sin e^{t}= e^{4t}\left[\frac{1}{(D+1)}\sin e^{t}-\frac{1}{(D+2)}\sin e^{t}\right]$

$=e^{4t}[e^{-t}\int e^{t}\sin e^{t}\,dt-e^{-2t}\int e^{2t}\sin e^{t}\,dt]$

$\because \frac{1}{(D+a)}F(t)=e^{-at}\int e^{at}F(t)dt$

$= e^{4t}[e^{-t}(-cose^{t})-e^{-2t}(-e^{t}\cos e^{t}+\sin e^{t}]$

Solving the two integrals by putting $e^{t}=u, \therefore e^{t}dt=du$

$\therefore$ P. I = $-e^{2t}\sin e^{t}$= $-x^{2}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1x^2+c_2x^3-x^2\sin x$

UNIT-2

2.2. First Order Linear Partial Differential Equations, Lagrange's Method

Let $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ be continuous differentiable functions with respect to each of the variables. Being x, y independent variables and $z = z(x,y)$ dependent variable, consider

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z) \tag{1}$$

This equation is called the first order quasi-linear partial differential equation. A method for solving such an equation was first given by Lagrange. For this reason, equation (1) is also called the Lagrange linear equation. If P and Q are independent of z and

$$R(x,y,z) = G(x,y) - C(x,y)z,$$

(1) gives the equation with linear partial differential, so a linear partial differential equation can also be considered as a quasi-linear partial differential equation. Therefore, the Lagrange method is also valid for linear partial differential equations.

Lagrange's Method

Let's assume that in a region of three-dimensional space, the functions P and Q are not both zero, and that the function $z = f(x,y)$ has a solution to the equation (1). Considering a fixed point $M(x,y,z)$ on the S surface defined by $z = f(x,y)$, we can give a simple geometrical meaning to equation (1).

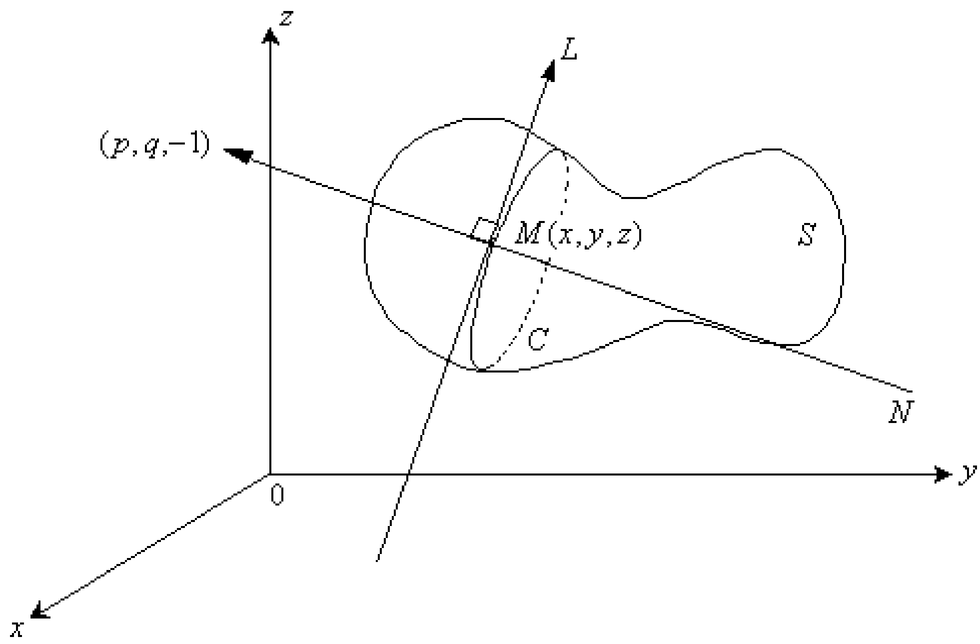


Figure 2.2.1

The normal vector N of surface S at point M is given by

$$\begin{aligned} \vec{n} &= grad\{f(x,y) - z\} \\ &= (f_x, f_y, -1) \\ &= (p, q, -1). \end{aligned}$$

If we write the equation (1) in the form

$$Pp + Qq - R = 0, \tag{2}$$

it is seen that the scalar product of the vectors $(p, q, -1)$ and (P, Q, R) is zero. These two vectors are perpendicular to each other. This means that there is a line L that passes through the point M and is perpendicular to the normal vector n, such that the direction cosines (P, Q, R) of L is tangent to the surface S. Let the plane passing through N and L cut the surface S along a curve C. The direction cosines of the tangent of C on M is (dx, dy, dz) and this tangent is parallel to L. Therefore, the direction cosines of these two lines must be proportional. That is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{3}$$

The first order ordinary differential equation system formed by the equations (3) is called the auxiliary system of the Lagrange equation or the Lagrange system. A system equivalent to system (3), being x independent variable, is

$$\frac{dy}{dx} = \frac{Q}{P} \quad , \quad \frac{dz}{dx} = \frac{R}{P}. \tag{4}$$

The general solution of (4) is

$$y = y(x, c_1, c_2) \quad , \quad z = z(x, c_1, c_2) \tag{5}$$

where c_1 and c_2 are arbitrary constants. If these equations are solved according to c_1 and c_2 , the general solution of the system (3) can be as follows

$$u(x, y, z) = c_1 \ , \ v(x, y, z) = c_2. \tag{6}$$

Each of $u = c_1$ and $v = c_2$ is called first integral of Lagrange system. The functions u and v must also be functionally independent. So at any point $M(x, y, z) \in \Omega$, all Jacobians

$$\frac{\partial(u, v)}{\partial(x, y)} \quad , \quad \frac{\partial(u, v)}{\partial(x, z)} \quad , \quad \frac{\partial(u, v)}{\partial(y, z)}$$

should not be zero at once. Each of the first integrals obtained by (6) is a surface family of one-parameter. Intersection curves of surfaces defined by (6) form the surfaces

$$F(u, v) = 0 \tag{7}$$

The equation (7), where F is an arbitrary function, gives the general solution to the partial differential equation (1).

It is also possible to explain this situation as follows: exact differential of (6) is in the form

$$\left.\begin{aligned} u_x dx + u_y dy + u_z dz = 0 \\ v_x dx + v_y dy + v_z dz = 0 \end{aligned}\right\} \tag{8}$$

Since u and v are the solutions of the system (3), the equations (3) and (8) show that u and v functions satisfy

$$\left.\begin{array}{l} u_x P + u_y Q + u_z R = 0 \\ v_x P + v_y Q + v_z R = 0 \end{array}\right\}. \tag{9}$$

If we solve the system (9) according to P, Q and R, we obtain

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}} \tag{10}$$

On the other hand, from the equation $F(u, v) = 0$, we eliminate the arbitrary function F, we obtain the partial differential equation

$$\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} q = \frac{\partial(u,v)}{\partial(x,y)}. \tag{11}$$

If the expressions P, Q, R in (10), which are proportional with Jacobians, are written in (11), we have

$$Pp + Qq = R,$$

which shows that (7) is the solution of (1). Since F is arbitrary in this solution, it is the general solution.

Example 1. Find the general solution of the equation $x^2 \dfrac{\partial z}{\partial x} + y^2 \dfrac{\partial z}{\partial y} = (x + y)z$.
Solution: The corresponding Lagrange system is in the form

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}.$$

From this system, the first integrals are obtained as follows:

i) From $\dfrac{dx}{x^2} = \dfrac{dy}{y^2}$,we have $-\dfrac{1}{x} = -\dfrac{1}{y} + c_1$ or $u = \dfrac{1}{y} - \dfrac{1}{x} = c_1$

ii) From $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dx - dy}{x^2 - y^2} = \dfrac{dx - dy}{(x-y)(x+y)} = \dfrac{dz}{(x+y)z}$ it follows $\dfrac{d(x-y)}{(x-y)} = \dfrac{dz}{z}$.

$$\Rightarrow \ln(x - y) = \ln z + \ln c_2 \;\Rightarrow\; v = \frac{x - y}{z} = c_2.$$

So, the general solution of the given equation is

$$F(\frac{1}{y}-\frac{1}{x}, \ \frac{x-y}{z})=0$$

where F is arbitrary function.

Remark: The general solution given above is also written as

$$z=(x-y)f(\frac{1}{y}-\frac{1}{x})$$

where f is arbitrary function.

Example 2. Find the general solution of the equation $xzp+yzq=-(x^2+y^2)$.

Solution: The corresponding Lagrange system is

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{-(x^2+y^2)}.$$

The first integrals:

$i)$ $\frac{dx}{xz}=\frac{dy}{yz} \Rightarrow \frac{dx}{x}=\frac{dy}{y} \Rightarrow \ln x-\ln y=\ln c_1 \Rightarrow u=\frac{x}{y}=c_1.$

$ii)$ $\frac{dx}{xz}=\frac{dy}{yz}=\frac{xdx}{x^2z}=\frac{ydy}{y^2z}=\frac{xdx+ydy}{z(x^2+y^2)}=\frac{dz}{-(x^2+y^2)}$

$\Rightarrow xdx+ydy=-zdz \Rightarrow xdx+ydy+zdz \Rightarrow \ v=x^2+y^2+z^2=c_2.$

The general solution of the given equation is

$$F(\frac{x}{y}, \ x^2+y^2+z^2)=0$$

where F is arbitrary function.

Example 3. Find the general solution of the equation $(y+x)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=\frac{x^2+y^2}{z}$..

Solution: The corresponding Lagrange system is

$$\frac{dy}{x-y} = \frac{dx}{y+x} = \frac{dz}{\dfrac{x^2+y^2}{z}}.$$

The first integrals:

$i)$ From $\dfrac{dy}{x-y} = \dfrac{dx}{y+x} = \dfrac{dx+dy}{2x}$ we have $\dfrac{dx}{y+x} = \dfrac{dx+dy}{2x}$.

$(x+y)d(x+y) - 2xdx = 0 \Rightarrow \; d\left[\dfrac{(x+y)^2}{2} - x^2\right] = 0 \Rightarrow (x+y)^2 - 2x^2 = c_1$ or
$\Rightarrow u(x,y,z) = y^2 + 2xy - x^2 = c_1.$

$ii)$ From $\dfrac{ydy - xdx}{y(x-y) - x(x+y)} = \dfrac{zdz}{x^2+y^2}, \; \dfrac{ydy - xdx}{-(x^2+y^2)} = \dfrac{zdz}{x^2+y^2} \Rightarrow ydy - xdx + zdz = 0$

$\Rightarrow v(x,y,z) = y^2 - x^2 + z^2 = c_2$

The general solution is

$$F(y^2 + 2xy - x^2, \;\; y^2 - x^2 + z^2) = 0$$

where F is arbitrary function.

Remark: It should be noted that the first independent pair of integrals obtained above is not the only pair used to write the general solution. In the last example, the pair of first integrals

$$\begin{aligned} w(x,y,z) &= z^2 + 2xy = c_1 \\ v(x,y,z) &= y^2 - x^2 + z^2 = c_2 \end{aligned}$$

form an independent pair of the first integrals of the auxiliary equation system. Hence the general solution can writen as

$$F(z^2 + 2xy, \;\; y^2 - x^2 + z^2) = 0$$

where F is an arbitrary function.

Formation of Partial Differential Equations

Partial differential equations can be obtained by the elimination of arbitrary constants or by the elimination of arbitrary functions.

By the elimination of arbitrary constants
Let us consider the function
$\phi(x, y, z, a, b) = 0$ ------------- (1)
where a & b are arbitrary constants
Differentiating equation (1) partially w.r.t x & y, we get

$$\frac{\partial\phi}{\partial x} + p\frac{\partial\phi}{\partial z} = 0 \qquad \_\_\_\_\_\_\_\_\_\_ (2)$$

$$\frac{\partial\phi}{\partial y} + q\frac{\partial\phi}{\partial z} = 0 \qquad \_\_\_\_\_\_\_\_\_\_ (3)$$

Eliminating a and b from equations (1), (2) and (3), we get a partial differential equation of the first order of the form f (x,y,z, p, q) = 0

Example 1
Eliminate the arbitrary constants a & b from $z = ax + by + ab$
Consider $z = ax + by + ab$ _____________ (1)
Differentiating (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = a \qquad \text{i.e, } p = a \qquad \_\_\_\_\_\_\_\_ (2)$$

$$\frac{\partial z}{\partial y} = b \qquad \text{i.e, } q = b \qquad \_\_\_\_\_\_\_ (3)$$

Using (2) & (3) in (1), we get

$$z = px + qy + pq$$

which is the required partial differential equation.
Example 2
Form the partial differential equation by eliminating the arbitrary constants a and b from
$z = (x^2 + a^2) (y^2 + b^2)$
Given $z = (x^2 + a^2) (y^2 + b^2)$ ……..(1)
Differentiating (1) partially w.r.t x & y , we get

p = 2x $(y^2 + b^2)$
q = 2y (x + a)

Substituting the values of p and q in (1), we get
4xyz = pq
which is the required partial differential equation.

Example 3

Find the partial differential equation of the family of spheres of radius one whose centre lie in the xy - plane.

The equation of the sphere is given by

$$(x - a)^2 + (y - b)^2 + z^2 = 1 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_} (1)$$

Differentiating (1) partially w.r.t x & y , we get
2 (x-a) + 2 zp = 0
2 (y-b) + 2 zq = 0

From these equations we obtain
x-a = -zp __________ (2)
y -b = -zq __________ (3)
Using (2) and (3) in (1), we get

$$z^2p^2 + z^2q^2 + z^2 = 1$$

or $z^2(p^2 + q^2 + 1) = 1$

Example 4

Eliminate the arbitrary constants a, b & c from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and form the partial differential equation.
The given equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (1)}$$

Differentiating (1) partially w.r.t x & y, we get

$$\frac{2x}{a^2} + \frac{2zp}{c^2} = 0$$

$$\frac{2y}{b^2} + \frac{2zq}{c^2} = 0$$

Therefore we get

$$\frac{x}{a^2} + \frac{zp}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$\frac{y}{b^2} + \frac{zq}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Again differentiating (2) partially w.r.t 'x', we set

$(1/a^2) + (1/c^2)(zr + p^2) = 0$ ____________________ (4)

Multiplying (4) by x, we get

$$\frac{x}{a^2} + \frac{xz\,r}{c^2} + \frac{p^2x}{c^2} = 0$$

From (2), we have

$$\frac{-zp}{c^2} + \frac{xzr}{c^2} + \frac{p^2x}{c^2} = 0$$

or $-zp + xzr + p^2x = 0$

or $-zp + xzr + p2x = 0$

By the elimination of arbitrary functions

Let u and v be any two functions arbitrary function. This relation can be expressed as

$u = f(v)$ _______________ (1)

Differentiating (1) partially w.r.t x & y and eliminating the arbitrary functions from these relations, we get a partial differential equation of the first order of the form

$f(x, y, z, p, q) = 0.$

Example 5

Obtain the partial differential equation by eliminating „f„from $z = (x+y) f (x^2 - y^2)$

Let us now consider the equation

$z = (x+y) f(x^2- y^2)$ ______________ (1)

Differentiating (1) partially w.r.t x & y , we get

$p = (x + y) f ' (x^2 - y^2) . 2x + f (x^2 - y^2)$

$q = (x + y) f ' (x^2 - y^2) . (-2y) + f (x^2 - y^2)$

$$p - f(x^2 - y^2) = (x+y) f'(x^2 - y^2) . 2x \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$q - f(x^2 - y^2) = (x+y) f'(x^2 - y^2) .(-2y) \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Hence, we get

$$\frac{p - f(x^2 - y^2)}{q - f(x^2 - y^2)} = - \frac{x}{y}$$

i.e, $py - yf(x^2 - y^2) = -qx +xf (x^2 - y^2)$

i.e, $py +qx = (x+y) f (x2 - y2)$

Therefore, we have by(1), $py +qx = z$

Example 6

Form the partial differential equation by eliminating the arbitrary function f from

$z = e^y f(x + y)$

Consider $z = e^y f(x + y)$ ____________ (1)

Differentiating (1) partially w.r.t x & y, we get

$p = e^y f'(x + y)$

$q = e^y f'(x + y) + f(x + y).e^y$

Hence, we have

$q = p + z$

Example 7

Form the PDE by eliminating f & Φ from $z = f(x + ay) + \Phi(x - ay)$

Consider $z = f(x + ay) + \Phi(x - ay)$ _______________ (1)

Differentiating (1) partially w.r.t x &y, we get

$p = f'(x + ay) + \Phi'(x - ay)$ ____________________ (2)

$q = f'(x + ay).a + \Phi'(x - ay)(-a)$ _______________ (3)

Differentiating (2) & (3) again partially w.r.t x & y, we get

$r = f''(x + ay) + \Phi''(x - ay)$

$t = f''(x + ay).a^2 + \Phi''(x - ay)(-a)^2$

i.e, $t = a^2\{f''(x + ay) + \Phi''(x - ay)\}$

or $t = a^2 r$

Exercises:

1. 1. Form the partial differential equation by eliminating the arbitrary constants „a" & „b" from the following equations.

(i) $z = ax + by$

(ii) $\dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{b^2} = 1$

(iii) $z = ax + by + \sqrt{a^2 + b^2}$

(iv) $ax^2 + by^2 + cz^2 = 1$

(v) $z = a^2x + b^2y + ab$

2. Find the PDE of the family of spheres of radius 1 having their centres lie on the xy plane{Hint: $(x - a)^2 + (y - b)^2 + z^2 = 1$}

3. Find the PDE of all spheres whose centre lie on the (i) z axis (ii) x-axis

4. Form the partial differential equations by eliminating the arbitrary functions in the following cases.

(i) $z = f(x + y)$

(ii) $z = f(x^2 - y^2)$

(iii) $z = f(x^2 + y^2 + z^2)$

(iv) $\phi(xyz, x + y + z) = 0$

(v) $F(xy + z^2, x + y + z) = 0$

(vi) $z = f(x + iy) + f(x - iy)$

(vii) $z = f(x^3 + 2y) + g(x^3 - 2y)$

UNIT-3

Lecture 21: The one dimensional Wave Equation: D'Alembert's Solution

(*Compiled 3 March 2014*)

In this lecture we discuss the one dimensional wave equation. We review some of the physical situations in which the wave equations describe the dynamics of the physical system, in particular, the vibrations of a guitar string and elastic waves in a bar. We describe the relationship between solutions to the the wave equation and transformation to a moving coordinate system known as the Galilean Transformation. The galilean transformation can be used to identify a general class of solutions to the wave equation requiring only that the solution be expressed in terms of functions that are sufficiently differentiable. We show how the second order wave equation can be decomposed into two first order wave operators, one representing a left-moving and the other a right moving wave. This decomposition is used to derive the classical D'Alembert Solution to the wave equation on the domain $(-\infty, \infty)$ with prescribed initial displacements and velocities. This solution fully describes the equations of motion of an infinite elastic string that has a prescribed shape and initial velocity.

Key Concepts: The one dimensional Wave Equation; Characteristics; Traveling Wave Solutions; Vibrations in a Bar; a Guitar String; Galilean Transformation; D'Alembert's Solution.

Reference Section: Boyce and Di Prima Section 10.7

21 The one dimensional Wave Equation

21.1 Types of boundary and initial conditions for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{21.1}$$

$$\begin{array}{lll} \dfrac{\partial^2 u}{\partial t^2} & \rightarrow \quad \text{expect 2 initial conditions} & u(x,0) = f(x) \\ & & \frac{\partial u}{\partial t}(x,0) = g(x) \\ \dfrac{\partial^2 u}{\partial x^2} & \rightarrow \quad \text{expect 2 boundary conditions} & u(0,t) = 0 \\ & & u(L,t) = 0. \end{array} \tag{21.2}$$

21.2 Some examples of physical systems in which the wave equation governs the dynamics

21.2.1 *The Guitar String*

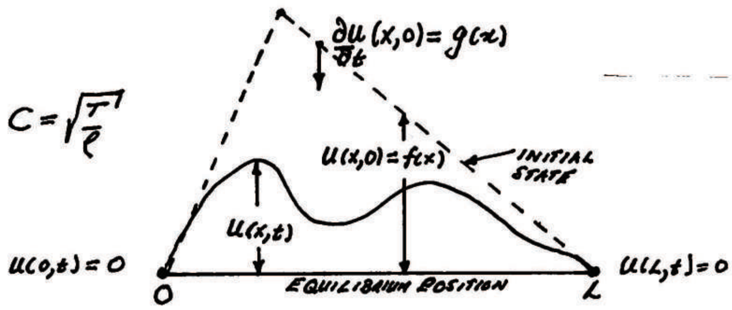


FIGURE 1. Initial condition and transient solution of the plucked guitar string, whose dynamics is governed by (21.1).

21.2.2 *Longitudinal Vibrations of an elastic bar*

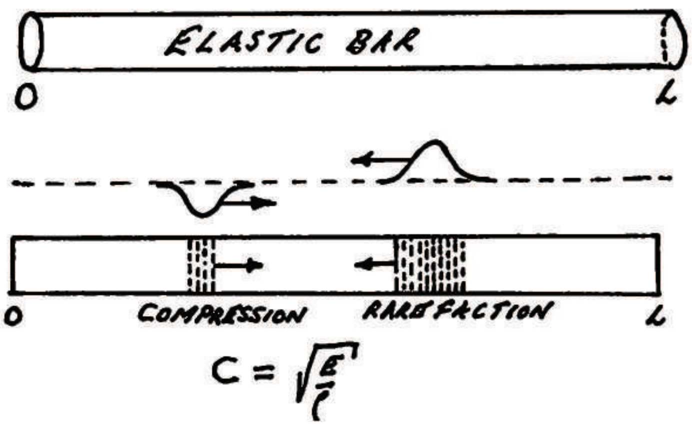


FIGURE 2. Compression and rarefaction waves in an elastic bar, whose dynamics is governed by (21.1).

21.3 A sneak preview - exponential solutions and the dispersion relation

To investigate the nature of the solutions to the wave equation that we might expect, let us look for exponential solutions of the form:

$$u = e^{ikx+\sigma t}$$

Substituting this trial solution into (21.1) yields

$$u_{tt} - c^2 u_{xx} = \left[\sigma^2 - c^2 (ik)^2\right] e^{ikx+\sigma t} = 0$$

Therefore in order that the exponential function (21.3 be a solution of (21.1), we require that σ satisfy the dispersion relation

$$\sigma^2 = -c^2 k^2$$

or

$$\sigma = \pm kc$$

which implies that there are two solutions of the form

$$u = e^{ik(x \pm ct)} = e^{\pm ikct} e^{ikx}$$

We will now demonstrate physical significance of the argument $(x \pm ct)$ of the exponential and show that this leads to a much more general class of solutions. The products of time varying sinusoids with arguments $ikct$ with spatially varying sinusoids with arguments kx are precisely the same form as the solutions one would obtain by separation of variables for the wave equation defined on a finite domain. The selection of permissable wavenumbers k that apply in a particular problem will be determined by solving the appropriate eigenvalue problem.

21.4 The Galilean Transformation and solutions to the wave equation

Claim 1 *The Galilean transformation $x' = x + ct$ associated with a coordinate system $O'x'$ moving to the left at a speed c relative to the coordinates Ox, yields a solution to the wave equation: i.e., $u(x,t) = G(x+ct)$ is a solution to (21.1)*

$$u_t = cG' \quad u_{tt} = c^2 G'' \tag{21.3}$$

$$u_x = G' \quad u_{xx} = G''. \tag{21.4}$$

Therefore

$$u_{tt} - c^2 u_{xx} = c^2 G'' - c^2 G'' = 0. \tag{21.5}$$

Similarly $u(x,t) = F(x - ct)$ is also a solution to (21.1) associated with a right moving coordinate system $O'x'$ such that $x' = x - ct$. Is the sum of two solutions also a solution?

Claim 2 *Because the wave equation is linear, superposition applies: i.e., If u_1 and u_2 are solutions to (21.1) then $u(x,t) = \alpha_1 u_1(x,t) + \alpha_2 u_2(x,t)$ is also a solution.*

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(\alpha_1 u_1 + \alpha_2 u_2) &= \alpha_1 \frac{\partial^2 u_1}{\partial t^2} + \alpha_2 \frac{\partial^2 u_2}{\partial t^2} \\ &= \alpha_1 c^2 \frac{\partial^2 u_1}{\partial x^2} + \alpha_2 c^2 \frac{\partial^2 u_2}{\partial x^2} \quad \text{since } u_1 \text{ and } u_2 \text{ solve (21.1)} \end{aligned}$$

Thus

$$\frac{\partial^2}{\partial t^2}(\alpha_1 u_1 + \alpha_2 u_2) = c^2 \frac{\partial^2}{\partial x^2}(\alpha_1 u_1 + \alpha_2 u_2).$$

Therefore, the general solution to the one dimensional wave equation (21.1) can be written in the form

$$u(x,t) = F(x - ct) + G(x + ct) \tag{21.6}$$

provided F and G are sufficiently differentiable functions.

Observations:

(1) This property is due to the linearity of $u_{tt} = c^2 u_{xx}$ (21.1).

(2) Every solution for (21.1) on $(-\infty, \infty)$ is of this form.

21.4.1 *Decomposition of the wave operator into left and right moving waves*

We observe that the wave operator can be decomposed as follows:

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u(x,t) = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t) = 0. \tag{21.7}$$

Let $w = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u$ then solving the wave equation can be reduced to solving the following system of first order wave equations:

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w \text{ and } \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0. \tag{21.8}$$

In Lecture 2 we used the Galilean Transformation to interpret and identify solutions to these two first order wave operators.

In particular,

$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \to \{\text{right moving pulse}\} \implies$

and

$\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \to \{\text{left moving pulse}\} \implies$

21.5 D'Alembert's Solution

Motivated by the left and right moving coordinate systems we consider the following change of variables.

$$\begin{array}{rclcrcl} r &=& x+ct & \quad & s &=& x-ct \\ x &=& \frac{1}{2}(r+s) & & t &=& \frac{1}{2c}(r-s) \end{array}. \tag{21.9}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial t}\frac{\partial t}{\partial r} = \frac{1}{2c}\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \tag{21.10}$$

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial}{\partial t}\frac{\partial t}{\partial s} = -\frac{1}{2c}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) \tag{21.11}$$

Therefore

$$-4c^2\frac{\partial^2 u}{\partial r \partial s} = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0. \tag{21.12}$$

Therefore

$$\frac{\partial^2 u}{\partial r \partial s}(r,s) = 0 \tag{21.13}$$

$$\Rightarrow \frac{\partial u}{\partial s}(r,s) = \bar{\phi}_1(s) \tag{21.14}$$

$$\Rightarrow u(r,s) = \int \bar{\phi}_1(s)\, ds + \phi_2(r) = \phi_1(s) + \phi_2(r). \tag{21.15}$$

Say we have the IC:

$$u(x,0) = u_0(x) \quad \text{displacement} \tag{21.16}$$

$$\frac{\partial u}{\partial t}(x,0) = v_0(x) \quad \text{velocity} \tag{21.17}$$

$$u(x,t) = F(x-ct) + G(x+ct) \tag{21.18}$$

$$u(x,0) = F(x) + G(x) = u_0(x) \tag{21.19}$$

$$\frac{\partial u}{\partial t}(x,0) = -cF'(x) + cG'(x) = v_0(x) \tag{21.20}$$

$$-cF(x) + cG(x) = \int_0^x v_0(\xi)\, d\xi + A \tag{21.21}$$

$$\begin{bmatrix} 1 & 1 \\ -c & c \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} u_0 \\ \int_0^x v_0(\xi)\, d\xi + A \end{bmatrix} \tag{21.22}$$

$$F = \frac{1}{2c}\left\{ cu_0 - \left(\int_0^x v_0(\xi)\, d\xi + A \right) \right\} \tag{21.23}$$

$$G = \frac{1}{2c}\left\{ \int_0^x v_0(\xi)\, d\xi + A + cu_0 \right\} \tag{21.24}$$

Therefore

$$\boxed{u(x,t) = \frac{1}{2}\left[u_0(x-ct) + u_0(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\, d\xi} \tag{21.25}$$

D'Alembert's Solution to the wave equation on $(-\infty, \infty)$.

Classification Of Partial Differential Equations And Their Solution Characteristics

Contents

- Partial Differential Equations
- Linear Partial Differential Equations
- Classification of second order linear PDEs
- Canonical Forms
- Characteristics
- Initial and Boundary Conditions
- Elliptic Partial Differential Equations
- Parabolic Partial Differential Equations
- Hyperbolic Partial Differential Equations
- Summary

Partial Differential Equations

- An equation which involves several independent variables (usually denoted *x, y, z, t*, ...), a dependent function *u* of these variables, and the partial derivatives of the dependent function *u* with respect to the independent variables such as

  $F(x, y, z, t, \ldots.., u_x, u_y, u_z, u_t, \ldots.., u_{xx}, u_{yy}, \ldots.., u_{xy}, \ldots..)=0$

  is called a partial differential equation.
- Partial differential equations are used to formulate, and thus aid the solution of, problems involving functions of several variables; such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, and elasticity.

Partial Differential Equations cont.

- Examples:

i. $u_t=k(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional heat equation]

ii. $u_{xx} + u_{yy} + u_{zz}=0$ [Laplace equation in three dimensions]

iii. $u_{tt}=c^2(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional wave equation]

iv. $u_t + uu_x=\mu u_{xx}$ [nonlinear one-dimensional Burger equation]

Partial Differential Equations cont.

- The order of a partial differential equation is the order of the highest derivative occurring in the equation.
- All the above examples are second order partial differential equations.
- $u_t = uu_{xxx} + \sin x$ is an example for third order partial differential equation.

Ordinary Differential Equations vs. Partial Differential Equations

Partial Differential Equations

- A relatively simple partial differential equation is

$$u_x(x, y)=0$$

- General solution of the above equation is

$$u(x, y)=f(y)$$

- General solution involves arbitrary functions

Ordinary Differential Equations

- The analogous ordinary differential equation is

$$u'(x)=0$$

- General solution of the above equation is

$$u(x)=c$$

- General solution involves arbitrary constants

Linear Partial Differential Equations

- The equation is called linear if the unknown function only appears in a linear form.

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$$

- Almost linear partial differential equations

$$P(x, y)u_x + Q(x, y)u_y = R(x, y, u)$$

- Quasi-linear partial differential equations

$$P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u)$$

Classification of second order linear PDEs

Consider the second order linear PDE in two variables

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ (1)

The discriminant

$$d = B^2(x_0, y_0) - 4A(x_0, y_0)C(x_0, y_0)$$

At (x_0, y_0), the equation is said to be

- Elliptic if $d<0$
- Parabolic if $d=0$
- Hyperbolic if $d>0$

If this is true at all points in a domain Ω, then the equation is said to be elliptic, parabolic, or hyperbolic in that domain

Classification of second order linear PDEs cont.

- If there are *n* independent variables x_1, x_2 , ..., x_n, a general linear partial differential equation of second order has the form
- $\sum\sum a_{i,j} u_{x_i x_j}$ plus lower order terms =0
- The classification depends upon the signature of the eigenvalues of the coefficient matrix.

Classification of second order linear PDEs cont.

i. Elliptic: The eigenvalues are all positive or all negative.
ii. Parabolic : The eigenvalues are all positive or all negative, save one which is zero.
iii. Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.

Canonical Forms

- Transformation of independent variables x and y of eq.(1) to new variables ξ, η, where

$$\xi = \xi(x, y),\ \eta = \eta(x, y)$$

i. Elliptic: $u_{\xi\xi} + u_{\eta\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$

ii. Parabolic: $u_{\xi\xi} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$ or $u_{\eta\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$

iii. Hyperbolic: $u_{\xi\xi} - u_{\eta\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$ or $u_{\xi\eta} = \varphi(\xi, \eta, u, u_{\xi}, u_{\eta})$

Characteristics

- Consider $L[u]=f(x, y, u, u_x, u_y)$ --(2) where $L[u]=a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy}$
- $L[u]$ is the principle part of the equation
- $\xi=\xi(x, y), \eta=\eta(x, y)$
- Transformed equation: $M[u]=g(\xi, \eta, u, u_\xi, u_\eta)$ with principle part

$M[u]=A(\xi, \eta)u_{\xi\xi} + B(\xi, \eta)u_{\xi\eta} + C(\xi, \eta)u_{\eta\eta}$ where

$$A=a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2$$

$$B=2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y$$

$$C=a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2$$

Characteristics cont.

- An integral of an ordinary differential equation is a function φ whose level curves, $\varphi(x, y)=k$, characterize solutions of the equation implicitly.
- $a(x, y)\xi_x^2 + b(x, y)\xi_x\xi_y + c(x, y)\xi_y^2=0$ iff ξ is an integral of the ordinary differential equation

$$a(x, y)y'^2 - b(x, y)y' + c(x, y)=0 \quad \text{--(3)}$$

$$\Rightarrow y'=[b(x, y) \pm \{b^2(x, y) - 4a(x, y)c(x, y)\}^{1/2}]/2a$$

- An integral curve, $\varphi(x, y)=k$, of (3) is a characteristic curve, and (3) is called the characteristic equation for the partial differential equation (2)

Characteristics cont.

- Therefore,

i. Elliptic partial differential equations have no characteristic curves

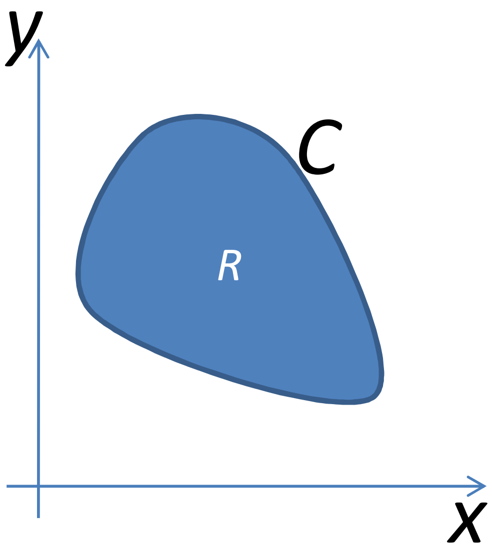
ii. Parabolic partial differential equations have a single characteristic curve

iii. Hyperbolic partial differential equations have two characteristic curves

Initial and Boundary Conditions

(a) Elliptic Equations: Boundary conditions

e.g. $u_{xx} + u_{yy} = G$ in a finite region R bounded by a closed curve C.



Initial and Boundary Conditions cont.

We must specify

(i) u on curve C or

(ii) u_n on C (***n*** is outward normal to C) or

(iii) $\alpha u + \beta u_n$ on C (α and β are given constants) or

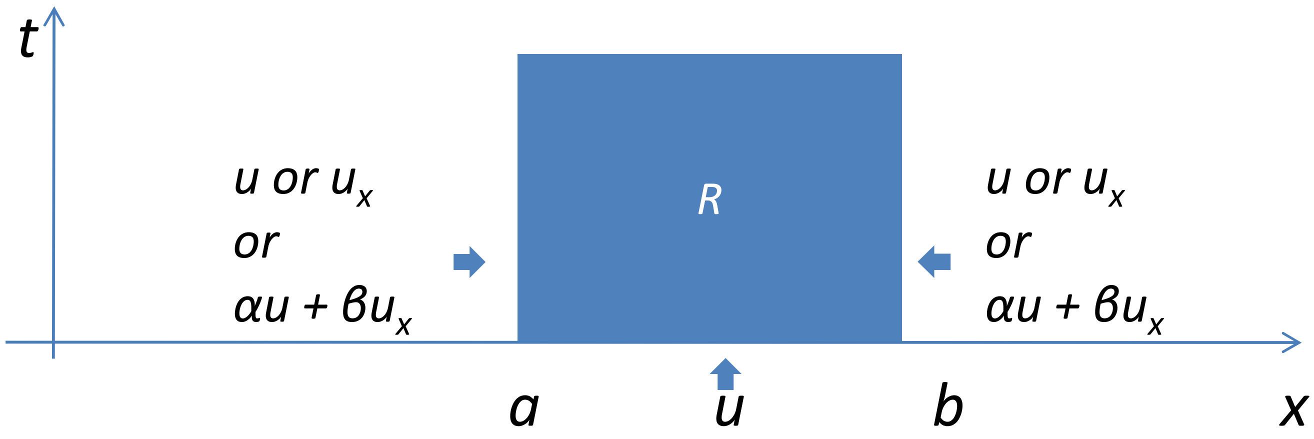
(iv) a combination of (i), (ii) and (iii) on different parts of C

- In Cartesian coordinates the simplest case is if R is rectangular with boundary condition (i).
- R can extend to infinity, in which case we must specify how the solution behaves as x or y (or both x and y) tend to infinity.

Initial and Boundary Conditions cont.

(b) Parabolic Equations: Initial conditions and boundary conditions.

e.g. $u_{xx}=u_t$ in the open region R in the (x, t) plane. R is the region $a \le x \le b$, $0 \le t < \infty$



Initial and Boundary Conditions cont.

- We must specify u on $t=0$ (i.e. $u(x, 0)$) for $a \leq x \leq b$. This is an initial condition (e.g. an initial temperature distribution) and suitable boundary conditions x on a and b are as shown.

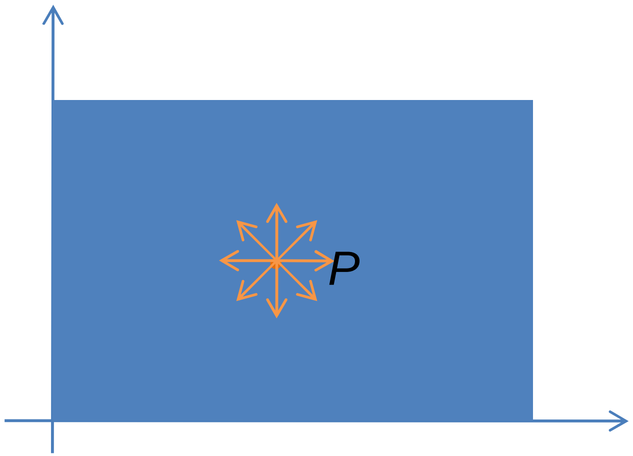
(c) Hyperbolic Equations: e.g. $u_{xx}=u_{tt}$ Initial conditions and boundary conditions as for (b) except that we must also specify u_t at $t=0$ for $a \leq x \leq b$ (in addition to u) and R is the region $a \leq x \leq b$, $-\infty < t < \infty$

Elliptic Partial Differential Equations

- The discriminant $B^2 - 4AC < 0$
- Solutions of elliptic PDEs are as smooth as the coefficients allow, within the interior of the region where the equation and solutions are defined.
- For example, solutions of Laplace's equation are analytic within the domain where they are defined, but solutions may assume boundary values that are not smooth.

Elliptic Partial Differential Equations cont.

- Region of Influence: Entire domain
- Region of Dependence: Entire domain



- Any disturbance at *P* is felt throughout the domain

Elliptic Partial Differential Equations cont.

Examples:

(i) Laplace Equation: $\Delta u=0$

- The Laplace equation is often encountered in heat and mass transfer theory, fluid mechanics, elasticity, electrostatics, and other areas of mechanics and physics.
- The two dimensional Laplace equation has the following form:

$u_{xx} + u_{yy}=0$ in the Cartesian coordinate system,

$(1/r)(ru_r)_r +(1/r^2)u_{\vartheta\vartheta}=0$ in the polar coordinate system

Laplace Equation cont.

- A function which satisfies Laplace's equation is said to be harmonic.
- A solution to Laplace's equation has the property that the average value over a spherical surface is equal to the value at the center of the sphere (Gauss' harmonic function theorem).
- Solutions have no local maxima or minima.
- Because Laplace's equation is linear and homogeneous, the superposition of any two solutions is also a solution

Laplace Equation cont.

Solution of Laplace's equation:

Consider $u_{xx} + u_{yy} = 0$ (2)

Solve by separation of variables

Let $u = X(x)Y(y)$

Substituting it in (2), we get

$$(1/X)X'' = -(1/Y)Y'' = k$$

Solution of Laplace Equation cont.

i. $k=p^2$: $X=c_1e^{px} + c_2e^{-px}$, $Y=c_3\cos py + c_4\sin py$

ii. $k=-p^2$: $X=c_5\cos px + c_6\sin px$, $Y=c_7e^{py} + c_8e^{-py}$

iii. $k=0$: $X=c_9x + c_{10}$, $Y=c_{11}y + c_{12}$

Thus, various possible solutions are:

$u=(c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py)$

$u=(c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py})$

$u=(c_9x + c_{10})(c_{11}y + c_{12})$

Laplace Equation cont.

Analytic functions:

- The real and imaginary parts of a complex analytic function both satisfy the Laplace equation.
- If $f(x + iy)=u(x, y) + iv(x, y)$ is an analytic function, then $u_{xx} + u_{yy}=0$, $v_{xx} + v_{yy}=0$
- The close connection between the Laplace equation and analytic functions implies that any solution of the Laplace equation has derivatives of all orders, and can be expanded in a power series, at least inside a circle that does not enclose a singularity.

Elliptic Partial Differential Equations cont.

(ii) Poisson Equation: $\Delta u + \Phi = 0$

- The two dimensional Poisson equation has the following form:

$u_{xx} + u_{yy} + f(x, y) = 0$ in the Cartesian coordinate system,

$(1/r)(ru_r)_r + (1/r^2)u_{\vartheta\vartheta} + g(r, \vartheta) = 0$ in the polar coordinate system

- Poisson's equation is a partial differential equation with broad utility in electrostatics, mechanical engineering and theoretical physics.
- E.g. In electrostatics: $\Delta V = -\rho/\varepsilon$

Elliptic Partial Differential Equations cont.

(iii) Helmholtz Equation: $\Delta u + \lambda u = -\Phi$

- Many problems related to steady state oscillations (mechanical, acoustical, thermal, electromagnetic) lead to the two dimensional Helmholtz equation. For ¸ $\lambda < 0$, this equation describes mass transfer processes with volume chemical reactions of the first order.

Helmholtz Equation cont.

- The two dimensional Helmholtz equation has the following form:

$u_{xx} + u_{yy} + \lambda u=-f(x, y)$ in the Cartesian coordinate system,

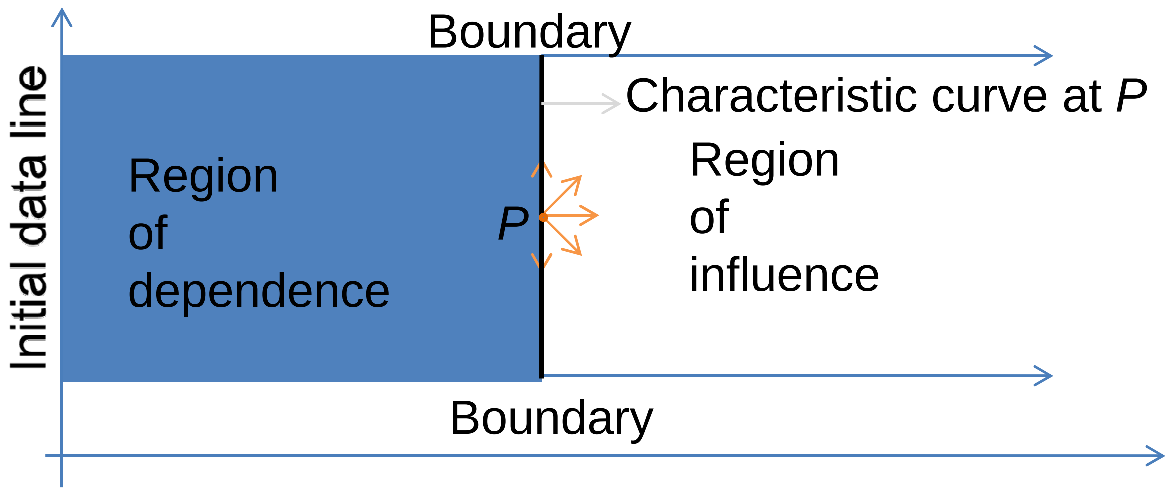
$(1/r)(ru_r)_r +(1/r^2)u_{\vartheta\vartheta} + \lambda u=-g(r, \vartheta)$ in the polar coordinate system

Parabolic Partial Differential Equations

- The discriminant $B^2 - 4AC = 0$
- Equations that are parabolic at every point can be transformed into a form analogous to the heat equation by a change of independent variables.
- Solutions smooth out as the transformed time variable increases

Parabolic Partial Differential Equations cont.

- Region of influence: Part of domain away from initial data line from the characteristic curve
- Region of dependence: Part of domain from the initial data line to the characteristic curve



Parabolic Partial Differential Equations cont.

Examples:

i. $u_t=au_{xx}$ heat equation (linear heat equation)

ii. $u_t=au_{xx} + f(x, t)$ non-homogeneous heat equation

iii. $u_t=au_{xx} + bu_x + cu + f(x, t)$ convective heat equation with a source

iv. $u_t=a(u_{rr} + (1/r)u_r)$ heat equation with axial symmetry

Parabolic Partial Differential Equations cont.

v. $u_t=a(u_{rr} + (1/r)u_r) + g(r, t)$ heat equation with axial symmetry (with a source)

vi. $u_t=a(u_{rr} + (2/r)u_r)$ heat equation with central symmetry

vii. $u_t=a(u_{rr} + (2/r)u_r) + g(r, t)$ heat equation with central symmetry (with a source)

viii. $i\hbar u_t=-(\hbar^2/2m)u_{xx} + h(x)u$ Schrodinger equation (linear schrodinger equation)

Parabolic Partial Differential Equations cont.

- Heat equation: $u_t = a\Delta u$
- The maximum value of *u* is either earlier in time than the region of concern or on the edge of the region of concern.
- even if *u* has a discontinuity at an initial time $t = t_0$, the temperature becomes smooth as soon as $t > t_0$. For example, if a bar of metal has temperature 0 and another has temperature 100 and they are stuck together end to end, then very quickly the temperature at the point of connection is 50 and the graph of the temperature is smoothly running from 0 to 100.

Parabolic Partial Differential Equations cont.

Solution of the heat equation:

Consider $u_t = au_{xx}$ (3)

- In plain English, this equation says that the temperature at a given time and point will rise or fall at a rate proportional to the difference between the temperature at that point and the average temperature near that point.

Solve by separation of variables

Let $u(x, t) = X(x)T(t)$

Substituting this in (3), we get

$$X''/X = T'/aT = k$$

Solution of heat equation cont.

i. $k=p^2$: $X=c_1e^{px} + c_2e^{-px}$, $T=c_3e^{ap^2t}$

ii. $k=-p^2$: $X=c_4\cos px + c_5\sin px$, $T=c_6e^{-ap^2t}$

iii. $k=0$: $X=c_7x + c_8$, $T=c_9$

Thus, various possible solutions are:

$u=(c_1e^{px} + c_2e^{-px})(c_3e^{ap^2t})$

$u=(c_4\cos px + c_5\sin px)(c_6e^{-ap^2t})$

$u=(c_7x + c_8)c_9$

Parabolic Partial Differential Equations cont.

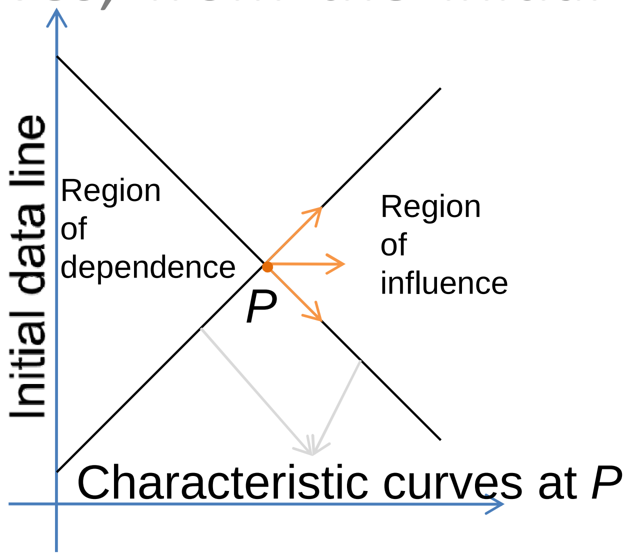
- Let $u(x, t)$ be a continuous function and a solution of $u_t = au_{xx}$ for $0 \leq x \leq l$, $0 \leq t \leq T$, where $T > 0$ is a fixed time. Then the maximum and minimum values of u are attained either at time $t=0$ or at the end points $x=0$ and $x=l$ at some time in the interval $0 \leq t \leq T$

Hyperbolic Partial Differential Equations

- The discriminant $B^2 - 4AC > 0$
- Hyperbolic equations retain any discontinuities of functions or derivatives in the initial data
- If a disturbance is made in the initial data of a hyperbolic differential equation, then not every point of space feels the disturbance at once. Relative to a fixed time coordinate, disturbances have a finite propagation speed. They travel along the characteristics of the equation
- An example is the wave equation

Hyperbolic Partial Differential Equations cont.

- **Region of influence:** Part of domain, between the characteristic curves, from point *P* to away from the initial data line
- **Region of dependence:** Part of domain, between the characteristic curves, from the initial data line to the point *P*



Hyperbolic Partial Differential Equations cont.

Examples:

i. $u_{tt}=a^2u_{xx}$ wave equation (linear wave equation)

ii. $u_{tt}=a^2u_{xx} + f(x, t)$ non-homogeneous wave equation

iii. $u_{tt}=a^2u_{xx} - bu$ Klein-Gordon equation

iv. $u_{tt}=a^2u_{xx} - bu + f(x, t)$ non-homogeneous Klein-Gordon equation

Hyperbolic Partial Differential Equations cont.

v. $u_{tt}=a^2(u_{rr} + (1/r)u_r) + g(r, t)$ non-homogeneous wave equation with axial symmetry

vi. $u_{tt}=a^2(u_{rr} + (2/r)u_r) + g(r, t)$ non-homogeneous wave equation with central symmetry

vii. $u_{tt} + ku_t=a^2u_{xx} + bw$ Telegraph equation

Hyperbolic Partial Differential Equations cont.

Solution of the wave equation:

Consider $u_{tt}=a^2u_{xx}$ (4)

- The equation has the property that, if u and its first time derivative are arbitrarily specified initial data on the initial line $t = 0$ (with sufficient smoothness properties), then there exists a solution for all time.

D'Alembert's solution

Introduce new independent variables:

$$y=x + at, \; z=x - at$$

Substituting these in (4), we get

$$u_{yz}=0 \quad (5)$$

Solution of Wave Equation cont.

Integrating (5) w.r.t. z, we get $u_y = f(y)$ (6)

Integrating (6) w.r.t. y, we obtain

$$u = \varphi(y) + \psi(z), \text{ where } \varphi(y) = \int f(y)dy$$

Thus, $u(x, t) = \varphi(x + at) + \psi(x - at)$ (7) is the general solution of (4)

Now suppose, $u(x, 0) = g(x)$ and $u_t(x, 0) = 0$, then (7) takes the form $u(x, t) = g(x + at) + g(x - at)$

which is the d'Alembert's solution of the wave equation (4)

Summary

- Second order semi-linear equation in two variables: $A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = \varphi(x, y, u, u_x, u_y)$ classified as

i. Elliptic: $B^2 - 4AC < 0$

ii. Parabolic: $B^2 - 4AC = 0$

iii. Hyperbolic: $B^2 - 4AC > 0$

Summary & Conclusion

General relation between the physical problems and the type of PDEs

- Propagation problems lead to parabolic or hyperbolic PDEs.
- Equilibrium equations lead to elliptic PDE.
- Most fluid equations with an explicit time dependence are Hyperbolic PDEs
- For dissipation problem, Parabolic PDEs

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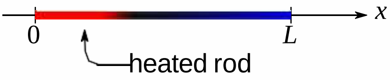
- Previous years’ talks

The One-Dimensional Heat Equation

Introduction

Goal: Model heat flow in a one-dimensional object (thin rod).

Set up: Place rod of length L along x-axis, one end at origin:



Let $u(x,t)$ = temperature in rod at position x, time t.

(Ideal) Assumptions:

- Rod is given some initial temperature distribution $f(x)$ along its length.
- Rod is perfectly insulated, i.e. heat only moves horizontally.
- No internal heat sources or sinks.

The Heat Equation

One can show that u satisfies the *one-dimensional heat equation*

$$u_t = c^2 u_{xx}.$$

Remarks:

- This can be derived via conservation of energy and Fourier's law of heat conduction (see textbook pp. 143-144).
- The constant c^2 is the *thermal diffusivity*:

$$c^2 = \frac{K_0}{s\rho}, \qquad \begin{aligned} K_0 &= \text{ thermal conductivity,} \\ s &= \text{ specific heat,} \\ \rho &= \text{ density.} \end{aligned}$$

Initial and Boundary Conditions

To completely determine u we must also specify:

Initial conditions: The initial temperature profile

$$u(x,0) = f(x) \text{ for } 0 < x < L.$$

Boundary conditions: Specific behavior at $x_0 = 0, L$:

1. Constant temperature: $u(x_0, t) = T$ for $t > 0$.
2. Insulated end: $u_x(x_0, t) = 0$ for $t > 0$.
3. Radiating end: $u_x(x_0, t) = Au(x_0, t)$ for $t > 0$.

Solving the Heat Equation
Case 1: homogeneous Dirichlet boundary conditions

We now apply separation of variables to the heat problem

$$\begin{aligned} u_t &= c^2 u_{xx} && (0 < x < L,\ \ t > 0),\\ u(0,t) &= u(L,t) = 0 && (t > 0),\\ u(x,0) &= f(x) && (0 < x < L). \end{aligned}$$

We seek separated solutions of the form $u(x,t) = X(x)T(t)$. In this case

$$\begin{aligned} u_t &= XT' \\ u_{xx} &= X''T \end{aligned} \quad \Rightarrow \quad XT' = c^2 X'' T \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{c^2 T} = k.$$

Together with the boundary conditions we obtain the system

$$\begin{aligned} X'' - kX &= 0,\ \ X(0) = X(L) = 0,\\ T' - c^2 kT &= 0. \end{aligned}$$

Already know: up to constant multiples, the only solutions to the BVP in X are

$$k = -\mu_n^2 = -\left(\frac{n\pi}{L}\right)^2,$$
$$X = X_n = \sin(\mu_n x) = \sin\left(\frac{n\pi x}{L}\right), \quad n \in \mathbb{N}.$$

Therefore T must satisfy

$$T' - c^2 k T = T' + \underbrace{\left(\frac{c n\pi}{L}\right)^2}_{\lambda_n} T = 0$$
$$T' = -\lambda_n^2 T \quad \Rightarrow \quad T = T_n = b_n e^{-\lambda_n^2 t}.$$

We thus have the *normal modes* of the heat equation:

$$u_n(x,t) = X_n(x)T_n(t) = b_n e^{-\lambda_n^2 t}\sin(\mu_n x), \quad n \in \mathbb{N}.$$

Superposition and initial condition

Applying the principle of superposition gives the general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin(\mu_n x).$$

If we now impose our initial condition we find that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

which is the sine series expansion of $f(x)$. Hence

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Remarks

- As before, if the sine series of $f(x)$ is already known, solution can be built by simply including exponential factors.
- One can show that this is the *only* solution to the heat equation with the given initial condition.
- Because of the decaying exponential factors:
  - ∗ The normal modes tend to zero (exponentially) as $t \to \infty$.
  - ∗ Overall, $u(x,t) \to 0$ (exponentially) *uniformly in* x as $t \to \infty$.
  - ∗ As c increases, $u(x,t) \to 0$ more rapidly.

This agrees with intuition.

Example

Solve the heat problem

$$\begin{aligned} u_t &= 3u_{xx} && (0 < x < 2,\ t > 0), \\ u(0,t) = u(2,t) &= 0 && (t > 0), \\ u(x,0) &= 50 && (0 < x < 2). \end{aligned}$$

We have $c = \sqrt{3}$, $L = 2$ and, by exercise 2.3.1 (with $p = L = 2$)

$$f(x) = 50 = \frac{200}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Since $\lambda_{2k+1} = \dfrac{c(2k+1)\pi}{L} = \dfrac{\sqrt{3}(2k+1)\pi}{2}$, we obtain

$$u(x,t) = \frac{200}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} e^{-3(2k+1)^2\pi^2 t/4} \sin\left(\frac{(2k+1)\pi x}{2}\right).$$

Solving the Heat Equation
Case 2a: steady state solutions

Definition: We say that $u(x,t)$ is a *steady state solution* if $u_t \equiv 0$ (i.e. u is time-independent).

If $u(x,t)$ is a steady state solution to the heat equation then

$$u_t \equiv 0 \;\Rightarrow\; c^2 u_{xx} = u_t = 0 \;\Rightarrow\; u_{xx} = 0 \;\Rightarrow\; u = Ax + B.$$

Steady state solutions can help us deal with inhomogeneous Dirichlet boundary conditions. Note that

$$\begin{aligned} u(0,t) &= T_1 \\ u(L,t) &= T_2 \end{aligned} \;\Rightarrow\; \begin{aligned} B &= T_1 \\ AL + B &= T_2 \end{aligned} \;\Rightarrow\; u = \frac{T_2 - T_1}{L} x + T_1.$$

Solving the Heat Equation
Case 2b: inhomogeneous Dirichlet boundary conditions

Now consider the heat problem

$$\begin{aligned} u_t &= c^2 u_{xx} && (0 < x < L,\ t > 0), \\ u(0,t) &= T_1,\ u(L,t) = T_2 && (t > 0), \\ u(x,0) &= f(x) && (0 < x < L). \end{aligned}$$

Step 1: Let u_1 denote the steady state solution from above:

$$u_1 = \frac{T_2 - T_1}{L} x + T_1.$$

Step 2: Let $u_2 = u - u_1$.

Remark: By superposition, u_2 still solves the heat equation.

The boundary and initial conditions satisfied by u_2 are

$$\begin{aligned} u_2(0,t) &= u(0,t) - u_1(0) = T_1 - T_1 = 0, \\ u_2(L,t) &= u(L,t) - u_1(L) = T_2 - T_2 = 0, \\ u_2(x,0) &= f(x) - u_1(x). \end{aligned}$$

Step 3: Solve the heat equation with homogeneous Dirichlet boundary conditions and initial conditions above. This yields u_2.

Step 4: Assemble $u(x,t) = u_1(x) + u_2(x,t)$.

Remark: According to our earlier work, $\lim_{t\to\infty} u_2(x,t) = 0$.

- We call $u_2(x,t)$ the *transient* portion of the solution.
- We have $u(x,t) \to u_1(x)$ as $t \to \infty$, i.e. the solution tends to the steady state.

Example

Solve the heat problem.

$$\begin{aligned} u_t &= 3u_{xx} && (0 < x < 2,\ t > 0),\\ u(0,t) &= 100,\quad u(2,t) = 0 && (t > 0),\\ u(x,0) &= 50 && (0 < x < 2). \end{aligned}$$

We have $c = \sqrt{3}$, $L = 2$, $T_1 = 100$, $T_2 = 0$ and $f(x) = 50$. The steady state solution is

$$u_1 = \frac{0 - 100}{2} x + 100 = 100 - 50x.$$

The corresponding homogeneous problem for u_2 is thus

$$\begin{aligned} u_t &= 3u_{xx} && (0 < x < 2,\ t > 0),\\ u(0,t) &= u(2,t) = 0 && (t > 0),\\ u(x,0) &= 50 - (100 - 50x) = 50(x - 1) && (0 < x < 2). \end{aligned}$$

According to exercise 2.3.7 (with $p = L = 2$), the sine series for $50(x-1)$ is

$$\frac{-100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{2k\pi x}{2}\right),$$

i.e. only *even* modes occur. Since $\lambda_{2k} = \frac{c2k\pi}{L} = \sqrt{3}k\pi$,

$$u_2(x,t) = \frac{-100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-3k^2\pi^2 t}\sin(k\pi x).$$

Hence

$$u(x,t) = u_1(x)+u_2(x,t) = 100-50x-\frac{100}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-3k^2\pi^2 t}\sin(k\pi x).$$

UNIT-4

UNIT - II

VECTOR CALCULUS

INTRODUCTION

In this chapter we study the basics of vector calculus with the help of a standard vector differential operator. Also we introduce concepts like gradient of a scalar valued function, divergence and curl of a vector valued function, discuss briefly the properties of these concepts and study the applications of the results to the evaluation of line and surface integrals in terms of multiple integrals.

2.1 GRADIENT – DIRECTIONAL DERIVATIVE

Vector differential operator

The vector differential operator ∇ (read as Del) is denoted by $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes $OX, OY \text{ and } OZ$.

It is also called Hamiltonian operator and it is neither a vector nor a scalar, but it behaves like a vector.

The gradient of a scalar function

If $\varphi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of φ is defined as $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

It is also denoted as Grad φ.

Note

(i) $\nabla\varphi$ is a vector quantity.

(ii) $\nabla\varphi = 0$ if φ is constant.

(iii) $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

(iv) $\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{\varphi^2{}_2}$ if $\varphi_2 \neq 0$

(v) $\nabla(\varphi \pm \chi) = \nabla\varphi \pm \nabla\chi$

Problems based on Gradient

Example: 2.1 Find the gradient of φ where φ is $3x^2y - y^3z^2$ at $(1, -2, 1)$.

Solution:

Given $\varphi = 3x^2y - y^3z^2$

Grad $\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

Now $\frac{\partial\varphi}{\partial x} = 6xy$, $\frac{\partial\varphi}{\partial y} = 3x^2 - 3y^2z^2$, $\frac{\partial\varphi}{\partial z} = -2y^3z$

$\therefore$ grad $\varphi = \vec{i}\,6xy + \vec{j}(3x^2 - 3y^2z^2) - \vec{k}2y^3z$

$\therefore (grad\ \varphi)_{(1,-2,\ 1)} = -12\vec{i} - 9\vec{j} + 16\vec{k}$

Example: 2.2 If $\varphi = \log(x^2 + y^2 + z^2)$ then find $\nabla\varphi$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{2}{r^2}\vec{r}$$

Example: 2.3 Find $\nabla(r), \nabla\left(\frac{1}{r}\right), \nabla(\log r)$ where $r = |\vec{r}|$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla(r) = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{1}{r}\vec{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial\left(\frac{1}{r}\right)}{\partial x} + \vec{j}\frac{\partial\left(\frac{1}{r}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r}\right)}{\partial z}$

$$= \vec{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \vec{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \left(-\frac{1}{r^2}\right)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= -\frac{1}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = -\frac{1}{r^3}\vec{r}$$

(iii) $\nabla(\log r) = \sum\vec{i}\frac{\partial(\log r)}{\partial x}$

$$= \sum\vec{i}\frac{1}{r}\frac{\partial r}{\partial x}$$

$$= \sum\vec{i}\frac{1}{r}\frac{x}{r}$$

$$= \sum\vec{i}\frac{x}{r^2}$$

$$= \frac{1}{r^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{1}{r^2}\vec{r}$$

Example: 2.4 Prove that $\nabla(r^n) = nr^{n-2}\ \vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla(r^n) = \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\right]$$

$$= \frac{nr^{n-1}}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = nr^{n-2}\vec{r}$$

Example: 2.5 Find $|\nabla\varphi|$ if $\varphi = 2xz^4 - x^2y$ at $(2, -2, -1)$

Solution:

Given $\varphi = 2xz^4 - x^2y$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 2z^4 - 2xy$, $\frac{\partial\varphi}{\partial y} = -x^2$, $\frac{\partial\varphi}{\partial z} = 8xz^3$

$$\therefore \nabla\varphi = \vec{i}\,(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$

$$\therefore (\nabla\varphi)_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$|\nabla\varphi| = \sqrt{100 + 16 + 256} = \sqrt{372}$$

Directional Derivative (D.D) of a scalar point function

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar function φ in a given direction $\vec{a}$ is the rate of change of φ in that direction. It is given by the component of $\nabla\varphi$ in the direction of $\vec{a}$.

The directional derivative of a scalar point function in the direction of $\vec{a}$ is given by

$$\textbf{D.D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

The maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad}\ \varphi|$.

Problems based on Directional Derivative

Example: 2.6 Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1, -2, 1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given $\varphi = 4xz^2 + x^2yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xyz + 4z^2) + \vec{j}\,(x^2z) + \vec{k}\,(x^2y + 8xz)$$

$$\therefore (\nabla\varphi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Given $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$$|\vec{a}| = \sqrt{4 + 1 + 4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \left(8\vec{i} - \vec{j} - 10\vec{k}\right) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}\,(16 + 1 + 20) = \frac{37}{3}$$

Example: 2.7 Find the directional derivative of $\varphi(x, y, z) = xy^2 + yz^3$ at the point P$(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$

Solution:

Given $\varphi = xy^2 + yz^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2) + \vec{j}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$\therefore (\nabla\varphi)_{(2,-1,\ 1)} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Given $\vec{a} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \left(3\vec{i} + \vec{j} + 3\vec{k}\right) - \left(2i - \vec{j} + \vec{k}\right)$$

$$= \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1 + 4 + 4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{\left(\vec{i} - 3\vec{j} - 3\vec{k}\right) \cdot \left(\vec{i} + 2\vec{j} + 2\vec{k}\right)}{3}$$

$$= \frac{1}{3}\,(1 - 6 - 6) = -\frac{11}{3}$$

Example: 2.8 In what direction from $(-1, 1, 2)$ is the directional derivative of $\varphi = xy^2\,z^3$ a maximum? Find also the magnitude of this maximum.

Solution:

Given $\varphi = xy^2\,z^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2z^3) + \vec{j}\,(2xy\,z^3) + \vec{k}\,(3xy^2z^2)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 2)} = 8\vec{i} - 16\vec{j} - 12\vec{k}$$

The maximum directional derivative occurs in the direction of $\nabla\varphi = 8\vec{i} - 16\vec{j} - 12\vec{k}$.

∴ The magnitude of this maximum directional derivative

$$|\nabla\varphi| = \sqrt{64 + 256 + 144} = \sqrt{464}$$

Example: 2.9 Find the directional derivative of the scalar function $\varphi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point$(3, 1, 3)$.

Solution:

Given $\varphi = xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$\therefore (\nabla \varphi)_{(3,\ 1,\ 3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$

Given surface is $z = xy \Rightarrow z - xy = 0$

$\nabla\chi = \vec{i}\frac{\partial\chi}{\partial x} + \vec{j}\frac{\partial\chi}{\partial y} + \vec{k}\frac{\partial\chi}{\partial z}$

$= \vec{i}\,(-y) + \vec{j}\,(-x) + \vec{k}\,(1)$

Let $\vec{a} = \nabla\chi_{(3,1,3)} = -\vec{i} - 3\vec{j} + \vec{k}$

$\Rightarrow |\vec{a}| = \sqrt{1+9+1} = \sqrt{11}$

D. D $= \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$

$= \frac{(3\vec{i}+9\vec{j}+3\vec{k})\cdot(-\vec{i}-3\vec{j}+\vec{k})}{\sqrt{11}}$

$= \frac{1}{\sqrt{11}}\,(-3-27+3) = -\frac{27}{\sqrt{11}}$

Example: 2.10 Find the directional derivative of $\varphi = xy + yz + zx$ at $(1, 2, 0)$ in the direction of $\vec{i} + 2\,\vec{j} + 2\vec{k}$. Find also its maximum value.

Solution:

Given $\varphi = xy + yz + zx$

$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$= \vec{i}\,(y+z) + \vec{j}\,(x+z) + \vec{k}\,(y+x)$

$\therefore (\nabla \varphi)_{(1,\ 2,\ 0)} = 2\vec{i} + \vec{j} + 3\vec{k}$

Given $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$|\vec{a}| = \sqrt{1+4+4} = 3$

D. D $= \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$

$= \frac{(2\vec{i}+\vec{j}+3\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})}{3}$

$= \frac{1}{3}\,(2+2+6) = \frac{10}{3}$

Maximum value is $|\nabla\varphi| = \sqrt{4+1+9} = \sqrt{14}$

Unit normal vector to the surface

If $\varphi(x, y, z)$ be a scalar function, then $\varphi(x, y, z) = c$ represents a surface and the unit normal vector to the surface φ is given by $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$

Normal Derivative $= |\nabla\varphi|$

Problems based on unit normal vector

Example: 2.11 Find the unit normal to the surface $x^2 + y^2 = z$ at the point $(1, -2, 5)$.

Solution:

Given $\varphi = x^2 + y^2 - z$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\,\varphi)_{(1,-2,\ 5)} = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\varphi| = \sqrt{4+16+1} = \sqrt{21}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i}-4\vec{j}-\vec{k}}{\sqrt{21}}$

Example: 2.12 Find the unit normal to the surface $x^2 + xy + y^2 + xyz$ at the point $(1, -2, 1)$.

Solution:

Given $\varphi = x^2 + xy + y^2 + xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x + y + yz) + \vec{j}\,(x + 2y + xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla\,\varphi)_{(1,-2,\ 1)} = -2\vec{i} - 2\vec{j} - 2\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i}-2\vec{j}-2\vec{k}}{2\sqrt{3}}$

$$= \frac{-1}{\sqrt{3}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

Example: 2.13 Find the normal derivative to the surface $x^2y + xz^2$ at the point $(-1, 1, 1)$.

Solution:

Given $\varphi = x^2y + xz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xy + z^2) + \vec{j}\,(x^2) + \vec{k}\,(2xz)$$

$$\therefore (\nabla\,\varphi)_{(-1,\ 1,\ 1)} = -\vec{i} + \vec{j} - 2\vec{k}$$

Normal derivative $|\nabla\varphi| = \sqrt{1+1+4} = \sqrt{6}$

Example: 2.14 What is the greatest rate of increase of $\varphi = xyz^2$ at the point $(1, 0, 3)$.

Solution:

Given $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y\,z^2) + \vec{j}\,(xz^2) + \vec{k}\,(2xyz)$$

$$\therefore (\nabla\,\varphi)_{(1,\ 0,\ 3)} = 0\vec{i} + 9\vec{j} + 0\vec{k}$$

$\therefore$ Greatest rate of increase $|\nabla\varphi| = \sqrt{9^2} = 9$

Angle between the surfaces

$$\cos\theta = \frac{\nabla\,\varphi_1 \cdot \nabla\,\varphi_2}{|\nabla\varphi_1\,||\nabla\varphi_2|}$$

$\Rightarrow \theta = cos^{-1}\left[\frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}\right]$

Problems based on angle between two surfaces

Example: 2.15 Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, 2)$.

Solution:

Given $\varphi = x^2 + y^2 - z - 3$

$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$

$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$

$\therefore (\nabla\varphi_1)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} - \vec{k}$

$|\nabla\varphi_1| = \sqrt{16+4+1} = \sqrt{21}$

$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(2z)$

$\therefore (\nabla\varphi_2)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$

$|\nabla\varphi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$

The angle between the surfaces is $\cos\theta = \frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

$= \frac{(4\vec{i}-2\vec{j}-\vec{k})\,(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}(6)}$

$= \frac{16+4-4}{\sqrt{21}(6)}$

$= \frac{16}{\sqrt{21}(6)} = \frac{8}{3\sqrt{21}}$

$\Rightarrow \theta = cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$

Example: 2.16 Find the angle between the normals to the surfaces $x^2 = yz$ at the point $(1, 1, 1)$ *and* $(2, 4, 1)$.

Solution:

Given $\varphi = x^2 - yz$

$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$= \vec{i}\,(2x) + \vec{j}\,(-z) + \vec{k}\,(-y)$

$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 2\vec{i} - \vec{j} - \vec{k}$

$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6}$

$\therefore (\nabla\varphi_2)_{(2,\ 4,\ 1)} = 4\vec{i} - \vec{j} - 4\vec{k}$

$|\nabla\varphi_2| = \sqrt{16+1+16} = \sqrt{33}$

The angle between the surfaces is $\cos\theta = \frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$

$$= \frac{(2\vec{i}-\vec{j}-\vec{k})\,(4\vec{i}-\vec{j}-4\vec{k})}{\sqrt{6}\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}}$$

$$= \frac{13}{\sqrt{2(3)}\sqrt{11(3)}} = \frac{13}{3\sqrt{22}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{13}{3\sqrt{22}}\right]$$

Example: 2.17 Find the angle between the surfaces $x\,log\,z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1, 1, 1)$.

Solution:

Given $\varphi_1 = y^2 - x\,log\,z - 1$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(-\log z) + \vec{j}\,(2y) + \vec{k}\left(-\frac{x}{z}\right)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 0\vec{i} + 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{0+4+1} = \sqrt{5}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2xy) + \vec{j}\,(x^2) + \vec{k}\,(1)$$

$$\therefore (\nabla\varphi_2)_{(1,\ 1,\ 1)} = 2\vec{i} + \vec{j} + \vec{k}$$

$$|\nabla\varphi_2| = \sqrt{4+1+1} = \sqrt{6}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\ |\nabla\varphi_2|}$

$$= \frac{(0\vec{i}+2\vec{j}-\vec{k})\cdot(2\vec{i}+\vec{j}+\vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{0+2-1}{\sqrt{30}}$$

$$= \frac{1}{\sqrt{30}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$$

Problems based on orthogonal surfaces

Two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

Example: 2.18 Find a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.

Solution:

Given $ax^2 - byz = (a+2)x$

Let $\varphi_1 = ax^2 - byz - (a+2)x$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$= \vec{i}\left(2ax - (a+2)\right) + \vec{j}\,(-bz) + \vec{k}\,(-by)$

$\therefore (\nabla \varphi_1)_{(1,-1,\ 2)} = \vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)$

Let $\varphi_2 = 4x^2y + z^3 - 4$

$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(3z^2)$

$\therefore (\nabla \varphi_2)_{(1,-1,\ 2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$

Since the two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

$\Rightarrow \left(\vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)\right) \cdot \left(-8\vec{i} + 4\vec{j} + 12\vec{k}\,\right) = 0$

$\Rightarrow -8\,(a-2) - 8b + 12b = 0$

$\Rightarrow -8a + 16 - 8b + 12b = 0$

$\Rightarrow -8a + 16 + 4b = 0$

$\div$ by 4 $\Rightarrow -2a + 4 + b = 0$

$\Rightarrow 2a - b - 4 = 0 \ \ldots (1)$

To find a and b we need another equation in a and b.

The point $(1, -1, 2)$ lies in $ax^2 - byz - (a+2)x = 0$

$\therefore a - b\,(-1)\,(2) - (a+2)\,(1) = 0$

$\Rightarrow a + 2b - a - 2 = 0$

$\Rightarrow 2b - 2 = 0$

$\Rightarrow b = 1$

Substitute $b = 1$ in (1) we get

$\Rightarrow 2a - 1 - 4 = 0$

$\Rightarrow 2a - 5 = 0$

$\Rightarrow a = \frac{5}{2}$

Example: 2.19 Find the values of a and b so that the surfaces $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2, -1, -3)$.

Solution:

Given $ax^3 - by^2z = (a+3)x^2$

Let $\varphi_1 = ax^3 - by^2z - (a+3)x^2$

$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$

$= \vec{i}\left(3ax^2 - 2x(a+3)\right) + \vec{j}\,(-2byz) + \vec{k}\,(-by^2)$

$\therefore (\nabla \varphi_1)_{(2,-1,-3)} = \vec{i}(8a - 12) + \vec{j}\,(-6b) + \vec{k}(-b)$

Let $\varphi_2 = 4x^2y - z^3 - 11$

$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$

$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(-3z^2)$

$\therefore (\nabla\varphi_2)_{(2,-1,-3)} = -16\vec{i} + 16\vec{j} - 27\vec{k}$

Given the two surfaces cut orthogonally if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

$\Rightarrow \left(\vec{i}(8a-12) + \vec{j}\,(-6b) - \vec{k}(b)\right) \cdot \left(-16\vec{i} + 16\vec{j} - 27\vec{k}\right) = 0$

$\Rightarrow -16\,(8a-12) - 16(6b) + 27b = 0$

$\Rightarrow -128a + 192 - 69b = 0$

$\Rightarrow 128a + 69b - 192 = 0 \;\dots (1)$

To find a and b we need another equation in a and b.

The point $(2,-1,-3)$ lies in $ax^3 - by^2z - (a+3)x^2 = 0$

$\therefore 8a - b\,(1)\,(-3) - (a+3)\,(4) = 0$

$\Rightarrow 4a + 3b - 12 = 0 \dots (2)$

Solving (1) and (2) we get, $a = -\frac{7}{3}$ & $b = \frac{64}{9}$

Equation of the tangent plane and normal to the surface

Equation of the tangent plane is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

Equation of the normal line is $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

Problems based on tangent plane

Example: 2.20 Find the equation of the tangent plane and normal line to the surface $xyz = 4$ at the point $\vec{i} + 2\vec{j} + 2\vec{k}$.

Solution:

Given $\varphi = xyz - 4$

$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$

$\therefore (\nabla\varphi)_{(1,\ 2,\ 2)} = 4\vec{i} + 2\vec{j} + 2\vec{k}$

Equation of the tangent plane at the point $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$ is $(\vec{r} - \vec{a}) \cdot \nabla\varphi = 0$

$\Rightarrow \left[\left(x\vec{i} + y\vec{j} + z\vec{k}\right) - \vec{i} + 2\vec{j} + 2\vec{k}\right] \cdot \left(4\vec{i} + 2\vec{j} + 2\vec{k}\right) = 0$

$\Rightarrow \left[(x-1)\,\vec{i} + (y-2)\,\vec{j} + (z-2)\,\vec{k}\right] \cdot \left(4\vec{i} + 2\vec{j} + 2\vec{k}\right) = 0$

$\Rightarrow 4(x-1) + 2(y-2) + 2(z-2) = 0$

$\Rightarrow 4x - 4 + 2y - 4 + 2z - 4 = 0$

$\Rightarrow 4x + 2y + 2z = 12$

$\Rightarrow 2x + y + z = 6$

Equation of the normal line $(\vec{r} - \vec{a}) \times \nabla\varphi = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\,[2\,(y-2) - 2\,(z-2)] - \vec{j}\,[2\,(x-1) - 4\,(z-2)] + \vec{k}\,[2\,(x-1) - 4\,(y-2)]$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\vec{k}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ we get

$$\Rightarrow 2\,(y-2) - 2\,(z-2) = 0$$

$$\Rightarrow (y-2) = (z-2) \quad \ldots (1)$$

$$\Rightarrow 2\,(x-1) - 4\,(z-2) = 0$$

$$\Rightarrow (x-1) = 2\,(z-2)$$

$$\Rightarrow \frac{x-1}{2} = (z-2) \qquad \ldots (2)$$

$$\Rightarrow 2\,(x-1) - 4\,(y-2) = 0$$

$$\Rightarrow (x-1) = 2\,(y-2)$$

$$\Rightarrow \frac{x-1}{2} = (y-2) \qquad \ldots (3)$$

From $(1), (2)$ *and* (3) we get $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$

Which is the required equation of the normal line.

Exercise: 2.1

1. Find $\nabla\varphi$ if $\varphi = \frac{1}{2}\log(x^2 + y^2 + z^2)$ **Ans:** $\frac{\vec{r}}{r^2}$

2. Find the directional derivative of

   (i) $\varphi = 2xy + z^2$ at the point $(1, -1, 3)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$. **Ans:** $\frac{14}{3}$

   (ii) $\varphi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$. **Ans:** $\frac{-11}{3}$

3. Prove that the directional derivative of $\varphi = x^3y^2z$ at $(1, 2, 3)$is maximum along the direction $9\vec{i} + 3\vec{j} + \vec{k}$. Also, find the maximum directional derivative. **Ans:** $4\sqrt{91}$

4. Find the unit tangent vector to the curve $\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2 - 65)\vec{k}$ at $t = 1$. **Ans:** $\frac{\vec{i}+2\vec{j}-\vec{k}}{\sqrt{6}}$

5. Find a unit normal to the following surfaces at the specified points.

   (i) $x^2y + 2xz = 4$ at $(2, -2, 3)$ **Ans:** $\pm\frac{1}{3}(\vec{i} - 2\vec{j} - 2\vec{k})$

   (ii) $x^2 + y^2 = z$ at $(1, -2, 5)$ **Ans:** $\frac{1}{\sqrt{21}}(2\vec{i} - 4\vec{j} - \vec{k})$

   (ii) $xy^3z^2 = 4$ at $(-1, -1, 2)$ **Ans:** $\frac{1}{\sqrt{11}}(-\vec{i} - 3\vec{j} + \vec{k})$

   (iv) $x^2 + y^2 = z$ at $(1, 1, 2)$ **Ans:** $\frac{1}{3}(2\vec{i} + 2\vec{j} - \vec{k})$

6. Find the angle between the surfaces $x^2 - y^2 - z^2 = z$ and $xy + yz - zx - 18 = 0$ at the

point $(6,4,3)$. **Ans:** $cos^{-1}\left[\frac{-24}{\sqrt{86}\sqrt{61}}\right]$

7. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1,-2,1)$. **Ans:** $cos^{-1}\left[\frac{-3}{7\sqrt{6}}\right]$

8. Find the equation of the tangent plane to the surfaces $2xz^2 - 3xy - 4x = 7$ at the point $(1,-1,2)$. **Ans:** $7x - 3y + 8z - 26 = 0$

9. Find the equation of the tangent plane to the surfaces $xz^2 + x^2y = z - 1$ at the point $(1,-3,2)$. **Ans:** $2x - y - 3z + 1 = 0$

10. Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1,1,1)$. **Ans:** $cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$

2.2 DIVERGENCE, CURL – IRROTATIONAL AND SOLENOIDAL VECTORS

Divergence of a vector function

If $\vec{F}(x,y,z)$ is a continuously differentiable vector point function in a given region of space, then the divergence of $\vec{F}$ is defined by

$$\nabla.\vec{F} = \text{div}\,\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \text{where} \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Note: $\nabla.\vec{F}$ Is a scalar point function.

Solenoidal vector

A vector $\vec{F}$ is said to be solenoidal if $div\,\vec{F} = 0$ (i.e)$\nabla.\vec{F} = 0$

Curl of a vector function

If $\vec{F}(x,y,z)$ is a differentiable vector point function defines at each point (x,y,z) in some region of space, then the curl of $\vec{F}$ is defined by

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Note: $\nabla \times \vec{F}$ Is a vector point function.

Irrotational vector

A vector is said to be irrotational if Curl $\vec{F} = 0$ $(i.e)$ $\nabla \times \vec{F} = 0$

Scalar potential

If $\vec{F}$ is an irrotational vector, then there exists a scalar function φ such that $\vec{F} = \nabla\varphi$. Such a scalar function is called scalar potential of $\vec{F}$.

Problems based on Divergence and Curl of a vector

Example: 2.21 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then find div $\vec{r}$ and curl$\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now div $\vec{r} = \nabla \cdot \vec{r}$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

And curl $\vec{r} = \nabla \times \vec{r}$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right) - \vec{j}\left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}.$$

Example: 2.22 If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find $\nabla.\vec{F}$ and $\nabla \times \vec{F}$ at the point (1,-1, 1).

Solution:

Given $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$

(i) $\nabla.\vec{F} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2)$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla.\vec{F}_{(1,-1,1)} = 1 + 2 + 6 = 9$$

(ii) $$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & 3yz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial(-3yz^2)}{\partial y} - \frac{\partial(2x^2yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(-3yz^2)}{\partial x} - \frac{\partial(xy^2)}{\partial z}\right] + \vec{k}\left[\frac{\partial(2x^2yz)}{\partial x} - \frac{\partial(xy^2)}{\partial y}\right]$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0) + \vec{k}(4xyz - 2xy)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = \vec{i}(-3 + 2) + \vec{k}(-4 + 2)$$

$$= -\vec{i} - 2\vec{k}$$

Example: 2.23 If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, then find $\nabla \cdot \vec{F}$, $\nabla(\nabla \cdot \vec{F})$, $\nabla \times \vec{F}$, $\nabla \cdot (\nabla \times \vec{F})$, and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$

$= (2x + 2z) + (-x + z) + 2z$

$= x + 5z$

$\therefore \nabla \cdot \vec{F}_{(1,1,1)} = 6$

(ii) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$

$= \vec{i}\left[\frac{\partial(z^2+x^2)}{\partial y} - \frac{\partial(xz-xy+yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(z^2+x^2)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial z}\right] + \vec{k}\left[\frac{\partial(xz-xy+yz)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial y}\right]$

$= -(x + y)\vec{\imath} - (2x - 2x)\vec{\jmath} + (y + z)\vec{k}$

$\therefore \nabla \times \vec{F}_{(1,1,1)} = -2\vec{\imath} + 2\vec{k}$

(iii) $\nabla(\nabla \cdot \vec{F}) = \vec{i}\frac{\partial}{\partial x}(x + 5z) + \vec{j}\frac{\partial}{\partial y}(x + 5z) + \vec{k}\frac{\partial}{\partial z}(x + 5z)$

$= \vec{\imath} + 5\vec{k}$

$\therefore \nabla(\nabla \cdot \vec{F})_{(1,1,1)} = \vec{\imath} + 5\vec{k}$

(iv) $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x + y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y + z)$

$= -1 + 0 + 1$

$\nabla \cdot (\nabla \times \vec{F})_{(1,\,1,\,1)} = 0$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x + y) & 0 & y + z \end{vmatrix}$

$\therefore \nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{\imath} + \vec{k}$

Example: 2.24 Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F}$ = grad($x^3+y^3+z^3-3xyz$)

Solution:

Given $\vec{F}$ = grad($x^3+y^3+z^3-3xyz$)

$= \vec{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \vec{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz) + \vec{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$

$\vec{F} = \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$

Now div $\vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$

$= 6x + 6y + 6z$

$= 6(x + y + z)$

Curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$

$= \vec{i}[-3x + 3x] - \vec{j}[-3y + 3y] + \vec{k}\,[-3z + 3z]$

$= \vec{0}$

Example: 2.25 Find div(grad φ) and curl(grad φ) at (1,1,1) for $\varphi = x^2y^3z^4$

Solution:

Given $\varphi = x^2y^3z^4$

$$\text{grad } \varphi = \nabla \varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xy^3z^4) + \vec{j}(x^2 3y^2z^4) + \vec{k}(x^2y^3 4z^3)$$

$$\text{Div(grad } \varphi) = \nabla \cdot (\text{grad } \varphi)$$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(x^2 3y^2z^4) + \frac{\partial}{\partial z}(x^2y^3 4z^3)$$

$$= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^4$$

$$\therefore \text{Div(grad } \varphi)_{(1,1,1)} = 2 + 6 + 12 = 20$$

$$\text{Curl(grad } \varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & x^2 3y^2z^4 & x^2y^3 4z^3 \end{vmatrix}$$

$$= \vec{i}(12x^2y^2z^3 - 12x^2y^2z^3) - \vec{j}(8xy^3z^3 - 8xy^3z^3) + \vec{k}(6xy^2z^4 - 6xy^2z^4)$$

$$= \vec{0}$$

$\therefore \text{Curl grad}\varphi_{(1,1,1)} = \vec{0}$

Vector Identities

1) $\nabla \cdot (\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$
2) $\nabla \times (\varphi \vec{F}) = \varphi(\nabla \times \vec{F}) + (\nabla\varphi) \times \vec{F}$
3) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
4) $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$
5) $\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) - (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) - (\vec{B} \cdot \nabla)\vec{A}$
6) $\nabla \cdot (\nabla\varphi) = \vec{0}$
7) $\nabla \cdot (\nabla \times \vec{F}) = 0$
8) $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$
9) $\nabla \cdot \nabla\varphi = (\nabla \cdot \nabla)\varphi = \nabla^2\varphi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a laplacian operator

1) If φ is a scalar point function, $\vec{F}$ is a vector point function, then $\nabla.(\varphi \vec{F}) = \varphi(\nabla \cdot \vec{F}) + \vec{F} \cdot \nabla\varphi$

Proof:

$$\nabla \cdot (\varphi \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \frac{\partial}{\partial x}(\varphi \vec{F})$$

$$= \sum \vec{i} \cdot \left(\varphi\frac{\partial\vec{F}}{\partial x} + \vec{F}\frac{\partial\varphi}{\partial x}\right)$$

$$= \varphi\left(\sum \vec{i}\cdot\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right) + \vec{F}\cdot\left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right)$$

$$\therefore \nabla\cdot(\varphi\,\vec{F}) = \varphi\left(\nabla\cdot\vec{F}\right) + \vec{F}\cdot\nabla\varphi$$

2) If $\boldsymbol{\varphi}$ is a scalar point fuction, $\vec{\mathbf{F}}$ is a vector point function, then $\nabla\times(\boldsymbol{\varphi}\,\vec{\mathbf{F}}) = \boldsymbol{\varphi}\left(\nabla\times\vec{\mathbf{F}}\right) + (\nabla\boldsymbol{\varphi})\times\vec{\mathbf{F}}$

Proof:

$$\nabla\times(\varphi\,\vec{F}) = \sum \vec{i}\times\frac{\partial}{\partial x}(\varphi\,\vec{F})$$

$$=\sum \vec{i}\times\left[\varphi\frac{\partial \vec{F}}{\partial x} + \vec{F}\frac{\partial \varphi}{\partial x}\right]$$

$$=\sum \vec{i}\times\left(\frac{\partial \varphi}{\partial x}\vec{F} + \varphi\frac{\partial \vec{F}}{\partial x}\right)$$

$$=\left(\sum \vec{i}\frac{\partial \varphi}{\partial x}\right)\times\vec{F} + \varphi\left[\sum \vec{i}\times\frac{\partial \vec{F}}{\partial x}\right]$$

$$\therefore \nabla\times(\varphi\,\vec{F}) = \nabla\varphi\times\vec{F} + \varphi(\nabla\times\vec{F})$$

3) If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are vector point functions, then $\nabla\cdot(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}\cdot(\nabla\times\vec{\mathbf{A}}) - \vec{\mathbf{A}}\cdot(\nabla\times\vec{\mathbf{B}})$

Proof:

$$\nabla.(\vec{A}\times\vec{B}) = \sum \vec{i}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$=\sum \vec{i}\cdot\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)$$

$$=\sum \vec{i}\cdot\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right) + \sum \vec{i}\cdot\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right)$$

$$= -\left(\sum \vec{i}\times\frac{\partial \vec{B}}{\partial x}\right)\cdot\vec{A} + \left(\sum \vec{i}\times\frac{\partial \vec{A}}{\partial x}\right).\vec{B}$$

$$= -\left(\nabla\times\vec{B}\right).\vec{A} + (\nabla\times\vec{A})\cdot\vec{B}$$

$$\therefore \nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \qquad \left[\because \left(\nabla\times\vec{A}\right)\cdot\vec{B} = \vec{B}\cdot(\nabla\times\vec{A})\right]$$

(4) If $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are vector point functions, then

$\nabla\times(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{A}}(\nabla\cdot\vec{\mathbf{B}}) - \vec{\mathbf{B}}(\nabla\cdot\vec{\mathbf{A}}) + \left(\vec{\mathbf{B}}\cdot\nabla\right)\vec{\mathbf{A}} - (\vec{\mathbf{A}}\cdot\nabla)\vec{\mathbf{B}}$

Proof:

$$\nabla\times(\vec{A}\times\vec{B}) = \sum \vec{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$=\sum \vec{i}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B} + \vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)$$

$$=\sum \vec{i}\times\left(\frac{\partial \vec{A}}{\partial x}\times\vec{B}\right) + \sum \vec{i}\times\left(\vec{A}\times\frac{\partial \vec{B}}{\partial x}\right)$$

We know that $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = (\vec{a}\cdot\vec{c})\,\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$$\nabla\times(\vec{A}\times\vec{B}) = \sum\left[\left(\vec{i}\cdot\vec{B}\right)\frac{\partial \vec{A}}{\partial x} - \left(\vec{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right] + \sum\left[\left(\vec{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\vec{i}\cdot\vec{A}\right)\frac{\partial \vec{B}}{\partial x}\right]$$

$$=\left(\sum \vec{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \sum\left(\vec{B}\cdot\vec{i}\right)\frac{\partial \vec{A}}{\partial x} - \sum\left(\vec{A}\cdot\vec{i}\right)\frac{\partial \vec{B}}{\partial x}$$

$$=\left(\sum \vec{i}\cdot\frac{\partial \vec{B}}{\partial x}\right)\vec{A} - \left(\sum \vec{i}\cdot\frac{\partial \vec{A}}{\partial x}\right)\vec{B} + \left(\vec{B}\cdot\sum \vec{i}\frac{\partial}{\partial x}\right)\vec{A} - \left(\vec{A}\cdot\sum \vec{i}\frac{\partial}{\partial x}\right)\vec{B}$$

$$\therefore \nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + \left(\vec{B}\cdot\nabla\right)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$$

(5) If $\vec{A}$ and $\vec{B}$ are vector point functions, then

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{B}\cdot\nabla)\vec{A}$$

Proof:

$$\nabla(\vec{A}\cdot\vec{B}) = \sum \vec{i}\,\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$$

$$= \sum \vec{i}\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{B} + \vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum \vec{i}\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right) + \sum \vec{i}\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)$$

$$= \sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{i} + \sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} \qquad \dots (1)$$

We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

$\therefore (\vec{a}\cdot\vec{b})\,\vec{c} = (\vec{a}\cdot\vec{c})\,\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$

Consider $\sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{i} = \sum\left[(\vec{B}\cdot\vec{i})\frac{\partial\vec{A}}{\partial x} - \vec{B}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{i}\right)\right]$

$$= \sum\left(\vec{B}\cdot\vec{i}\,\frac{\partial}{\partial x}\right)\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\,\vec{A} + \sum\left[\vec{B}\times\left(\vec{i}\,\frac{\partial}{\partial x}\times\vec{A}\right)\right]$$

$$= (\vec{B}\cdot\nabla)\,\vec{A} + \vec{B}\times(\nabla\times\vec{A}) \qquad \dots (2)$$

In (2) interchanging $\vec{A}$ and $\vec{B}$ we get,

$$\sum\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{i} = (\vec{A}\cdot\nabla)\,\vec{B} + \vec{A}\times(\nabla\times\vec{B}) \qquad \dots (3)$$

Substitute in equation (1)

$$(1) \Rightarrow \nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\,\vec{A} + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\,\vec{B} + \vec{A}\times(\nabla\times\vec{B})$$

(6) If φ is a scalar point function, then $\nabla\times(\nabla\varphi) = \vec{0}$.

(or)

Prove that $curl(grad\ \varphi) = 0$.

Solution:

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum\vec{i}\left[\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right]$$

$$= \sum\vec{i}\,(\vec{0}) = \vec{0}$$

(7) If $\vec{F}$ is a vector point function, then $\nabla\cdot(\nabla\times\vec{F}) = 0$.

(or)

Prove that $\boldsymbol{div(curl\,\vec{F}) = 0}$.

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot$$

$$\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_1}{\partial y\partial z} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$$= 0$$

(8) If is a vector point function, then $\boldsymbol{\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}}$

(or)

Prove that $\boldsymbol{curl\,(curl\,\vec{F}) = grad\,(div\,\vec{F}) - \nabla^2\vec{F}}$

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times (\nabla \times \vec{F}) = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

And $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

L.H.S $\quad \nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial y\partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_3}{\partial z\partial x} + \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial z\partial y} + \frac{\partial^2 F_2}{\partial z^2}\right]$$

$$+\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y\partial z}\right]$$

R.H.S $\quad \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x\partial y} + \frac{\partial^2 F_3}{\partial x\partial z}\right] + \vec{j}\left[\frac{\partial^2 F_1}{\partial y\partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y\partial z}\right] + \vec{k}\left[\frac{\partial^2 F_1}{\partial z\partial x} + \frac{\partial^2 F_2}{\partial z\partial y} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial x\partial y} + \frac{\partial^2 F_3}{\partial x\partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial z\partial y} + \frac{\partial^2 F_2}{\partial z^2}\right] +$$

$$\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y\partial z}\right]$$

L.H.S = R.H.S

$$\therefore \nabla \times \left(\nabla \times \vec{F}\right) = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla^2 \vec{F}$$

(9) $\boldsymbol{\nabla \cdot (\nabla \varphi) = (\nabla \cdot \nabla)\, \varphi = \nabla^2 \varphi}$

Proof:

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\therefore \nabla \cdot (\nabla\varphi) = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right)$$

$$= \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot (\nabla\varphi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = \nabla^2\varphi$$

Example: 2.26 Find (i) $\boldsymbol{\nabla \cdot \vec{r}}$ **(ii)** $\boldsymbol{\nabla \times \vec{r}}$

Solution:

Let $\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$

(i) $\nabla \cdot \vec{r} = \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{\imath}(0) + \vec{\jmath}\,(0) + \vec{k}\,(0) = \vec{0}$$

Example: 2.27 Find $\boldsymbol{\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right)}$ **where** $\boldsymbol{\vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}}$

Solution:

$$\nabla \cdot \left(\frac{1}{r}\,\vec{r}\right) = \nabla \cdot \left[\frac{1}{r}\left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)\right]$$

$$= \left(\vec{\imath}\,\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\,\vec{\imath} + \frac{y}{r}\,\vec{\jmath} + \frac{z}{r}\,\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum\left[\frac{1}{r}(1) + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right]$$

$$= \sum\left[\frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right)\right] \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

`

$$= \sum\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \frac{3}{r} - \frac{1}{r^3}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} \qquad \because r^2 = \left(x^2 + y^2 + z^2\right)$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

Example: 2.28 If is a constant vector and is the position vector of any point, prove that

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = 0$ (ii) $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$$\vec{a} = a_1\,\vec{i} + a_2\,\vec{j} + a_3\,\vec{k}$$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(a_2 z - a_3 y) - \vec{j}(a_1 z - a_3 x) + \vec{k}(a_1 y - a_2 x)$$

(i) $\nabla \cdot (\vec{a} \times \vec{r}) = \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(-a_1 z + a_3 x) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$

$$= 0 + 0 + 0 = 0$$

(ii) $\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & -a_1 z + a_3 x & a_1 y - a_3 x \end{vmatrix}$

`$= \vec{i}(a_1 + a_1) - \vec{j}(-a_2 - a_2) + \vec{k}(a_3 + a_3)$

$$= 2a_1\vec{i} + 2a_2\vec{j} + 2a_3\vec{k}$$

$$= 2\left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right) = 2\vec{a}$$

Example: 2.29 Prove that $curl(f(r)\vec{r}) = \vec{0}$

Solution:

Let $f(r)\vec{r} = f(r)\left[x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right]$

$$= xf(r)\vec{i} + yf(r)\vec{j} + zf(r)\vec{k}$$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{i}\left[zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[zf'(r)\left(\frac{y}{r}\right) - yf'(r)\left(\frac{z}{r}\right)\right]$$

$$= \sum \vec{i}\left[\frac{zy}{r}f'(r) - \frac{zy}{r}f'(r)\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

Example: 2.30 Prove that $curl[\varphi\,\nabla\varphi] = \vec{0}$

(or)

Prove that $\nabla \times [\varphi\,\nabla\varphi] = \vec{0}$

Solution:

$$\varphi\nabla\varphi = \varphi\left[\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right]$$

$$= \vec{i}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \vec{j}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \vec{k}\left(\varphi\frac{\partial\varphi}{\partial z}\right)$$

$$\nabla \times (\varphi\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial\varphi}{\partial x} & \varphi\frac{\partial\varphi}{\partial y} & \varphi\frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right]$$

$$= \sum \vec{i}\left[\varphi\frac{\partial^2\varphi}{\partial y\partial z} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z}\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

Example: 2.31 If $\vec{\omega}$ is a constant vector and $\vec{v} = \vec{\omega} \times \vec{r}$, then prove that $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$.

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$$\vec{\omega} = \omega_1\,\vec{i} + \omega_2\,\vec{j} + \omega_3\,\vec{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(\omega_2 z - \omega_3 y) - \vec{j}(\omega_1 z - \omega_3 x) + \vec{k}(\omega_1 y - \omega_2 x)$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \vec{i}(\omega_1 + \omega_1) - \vec{j}(-\omega_2 - \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1\vec{i} + 2\omega_2\vec{j} + 2\omega_3\vec{k}$$

$$= 2\left(\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}\right) = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$$

Problems based on solenoidal vector and irrotational vector and scalar potential

Example: 2.32 Prove that the vector $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.33 Show that the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

$\therefore\ \vec{F}$ is solenoidal.

Example: 2.34 If $\vec{F} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + \lambda z)\vec{k}$ is solenoidal, then find the value of λ.

Solution:

Given $\vec{F}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + \lambda z) = 0$$

$$\Rightarrow 1 + 1 + \lambda = 0$$

$$\therefore\ \lambda = -2$$

Example: 2.35 Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

Solution:

Given $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

$$(ie)\nabla \cdot \vec{F} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$$\Rightarrow 3 + a + 2 = 0$$

$$\therefore\ a = -5$$

Example: 2.36 Show that the vector $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution:

Given $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

To prove $curl\ \vec{F} = 0$

$$(i.e) To\ prove\ \nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - 3z^2 & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}(-1 + 1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

Example: 2.37 Find the constants a, b, c so that the vectors is irrotational

UNIT-5

$\vec{F} = (x + 2y + az)\vec{i} + (bx + 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ **.**

Solution:

Given $\vec{F} = (x + 2y + az)\vec{i} + (bx + 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ **is irrotational.**

$$(ie)\nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx + 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}(c + 1) - \vec{j}(4 - a) + \vec{k}(b - 2) = \vec{0}$$

$$\Rightarrow c + 1 = 0 \ ; \qquad 4 - a = 0 \ ; \qquad b - 2 = 0$$

$$\Rightarrow c = -1 \ ; \qquad 4 = a \ ; \qquad b = 2$$

Example: 2.38 Prove that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ **is irrotational and find** φ **such that** $\vec{F} = \nabla\varphi$**.**

Solution:

Given $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}(-1 + 1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}$ and $\vec{k}$ we get,

$$\frac{\partial\varphi}{\partial x} = 6xy + z^3; \qquad \frac{\partial\varphi}{\partial y} = 3x^2 - z; \qquad \frac{\partial\varphi}{\partial z} = 3xz^2 - y$$

Integrating the above equations partially with respect to x, y, z respectively

$$\varphi = 3x^2y + xz^3 + f_1(y, z)$$

$$\varphi = 3x^2y - yz + f_2(x, z)$$

$$\varphi = xz^3 - yz + f_3(x, y)$$

$\therefore \varphi = 3x^2y + xz^3 - yz + c$ where c is constant.

Example: 2.39 Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y\,\sin z - 4)\vec{j} + (3xz^2)\vec{k}$ **is irrotational and find its scalar potential.**

Solution:

Given $\vec{F} = (y^2 cos\, x + z^3)\vec{\imath} + (2y\, sinz - 4)\vec{\jmath} + (3xz^2)\vec{k}$

To prove $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 cos\, x + z^3 & 2y\, sinz - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{\imath}(0-0) - \vec{\jmath}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2ycos\, x)$$

$$= \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

To find φ such that $\vec{F} = \nabla\varphi$.

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = y^2\cos x + z^3;\qquad \frac{\partial\varphi}{\partial y} = 2ysin\, x - 4;\qquad \frac{\partial\varphi}{\partial z} = 3xz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = y^2\sin x + z^3x + f_1(y,z)$

$\varphi = y^2\sin x - 4y + f_2(x,z)$

$\varphi = xz^3 + f_3(x,y)$

$\therefore \varphi = y^2\sin x + z^3x - 4y + c$ is scalar potential.

Example: 2.40 Prove that $\vec{F} = (2x + yz)\vec{\imath} + (4y + zx)\vec{\jmath} + (6z - xy)\vec{k}$ is solenoidal as well as irrotational also find the scalar potential of $\vec{F}$.

Solution:

Given $\vec{F} = (2x + yz)\vec{\imath} + (4y + zx)\vec{\jmath} + (6z - xy)\vec{k}$

(i) To prove $\vec{F}$ is solenoidal.

$(ie)\ To\ prove\ \nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x + yz) + \frac{\partial}{\partial y}(4y + zx) + \frac{\partial}{\partial z}(-6z + xy)$$

$$= 2 + 4 - 6 = 0$$

$\therefore\ \vec{F}$ is solenoidal.

(ii) To prove $\vec{F}$ is irrotational.

$(ie)\ To\ prove\ \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + yz & 4y + zx & -6z + xy \end{vmatrix}$$

$$= \vec{\imath}(x - x) - \vec{\jmath}(y - y) + \vec{k}(z - z)$$

$= \vec{0}$

$\therefore\ \vec{F}$ is irrotational.

(iii) To find φ such that $\vec{F} = \nabla\varphi$.

$$(2x + yz)\vec{\imath} + (4y + zx)\vec{\jmath} + (6z - xy)\vec{k} = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Equating the coefficients of $\vec{\imath}, \vec{\jmath}$ and $\vec{k}$ we get,

$\frac{\partial\varphi}{\partial x} = 2x + yz;\quad \frac{\partial\varphi}{\partial y} = 4y + zx;\quad \frac{\partial\varphi}{\partial z} = -6z + xy$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2 + xyz + f_1(y,z)$

$\varphi = 2y^2 + xyz + f_2(x,z)$

$\varphi = -3z^2 + xyz + f_3(x,y)$

$\therefore \varphi = x^2 + 2y^2 - 3z^2 + xyz + c$ where c is a constant.

$\therefore\ \varphi$ is a scalar potential of $\vec{F}$.

Example: 2.41 If $\nabla\varphi = 2xyz^3\vec{\imath} + x^2z^3\vec{\jmath} + 3x^2yz^2\vec{k}$ find φ if $\varphi(-1,2,2) = 4$

Solution:

Given $\nabla\varphi = 2xyz^3\vec{\imath} + x^2z^3\vec{\jmath} + 3x^2yz^2\vec{k}$ ... (1)

We know that $\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$ ... (2)

Comparing (1) and (2)

$\frac{\partial\varphi}{\partial x} = 2xyz^3;\quad \frac{\partial\varphi}{\partial y} = x^2z^3;\quad \frac{\partial\varphi}{\partial z} = 3x^2yz^2$

Integrating the above equations partially with respect to x, y, z respectively

$\varphi = x^2yz^3 + f_1(y,z)$

$\varphi = x^2yz^3 + f_2(x,z)$

$\varphi = x^2yz^3 + f_3(x,y)$

$\therefore \varphi = x^2yz^3 + c$ where c is a constant.

Given $\varphi(-1,2,2) = 4$

$\Rightarrow 16 + c = 4$

$\Rightarrow c = -12$

$\therefore\ \varphi = x^2yz^3 - 12$

Example: 2.42 If $\vec{A}$ and $\vec{B}$ are irrotational, then prove that $\vec{A} \times \vec{B}$ is solenoidal.

Solution:

Given $\vec{A}$ and $\vec{B}$ are irrotational.

$(ie)\nabla \times \vec{A} = 0$ and $\nabla \times \vec{B} = 0$

We know that $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$

$= 0 \cdot \vec{A} - 0 \cdot \vec{B}$

$$= 0$$

Hence $\vec{A} \times \vec{B}$ is solenoidal.

Example: 2.43 if $\vec{A}$ is a constant vector, then prove that (i) $div\,\vec{A} = 0$ and (ii) $curl\,\vec{A} = 0$

Solution:

Let $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

$\frac{\partial A_1}{\partial x} = 0; \quad \frac{\partial A_2}{\partial y} = 0; \quad \frac{\partial A_3}{\partial z} = 0$

(i) $\nabla \cdot \vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

$= 0 + 0 + 0 = 0$

Hence $div\,\vec{A} = 0$.

(ii) $\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$

$= \vec{0}$

$\therefore\ curl\,\vec{F} = \vec{0}$

Example: 2.44 If φ and χ are differentiable scalar fields, prove $\nabla\varphi \times \nabla\chi$ is solenoidal.

Solution:

Consider $\nabla \cdot (\nabla\varphi \times \nabla\chi)$

$= \nabla\chi \cdot \nabla \times (\nabla\varphi) - \nabla\varphi \cdot [\nabla \times (\nabla\chi)] \qquad [\because \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})]$

$= \nabla\chi \cdot 0 - \nabla\varphi \cdot 0$

$= 0$

$\therefore \nabla\varphi \times \nabla\chi$ is solenoidal.

Example: 2.45 Find $f(r)$ if the vector $f(r)\vec{r}$ is both solenoidal and irrotational.

Solution:

(i) Given $f(r)\vec{r}$ is solenoidal.

$$\therefore\ \nabla \cdot (f(r)\vec{r}) = 0$$

We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore\ f(r)\vec{r} = f(r)\,x\vec{i} + f(r)\,y\vec{j} + f(r)\,z\vec{k}$

Now $\nabla \cdot (f(r)\vec{r}) = 0$

$\Rightarrow \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \cdot \left(f(r)\,x\,\vec{i} + f(r)\,y\,\vec{j} + f(r)\,z\,\vec{k}\right) = 0$

$\Rightarrow \frac{\partial}{\partial x}(f(r)\,x) + \frac{\partial}{\partial y}(f(r)\,y) + \frac{\partial}{\partial z}(f(r)\,z) = 0$

$\Rightarrow \sum \frac{\partial}{\partial x}(f(r)\,x) = 0$

$\Rightarrow \sum \left[f(r)\ \cdot 1 + x\, f'(r) \frac{\partial r}{\partial x}\right] = 0$

$\Rightarrow \sum \left[f(r) + x\, f'(r) \frac{x}{r}\right] = 0$

$\Rightarrow \sum \left[f(r) + \frac{x^2}{r}\ f'(r)\right] = 0$

$\Rightarrow 3f(r) + f'(r)\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] = 0$

$\Rightarrow 3f(r) + \frac{f'(r)}{r}[r^2] = 0 \quad [\because\ x^2 + y^2 + z^2 = r^2]$

$\Rightarrow 3f(r) + f'(r)\, r = 0$

$\Rightarrow f'(r)\, r = -3f(r)$

$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$

Integrating with respect to r, we get

$\Rightarrow \int \frac{f'(r)}{f(r)}\, dr = \int \frac{-3}{r}\, dr$

$\Rightarrow \log f(r) = -3 \log r + \log c$

$= \log r^{-3} + \log c$

$= \log\left(\frac{1}{r^3}\right) + \log c$

$= \log\left(\frac{c}{r^3}\right)$

$\therefore f(r) = \frac{c}{r^3}$

(ii) Given $f(r)\vec{r}$ is irrotational.

$$\nabla \times f(r)\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$= \sum \vec{i}\left[z \frac{\partial}{\partial y} f(r) - y \frac{\partial}{\partial z} f(r)\right]$

$= \sum \vec{i}\left[z f'(r) \frac{\partial r}{\partial y} - y\, f'(r) \frac{\partial r}{\partial z}\right]$

$= \sum \vec{i}\ \left[z f'(r) \frac{y}{r} - y\, f'(r) \frac{z}{r}\right]$

$= \sum \vec{i} f'(r)\ \left[\frac{zy}{r} - \frac{zy}{r}\right]$

$= \vec{0}$ for all $f(r)$

Example: 2.46 Prove that $r^n\vec{r}$ is irrotational for every n and solenoidal only for $n = -3$.

Solution:

We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore\ r^n\vec{r} = r^n\, x\vec{i} + r^n\, y\vec{j} + r^n\, z\vec{k}$

(i) To prove $r^n\vec{r}$ is irrotational.

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \sum \vec{i} \left[z\, nr^{n-1} \frac{\partial r}{\partial y} - y\, nr^{n-1} \frac{\partial r}{\partial z} \right]$$

$$= \sum \vec{i} \left[z\, nr^{n-1} \frac{y}{r} - y\, nr^{n-1} \frac{z}{r} \right]$$

$$= \sum \vec{i} \left[nr^{n-1} \frac{zy}{r} - nr^{n-1} \frac{zy}{r} \right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

$\therefore$ $r^n \vec{r}$ is irrotational for every n.

(ii) To prove $r^n \vec{r}$ is solenoidal.

$$\nabla \cdot (r^n \vec{r}) = \nabla \cdot \left(r^n x\vec{i} + r^n y\,\vec{j} + r^n z\,\vec{k} \right)$$

$$= \sum \frac{\partial}{\partial x} (r^n x)$$

$$= \sum \left[r^n (1) + x n r^{n-1} \frac{\partial r}{\partial x} \right]$$

$$= \sum \left[r^n + x n r^{n-1} \frac{x}{r} \right]$$

$$= \sum [r^n + x^2 n r^{n-2}]$$

$$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2)$$

$$= 3r^n + nr^{n-2}(r^2)$$

$$= 3r^n + nr^n$$

$$= r^n(3 + n)$$

When $n = -3$, we get $\nabla \cdot (r^n \vec{r}) = 0$

$\therefore$ $r^n \vec{r}$ is solenoidal only if $n = -3$.

Problems based on Laplace operator

Example: 2.47 Find $\nabla^2(\log r)$

Solution:

$$\nabla^2(\log r) = \sum \frac{\partial^2}{\partial x^2} (\log r)$$

$$= \sum \frac{\partial}{\partial x} \left(\frac{1}{r} \frac{\partial r}{\partial x} \right)$$

$$= \sum \frac{\partial}{\partial x} \left(\frac{1}{r^2}\, x \right)$$

$$= \sum \left[\frac{1}{r^2}(1) + x \left(-\frac{2}{r^3} \right) \frac{\partial r}{\partial x} \right]$$

$$= \sum \left[\frac{1}{r^2} - x \left(\frac{2}{r^3} \right) \frac{x}{r} \right]$$

$$= \sum \left[\frac{1}{r^2} - \frac{2x^2}{r^4} \right]$$

$$= \frac{3}{r^2} - \frac{2}{r^4} (x^2 + y^2 + z^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(r^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

Example: 2.48 Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ and hence deduce $\nabla^2\left(\frac{1}{r}\right)$.

(or)

Prove that $div\,(grad\, r^n) = n(n+1)\, r^{n-2}$

Solution:

Let $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$; $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n) = \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$= \sum \frac{\partial}{\partial x}\left[nr^{n-1}\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[nr^{n-1}\frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}[n\, xr^{n-2}]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{\partial r}{\partial x} + r^{n-2}(1)\right]$$

$$= \sum n\left[x(n-2)r^{n-3}\frac{x}{r} + r^{n-2}\right]$$

$$= \sum[n[(n-2)r^{n-4}x^2 + r^{n-2}]]$$

$$= \sum[n(n-2)r^{n-4}x^2 + n\, r^{n-2}]$$

$$= n(n-2)r^{n-4}(x^2 + y^2 + z^2) + 3\, n\, r^{n-2}$$

$$= n(n-2)r^{n-4}r^2 + 3\, n\, r^{n-2}$$

$$= n(n-2)r^{n-2} + 3\, n\, r^{n-2}$$

$$= n\, r^{n-2}(n-2+3)$$

$$= n\, r^{n-2}(n+1) \quad \ldots (1)$$

(ii) $\nabla^2\left(\frac{1}{r}\right) = \nabla^2(r^{-1})$

$$= (-1)(-1+1)\, r^{-1-2} \quad \text{by (1)}$$

$$= (-1)(0)\, r^{-3} = 0$$

Example: 2.49 Prove that $\nabla^2(r^n\vec{r}) = n\,(n+3)r^{n-2}\vec{r}$

Solution:

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Hence $\frac{\partial \vec{r}}{\partial x} = \vec{i}$; $\quad \frac{\partial \vec{r}}{\partial y} = \vec{j}$; $\quad \frac{\partial \vec{r}}{\partial z} = \vec{k}$

Also $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Hence $\frac{\partial r}{\partial x} = \frac{x}{r}$; $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla^2(r^n\vec{r}) = \sum \frac{\partial^2}{\partial x^2}(r^n\vec{r})$$

$$= \sum \frac{\partial}{\partial x}\left[r^n \frac{\partial \vec{r}}{\partial x} + n\, r^{n-1} \frac{\partial r}{\partial x}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\, \vec{i} + n\, r^{n-1} \frac{x}{r}\vec{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[r^n\, \vec{i} + n\, r^{n-2}\, x\, \vec{r}\right]$$

$$= \sum\left[n\, r^{n-1} \frac{\partial r}{\partial x}\vec{i} + n\left[r^{n-2}x\left(\frac{\partial \vec{r}}{\partial x}\right) + r^{n-2}(1)\vec{r} + \left[(n-2)\, r^{n-3} \frac{\partial r}{\partial x}\right] x\vec{r}\right]\right]$$

$$= \sum\left[n\, r^{n-1} \frac{x}{r}\vec{i} + n\, r^{n-2}\, x\vec{i} + n\, r^{n-2}\vec{r} + n(n-2)\, r^{n-3} \frac{x}{r}\, x\vec{r}\right]$$

$$= \sum[n\, r^{n-2}x\vec{i} + n\, r^{n-2}\, x\vec{i} + n\, r^{n-2}\vec{r} + n(n-2)\, r^{n-4}\, x^2\vec{r}]$$

$$= n\, r^{n-2}\left(x\vec{i} + y\,\vec{j} + z\,\vec{k}\right) + n\, r^{n-2}\left(x\vec{i} + y\,\vec{j} + z\,\vec{k}\right) + 3n\, r^{n-2}\vec{r}$$

$$n(n-2)\, r^{n-4}\vec{r}(x^2 + y^2 + z^2)$$

$$= n\, r^{n-2}\vec{r} + n\, r^{n-2}\vec{r} + 3n\, r^{n-2}\vec{r} + n(n-2)r^{n-4}\vec{r}r^2$$

$$= 5n\, r^{n-2}\vec{r} + n(n-2)r^{n-2}\vec{r}$$

$$= n\, r^{n-2}\vec{r}(5 + n - 2)$$

$$= n\, r^{n-2}\vec{r}(n+3)$$

$$= n(n+3)r^{n-2}\vec{r}$$

Example: 2.50 Prove that $\nabla^2 f(r) = f''(r) + \left(\frac{2}{r}\right) f'(r)$

Solution:

$$\nabla^2 f(r) = \sum \frac{\partial^2}{\partial x^2} f(r)$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\, \frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)\, \frac{x}{r}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[f'(r)x\, \frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\left[\frac{-1}{r^2}\frac{\partial r}{\partial x}\right] + f'(r)\,(1)\,\frac{1}{r} + f''(r)\,\frac{\partial r}{\partial x}\, x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)x\, \frac{-1}{r^2}\frac{x}{r} + f'(r)\, \frac{1}{r} + f''(r)\, \frac{x}{r}\, x\frac{1}{r}\right]$$

$$= \sum\left[f'(r)\, \frac{-1}{r^3}x^2 + f'(r)\, \frac{1}{r} + f''(r)\, \frac{1}{r^2}\, x^2\right]$$

$$= f'(r)\, \frac{-1}{r^3}(x^2 + y^2 + z^2) + \frac{3}{r}f'(r) + f''(r)\, \frac{1}{r^2}\,(x^2 + y^2 + z^2)$$

$$= -f'(r)\, \frac{1}{r^3}(r^2) + \frac{3}{r}f'(r) + f''(r)\, \frac{1}{r^2}\,(r^2)$$

$$= -f'(r)\, \frac{1}{r} + \frac{3}{r}f'(r) + f''(r)$$

$$= f''(r) + \frac{2}{r}f'(r)$$

Exercise: 2.2

1. When $\varphi = x^3 + y^3 + z^3 - 3xyz$, find $\nabla\varphi, \nabla \cdot \nabla\varphi, \nabla \times \nabla\varphi$ at the point $(1, 2, 3)$.

Ans: $(\nabla\varphi)_{(1,2,3)} = -15\vec{\imath} + 3\vec{\jmath} + 21\vec{k}$

$(\nabla \cdot \varphi)_{(1,2,3)} = 36$

$(\nabla \times \nabla\varphi)_{(1,2,3)} = \vec{0}$

2. Show that, $div\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$

3. Find $\nabla \cdot \vec{F}\ and\ \nabla \times \vec{F}$ of the vector point function $\vec{F} = xz^3\,\vec{\imath} - 2x^2yz\,\vec{\jmath} + 2yz^4\,\vec{k}$ at $(1, -1, 1)$. **Ans:** $\left(\nabla \cdot \vec{F}\right)_{(1,-1,1)} = -9, \left(\nabla \times \vec{F}\right)_{(1,-1,1)} = 3\vec{\jmath} + 4\vec{k}$

4. Show that the vector $\vec{F} = (\sin y + z)\vec{\imath} + (x\cos y - z)\,\vec{\jmath} + (x - y)\,\vec{k}$ is irrotational.

5. Show that the vector $\vec{F} = (2xy - z^2)\vec{\imath} + (x^2 + 2yz)\,\vec{\jmath} + (y^2 - 2zx)\,\vec{k}$ is irrotational and find its scalar potential. **Ans:** $x^2y - xz^2 + y^2z + c$

6. Show that the vector $\vec{F} = (3x^2 + 2y^2 + 1)\vec{\imath} + (4xy - 3y^2z - 3)\,\vec{\jmath} + (2 - y^3)\,\vec{k}$ is irrotational and find its scalar potential. **Ans:** $x^3 + 2y^2x + x - y^3z - 3y + 2z + c$

7. Show that the vector $\vec{F} = (y^2 + 2xz^2)\vec{\imath} + (2xy - z)\,\vec{\jmath} + (2x^2z - y + 2z)\,\vec{k}$ is irrotational and find its scalar potential. **Ans:** $xy^2 + x^2z^2 - yz + z^2 + c$

8. Prove that $\vec{F} = (x^2 - y^2 + x)\vec{\imath} - (2xy + z)\,\vec{\jmath}$ is irrrotational and hence, find its scalar potential. **Ans:** $\frac{x^3}{3} - xy^2 + \frac{x^2}{2} - \frac{y^2}{2} + c$

9. Find the constants a, b, c so that the following vector is irrotational.

(i) $\vec{F} = (axy + bz^3)\vec{\imath} + (3x^2 - cz)\,\vec{\jmath} + (3xz^2 - y)\,\vec{k}$ **Ans:** $a = 6, b = 1, c = 1$

(ii) $\vec{A} = (axy - z^3)\vec{\imath} + (a - 2)x^2\,\vec{\jmath} + (1 - a)xz^2\,\vec{k}$ **Ans:** $a = 4$

10. Show that the following vectors are solenoidal.

(i) $\vec{a} = (x + 3y)\vec{\imath} + (y - 3z)\,\vec{\jmath} + (x - 2z)\,\vec{k}$

(ii) $\vec{a} = 5y^4z^3\vec{\imath} + 8xz^2\,\vec{\jmath} - y^2x\,\vec{k}$

2.3 VECTOR INTEGRATION

Line Integral

An integral which is evaluated along a curve then it is called line integral.

Let C be the curve in same region of space described by a vector valued function $\vec{r} = x\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$ of a point (x, y, z) and let $\vec{F} = F_1\vec{\imath} + F_2\,\vec{\jmath} + F_3\,\vec{k}$ be a continuous vector valued function defined along a curve C. Then the line integral $\vec{F}$ over C is denoted by

$$\int_c \vec{F} \cdot d\vec{r}.$$

Work done by a Force

If $\vec{F}(x, y, z)$ is a force acting on a particle which moves along a given curve C, then

$\int_c \vec{F} \cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in the displacement along C.

Thus work done by force $\vec{F} = \int_c \vec{F} \cdot d\vec{r}$

Conservative force field

The line integral $\int_A^B \vec{F} \cdot d\vec{r}$ depends not only on the path C but also on the end points A and B.

If the integral depends only on the end points but not on the path C, then $\vec{F}$ is said to be conservative vector field.

If $\vec{F}$ is conservative force field, then it can be expressed as the gradient of some scalar function φ.

$(ie)\ \vec{F} = \nabla\varphi$

$\vec{F} = \nabla\varphi = \left(\vec{\imath}\,\frac{\partial\varphi}{\partial x} + \vec{\jmath}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}\right)$

$\vec{F} \cdot d\vec{r} = \left(\vec{\imath}\,\frac{\partial\varphi}{\partial x} + \vec{\jmath}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}\right) \cdot \left(dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}\right)$

$= \frac{\partial\varphi}{\partial x}\,dx + \frac{\partial\varphi}{\partial y}\,dy + \frac{\partial\varphi}{\partial z}\,dz = \partial\varphi$

$\int_c \vec{F} \cdot d\vec{r} = \int_A^B \partial\varphi$

$= [\varphi]_A^B$

$= \varphi[B] - \varphi[A]$

$\therefore$ work done by $\vec{F} = \varphi[B] - \varphi[A]$

Note:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \nabla \times (\nabla\varphi) = \vec{0}$ and hence $\vec{F}$ is irrotational.

Problems based on line integral

Example: 2.51 If $\vec{F} = 3xy\vec{\imath} - y^2\vec{\jmath}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ where c is the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution:

Given $\vec{F} = 3xy\vec{\imath} - y^2\vec{\jmath}$

$d\vec{r} = dx\vec{\imath} + dy\,\vec{\jmath}$

$\vec{F} \cdot d\vec{r} = 3xy\,dx - y^2 dy$

Given C is $y = 2x^2$

$\therefore dy = 4xdx$

Along C, x varies from 0 to 1.

$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 3x\,(2x^2)dx - 4x^4(4xdx)$

$= \int_0^1 6x^3 - 16x^5\,dx$

$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]$

$= \frac{6}{4} - \frac{16}{6} = -\frac{7}{6}$ units.

Example: 2.52 Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ moves a particle from the origin to the point $(1, 1)$ along $y^2 = x$.

Solution:

Given $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

$\vec{F} \cdot d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Given $y^2 = x \Rightarrow 2ydy = dx$

Along the curve C, y varies from 0 to 1.

$\int_c \vec{F} \cdot d\vec{r} = \int_0^1((y^2)^2 - y^2 + y^2)\, 2ydy - (2(y^2)y + y)dy$

$= \int_0^1(2y^5 - 2y^3 + 2y^3 - 2y^3 - y)\, dy$

$= \int_0^1(2y^5 - 2y^3 - y)\, dy$

$= \left[2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\right]_0^1$

$= \frac{2}{6} - \frac{2}{4} - \frac{1}{2} = -\frac{2}{3}$

Example: 2.53 Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ from $t = 0$ to $t = 1$ along the curve $x = 2t^2$, $y = t, z = 4t^3$.

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = 3x^2dx + (2xz - y)dy - zdz$

Given $x = 2t^2$, $\quad y = t, \quad z = 4t^3$

$dx = 4tdt$, $\quad dy = dt, \quad dz = 12t^2dt$

$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 48t^5dt + (16t^5 - t)dt - 48t^5\, dt$

$= \int_0^1(16t^5 - t)dt$

$= \left[\frac{16\, t^6}{6} - \frac{t^2}{2}\right]_0^1 = \frac{16}{6} - \frac{1}{2} = \frac{13}{6}$

Example: 2.54 If $\vec{F} = (3x^2 + 6y)\vec{i} + 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t$, $y = t^2$, $z = t^3$.

Solution:

Given $\vec{F} = (3x^2 + 6y)\vec{\imath} + 14yz\vec{\jmath} + 20xz^2\vec{k}$

$d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (3x^2 + 6y)dx + 14yzdy + 20xz^2dz$

Given $x = t$, $y = t^2$, $z = t^3$

$dx = dt$, $d\,y = 2tdt$, $dz = 3t^2dt$

The point $(0,0,0)$ to $(1,1,1)$ on the curve correspond to $t = 0$ and $t = 1$.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + 6t^2)dt + 14t^5(2t\,dt) + 20t^7(3t^2)dt$$

$$= \int_0^1 (9t^2 + 28\,t^6 + 60t^9)\,dt$$

$$= \left[9\frac{t^3}{3} + 28\frac{t^7}{7} + 60\,\frac{t^9}{9}\right]_0^1$$

$$= \frac{9}{3} + \frac{28}{7} + \frac{60}{10} = 3 + 4 + 6 = 13\text{units.}$$

Example: 2.55 Find $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = (2y + 3)\vec{\imath} + xz\vec{\jmath} + (yz - x)\vec{k}$ along the line joining the points $(0,0,0)$ to $(2,1,1)$.

Solution:

Given $\vec{F} = (2y + 3)\vec{\imath} + xz\vec{\jmath} + (yz - x)\vec{k}$

$d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (2y + 3)dx + xzdy + (yz - x)dz$

Equation of Straight line $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

The equation of the straight line joining $(0,0,0)$ to $(2,1,1)$.

$$\Rightarrow \frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{1 - 0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t \text{ (say)}$$

$x = 2t$, $y = t$, $z = t$

$dx = 2dt$, $d\,y = dt$, $dz = dt$

When t = 0 we get $(0,0,0)$

When t = 1 we get $(2,1,1)$

$\therefore t$ varies from 0 to 1.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 (2t + 3)2dt + (2t)tdt + (t^2 - 2t)dt$$

$$= \int_0^1 (4t + 6 + 2t^2 + t^2 - 2t)\,dt$$

$$= \int_0^1 (3t^2 + 2t + 6)\,dt$$

$$= \left[3\frac{t^3}{3} + 2\frac{t^2}{2} + 6t\right]_0^1$$

$$= \frac{3}{3} + \frac{2}{2} + 6 = 8 \text{ units}$$

Example: 2.56 Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the straight line $(0, 0, 0)$ to $(2, 1, 3)$.

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 3x^2dx + (2xz - y)dy + zdz$$

Equation of Straight line $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

The equation of the line joining two points $(0, 0, 0)$ to $(2, 1, 3)$ is

$$\Rightarrow \frac{x - 0}{2 - 0} = \frac{y - 0}{1 - 0} = \frac{z - 0}{3 - 0}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$$x = 2t, \quad y = t, \quad z = 3t$$

$$dx = 2dt, \quad dy = dt, \quad dz = 3dt$$

When $t = 0$ we get $(0, 0, 0)$

When $t = 1$ we get $(2, 1, 3)$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 3(4t^2)2dt + [2(2t)(3t) - t]dt + (3t)3dt$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)\, dt$$

$$= \int_0^1 (36t^2 + 8t)\, dt$$

$$= \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1$$

$$= 12 + 4 = 16 \text{ units}$$

Example: 2.57 Find $\int_c \vec{F} \cdot d\vec{r}$ where c is the circle $x^2 + y^2 = 4$ in the xy plane where $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$.

Solution:

Given $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$

In xy plane $z = 0 \Rightarrow dz = 0$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{r} = 2xydx + x^2dy$$

Given C is $x^2 + y^2 = 4$

The parametric form of circle is

$$x = 2\cos\theta, \qquad y = 2\sin\theta$$

$$dx = -2\sin\theta d\theta, \quad dy = 2\cos\theta d\theta$$

And θ varies from 0 to 2π

$$\int_c \vec{F}\cdot d\vec{r} = \int_0^{2\pi}[2(2\cos\theta)(2\sin\theta)]\,(-2\sin\theta d\theta) + (2\cos\theta)^2 2\cos\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta sin^2\theta + 8\cos^3\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta(1 - cos^2\theta) + 8\cos^3\theta\, d\theta$$

$$= \int_0^{2\pi} -16\cos\theta + 16\, cos^3\theta + 8\cos^3\theta\, d\theta$$

$$= -16\int_0^{2\pi}\cos\theta\, d\theta + 24\int_0^{2\pi}\cos^3\theta\, d\theta$$

$$= -16\int_0^{2\pi}\cos\theta\, d\theta + 24\int_0^{2\pi}\frac{3\cos\theta + \cos 3\theta}{4}\, d\theta$$

$$= 16\,[\sin\theta]_0^{2\pi} + \frac{24}{4}\left[3\sin\theta + \frac{sin3\theta}{3}\right]_0^{2\pi}$$

$$= 0 \qquad [\because \sin n\pi = 0, sin0 = 0]$$

Example: 2.58 State the physical interpretation of the line integral $\int_A^B \vec{F}\cdot d\vec{r}$.

Solution:

Physically $\int_A^B \vec{F}\cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$, displacing a particle from A to B along the curve C.

Example: 2.59 If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$, check whether the integral $\int_c \vec{F}\cdot d\vec{r}$ is independent of the path C.

Solution:

Given $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (4xy - 3x^2z^2)dx + 2x^2dy - 2x^2zdz$$

Then $\int_c \vec{F}\cdot d\vec{r}$ is independent of path C if $\nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-6x^2z + 6x^2z) + \vec{k}(4x - 4x)$$

$$= \vec{0}$$

Hence the line integral is independent of path.

Example: 2.60 Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

Solution:

If $\vec{F}$ is conservative, then $\nabla\times\vec{F} = \vec{0}$.

$$\text{Now, } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is a conservative vector field.

Surface Integral

An integral which is evaluated over a surface is called a surface integral.

Consider a surface S. Let $\vec{F}$ be a vector valued function which is defined at each point on the surface and let P be any point on the surface and $\vec{n}$ be the unit outward normal to the surface at P. The normal component of $\vec{F}$ at P is $\vec{F} \cdot \vec{n}$.

The integral of the normal component of $\vec{F}$ is denoted by $\iint_S \vec{F} \cdot \vec{n}\, ds$ and is called the surface integral.

Evaluation of surface integral

Let R_1 be the projection of S on the $xy-$ plane, $\vec{k}$ is the unit vector normal to the $xy-$ plane then $ds = \frac{dx\, dy}{|\vec{n} \cdot \vec{k}|}$

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R1} \vec{F} \cdot \vec{n} \frac{dx\, dy}{|\vec{n} \cdot \vec{k}|}$$

If R_2 be the projection of s on $yz-$ plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R2} \vec{F} \cdot \vec{n} \frac{dx\, dy}{|\vec{n} \cdot \vec{i}|}$$

If R_3 be the projection of s on $xz-$ plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R3} \vec{F} \cdot \vec{n} \frac{dx\, dy}{|\vec{n} \cdot \vec{j}|}$$

Problems based on surface integral

Example: 2.61 Evaluate $\iint_S \vec{F} \cdot \vec{n}\, ds$ if $\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and s is the surface of the plane $2x + y + 2z = 6$ in the first octant.

Solution:

Given $\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$

Let $\varphi = 2x + y + 2z - 6$

Then $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$$= 2\vec{i} + 1\vec{j} + 2\vec{k}$$

$|\nabla\varphi| = \sqrt{4+1+4} = \sqrt{9} = 3$

$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i}+1\vec{j}+2\vec{k}}{3}$

$\vec{F}\cdot\hat{n} = \left[(x+y^2)\vec{i} - 2x\vec{j} + 2yz\,\vec{k}\right]\cdot\left(\frac{2\vec{i}+1\vec{j}+2\vec{k}}{3}\right)$

$= \frac{1}{3}\,[2(x+y^2) - 2x + 4yz]$

$= \frac{2}{3}\,[y^2 + 2yz]$

$= \frac{2}{3}\,y[y+2z]$

$= \frac{2}{3}\,y[y+6-2x-y]$ $\qquad [\because 2z = 6-2x-y]$

$= \frac{2}{3}\,y[6-2x]$

$= \frac{4}{3}\,y[3-x]$

Let R be the projection of S on the $xy -$ plane

$$\therefore ds = \frac{dx\,dy}{\left|\hat{n}\cdot\vec{\mathrm{k}}\right|}$$

$\hat{n}\cdot\vec{\mathrm{k}} = \left(\frac{2\vec{i}+1\vec{j}+2\vec{k}}{3}\right)\cdot\vec{\mathrm{k}} = \frac{2}{3}$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{\left|\hat{n}\cdot\vec{\mathrm{k}}\right|}$

$= \iint_R \frac{4}{3}\,y(3-x)\,\frac{dx\,dy}{\left(\frac{2}{3}\right)}$

$= 2\int\int(3-x)y\,dxdy$

In $R_1(2x+y=6)$, x varies from 0 to $\frac{6-y}{2}$

y varies from 0 to 6

$= 2\int_0^6\int_0^{\frac{6-y}{2}} y\,(3-x)\,dxdy$

$= 2\int_0^6 y\left[3x - \frac{x^2}{2}\right]_0^{\frac{6-y}{2}} dy$

$= 2\int_0^6 y\left[3\left(\frac{6-y}{2}\right) - \frac{1}{2}\left(\frac{6-y}{2}\right)^2\right]dy$

$= 2\int_0^6 \frac{1}{2}(18y-3y^2) - \frac{1}{8}(6-y)^2\,dy$

$= \frac{2}{2}\left[18\,\frac{y^2}{2} - \frac{3y^3}{3} - \frac{1}{8}\frac{(6-y)^3}{3(-1)}\right]$

$= \left[9(6)^2 - (6)^3 + \frac{1}{12}(0)\right] - \left[0 - 0 + \frac{1}{12}\,(6)^3\right]$

$= 81$ units

Example: 2.62 Show that $\iint_S \left(\boldsymbol{yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}}\right)\cdot\hat{n}\,\boldsymbol{ds} = \frac{3}{8}$ **where s is the surface of the sphere** $\boldsymbol{x^2+y^2+z^2=1}$ **in the first octant.**

Solution:

Given $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

Let $\varphi = x^2 + y^2 + z^2 - 1$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2(1)$$

$\therefore$ The unit outward normal is $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2}$

$\vec{F}\cdot\hat{n} = \left[yz\vec{i} + zx\vec{j} + xy\,\vec{k}\right]\cdot\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$

$= 3xyz$

Let R be the projection of S on xy −plane

$$\therefore ds = \frac{dx\,dy}{\left|\hat{n}\cdot\vec{\mathrm{k}}\right|}$$

$\left|\hat{n}\cdot\vec{\mathrm{k}}\right| = \left(x\vec{i} + y\vec{j} + z\vec{k}\right)\cdot\vec{k} \quad = z$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{\left|\hat{n}\cdot\vec{\mathrm{k}}\right|}$$

$$= \int\int 3xyz\,\frac{dxdy}{z}$$

$$= \int\int 3xy\ dxdy$$

In $R_1(x^2 + y^2 = 1)$, x varies from 0 to $\sqrt{1 - y^2}$

y varies from 0 to 1

$$= \int_0^1\int_0^{\sqrt{1-y^2}} 3xy\,dxdy$$

$$= 3\int_0^6 \left[y\,\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_0^1 y(1 - y^2)dy$$

$$= \frac{3}{2}\int_0^1 y - y^3 dy$$

$$= \frac{3}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}$$

Volume integral

An integral which is evaluated over a volume bounded by a surface is called a volume integral.

If $\vec{F} = F_1\vec{i} + F_2\,\vec{j} + F_3\,\vec{k}$ is a vector field in V, then the volume integral is defined by

$$\iiint_V \vec{F}\ dv$$

Problems based on volume integral

Example: 2.63 If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla \times \vec{F}\, dv$ where v is the volume of the region bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution:

Given $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-4+3) + \vec{k}(-2y-0)$$

$$= \vec{j} - 2y\vec{k}$$

For limits

Given $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$

$\therefore z : 0 \to 4 - 2x - 2y$

Put $z = 0 \Rightarrow 2x + 2y = 4\ (or)\ x + y = 4$

$\therefore y : 0 \to 2 - x$

Put $z = 0, y = 0 \Rightarrow 2x = 4\ (or)\ x = 2$

$\therefore x : 0 \to 2$

$$\therefore \int\int\int \nabla \times \vec{F}\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k})\, dz dy dx$$

$$= \int_0^2 \int_0^{2-x} (\vec{j} - 2y\vec{k})[z]_0^{4-2x-2y}\, dy dx$$

$$= \int_0^2 \int_0^{2-x} \left[(4 - 2x - 2y)\vec{j} - 2y(4 - 2x - 2y)\vec{k}\right] dy dx$$

$$= \int_0^2 \left\{\left[4y - 2xy - \frac{2y^2}{2}\right]\vec{j} - \left[4y^2 - 2xy^2 - \frac{4y^3}{3}\right]\vec{k}\right\}_0^{2-x} dx$$

$$= \int_0^2 \{[4(2-x) - 2x(2-x) - (2-x)^2]\,\vec{j} - \left[4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\vec{k}\right]\}\, dx$$

$$= \int_0^2 [8 - 4x - 4x + 2x^2 - 4 + 4x - x^2]\,\vec{j} - \left[16 - 16x + 4x^2 - 8x + 8x^2 - 2x^3 - \frac{4}{3}(8 - 12x + 6x^2 - x^3)\,\vec{k}\right] dx$$

$$= \int_0^2 \left[(4 - 4x + x^2)\,\vec{j} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3)\right] dx$$

$$= \left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2 \vec{j} + \frac{\vec{k}}{3}\left[16x - 12x^2 + 4x^3 - \frac{x^4}{2}\right]_0^2$$

$$= \left(8 - 8 + \frac{8}{3}\right)\vec{j} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8)$$

$$= \frac{8}{3}(\vec{j} - \vec{k})$$

Exercise: 2.3

1. If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, evaluate the line integral $\int_c \vec{F} \cdot d\vec{r}$ from $(0,0)\ to\ (1,1)$along the path $y = x$. **Ans:** $\frac{1}{2}$

2. Evaluate $\int_c \vec{F} \cdot d\vec{r}$,where $\vec{F} = x^2y^2\vec{i} + y\,\vec{j}$and C is $y^2 = 4x$ in the XY plane from $(0,0)\ to\ (4,4)$. **Ans:** 264

3. If $\vec{F} = xy\vec{i} + (x^2 + y^2)\,\vec{j}$, then find $\int_c \vec{F} \cdot d\vec{r}$, where C is the arc of the parabola $y = x^2 - 4$ from $(2,0)\ to\ (4,12)$ **Ans:** 732

4. If $\vec{F} = xy\vec{i} + z\,\vec{j} - xyz\,\vec{k}$, then evaluate $\int_c \vec{F} \cdot d\vec{r}$, from the point $(0,0,0)\ to\ (1,1,1)$ where C is the curve $x = t, y = t^2, z = t$ **Ans:** $\frac{67}{60}$

5. Find the work done in moving a particle in the field $\vec{F} = 3x^2\vec{i} + (2xz - y)\,\vec{j} + (x^2 + y^2)\,\vec{k}$ along the straight line from $(0,0,0)\ to\ (2,1,3)$. **Ans:** 16

6. Evaluate the line integral $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$, where C is the square formed by the lines $x = \pm 1\ and\ y = \pm 1$. **Ans:** 0

7. Find the total work done in moving a particle by a force field $\vec{F} = yz\vec{i} + xz\,\vec{j} + xy\,\vec{k}$ along the curve $x = t, y = t^2, z = t^3$ from $(0,0,0)\ to\ (2,4,8)$ **Ans:** 64

8. Evaluate $\iint_S \vec{F} \cdot \hat{n}\ ds$ where $\vec{F} = 18z\vec{i} - 12\,\vec{j} + 3y\,\vec{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ which is in the first order. **Ans:**24

9. Evaluate $\iint_S \vec{F} \cdot \hat{n}\ ds$ where $\vec{F} = (x + y^2)\vec{i} - 2x\,\vec{j} + 2yz\,\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ which is in the first order. **Ans:**24

10. Evaluate $\iiint_V \nabla \times \vec{F}\ dv$ where $\vec{F} = (2x^2 - 3z)\vec{i} - 2y\,\vec{j} - 4xz\,\vec{k}$ and V is bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$ **Ans:** $\frac{8}{3}$

2.4 Green's Theorem

Green's theorem relates a line integral to the double integral taken over the region bounded by the closed curve.

Statement

If $M(x,y)$ and $N(x,y)$ are continuous functions with continuous, partial derivatives in a region R of the xy – plane bounded by a simple closed curve C, then

$$\oint_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy,$$
where C is the curve described in the positive direction.

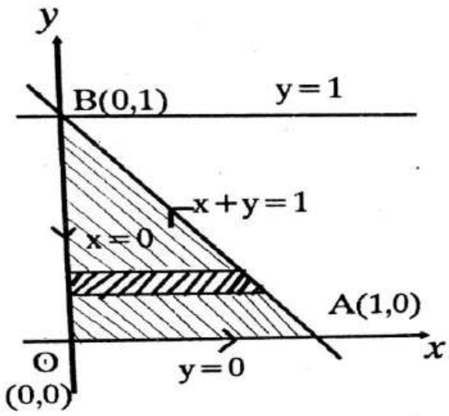
Vector form of Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \vec{k}\, dR$$

Problems based on Green's theorem

Example: 2.64 Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.

Solution:



We have to prove that $\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$

$\therefore \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_C M\,dx + N\,dy$

By Green's theorem in the plane,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (10y)\,dy\,dx$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2\,dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{5}{3} \dots (1)$$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

Along $OA, y = 0 \Rightarrow dy = 0, x$ varies from 0 to 1

$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$

Along $AB, y = 1 - x \Rightarrow dy = -dx \; and \; x$ varies from 1 to 0

$$\therefore \int_{AB} M\,dx + N\,dy = \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)]dx$$

$$= \left[\frac{3x^3}{3} - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + 3x^2 - 2x^3\right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = \frac{8}{3}$$

Along $BO, x = 0 \Rightarrow dx = 0 \; and \; y$ varies from 1 to 0

$$\therefore \int_{BO} M\,dx + N\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$$\therefore \int_{c} M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \dots(2)$$

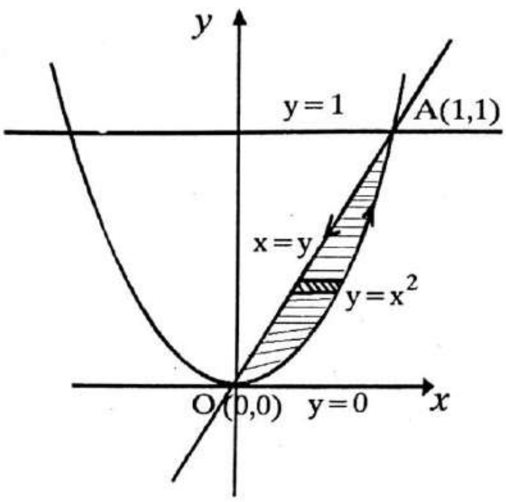
$\therefore$ From (1) and (2)

$$\therefore \int_{c} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

Example: 2.65 Verify Green's theorem in the XY −plane for $\int_c (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x, y = x^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = xy + y^2 \; and \; N = x^2$

$$\Rightarrow \frac{\partial M}{\partial y} = x + 2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

R.H.S $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Limits:

x varies from y to $\sqrt{y}$

y varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \int_0^1 \int_y^{\sqrt{y}} 2x - (x+2y)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y\sqrt{y}\right) - \left(\frac{y^2}{2} - 2y^2\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{\frac{3}{2}} + 3\frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{4y^{\frac{5}{2}}}{5} + \frac{y^3}{2}\right]_0^1$$

$$= \frac{1}{4} - \frac{4}{5} + \frac{1}{2} = -\frac{1}{20}$$

L.H.S = $\int_c M\, dx + N\, dy$

Consider $\int M\, dx + N\, dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \Rightarrow dy = 2x\, dx, x$ varies from 0 to 1

$$\therefore \int_{OA} M\, dx + N\, dy = \int_0^1 [(x(x^2) + (x^2)^2)dx + x^2 \cdot 2x\, dx]$$

$$= \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along $AO, y = x \Rightarrow dy = dx\ and\ x$ varies from 1 to 0

$$\therefore \int_{A0} M\, dx + N\, dy = \int_1^0 (x^2 + x^2)dx + x^2\, dx$$

$$= \int_1^0 3x^2 dx = [x^3]_1^0 = -1$$

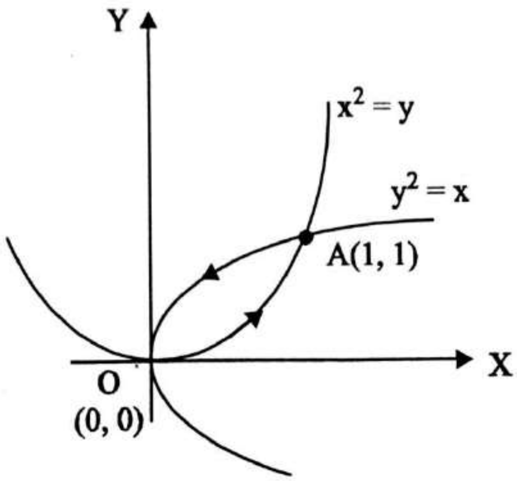
L.H.S = $\int_c M\, dx + N\, dy = \frac{19}{20} - 1 = -\frac{1}{20}$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.66 Verify Green's theorem in the plane for $\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $y = x^2, x = y^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2 \; and \; N = 4y - 6xy$

$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Limits:

x varies from y^2 to $\sqrt{y}$

y varies from 0 to 1

$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{y^2}^{\sqrt{y}} (-6y + 16y)\, dx\,dy$

$$= \int_0^1 [10xy]_{y^2}^{\sqrt{y}} dy$$

$$= 10\int_0^1 \left(y\sqrt{y} - y^3\right) dy$$

$$= 10\left[\frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left(\frac{2}{5} - \frac{1}{4}\right) = \frac{3}{2}$$

L.H.S = $\int_c M\,dx + N\,dy$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \;\; \Rightarrow dy = 2x\,dx, x$ varies from 0 to 1

$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)(2x)dx$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)dx$$

$$= \int_0^1 (-20x^4 + 8x^3 + 3x^2)\,dx$$

$$= \left[-20\frac{x^5}{5} + 8\frac{x^4}{4} + 3\frac{x^3}{3}\right]_0^1$$

$$= -4 + 2 + 1 = -1$$

Along $AO, x = y^2 \Rightarrow dx = 2ydy \ and \ y$ varies from 1 to 0

$$\therefore \int_{Ao} M\,dx + N\,dy = \int_1^0 (3y^4 - 8y^2)2ydy + (4y - 6yy^2)\,dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)dx$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dx$$

$$= \left[6\frac{y^6}{6} - 22\frac{y^4}{4} + 4\frac{y^2}{2}\right]_1^0$$

$$= 0 - \left[1 - \frac{11}{2} + 2\right]$$

$$= -\left(3 - \frac{11}{2}\right) = \frac{5}{2}$$

L.H.S = $\int_c M\,dx + N\,dy = -1 + \frac{5}{2} = \frac{3}{2}$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.67 Verify Green's theorem in the plane for the integral $\int_c (x - 2y)dx + xdy$ taken around the circle $x^2 + y^2 = 1$.

Solution:

We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = x - 2y \ and \ N = x$

$\Rightarrow \frac{\partial M}{\partial y} = -2 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 + 2)dxdy$$

$$= 3\iint_R dxdy$$

= 3 (Area of the circle)

$$= 3\pi r^2$$

$$= 3\pi \qquad (\because radius = 1)$$

L.H.S = $\int_c M\,dx + N\,dy$

Given C is $x^2 + y^2 = 1$

The parametric equation of circle is

$x = \cos\theta,\ y = \sin\theta$

$dx = -\sin\theta d\theta,\ dy = \cos\theta\, d\theta$

Where θ varies from 0 to 2π

$$\therefore \int_c M\,dx + N\,dy = \int_0^{2\pi} (\cos\theta - 2\sin\theta)\,(-\sin\theta\, d\theta) + \cos\theta\,(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + \sin^2\theta + 1)\, d\theta \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \int_0^{2\pi} \left(-\frac{\sin 2\theta}{2} + \frac{1-\cos 2\theta}{2} + 1\right) d\theta$$

$$= \left[-\frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right) + \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) + \theta\right]_0^{2\pi}$$

$$= \left[\frac{\cos(4\pi)}{4} + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} + 2\pi\right] - \left[\frac{\cos 0}{4} + \frac{0}{2} - \frac{\sin 0}{4} + 0\right]$$

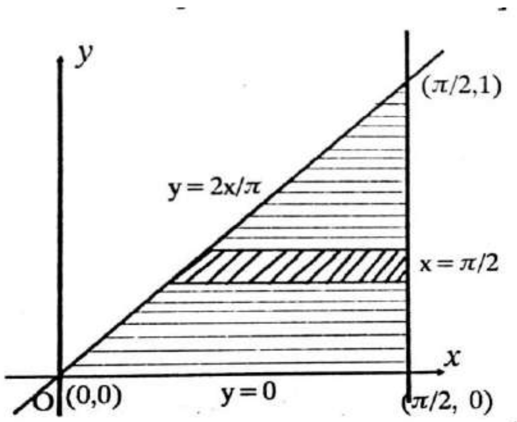
$$= \frac{1}{4} + \pi + 2\pi - \frac{1}{4} = 3\pi \quad [\because \sin n\pi = 0, \sin 0 = 0, \cos 0 = 1], [\cos n\pi = (-1)^n]$$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: 2.68 Using Green's theorem evaluate $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = y - \sin x$ and $N = \cos x$

$\Rightarrow \frac{\partial M}{\partial y} = 1 - 0 \qquad \Rightarrow \frac{\partial N}{\partial x} = -\sin x$

Limits:

x varies from $\frac{y\pi}{2}$ to $\frac{\pi}{2}$

y varies from 0 to 1

Hence $\int_c (y - sin\, x)dx + cos\, x\, dy = \int_0^1 \int_{\frac{y\pi}{2}}^{\frac{\pi}{2}} (-\sin x - 1)\ dx\, dy$

$$= \int_0^1 (\cos x - x)_{\frac{y\pi}{2}}^{\frac{\pi}{2}}\ dy$$

$$= \int_0^1 \left[\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\left(\frac{y\pi}{2}\right) - \frac{y\pi}{2}\right)\right] dy$$

$$= \int_0^1 \left[0 - \frac{\pi}{2} - \cos\frac{y\pi}{2} + \frac{y\pi}{2}\right] dy$$

$$= \left[-\frac{\pi}{2}y - \frac{\sin\frac{y\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{2}\frac{y^2}{2}\right]_0^1$$

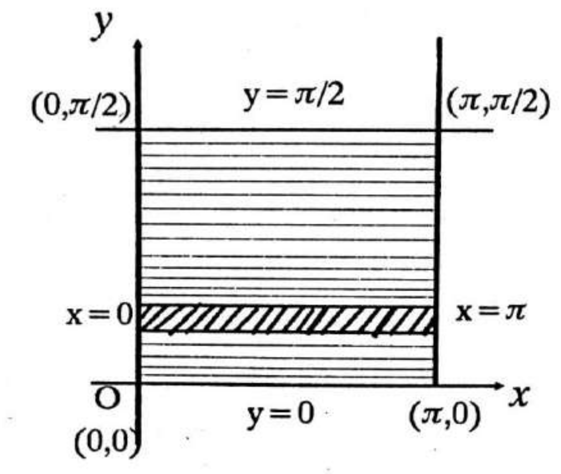
$$= -\frac{\pi}{2} - \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}$$

$$= -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

Example: 2.69 Evaluate by Green's theorem $\int_c [e^{-x}(sin\, y\, dx + cos\, y\, dy)]$ where C being the rectangle with vertices $(0,0), (\pi,0), \left(\pi,\frac{\pi}{2}\right)$ and $\left(0,\frac{\pi}{2}\right)$.

Solution:



We have to prove that $\int_c M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Here, $M = e^{-x}\sin y$ and $N = e^{-x}\cos y$

$\Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y \qquad \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$

Limits:

x varies from 0 to π

y varies from 0 to $\frac{\pi}{2}$

$\therefore \int_c [e^{-x}(sin\, y\, dx + cos\, y\, dy)] = \int_0^{\frac{\pi}{2}} \int_0^{\pi} (-e^{-x}\cos y - e^{-x}\cos y)\, dxdy$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\pi} -2\, e^{-x} \cos y \; dx\, dy$$

$$= -2 \int_0^{\frac{\pi}{2}} \left[\frac{e^{-x} \cos y}{-1}\right]_0^{\pi} dy$$

$$= 2 \int_0^{\frac{\pi}{2}} [e^{-\pi} \cos y - e^{0} \cos y] dy$$

$$= 2 \int_0^{\frac{\pi}{2}} [e^{-\pi} \cos y - \cos y] dy$$

$$= 2\, [e^{-\pi} \sin y - \sin y]_0^{\frac{\pi}{2}}$$

$$= 2 \left[\left(e^{-\pi} \sin\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - (e^{-\pi} \sin 0 - \sin 0)\right]$$

$$= 2\, [e^{-\pi} - 1]$$

Example: 2.70 Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2} \int_c (xdy - ydx)$. Hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using Green's theorem.

Solution:

By Green theorem, $\int_c M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

Let $M = -y$ and $N = x$

$$\Rightarrow \frac{\partial M}{\partial y} = -1 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$$

$$\therefore \int_c (xdy - ydx) = \iint_R (1 + 1)\, dx\, dy$$

$$= 2 \iint_R dx\, dy = 2 \quad \text{(Area enclosed by C)}$$

$\therefore$ Area enclosed by $C = \frac{1}{2} \int_c (xdy - ydx)$

Equation of ellipse in parametric form is $x = a \cos\theta$ and $y = b \sin\theta$ where $0 \le \theta \le 2\pi$.

$\therefore$ Area of the ellipse $= \frac{1}{2} \int_0^{2\pi} (a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\; d\theta$

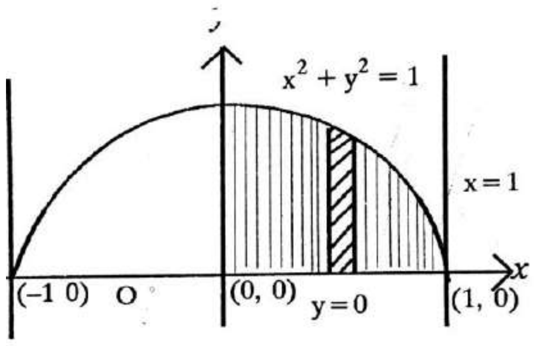
$$= \frac{1}{2}\, ab \int_0^{2\pi} (cos^2\theta + sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\, ab \int_0^{2\pi} d\theta = \frac{1}{2}\, ab\, [\theta]_0^{2\pi} = \pi ab$$

Example: 2.71 Evaluate the integral using Green's theorem

$\int_c (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary in the xy – plane of the area enclosed by the x – axis and the semicircle $x^2 + y^2 = a^2$ in the upper half xy – plane.

Solution:



In this figure $'a'$ is represented as 1

By Green theorem, $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = 2x^2 - y^2$ and $N = x^2 + y^2$

$$\Rightarrow \frac{\partial M}{\partial y} = -2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

Limits:

y varies from 0 to $\sqrt{a^2 - x^2}$

x varies from $-a$ to a

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (2x + 2y)\,dy\,dx$$

$$= 2\int_{-a}^{a} \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2}\right] dx$$

In the first integral, the function is odd function.

$\therefore$ The value is zero.

$\therefore$ we get $2\int_{-a}^{a} \frac{a^2 - x^2}{2} dx$

$$= \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)$$

$$= \frac{4a^3}{3}$$

Exercise: 2.4

1. Using Green's theorem in the plane, evaluate $\int_c (x^2 - y^2)dx + 2xydy$ where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$ **Ans:** $\frac{3}{5}$

2. Find by Green's theorem the value of $\int_c (x^2ydx + ydy)$ along the closed curve formed by $y = x^2$ and $y^2 = x$ between $(0,0)$ to $(1,1)$ **Ans:** $\frac{1}{28}$

3. Verify Green's theorem for the integral $\int_c [(x-y)dx + (x+y)dy]$ taken around the boundary area in the first quadrant between the curves $y = x^2 and\ y^2 = x$.

Ans: Common value $= \frac{2}{3}$

4. Find the area of a circle of radius 'a' using Green's theorem. **Ans:** πa^2

5. Evaluate $\int_c [(sinx - y)dx - \cos x\, dy]$, where C is the triangle with vertices $(0,0), \left(\frac{\pi}{2}, 0\right)\ and\ \left(\frac{\pi}{2}, 1\right)$ **Ans:** $\frac{2}{\pi} + \frac{\pi}{4}$

6. Using Green's theorem, find the value of $\int_c [(xy - x^2)dx + x^2ydy]$ along the closed curve C formed by $y = 0, x = 1\ and\ y = x$ **Ans:** $-\frac{1}{12}$

7. Verify Green's theorem for $\int_c [(x^2 - y^2)dx + 2xydy]$, where C is the boundary of the rectangle in the xoy – plane bounded by the lines $x = 0, x = a, y = 0\ and\ y = b$.

Ans: Common value $= 2ab^2$

8. Verify Green's theorem for $\int_c [(2x - y)dx + (x + y)dy]$, where C is the boundary of the Circle $x^2 + y^2 = a^2$ in the xoy – plane. **Ans:** $2\pi a^2$

2.5 STOKE'S THEOREM

Statement of Stoke's theorem

If S is an open surface bounded by a simple closed curve C if $\vec{F}$ is continuous having continuous partial derivatives in S and C, then

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\, ds$$

(or)

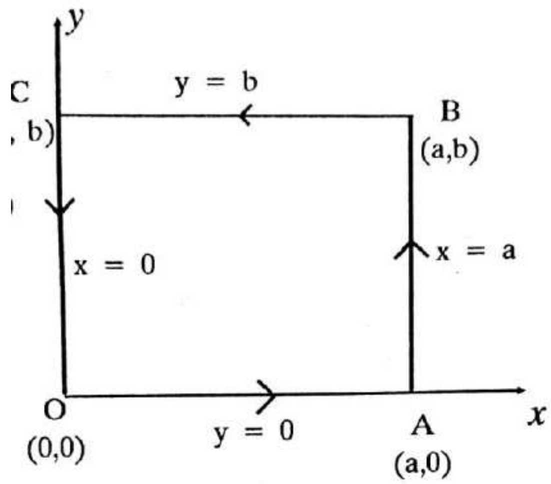
$$\int_c \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$\hat{n}$ is the outward unit normal vector and C is traversed in the anti – clockwise direction.

Problems based on Stoke's theorem

Example: 2.72 Verify stokes theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in a rectangular region in the xoy plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

To evaluate: $\iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2\,x\,y\vec{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$= \vec{i}\,(0) - \vec{j}\,(0 - 0) + \vec{k}[\,2y - (0 - 2y)]$

$= 4y\,\vec{k}$

Since the surface is a rectangle in the xy plane, $\hat{n} = \vec{k}$, $dS = dxdy$

Curl $\vec{F} \cdot \hat{n} = 4y\;\vec{k} \cdot \vec{k} = 4y$

Order of integration is $dxdy$

x varies from $x = 0$ to $x = a$

y varies from $y = 0$ to $y = b$

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = \int_0^b \int_0^a 4y\, dx\, dy$$

$$= \int_0^b 4y\, [x]_0^a dy$$

$$= \int_0^b 4ay dy$$

$$= \left[\frac{4ay^2}{2}\right]_0^b$$

$$= 2ab^2$$

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = 2ab^2 \qquad \ldots (1)$$

Here the line integral over the simple closed curve C bounding the surface $OABCO$ consisting of the edges OA, AB ,BC and CO.

Curve	Equation	Limit
OA	$y = 0$	$x = 0$ to $x = a$
AB	$x = a$	$y = 0$ to $y = b$
BC	$y = b$	$x = a$ to $x = 0$
CO	$x = 0$	$y = b$ to $y = 0$

Therefore, $\int_c \vec{F} \cdot d\vec{r} = \int_{OABCO} \vec{F} \cdot d\vec{r}$

$$\int_c \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2) + 2xy\mathrm{d}y \qquad \dots (2)$$

On OA: $y = 0, dy = 0, x$ varies from 0 to a

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = x^2 dx$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB: $x = a, dx = 0, y$ varies from 0 to b

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = 2ay\, dy$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2ay\, dy$$

$$= \left[\frac{2ay^2}{2}\right]_0^b = ab^2$$

On BC: $y = b, dy = 0, x$ varies from a to 0

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = (x^2 - b^2)dx$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 x^2 - b^2\, dx$$

$$= \left[\frac{x^3}{3} - b^2 x\right]_a^0$$

$$= -\frac{a^3}{3} + a\, b^2$$

On CO: $x = 0, dx = 0, y$ varies from b to 0

(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = 0$

$$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

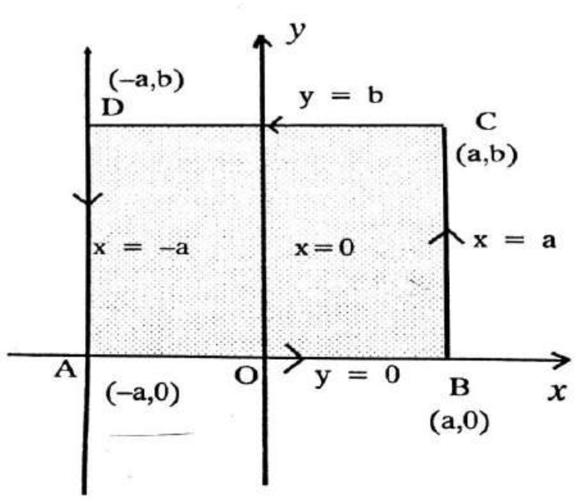
(2) $\Rightarrow \vec{F} . \mathrm{d}\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2 \qquad \dots (3)$

From (3) and (1) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Hence Stokes theorem is verified.

Example: 2.73 Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}\,[0-0] + \vec{k}\,[-2y - 2y]$$

$$= -4y\,\vec{k}$$

Since the region is in xoy plane we can take $\hat{n} = \vec{k}\ and\ dS = dx\,dy$

Limits:

x varies from $-a$ to a.

y varies from 0 to b.

$$\therefore \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = -4 \int_0^b \int_{-a}^{a} y\,dx\,dy$$

$$= -4 \int_0^b [xy]_{-a}^{a}\, dy$$

$$= -8a \left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad \dots(1)$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB: $y = 0, dy = 0, x$ varies from $-a\ to\ a$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along $BC, x = a, dx = 0,$ y varies from 0 to b

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_0^b (-2ay)\,dy$$

$$= -a[y^2]_0^b = -ab^2$$

Along CD: $y = b, dy = 0, x$ varies from a $to - a$

$$\int_{CD} \vec{F} \cdot d\vec{r} = \int_a^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2a^3}{3} - 2ab^2$$

Along DC: $x = -a, dx = 0, y$ varies from b to 0

$$\int_{DC} \vec{F} \cdot d\vec{r} = \int_b^0 2ay\,dy$$

$$= a[y^2]_b^0 = -b^2a$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - b^2a$$

$$= -4ab^2 \quad \ldots.(2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \vec{n}\,dS$

Hence Stoke's theorem is verified.

Example: 2.74 Verify Stoke's theorem for $\vec{F} = (2x - y)\vec{i} - yz^2\,\vec{j} - y^2z\vec{k}$ where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and c is the Circular boundary on $z = 0$ plane.

Solution:

By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\,dS$

Given $\vec{F} = (2x - y)\vec{i} - yz^2\,\vec{j} - y^2z\vec{k}$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}[-2yz + 2yz] - \vec{j}\,[0 - 0] + \vec{k}\,[0 + 1]$$

$$= \vec{k}$$

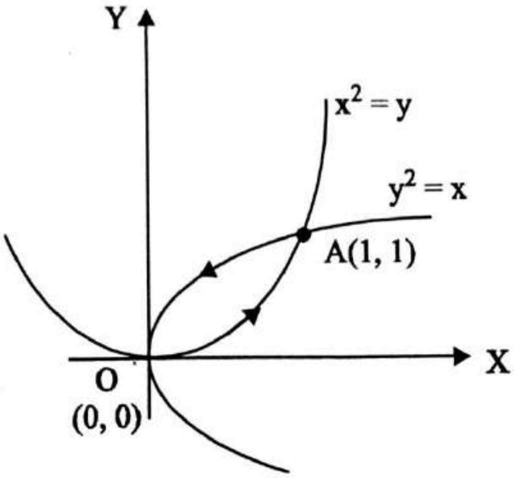
Here $\vec{n} = \vec{k}$ since C is the Circular boundary on $z = 0$ plane.

$\therefore \iint_S Curl\,\vec{F} \cdot \hat{n}\,dS = \iint_S dx\,dy$ = area of the circle

$$= \pi(1)^2 = \pi$$

Example: 2.75 Verify stokes theorem in a plane for $\vec{F} = (2xy - x^2)\vec{i} - (x^2 - y^2)\vec{j}$ Where C is the boundary of the region bounded by the parabolas $y^2 = x$ and $x^2 = y$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

To evaluate: $\iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (2xy - x^2)\vec{i} - (x^2 - y^2)\vec{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy - x^2) & -(x^2 - y^2) & 0 \end{vmatrix}$$

$$= \vec{i}\,[\tfrac{\partial}{\partial y}(0) - \tfrac{\partial}{\partial z}(-(x^2 - y^2))] - \vec{j}\,[\tfrac{\partial}{\partial x}(0) - \tfrac{\partial}{\partial z}(2xy - x^2)]$$

$$+ \vec{k}[\tfrac{\partial}{\partial x}(-(x^2 - y^2)) - \tfrac{\partial}{\partial y}(2xy - x^2)]$$

$$= \vec{i}\,(0) - \vec{j}\,(0 - 0) + \vec{k}\,(-2x - 2x)]$$

$$= -4x\vec{k}$$

Since the surface is a rectangle in the xy –plane, $\hat{n} = \vec{k}$, $dS = dxdy$

Curl $\vec{F} \cdot \hat{n} = -4x\vec{k} \cdot \vec{k} = -4x$

Order of integration is $dxdy$

Limits:

x varies from y^2 to $\sqrt{y}$.

y varies from 0 to 1

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = \int_0^1 \int_{y^2}^{\sqrt{y}} -4x\, dx\, dy$$

$$= -4\int_0^1 \left[\frac{x^2}{2}\right]_{y^2}^{\sqrt{y}} dy$$

$$= -2\int_0^1 (y - y^4)dy$$

$$= -2\left[\frac{y^2}{2} - \frac{y^5}{2}\right]_0^1$$

$$= -2\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$= -\frac{3}{5}$$

$$\therefore \iint_S Curl\,\vec{F}\cdot\hat{n}\,dS = -\frac{3}{5} \qquad \ldots(1)$$

To evaluate: $\iint_S Curl\,\vec{F}\cdot\hat{n}\,dS$

Here the line integral over the simple closed curve C bounding the surface OAO consisting of the curves OA and AO.

$$\int_c \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AO} \quad \ldots\ (2)$$

$$\vec{F}\cdot d\vec{r} = (2xy - x^2)\vec{\imath} - (x^2 - y^2)\vec{\jmath} \qquad \ldots(3)$$

On OA: $y = x^2, dy = 2xdx,$ x varies from $0\ to\ 1$

(3) $\Rightarrow \vec{F}\,.\,\mathrm{d}\vec{r} = (2xx^2 - x^2)dx - (x^2 - x^4)2xdx$

$$= (2x^3 - x^2 - 2x^3 + 2x^5)dx$$

$$= (2x^5 - x^2)dx$$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^1 (2x^5 - x^2)dx$$

$$= \left[\frac{2x^6}{3} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3} - \frac{1}{3} = 0$$

On AO: $x = y^2, dx = 2ydy,$ y varies from $1\ to\ 0$

(3) $\Rightarrow \vec{F}\,.\,\mathrm{d}\vec{r} = (2y^2y - y^4)2ydy - (y^4 - y^2)dy$

$$= (4y^4 - 2y^5)dy - (y^4 - y^2)dy$$

$$= (4y^4 - 2y^5 - y^4 + y^2)dy$$

$$= (3y^4 - 2y^5 + y^2)dy$$

$$\int_{AO} \vec{F}\cdot d\vec{r} = \int_1^0 (3y^4 - 2y^5 + y^2)dy$$

$$= \left[\frac{3y^5}{5} - \frac{2y^6}{6} + \frac{y^3}{3}\right]_1^0 = -\frac{3}{5} + \frac{1}{3} - \frac{1}{3} = -\frac{3}{5}$$

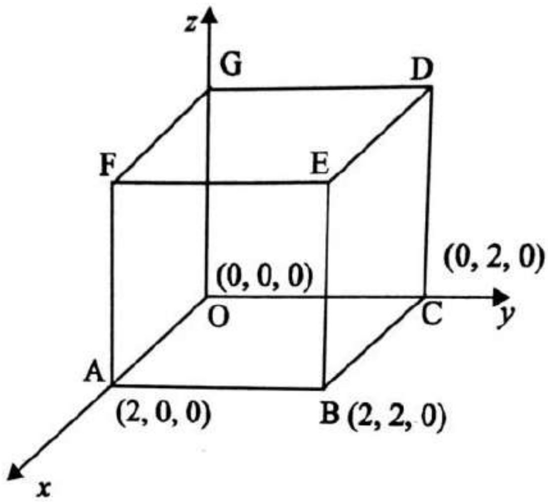
(2) $\Rightarrow \int_c \vec{F}\cdot d\vec{r} = 0 - \frac{3}{5} = -\frac{3}{5} \qquad \ldots\ (3)$

From (3) and (1) $\int_c \vec{F}\cdot d\vec{r} = \iint_S Curl\,\vec{F}\cdot\hat{n}\,dS$

Hence Stokes theorem is verified.

Example: 2.76 Verify Stoke's theorem in a plane for $\vec{F} = (y - z + 2)\vec{i} - (yz + 4)\vec{j} - xz\,\vec{k}$, where S is the open surface of the cube formed by the planes $x = 0, x = 2, y = 0, \quad y = 2,$ $z = 0, z = 2$ above the xy – plane.

Solution:



Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\, ds$

Given $\vec{F} = (y - z + 2)\vec{i} - (yz + 4)\vec{j} - xz\,\vec{k}$

$\vec{F} \cdot d\vec{r} = (y - z + 2)dx - (yz + 4)dy - xz\,dz$

L.H.S $= \int_c \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

In xy plane $z = 0 \Rightarrow dz = 0$

$\vec{F} \cdot d\vec{r} = (y + 2x)dx + 4dy$

On OA: $y = 0 \Rightarrow dy = 0$, x varies from 0 to 2.

$\Rightarrow \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2\,dx$

$= 2[x]_0^2 = 4$

On AB: $x = 2 \Rightarrow dx = 0$, y varies from 0 to 2.

$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4\,dy$

$= 4[y]_0^2 = 8$

On BC: $y = 2 \Rightarrow dy = 0$, x varies from 2 to 0.

$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_2^0 4\,dx$

$= 4[x]_2^0 = -8$

On CO: $x = 0 \Rightarrow dx = 0$, y varies from 2 to 0.

$$\Rightarrow \int_{CO} \vec{F} \cdot d\vec{r} = \int_0^2 4\, dy$$

$$= 4[y]_0^2 = -8$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = 4 + 8 - 8 - 8 = -4 \quad \dots (1)$$

R.H.S $= \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$$curl\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= \vec{i}\,(0 - y) - \vec{j}\,(-z + 1) + \vec{k}\,(0 - 1)$$

$$= y\vec{i} - (z - 1)\,\vec{j} - \vec{k}$$

Given S is an open surface consisting of the 5 faces of the cube except, $xy -$ plane.

$$\int\int curl\, \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \dots + \iint_{S5}$$

$$curl\, \vec{F} = y\vec{i} - (z - 1)\,\vec{j} - \vec{k}$$

Faces	Plane	ds	$\hat{n}$	$curl\, \vec{F} \cdot \hat{n}$	
Top (S_1)	xy	$dxdy$	$\vec{k}$	-1	$\int_0^2 \int_0^2 -1\, dxdy$
Left (S_2)	xz	$dxdz$	$-\vec{j}$	$-(z - 1)$	$\int_0^2 \int_0^2 (-z + 1)\, dxdz$
Right (S_3)	xz	$dxdz$	$\vec{j}$	$(z - 1)$	$\int_0^2 \int_0^2 (z - 1)\, dxdz$
Back (S_4)	yz	$dydz$	$-\vec{i}$	y	$\int_0^2 \int_0^2 y\, dydz$
Front (S_5)	yz	$dydz$	$\vec{i}$	$-y$	$\int_0^2 \int_0^2 -y\, dydz$

On S_1: $\int_0^2 \int_0^2 (-1)\, dxdy$

$$= -\int_0^2 [x]_0^2\, dy$$

$$= 2\int_0^2 dy$$

$$= -2[y]_0^2 = -4$$

On S_2: $\int_0^2 \int_0^2 (-z + 1)\, dxdz$

$$= \int_0^2 (-z+1)[x]_0^2\, dz$$

$$= 2\int_0^2 (-z+1)\, dz$$

$$= 2\left[-\frac{z^2}{2} + z\right]_0^2 = 2(0) = 0$$

On S_3: $\int_0^2 \int_0^2 (z-1)\, dxdz$

$$= \int_0^2 (z-1)[x]_0^2\, dz$$

$$= 2\int_0^2 (z-1)\, dz$$

$$= 2\left[\frac{z^2}{2} - z\right]_0^2 = 2(0) = 0$$

On S_4: $\int_0^2 \int_0^2 y\, dydz$

$$= \int_0^2 \left[\frac{y^2}{2}\right]_0^2 dy$$

$$= 2\int_0^2 dz$$

$$= 2[z]_0^2 = 4$$

On S_5: $\int_0^2 \int_0^2 -y\, dydz$

$$= -\int_0^2 \left[\frac{y^2}{2}\right]_0^2 dy$$

$$= -2\int_0^2 dz$$

$$= -2[z]_0^2 = -4$$

$$\therefore \iint_S curl\,\vec{F}\cdot\hat{n}\, ds = -4+0+0+4-4 = -4 \quad \dots (2)$$

From (1) and (2) $\int_c \vec{F}\cdot d\vec{r} = \iint_S curl\,\vec{F}\cdot\hat{n}\, ds$

Hence Stoke's theorem is verified.

Example: 2.77 Verify Stoke's theorem in a plane for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\,\vec{k}$, where S is the open surface of the rectangular parallelopiped formed by the planes $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ above the xoy – plane.

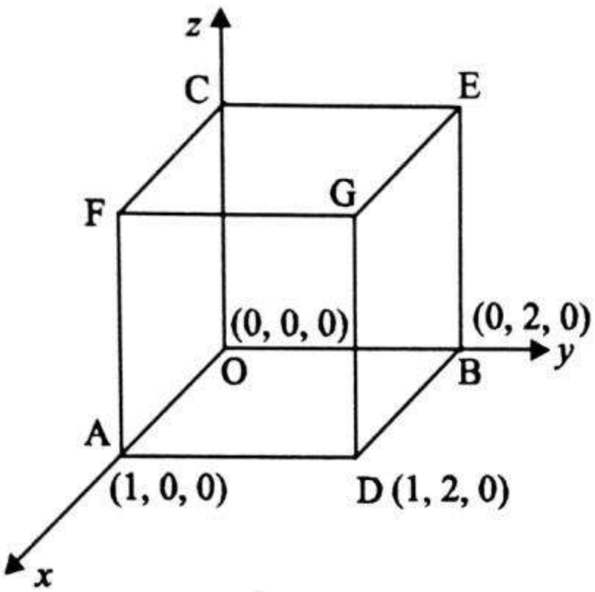
Solution:

Stoke's theorem is $\int_c \vec{F}\cdot d\vec{r} = \iint_S curl\,\vec{F}\cdot\hat{n}\, ds$

Given $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\,\vec{k}$

$\vec{F}\cdot d\vec{r} = xydx - 2yzdy - zx\,dz$

L.H.S $= \int_c \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$



In xy plane $z = 0 \Rightarrow dz = 0$

$\vec{F} \cdot d\vec{r} = xydx$

On OA: $y = 0 \Rightarrow dy = 0$, x varies from 0 to 1.

$$\Rightarrow \int_{OA} \vec{F} \cdot d\vec{r} = \int_0^1 0\, dx$$

$$= 0$$

On AB: $x = 1 \Rightarrow dx = 0$, y varies from 0 to 2.

$$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 0\, dy$$

$$= 0$$

On BC: $y = 2 \Rightarrow dy = 0$, x varies from 1 to 0.

$$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_1^0 2x\, dx$$

$$= 2\left[\frac{x^2}{2}\right]_1^0 = -1$$

On CO: $x = 0 \Rightarrow dx = 0$, y varies from 2 to 0.

$$\Rightarrow \int_{CO} \vec{F} \cdot d\vec{r} = \int_2^0 0\, dy$$

$$= 0$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = 0 + 0 - 1 + 0 = -1 \quad \ldots (1)$$

R.H.S $= \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix}$$

$$= \vec{i}\,(0 + 2y) - \vec{j}\,(-z - 0) + \vec{k}\,(0 - x)$$

$$= 2y\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Given S is an open surface consisting of the 5 faces of the cube except, $xy -$ plane.

$$\int\int curl\,\vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \cdots + \iint_{S5}$$

$$curl\,\vec{F} = 2y\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Faces	Plane	ds	$\hat{n}$	$curl\,\vec{F}\cdot\hat{n}$	
Top (S_1)	xy	$dxdy$	$\vec{k}$	$-x$	$\int_0^2\int_0^1 -x\,dxdy$
Left (S_2)	xz	$dxdz$	$-\vec{j}$	$-z$	$\int_0^3\int_0^1 -z\,dxdz$
Right (S_3)	xz	$dxdz$	$\vec{j}$	z	$\int_0^3\int_0^1 z\,dxdz$
Back (S_4)	yz	$dydz$	$-\vec{i}$	$-2y$	$\int_0^3\int_0^2 -2y\,dydz$
Front (S_5)	yz	$dydz$	$\vec{i}$	$2y$	$\int_0^3\int_0^2 2y\,dydz$

On S_1: $\int_0^2\int_0^1 (-1)\,dxdy$

$$= -\int_0^2 \left[\frac{x^2}{2}\right]_0^1 dy$$

$$= -\frac{1}{2}\int_0^2 dy$$

$$= -\frac{1}{2}[y]_0^2 = -1$$

On S_2: $\int_0^3\int_0^1 -z\,dxdz$

$$= -\int_0^3 [zx]_0^1\,dz$$

$$= -\int_0^2 z\,dz$$

$$= -\left[\frac{z^2}{2}\right]_0^3 = -\frac{9}{2}$$

On S_3: $\int_0^3\int_0^1 z\,dxdz$

$$= \int_0^3 [zx]_0^1 \, dz$$

$$= 2\int_0^3 z \, dz$$

$$= \left[\frac{z^2}{2}\right]_0^3 = \frac{9}{2}$$

On S_4: $\int_0^3 \int_0^2 -2y \, dydz$

$$= -2\int_0^3 \left[\frac{y^2}{2}\right]_0^2 dz$$

$$= -4\int_0^3 dz$$

$$= -4[z]_0^3 = -12$$

On S_5: $\int_0^3 \int_0^2 2y \, dydz$

$$= 2\int_0^3 \left[\frac{y^2}{2}\right]_0^2 dz$$

$$= 4\int_0^3 dz$$

$$= 4[z]_0^3 = 12$$

$$\therefore \iint_S curl\,\vec{F} \cdot \hat{n} \, ds = -1 - \frac{9}{2} + \frac{9}{2} - 12 + 12 = -1 \quad \ldots (2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n} \, ds$

Hence Stoke's theorem is verified.

Example: 2.78 Verify Stoke's theorem for $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\,\vec{k}$, where S is the open surface of the cube formed by the planes $x = \pm a$, $y = \pm a$, and $z = \pm a$ in which the plane $z = -a$ is a cut.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n} \, ds$

Given $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\,\vec{k}$

$\vec{F} \cdot d\vec{r} = y^2zdx + z^2xdy + x^2y \, dz$

This square ABCD lies in the plane $z = -a \Rightarrow dz = 0$

$\therefore \vec{F} \cdot d\vec{r} = -ay^2dx + a^2x \, dy$

L.H.S = $\int_c \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

On AB: $y = -a \Rightarrow dy = 0$, x varies from $-a$ to a.

$$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} -a^3 \, dx$$

$$= -a^3 [x]_{-a}^{a}$$

$$= -a^3(2a) = -2a^4$$

On BC: $x = a \Rightarrow dx = 0$, y varies from $-a$ to a.

$$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} a^3\, dy$$

$$= a^3\, [y]_{-a}^{a}$$

$$= a^3(2a) = 2a^4$$

On CD: $y = a \Rightarrow dy = 0$, x varies from a to $-a$.

$$\Rightarrow \int_{CD} \vec{F} \cdot d\vec{r} = \int_{a}^{-a} -a^3\, dx$$

$$= -a^3\, [x]_{a}^{-a}$$

$$= -a^3(-2a) = 2a^4$$

On DA: $x = -a \Rightarrow dx = 0$, y varies from a to $-a$.

$$\Rightarrow \int_{DA} \vec{F} \cdot d\vec{r} = \int_{a}^{-a} -a^3\, dy$$

$$= -a^3\, [y]_{a}^{-a}$$

$$= -a^3(-2a) = 2a^4$$

$$\therefore \int_{c} \vec{F} \cdot d\vec{r} = -2a^4 + 2a^4 + 2a^4 + 2a^4 = 4a^4 \quad \ldots (1)$$

R.H.S = $\iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$$curl\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \vec{i}\,(x^2 - 2xz) - \vec{j}\,(y^2 - 2xy) + \vec{k}\,(z^2 - 2yz)$$

Given S is an open surface consisting of the 5 faces of the cube except, $z = -a$.

$$\int\int curl\, \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \cdots + \iint_{S5}$$

$$curl\, \vec{F} = 2y\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Faces	Plane	ds	$\hat{n}$	Eqn	$curl\, \vec{F} \cdot \hat{n}$	$\nabla \times \vec{F} \cdot \hat{n}$
Top (S_1)	xy	$dxdy$	$\vec{k}$	$z = a$	$z^2 - 2yz$	$a^2 - 2ay$
Left (S_2)	xz	$dxdz$	$-\vec{j}$	$y = -a$	$y^2 - 2xy$	$a^2 + 2ax$
Right (S_3)	xz	$dxdz$	$\vec{j}$	$y = a$	$-(y^2 - 2xy)$	$-(a^2 - 2ax)$
Back (S_4)	yz	$dydz$	$-\vec{i}$	$x = -a$	$-(x^2 - 2xz)$	$-(a^2 + 2az)$
Front (S_5)	yz	$dydz$	$\vec{i}$	$x = a$	$x^2 - 2xz$	$a^2 - 2az$

On S_1: $\int_{-a}^{a} \int_{-a}^{a} (a^2 - 2ay)\, dxdy$

$$= \int_{-a}^{a} [(a^2 x - 2ayx)]_{-a}^{a}\, dy$$

$$= \int_{-a}^{a} (a^3 - 2a^2y) - (-a^3 + 2a^2y)\, dy$$

$$= \int_{-a}^{a} 2a^3 - 4a^2y\, dy$$

$$= \left[2a^3y - 4a^2\frac{y^2}{2}\right]_{-a}^{a}$$

$$= (2a^4 - 2a^4) - (-2a^4 - 2a^4)$$

$$= 2a^4 - 2a^4 + 2a^4 + 2a^4$$

$$= 4a^4$$

On $S_2 + S_3$: $\int_{-a}^{a}\int_{-a}^{a}(a^2 + 2ax)\, dxdz + \int_{-a}^{a}\int_{-a}^{a} -(a^2 - 2ax)\, dxdz$

$$= \int_{-a}^{a}\int_{-a}^{a}(a^2 + 2ax - a^2 + 2ax)\, dx\, dz$$

$$= \int_{-a}^{a}\int_{-a}^{a} 4ax\, dx\, dz$$

$$= 4a\int_{-a}^{a}\left[\frac{x^2}{2}\right]_{-a}^{a} dz$$

$$= 2a^3\int_{-a}^{a} dz$$

$$= 2a^3\,[z]_{-a}^{a}$$

$$= 2a^3(0) = 0$$

On $S_4 + S_5$: $\int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2az)\, dydz + \int_{-a}^{a}\int_{-a}^{a}(a^2 - 2az)\, dydz$

$$= \int_{-a}^{a}\int_{-a}^{a}(-a^2 - 2az + a^2 - 2az)\, dy\, dz$$

$$= \int_{-a}^{a}\int_{-a}^{a} -4az\, dy\, dz$$

$$= -4a\int_{-a}^{a}[zy]_{-a}^{a} dz$$

$$= -4a\int_{-a}^{a} z(2a) dz$$

$$= -6a^2\left[\frac{z^2}{2}\right]_{-a}^{a}$$

$$= -3a^2(a^2 - a^2) = 0$$

$\therefore \iint_S curl\, \vec{F} \cdot \hat{n}\, ds = 4a^4 + 0 + 0 = 4a^4 \quad \ldots (2)$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

Hence Stoke's theorem is verified.

Example: 2.79 Evaluate $\int_c \vec{F} \cdot d\vec{r}$ by stoke's theorem, where $\vec{F} = y^2\vec{i} + x^2\vec{j} + (x + z)\,\vec{k}$, and C is the boundary of the triangle with vertices at $(0,0,0), (1,0,0)$ and $(1,1,0)$.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds \quad \ldots (1)$

Given $\vec{F} = y^2\vec{i} + x^2\vec{j} + (x + z)\,\vec{k}$

And C is triangle $(0,0,0), (1,0,0)$ and $(1,1,0)$.

Since z −coordinate of each vertex is zero the triangle lies in xy − plane with corners $(0,0), (1,0)$ and $(1,1)$.

To evaluate : $\iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$In\ xy\ - plane\ \hat{n} = \vec{k}, ds = dxdy$

$$curl\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}\,(0) - \vec{j}\,(-1) + \vec{k}\,(2x - 2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

$$curl\, \vec{F} \cdot \hat{n} = \left(\vec{j} + 2(x-y)\vec{k}\right) \cdot \vec{k}$$

$$= 2(x-y)$$

Limits:

x varies from y to 1.

y varies from 0 to 1.

$$\therefore \iint_S curl\, \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_y^1 2(x-y)dxdy$$

$$= 2\int_0^1 \left[\frac{x^2}{2} - xy\right]_y^1 dy$$

$$= 2\int_0^1 \left(\frac{1}{2} - y - \frac{y^2}{2} + y^2\right) dy$$

$$= 2\left[\frac{y}{2} - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

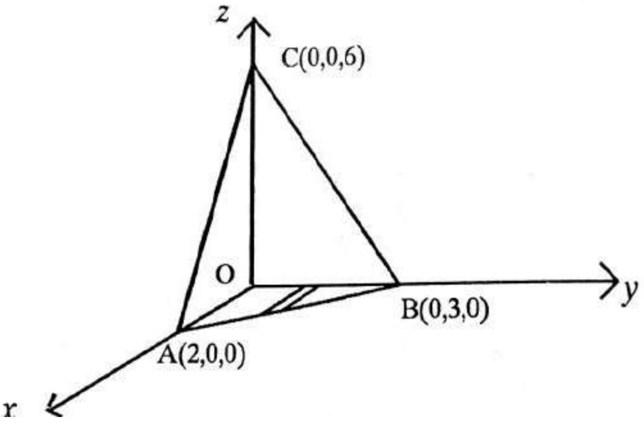
$$= 2\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6}\right] = \frac{1}{3}$$

From (1), $\int_c \vec{F} \cdot d\vec{r} = \frac{1}{3}$

Example: 2.80 Evaluate the integral $\int_c (x+y)dx + (2x-z)dy + (y+z)dz$, where C is the boundary of the triangle with vertices $(2,0,0), (0,3,0)$ and $(0,0,6)$ using stoke's theorem.

Solution:



Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\; ds$

Given $\vec{F} \cdot d\vec{r} = (x+y)dx + (2x-z)dy + (y+z)dz$

$\therefore \vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}\,(1-1) - \vec{j}\,(0) + \vec{k}\,(2-1)$$

$$= 2\vec{i} + \vec{k}$$

Given C is the triangle with vertices $(2,0,0), (0,3,0)\; and\; (0,0,6)$.

Equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\Rightarrow 3x + 2y + z = 6$$

Let $\varphi = 3x + 2y + z - 6$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 3\vec{i} + 2\vec{j} + \vec{k}$$

$$|\nabla\varphi| = \sqrt{9+4+1} = \sqrt{14}$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}$$

Let R be the projection on XY –plane.

$$\therefore ds = \frac{dx\,dy}{\left|\hat{n}\cdot\vec{k}\right|} = \frac{dx\,dy}{\left(\frac{1}{\sqrt{14}}\right)}$$

Where $\hat{n} \cdot \vec{k} = \left(\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}\right) \cdot \vec{k}$

$$= \frac{1}{\sqrt{14}}$$

Now $curl\,\vec{F} \cdot \hat{n} = \left(2\vec{i} + \vec{k}\right) \cdot \left(\frac{3\vec{i}+2\vec{j}+\vec{k}}{\sqrt{14}}\right)$

$$= \frac{6+1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$$

$$\Rightarrow \iint_S curl\,\vec{F}\cdot\,\hat{n}\,ds = \iint_R \frac{7}{\sqrt{14}}\frac{dxdy}{\left(\frac{1}{\sqrt{14}}\right)}$$

$$= 7\iint_R dxdy$$

$$= 7\,[Area\ of\ the\ triangle]$$

$$= 7\left[\frac{1}{2}\,(2)\,(3)\right] = 21\quad \left[\because Area\ of\ the\ triangle = \frac{1}{2}bh\right]$$

Example: 2.81 Evaluate by Stoke's theorem $\int_c (e^x dx + 2ydy - dz)$, where C is the Curve $x^2 + y^2 = 4, z = 2$.

Solution:

Stoke's theorem is $\int_c \vec{F}\cdot d\vec{r} = \iint_S curl\,\vec{F}\cdot\,\hat{n}\,ds$ ... (1)

Given $\vec{F}\cdot d\vec{r} = e^x dx + 2ydy - dz$

$\therefore \vec{F} = e^x\vec{i} + 2y\vec{j} - \vec{k}$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\,(0-1)$$

$$= \vec{0}$$

$\therefore$ (1) $\Rightarrow \int_c \vec{F}\cdot d\vec{r} = 0$

$(i.e)\int_c (e^x dx + 2ydy - dz) = 0$

Example: 2.82 Evaluate $\int_c \left(yz\vec{i} + xz\vec{j} + xy\vec{k}\right)\cdot d\vec{r}$, where C is the boundary of the surface S.

Solution:

Given $\vec{F} = yz\vec{i} + xz\vec{j} + xy\,\vec{k}$

Stoke's theorem is $\int_c \vec{F}\cdot d\vec{r} = \iint_S \left(\nabla\times\vec{F}\right)\cdot\,\hat{n}\,ds$ ... (1)

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \vec{i}\,(x-x) - \vec{j}\,(y-y) + \vec{k}\,(z-z)$$

$$= \vec{0}$$

$\therefore (1) \Rightarrow \int_c \vec{F} \cdot d\vec{r} = 0$

Exercise: 2.5

1. Verify Stoke's theorem for the function $\vec{F} = x^2\vec{i} + xy\vec{j}$, integrated round the square in the $z = 0$ plane whose sides are along the lines $x = 0, y = 0, x = a, y = a$. **Ans:** $\frac{a^3}{2}$

2. Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **Ans:** $-\pi$

3. Evaluate $\int_c [xydx + xy^2dy]$ by Stoke's theorem where C is the square in xy plane with vertices $(1,0), (-1,0), (0,1)$ *and* $(0,-1)$ **Ans:** $\frac{1}{2}$

4. Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$, where S is the open surface of the cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the xy plane.

   Ans: Common value $= -4$

5. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j} + xyz\vec{k}$, over the surface of the box bounded by the planes $x = 0, y = 0, x = a, y = b, z = c$ above the xy plane.

   Ans: Common value $2ab^2$

6. Verify Stoke's theorem for $\vec{F} = xy\vec{i} - 2yz\vec{j} - zx\vec{k}$, where S is the open surface of the rectangular parallelepiped formed by the planes $x = 0, x = 1, y = 0, y = 2, z = 3$ above the xoy plane.

   Ans: Common value -1

7. Verify Stoke's theorem for $\vec{F} = -y\vec{i} + 2yz\vec{j} + y^2\vec{k}$, where S is the half of the sphere $x^2 + y^2 + z^2 = a^2$ and C is the circular boundary on the xoy plane.

   Ans: Common value $= \pi a^2$

8. Using Stoke's theorem $\int_c \vec{F} \cdot d\vec{r}$ where $\vec{F} = (\sin x - y)\vec{i} - \cos x\,\vec{j}$ and C is the boundary of the triangle whose vertices $(0,0), \left(\frac{\pi}{2}, 0\right)$ *and* $\left(\frac{\pi}{2}, 1\right)$ **Ans:** $\frac{\pi}{4} + \frac{2}{\pi}$

2.6 GAUSS DIVERGENCE THEOREM

This theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

Statement of Gauss Divergence theorem

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S, then

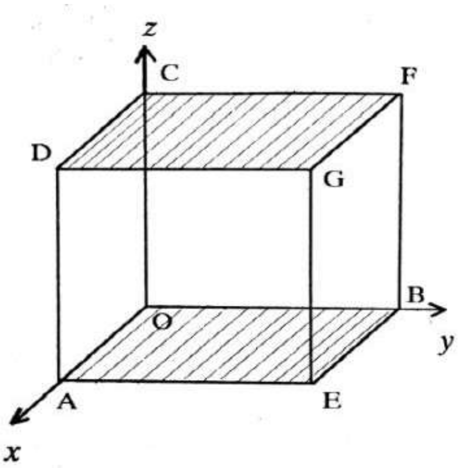
$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Where $\hat{n}$ is the unit outward normal to the surface S and $dV = dxdydz$

Problems based on gauss Divergence theorem

Example: 2.83 Verify the G.D.T for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0,\ x = 1,$ $y = 0,\ y = 1,\ z = 0,\ z = 1.$

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = 4z - 2y + y$$

$$= 4z - y$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\,dxdydz$$

$$= \int_0^1 \int_0^1 [(4xz - yz)]_0^1\ dydz$$

$$= \int_0^1 \int_0^1 (4z - y)\,dydz$$

$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1\ dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right)\ dz$$

$$= \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = \left(2 - \frac{1}{2}\right) - 0 = \frac{3}{2}$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F} \cdot \hat{n}$	Equation	$\vec{F} \cdot \hat{n}$ on S	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z = 0$	0	$\int_0^1 \int_0^1 0\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z = 1$	y	$\int_0^1 \int_0^1 y\, dxdy$

$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	y^2	$y=0$	0	$\int_0^1\int_0^1 0\,dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$-y^2$	$y=1$	-1	$\int_0^1\int_0^1 -1dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-4xz$	$x=0$	0	$\int_0^1\int_0^1 0\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$4xz$	$x=1$	$4z$	$\int_0^1\int_0^1 4z\,dydz$

$(i)\iint_{S1} \vec{F}\cdot\hat{n}\,ds + \iint_{S2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdy + \int_0^1\int_0^1 y\,dxdy$

$= 0 + \int_0^1\int_0^1 y\,dxdy$

$= \int_0^1 [yx]_0^1\, dy$

$= \int_0^1 y\,dy$

$= \left[\frac{y^2}{2}\right]_0^1 \quad = \frac{1}{2} - 0 \quad = \frac{1}{2}$

$(ii)\iint_{S3} \vec{F}\cdot\hat{n}\,ds + \iint_{S4} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdz + \int_0^1\int_0^1 -1\,dxdz$

$= 0 + \int_0^1\int_0^1 -1\,dxdz$

$= -\int_0^1 [x]_0^1\, dz$

$= -\int_0^1 dz$

$= -[z]_0^1 \quad = -[1]$

$(iii)\iint_{S5} \vec{F}\cdot\hat{n}\,ds + \iint_{S6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dydz + \int_0^1\int_0^1 4zdydz$

$= 0 + \int_0^1\int_0^1 4z\,dydz$

$= \int_0^1 [4zy]_0^1\, dz$

$= \int_0^1 4z\,dz$

$= 4\left[\frac{z^2}{2}\right]_0^1 \quad = 4\left(\frac{1}{2} - 0\right) \quad = 2$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

$= \quad (i) + (ii) + (iii)$

$= \frac{1}{2} - 1 + 2 \quad = \frac{3}{2}$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$

Hence Gauss divergence theorem is verified.

Example: 2.84 Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$. (OR)

Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped bounded by $x = 0, x = a, y = 0, y = b, z = 0, z = c$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= 2\int_0^c \int_0^b \int_0^a (x + y + z)dxdydz$$

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$$= 2\int_0^c \int_0^b \left[\left(\frac{x^2}{2} + xy + xz\right)\right]_0^a dydz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az\right) dydz$$

$$= 2\int_0^c \left(\frac{a^2 y}{2} + \frac{ay^2}{2} + azy\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + azb\right) dz$$

$$= 2\left[\frac{a^2 bz}{2} + \frac{ab^2 z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left(\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2}\right)$$

$$= abc(a + b + c)$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F} \cdot \hat{n}$	Eqn	$\vec{F} \cdot \hat{n}$ on S	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-(z^2 - xy)$	$z = 0$	xy	$\int_0^b \int_0^a xy\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$(z^2 - xy)$	$z = c$	$c^2 - xy$	$\int_0^b \int_0^a c^2 - xy\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-(y^2 - xz)$	$y = 0$	xz	$\int_0^c \int_0^a xz\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$(y^2 - xz)$	$y = b$	$b^2 - xz$	$\int_0^c \int_0^a b^2 - xz\, dxdz$

$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(x^2 - yz)$	$x = 0$	yz	$\int_0^c \int_0^b yz\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$(x^2 - yz)$	$x = a$	$a^2 - yz$	$\int_0^c \int_0^b a^2 - yz\, dydz$

$$(i) \iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_0^b \int_0^a xy\, dxdy + \int_0^b \int_0^a c^2 - xy\, dxdy$$

$$= \int_0^b \int_0^a c^2\, dxdy$$

$$= c^2 \int_0^a dx \int_0^b dy$$

$$= c^2 [x]_0^a [y]_0^b \quad = c^2 ab$$

$$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^c \int_0^a xz\, dxdz + \int_0^c \int_0^a b^2 - xz\, dxdz$$

$$= \int_0^c \int_0^a b^2\, dxdz$$

$$= b^2 \int_0^a dx \int_0^c dz$$

$$= b^2 [x]_0^a [z]_0^c \quad = b^2 ac$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^c \int_0^b yz\, dydz + \int_0^c \int_0^b a^2 - yz\, dydz$$

$$= \int_0^c \int_0^b a^2\, dydz$$

$$= a^2 \int_0^b dy \int_0^c dz$$

$$= a^2 [y]_0^b [z]_0^c \quad = a^2 bc$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= abc(a + b + c)$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.

Example: 2.85 Verify divergence theorem for $\vec{F} = (2x - z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (2x - z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$

$\nabla \cdot \vec{F} = 2 + x^2 - 2xz$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz)\, dxdydz$

$= \int_0^1 \int_0^1 \left[\left(2x + \frac{x^3}{3} - \frac{2zx^2}{2}\right)\right]_0^1 dydz$

$= \int_0^1 \int_0^1 \left(2 + \frac{1}{3} - z\right) dydz$

$= \int_0^1 \left(2y + \frac{1}{3}y - zy\right)_0^1 dz$

$= \int_0^1 \left(2 + \frac{1}{3} - z\right) dz$

$= \left[2z + \frac{1}{3}z - \frac{z^2}{2}\right]_0^1$

$= \left(2 + \frac{1}{3} - \frac{1}{2}\right) - 0 = \frac{11}{6}$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	dS	$\hat{n}$	$\vec{F} \cdot \hat{n}$	**Equation**	$\vec{F} \cdot \hat{n}$ **on S**	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	dxdy	$-\vec{k}$	xz^2	$z = 0$	0	$\int_0^1 \int_0^1 0\, dxdy$
S_2(Top)	xy	dxdy	$\vec{k}$	$-xz^2$	$z = 1$	$-x$	$\int_0^1 \int_0^1 (-x)\, dxdy$
$S_3(Left)$	xz	dxdz	$-\vec{j}$	$-x^2y$	$y = 0$	0	$\int_0^1 \int_0^1 0\, dxdz$
$S_4(Right)$	xz	dxdz	$\vec{j}$	x^2y	$y = 1$	x^2	$\int_0^1 \int_0^1 x^2 dxdz$
$S_5(Back)$	yz	dydz	$-\vec{i}$	$-(2x - z)$	$x = 0$	z	$\int_0^1 \int_0^1 z\, dydz$
$S_6(Front)$	yz	dydz	$\vec{i}$	$(2x - z)$	$x = 1$	$2 - z$	$\int_0^1 \int_0^1 2 - z\, dydz$

$(i) \iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdy + \int_0^1 \int_0^1 (-x)\, dxdy$

$= \int_0^1 \int_0^1 (-x)\, dxdy$

$$= -\int_0^1 \left[\frac{x^2}{2}\right]_0^1 dy$$

$$= -\int_0^1 \frac{1}{2}\, dy$$

$$= -\left[\frac{1}{2}y\right]_0^1 \quad = -\left(\frac{1}{2} - 0\right) \quad = \frac{-1}{2}$$

$$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdz + \int_0^1 \int_0^1 x^2\, dxdz$$

$$= \int_0^1 \int_0^1 x^2\, dxdz$$

$$= \int_0^1 \left[\frac{x^3}{3}\right]_0^1 dz$$

$$= \int_0^1 \frac{1}{3}\, dz$$

$$= \left[\frac{1}{3}z\right]_0^1 \quad = \left(\frac{1}{3} - 0\right) \quad = \frac{1}{3}$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 z\, dydz + \int_0^1 \int_0^1 (2 - z) dydz$$

$$= \int_0^1 \int_0^1 2\, dydz$$

$$= 2 \int_0^1 [y]_0^1\, dz$$

$$= 2 \int_0^1 dz$$

$$= 2\, [z]_0^1 \quad = 2$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= -\frac{1}{2} + \frac{1}{3} + 2 \quad = \frac{11}{6}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.

Example: 2.86 Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube bounded by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = 2x + y$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + y) dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[\left(2\frac{x^2}{2}+yx\right)\right]_{-1}^{1} dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[(1+y)-(1-y)]dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[2y]dydz$$

$$= \int_{-1}^{1}\left(2\frac{y^2}{2}\right)_{-1}^{1} dz$$

$$= \int_{-1}^{1}[(1)-((-1)^2)]\, dz$$

$$= \int_{-1}^{1}[0]\, dz$$

$$= 0$$

Now, L.H.S $= \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Equation	$\vec{F}\cdot\hat{n}$ on S	$= \iint_S \vec{F}\cdot\hat{n}\,ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z=-1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\,dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z=1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\,dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-z$	$y=-1$	$-z$	$\int_{-1}^{1}\int_{-1}^{1} -z\,dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	z	$y=1$	z	$\int_{-1}^{1}\int_{-1}^{1} zdxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-x^2$	$x=-1$	-1	$\int_{-1}^{1}\int_{-1}^{1} -1\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	x^2	$x=1$	1	$\int_{-1}^{1}\int_{-1}^{1} dydz$

$(i)\iint_{S1} \vec{F}\cdot\hat{n}\,ds + \iint_{S2} \vec{F}\cdot\hat{n}\,ds = \int_{-1}^{1}\int_{-1}^{1} y\,dxdy + \int_{-1}^{1}\int_{-1}^{1} y\,dxdy$

$$= \int_{-1}^{1}\int_{-1}^{1} 2y\,dxdy$$

$$= 2\int_{-1}^{1}[xy]_{-1}^{1}\, dy$$

$$= 2\int_{-1}^{1}[(y)-(-y)]\, dy$$

$$= 2\int_{-1}^{1} 2ydy$$

$$= 4\left[\frac{y^2}{2}\right]_{-1}^{1} = 4\left[\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)\right] = 0$$

$(ii)\iint_{S3} \vec{F}\cdot\hat{n}\,ds + \iint_{S4} \vec{F}\cdot\hat{n}\,ds = \int_{-1}^{1}\int_{-1}^{1} -z\,dxdz + \int_{-1}^{1}\int_{-1}^{1} z\,dxdz$

$$= \int_{-1}^{1}\int_{-1}^{1} 0\,dxdz$$

$$= 0$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = -\int_{-1}^{1}\int_{-1}^{1} dxdz + \int_{-1}^{1}\int_{-1}^{1} dxdz$$

$$= 0$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

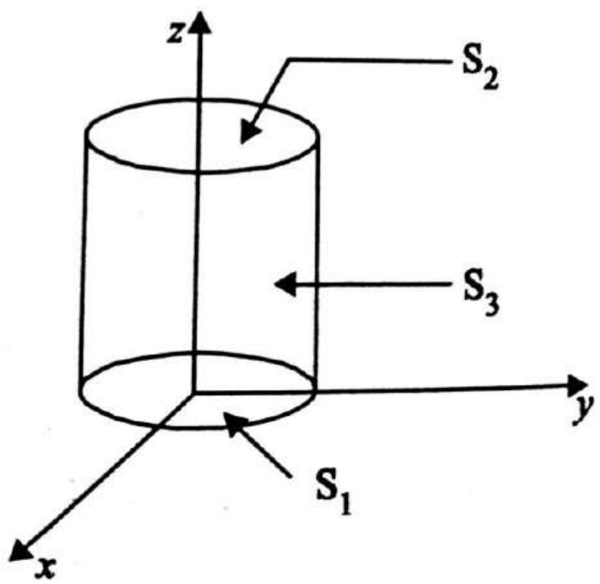
$$= 0$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iiint_{V} \nabla \cdot \vec{F}\, dv$$

Hence , Gauss divergence theorem is verified.

Example: 2.87 Verify divergence theorem for the function $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface bounded by the cylinder $x^2 + y^2 = 4$ and $z = 0, z = 3$.

Solution:



Gauss divergence theorem is $\iint_{S} \vec{F} \cdot \hat{n}\, ds = \iiint_{V} \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$$\nabla \cdot \vec{F} = 4 - 4y + 2z$$

Limits:

$z = 0$ to 3

$x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$

$$\Rightarrow y = \pm\sqrt{4 - x^2}$$

$\therefore y = -\sqrt{4 - x^2}$ to $\sqrt{4 - x^2}$

Put $y = 0 \Rightarrow x^2 = 4$

$$\Rightarrow \; x = \pm\, 2$$

$\therefore y = \; -2 \; to \; 2$

$\therefore$ R.H.S $= \iiint\limits_V \; \nabla \cdot \vec{F} \, dv$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4 - 4y + 2z) \, dzdydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3 \, dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) dydx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dydx$$

$$= 2\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} 21 \, dydx \qquad \left[\begin{matrix} \because \int_{-a}^{a} f(x)\,dx = 2 \int_0^a f(x) \; if \; f(x) \; is \; even \\ = 0 \; if \; f(x) \; is \; odd \end{matrix}\right]$$

$$= 42 \int_{-2}^{2} [y]_0^{\sqrt{4-x^2}} \, dx$$

$$= 42 \int_{-2}^{2} \sqrt{4 - x^2} \, dx$$

$$= 42 \times 2 \int_{0}^{2} \sqrt{4 - x^2} \, dx \qquad [\because even \; function]$$

$$= 84 \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2} sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84 \left[0 + 2 \, sin^{-1}(1)\right]$$

$$= 84 \left[2 \times \frac{\pi}{2}\right]$$

$$= 84\,\pi$$

L.H.S $= \iint\limits_S \; \vec{F} \cdot \hat{n} \, ds$

$$= \iint\limits_{S1} + \iint\limits_{S2} + \iint\limits_{S3}$$

Along S_1 (bottom):

xy −plane $\Rightarrow z = 0, dz = 0$

And $ds = dxdy, \;\; \hat{n} \; = \; -\vec{k}$

$\therefore \vec{F} \cdot \hat{n} \;= \left(4x\,\vec{\imath} - \; 2y^2\,\vec{\jmath} + \; z^2\,\vec{k}\right) \cdot \left(-\vec{k}\right)$

$$= \; -z^2 = 0$$

$\therefore \iint\limits_{S1} \; \vec{F} \cdot \hat{n}\,ds = \iint\limits_{S1} \; 0 = 0$

Along S_2 (top):

xy −plane $\Rightarrow z = 3, dz = 0$

And $ds = dxdy, \;\; \hat{n} \; = \; \vec{k}$

$\therefore \vec{F} \cdot \hat{n} \;= \left(4x\,\vec{\imath} - \; 2y^2\,\vec{\jmath} + \; z^2\,\vec{k}\right) \cdot \left(-\vec{k}\right)$

$$= \ z^2 = 9$$

$$\therefore \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \iint_{S2} 9\, dxdy$$

$$= \iint_{R} 9\, dxdy$$

$$= 9\ (Area\ of\ the\ circle)$$

$$= 9\ (\pi r^2) \qquad [\because r = 2]$$

$$= 36\,\pi$$

Along S_3 (curved surface):

Given $x^2 + y^2 = 4$

Let $\varphi = x^2 + y^2 - 4$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{4} = 4$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i}+y\vec{j})}{4}$$

$$= \frac{x\vec{i}+y\vec{j}}{2}$$

The cylindrical coordinates are

$x = 2\cos\theta,\ y = 2\sin\theta\ ds = 2dzd\theta$

Where z varies from 0 to 3

θ varies from 0 to 2π

Now $\vec{F} \cdot \hat{n} = \left(4x\,\vec{i} - 2y^2\,\vec{j} + z^2\,\vec{k}\right) \cdot \left(\frac{x\vec{i}+y\vec{j}}{2}\right)$

$$= 2x^2 - y^3$$

$$= 2(2\cos\theta)^2 - (2\sin\theta)^3$$

$$= 8\cos^2\theta - 8\sin^3\theta$$

$$= 8\left[\frac{1+\cos 2\theta}{2} - \left(\frac{3\sin\theta - \sin 3\theta}{4}\right)\right]$$

$$\therefore \iint_{S3} \vec{F} \cdot \hat{n}\, ds = 8\int_0^{2\pi}\int_0^3 \left(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right) 2dzd\theta$$

$$= 16\int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right) [z]_0^3\, d\theta$$

$$= 48\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{3\cos\theta}{4} - \frac{\cos 3\theta}{12}\right]_0^{2\pi}$$

$$= 48\left[\left(\frac{2\pi}{2} + \frac{3}{4} - \frac{1}{12}\right) - \left(\frac{3}{4} - \frac{1}{12}\right)\right]$$

$$= 48\,\pi$$

L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = 0 + 36\,\pi + 48\pi$

$= 84\,\pi$

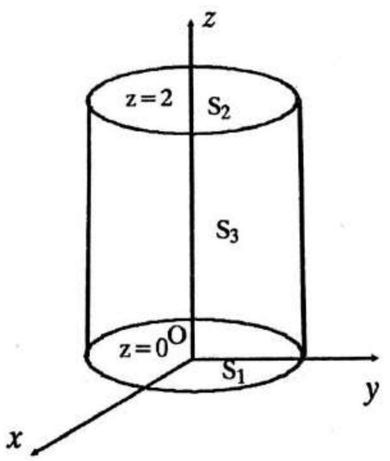
$\therefore$ L.H.S = R.H.S

$(i.e) \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Hence Gauss divergence theorem is verified.

Example: 2.88 Verify divergence theorem for the function $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$ and $z = 0, z = 2.$

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = y\vec{i} + x\vec{j} + z^2\vec{k}$

$\nabla \cdot \vec{F} = 2z$

Limits:

$z = 0$ to 2

$x^2 + y^2 = 9 \Rightarrow y^2 = 9 - x^2$

$\Rightarrow y = \pm\sqrt{9 - x^2}$

$\therefore y = -\sqrt{9 - x^2}$ to $\sqrt{9 - x^2}$

Put $y = 0 \Rightarrow x^2 = 9$

$\Rightarrow x = \pm 3$

$\therefore y = -3$ to 3

$\therefore$ R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^2 (2z)\, dzdydx$

$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[2\frac{z^2}{2}\right]_0^2 dydx$

$$= 4\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dydx$$

$$= 4\int_{-3}^{3}[y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dx$$

$$= 4\int_{-3}^{3} 2\sqrt{9-x^2}\, dx$$

$$= 8 \times 2\int_{0}^{3}\sqrt{9-x^2}\, dx \qquad [\because even\ function]$$

$$= 16\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}sin^{-1}\frac{x}{3}\right]_0^3$$

$$= 16\left[0 + \frac{9}{2}\, sin^{-1}(1)\right]$$

$$= 16\left[\frac{9}{2} \times \frac{\pi}{2}\right]$$

$$= 36\,\pi$$

L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds$

$$= \iint_{S1} + \iint_{S2} + \iint_{S3}$$

Along S_1 (bottom):

xy −plane $\Rightarrow z = 0, dz = 0$

And $ds = dxdy,\ \hat{n} = -\vec{k}$

$\therefore \vec{F} \cdot \hat{n} = (y\,\vec{\imath} + x\,\vec{\jmath} + z^2\,\vec{k}) \cdot (-\vec{k})$

$= -z^2 = 0$

$\therefore \iint_{S1} \vec{F} \cdot \hat{n}\, ds = \iint_{S1} 0 = 0$

Along S_2 (top):

xy −plane $\Rightarrow z = 2, dz = 0$

And $ds = dxdy,\ \hat{n} = \vec{k}$

$\therefore \vec{F} \cdot \hat{n} = (y\,\vec{\imath} + x\,\vec{\jmath} + z^2\,\vec{k}) \cdot (\vec{k})$

$= z^2 = 4$

$\therefore \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \iint_{S2} 4\, dxdy$

$$= \iint_R 4\, dxdy$$

$$= 4\,(Area\ of\ the\ circle)$$

$$= 4\,(\pi r^2) \qquad [\because r = 2]$$

$$= 36\,\pi$$

Along S_3 (curved surface):

Given $x^2 + y^2 = 9$

Let $\varphi = x^2 + y^2 - 9$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{9} = 6$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i}+y\vec{j})}{6}$$

$$= \frac{x\vec{i}+y\vec{j}}{3}$$

The cylindrical coordinates are

$x = 3\cos\theta,\ y = 3\sin\theta$

$ds = 3dzd\theta$

Where z varies from 0 to 2

θ varies from 0 to 2π

Now $\vec{F}\cdot\hat{n} = \left(y\,\vec{i} + x\,\vec{j} + z^2\,\vec{k}\right)\cdot\left(\frac{x\vec{i}+y\vec{j}}{3}\right)$

$$= \frac{xy}{3} + \frac{xy}{3} = \frac{2xy}{3}$$

$$= \frac{2}{3}(3\cos\theta)\,(3\sin\theta)$$

$$= 2\times 3\sin\theta\cos\theta$$

$$= 3\sin 2\theta$$

$$\therefore \iint_{S3} \vec{F}\cdot\hat{n}\,ds = 3\int_0^{2\pi}\int_0^2 (\sin 2\theta)\,3dzd\theta$$

$$= 9\int_0^{2\pi}(\sin 2\theta)\,[z]_0^2\,d\theta$$

$$= 9\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= -9\,[1 - 1]$$

$$= 0$$

$$\text{L.H.S} = \iint_S \vec{F}\cdot\hat{n}\,ds = 0 + 36\,\pi + 0$$

$$= 36\,\pi$$

$\therefore$ L.H.S = R.H.S

$$(i.e)\ \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Hence Gauss divergence theorem is verified.

Example: 2.89 If S is any closed surface enclosing a volume V and if $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$, prove that $\iint_S \vec{F} \cdot \hat{n}\, ds = (a+b+c)V$. Deduce that $\iint_S \vec{F} \cdot \hat{n}\, ds = \frac{4\pi}{3}(a+b+c)$ if S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = a + b + c$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V (\mathrm{a+b+c})\, dv$$

$$= (a+b+c)V$$

If S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ then $V = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = (\mathrm{a+b+c})\,\frac{4\pi}{3}$$

$$= \frac{4\pi}{3}(a+b+c)$$

Example: 2.90 Using the divergence theorem of Gauss evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, and S is the sphere $x^2 + y^2 + z^2 = a^2$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$$\nabla \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$= 3(x^2 + y^2 + z^2)$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = 3\iiint_V (x^2 + y^2 + z^2)\, dxdydz$$

Here we have to use spherical polar co – ordinates.

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$x^2 + y^2 + z^2 = a^2$ and $dxdydz = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

$$\therefore 3\iiint_V (x^2 + y^2 + z^2)\, dxdydz = 3\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\left[\frac{r^5}{5}\sin\theta\right]_0^a d\theta\, d\varphi$$

$$= \frac{3a^5}{5}\int_0^{2\pi}[-\cos\theta]_0^{\pi}\, d\varphi$$

$$= \frac{3a^5}{5}\int_0^{2\pi}(-\cos\pi + \cos 0)\, d\varphi$$

$$= \frac{6a^5}{5}[\varphi]_0^{2\pi}$$

$$= \frac{6a^5}{5}(2\pi) = \frac{12\pi a^5}{5}$$

Example: 2.91 Show that $\iint_S \boldsymbol{curl\,\vec{F}} \cdot \hat{n}\, \boldsymbol{ds} = \mathbf{0}$ where S is any closed surface.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot (\nabla \times \vec{F})\, dv$$

where V is the volume of the closed surface S.

Since $\nabla \cdot (\nabla \times \vec{F}) = 0, we\ get \iiint_V \nabla \cdot (\nabla \times \vec{F})\, dv = 0$

$$\therefore \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\; dv = 0\ (or) \iint_S curl\vec{F} \cdot \hat{n}\, ds = 0$$

Example: 2.92 Prove that $\iint_S \frac{\vec{r} \cdot \hat{n}}{r^2} \boldsymbol{ds} = \iiint_V \frac{dv}{r^2}$

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

$$\therefore \iint_S \frac{\vec{r}}{r^2} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \left(\frac{\vec{r}}{r^2}\right) dv$$

Now $\nabla \cdot \frac{\vec{r}}{r^2} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r^2}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^2}\right)$$

$$= \sum \frac{r^2(1) - x\, 2r\frac{\partial r}{\partial x}}{r^4}$$

$$= \sum \frac{r^2 - 2xr\left(\frac{x}{r}\right)}{r^4}$$

$$= \sum \frac{r^2 - 2x^2}{r^4}$$

$$= \frac{3r^2 - 2(x^2 + y^2 + z^2)}{r^4}$$

$$= \frac{3r^2 - 2r^2}{r^4} = \frac{r^2}{r^4} = \frac{1}{r^2}$$

$$\therefore \iint_S \frac{\vec{r}}{r^2} \cdot \hat{n}\, ds = \iiint_V \frac{1}{r^2}\, dv$$

Example: 2.93 Evaluate $\iint_S \vec{\boldsymbol{r}} \cdot \hat{n}\, \boldsymbol{ds}$ where S is a closed surface using Gauss divergence theorem.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

$$\therefore \iint_S \vec{r} \cdot \hat{n}\, ds = \iiint_V (\nabla \cdot \vec{r})\, dv$$

$$= \iiint_V [\nabla \cdot (x\vec{i} + y\vec{j} + z\vec{k})]\, dv$$

$$= \iiint_V (1 + 1 + 1)\, dv$$

$$= 3\iiint_V dv$$

$$= 3V$$

Exercise: 2.5

1. Verify divergence theorem for the function $\vec{F} = (x^2 - yz)\vec{i} - (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ over the surface bounded by $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ **Ans:** 36
2. Verify divergence theorem for the function $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ **Ans:** Common value $= \frac{3}{2}$
3. Verify divergence theorem for the function $\vec{F} = (2x - z)\vec{i} - x^2y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

   Ans: Common value $= \frac{11}{6}$
4. Verify divergence theorem for $\vec{F} = xy^2\vec{i} + yz^2\vec{j} + zx^2\vec{k}$ over the region $x^2 + y^2 = 4\ and\ z = 0, z = 3$ **Ans:** Common value $= 84\pi$
5. Using divergence theorem, prove that (i) $\iint_S \vec{R} \cdot d\vec{S} = 3V$ (ii) $\iint_S \nabla r^2 \cdot d\vec{S} = 6V$
6. $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0, z = a$

   Ans: Common value $= \frac{3a}{2}$
7. $\vec{F} = (x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}$ over the parallelopiped bounded by the planes $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ **Ans:** Common value $= 2$
8. $\vec{F} = 2xy\vec{i} + y\,z^2\vec{j} + xz\vec{k}$ over the parallelopiped bounded by the planes $x = 0, y = 0, z = 0, x = 2, y = 1, z = 3$ **Ans:** Common value $= 20$
9. $\vec{F} = 2x^2y\vec{i} - y^2\vec{j} + 4\,xz^2\vec{k}$ taken over the region in the first octant bounded $x^2 + y^2 = 9$ and $x = 2$ **Ans:** Common value $= 180$
10. $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cuboid formed by the planes $x = 0, x = a, y = 0, y = b, z = 0, z = c$ **Ans:** Common value $= abc(a + b + c)$