BETA AND GAMMA FUNCTIONS

Beta function

The first eulerian integral $\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ where m>0, n>0 is called a Beta function and is denoted by B(m,n).

The quantities m and n are positive but not necessarily integers.

Example:-

Properties of Beta Function

$$\mathrm{B}(x,y) = \mathrm{B}(y,x).$$

$$\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$$

B(x,y) = B(x,y+1) + B(x+1,y)

xB(x,y +1) =y B(x+1,y)

$$\mathrm{B}(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$$

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

$$\mathrm{B}(x,y)\cdot\mathrm{B}(x+y,1-y) = \frac{\pi}{x\sin(\pi y)},$$

Gamma function

The Eulerian integral $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$,n>0 is called gamma function and is denoted by $\Gamma(x)$

Example:- $\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}dt$

$$= \lim_{n\to\infty}\int_0^n e^{-t}dt$$

$$= \lim_{n\to\infty} -e^{-t}\Big|_0^n$$

$$= \lim_{n\to\infty}\frac{-1}{e^n} - \frac{-1}{e^0} = \lim_{n\to\infty} 1 - \frac{1}{e^n} = 1$$

Recurrence formulae for gamma function

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$$ Use integration by parts.

$$u = t^x \qquad dv = e^{-t}dt$$

$$du = xt^{x-1}dt \qquad v = -e^{-t}$$

$$\Gamma(x+1) = -t^x e^{-t}\Big|_0^\infty - \int_0^\infty \left(-e^{-t}\right) xt^{x-1}dt$$

$$\Gamma(x+1) = 0 + \int_0^\infty xt^{x-1}e^{-t}dt = x\int_0^\infty t^{x-1}e^{-t}dt$$

$$\therefore\ \Gamma(x+1) = x\Gamma(x)$$

Relation between gamma and factorial

$$\Gamma(n+1) = n!$$

Other results

$$\Gamma(1/2) = \int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$$

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi} \qquad n = 1, 2, 3, \ldots$$

Relation between beta and gamma function

$$\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx.$$

Setting $x = y + \frac{1}{2}$ gives the more symmetric formula

$$\mathrm{B}(a,b) = \int_{-1/2}^{1/2} (\frac{1}{2}+y)^{a-1}(\frac{1}{2}-y)^{b-1}\,dy.$$

Now let $y = \frac{t}{2s}$ to obtain

$$(2s)^{a+b-1}\mathrm{B}(a,b) = \int_{-s}^{s} (s+t)^{a-1}(s-t)^{b-1}\,dt.$$

Multiply by e^{-2s} then integrate with respect to s, $0 \le s \le A$, to get

$$\mathrm{B}(a,b)\int_0^A e^{-2s}(2s)^{a+b-1}\,ds = \int_0^A\int_{-s}^{s} e^{-2s}(s+t)^{a-1}(s-t)^{b-1}\,dt\,ds.$$

Take the limit as $A \to \infty$ to get

$$\frac{1}{2}\mathrm{B}(a,b)\Gamma(a+b) = \lim_{A\to\infty} \int_0^A \int_{-s}^{s} e^{-2s}(s+t)^{a-1}(s-t)^{b-1}\, dt\, ds.$$

Let $\sigma = s+t,\ \tau = s-t$, so we integrate over

$$R = \{(\sigma,\tau) : \sigma+\tau \le 2A,\ \sigma,\tau \ge 0\}.$$

Since $s = \frac{1}{2}(\sigma+\tau),\ t = \frac{1}{2}(\sigma-\tau)$ the Jacobian determinant of the change of variables is

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

so

$$\frac{1}{2}\mathrm{B}(a,b)\Gamma(a+b) = \lim_{A\to\infty} \iint_R \frac{1}{2} e^{-(\sigma+\tau)}\sigma^{a-1}\tau^{b-1}\, d\tau\, d\sigma.$$

Thus

$$
\begin{aligned}
B(a,b)\Gamma(a+b) &= \int_0^\infty \int_0^\infty e^{-(\sigma+\tau)} \sigma^{a-1} \tau^{b-1} \, d\tau \, d\sigma \\
&= \int_0^\infty \int_0^\infty e^{-\sigma} \sigma^{a-1} e^{-\tau} \tau^{b-1} \, d\tau \, d\sigma \\
&= \left(\int_0^\infty e^{-\sigma} \sigma^{a-1} \, d\sigma \right) \left(\int_0^\infty e^{-\tau} \tau^{b-1} \, d\tau \right).
\end{aligned}
$$

Thus, we have…

$$
\mathrm{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}
$$